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Faculty of Exact Sciences and Sciences of Nature and Life

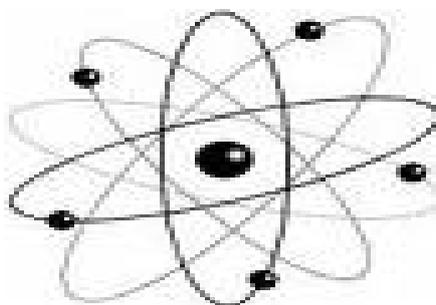


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Section of Physics

Specialty Physics of Photo-Voltaic



Final Dissertation in Master

Entitled:

**Spectrum of Multipolar Potential by  
Perturbation Theory in 2D Systems**

Presented by:

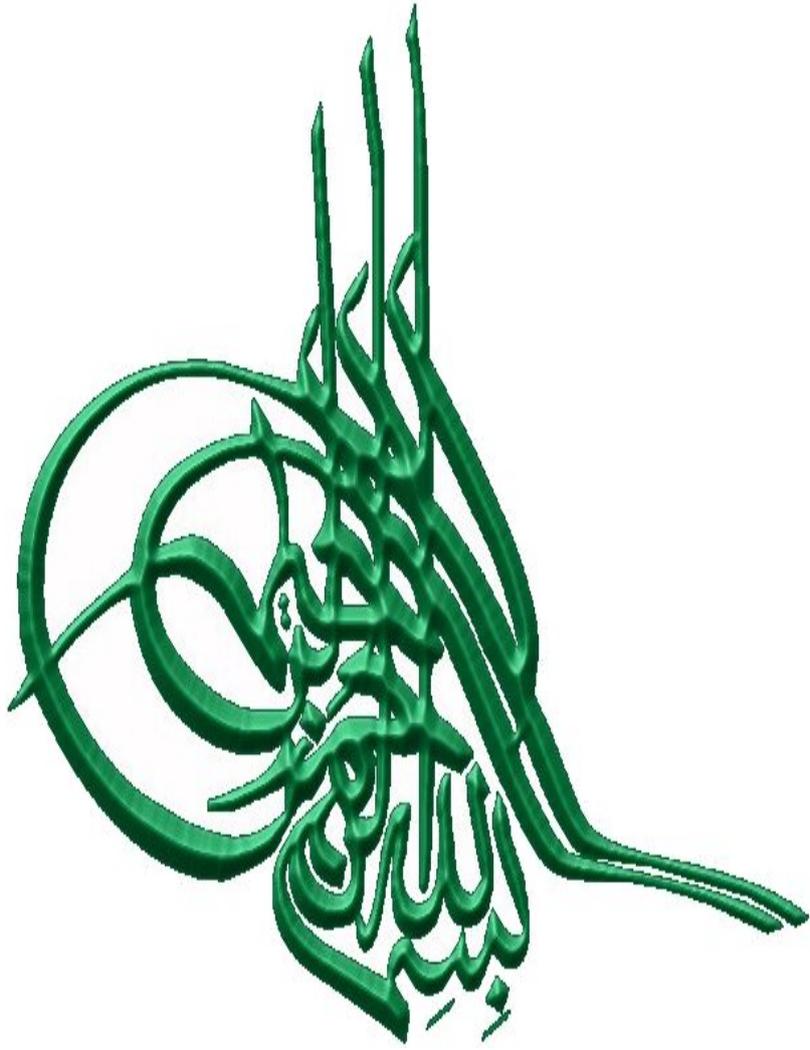
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*Academic Year*

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# Dedications

*In the Name of Allah, Most Gracious, Most Merciful All the Pries is due to*

*Allah alone, the sustainers of the entire world.*

*To my love my beloved father Houcine*

*To my precious mother Malika*

*To my Prather Yazid and his wife Souhila*

*To my Prather Zoubir and his wife Khaeola*

*To my Prather's: Tidjani, Elelmi, Ahmed and Rayane*

*To my sister's: Mofida, Hadjira, Nissa and Rania*

*To my aunt Saliha and her husband Ammar*

*To my friend's: Hadjer, Djihane, Khaeola Ch, Rayhana, Sabah, Imen, wanassa, Dihad ,Khaeola R, Sabrina ,Noura, Sara ,Meriem, Jasmine, Monira, Zoulikha, Asama, Zhour ,Manel ,Masouda ,Yousra.Mima,Firoze,Naziha.*

*To our prom physic Photo-Voltaïc*

*I dedicate this simple work*

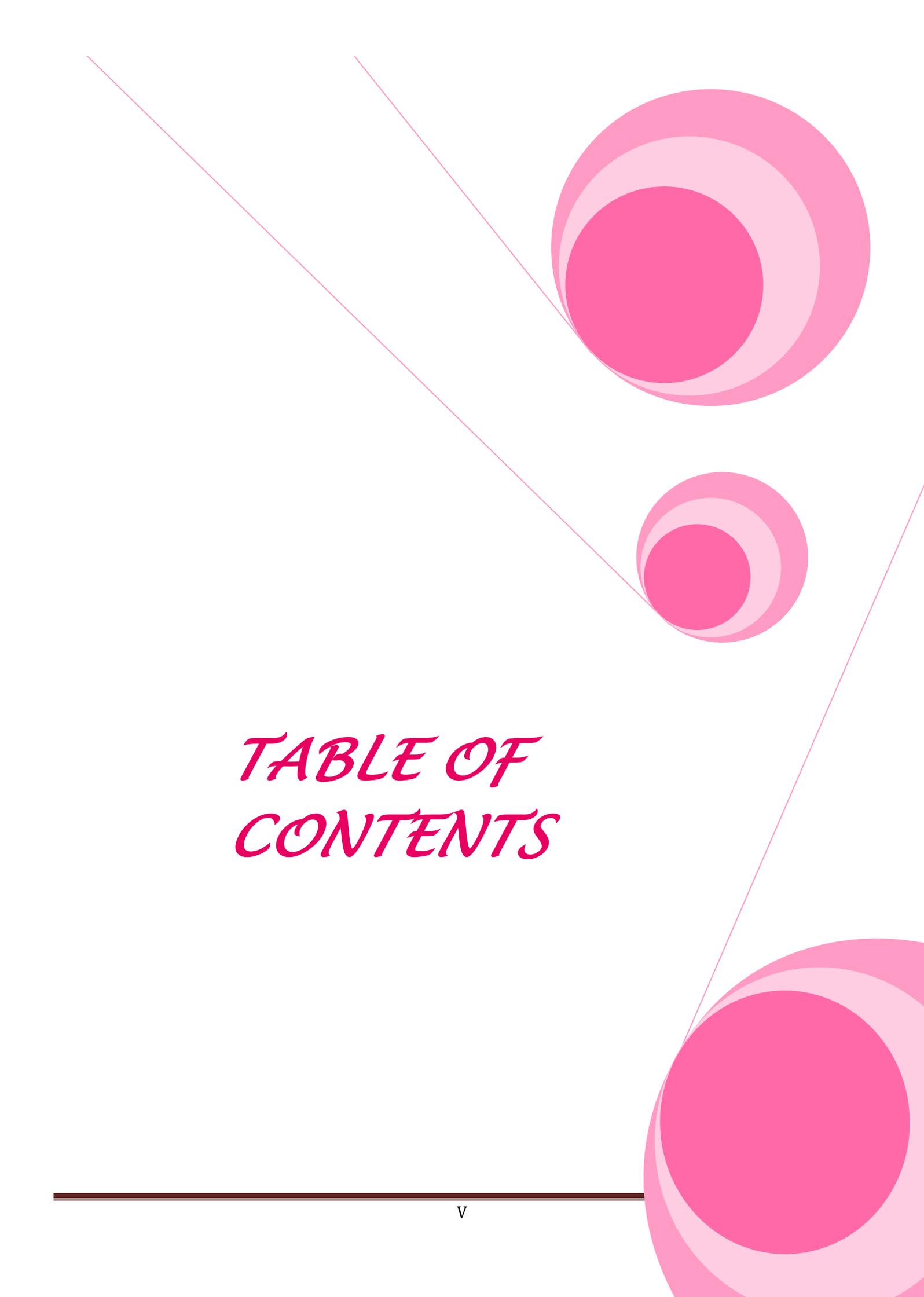
# Acknowledgement

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*My first duty of gratitude is to my thesis supervisor. For his understanding and offered many comments, he read my several revisions just to keep it in a better shape ,and for gave me his time and expertise. Shall I compare you to a candle that kept on enlightening the path of knowledge for me...you are more generous. **Mr. MOUMNI Mustapha**. You are a true teacher.*

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The page features a decorative graphic consisting of several overlapping pink circles of varying sizes and shades, arranged in a diagonal line from the top right towards the bottom right. Two thin pink lines intersect at the top left, forming a V-shape that frames the circles. The text 'TABLE OF CONTENTS' is centered in the middle of the page in a bold, italicized, pink font.

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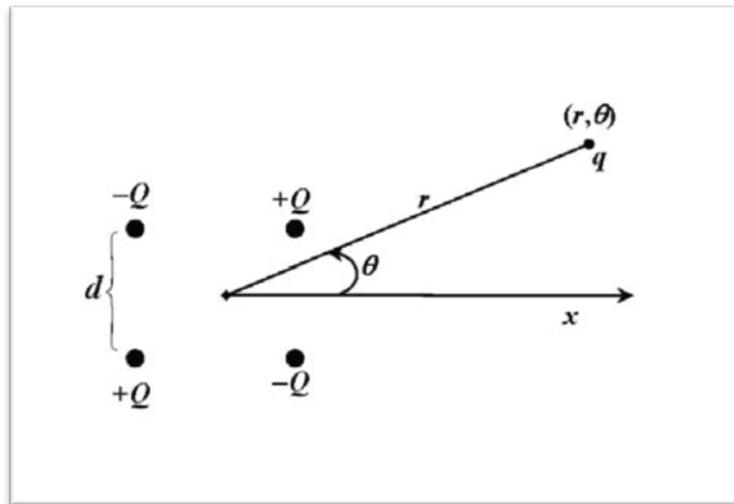
## Introduction

## Introduction

Quantum Mechanics is one the most important and the most difficult branch of physics [1]. Quantum mechanics (QM – also known as quantum physics, or quantum theory) is a branch of physics which deals with physical phenomena at nanoscopic scales where the action is on the order of the Planck constant [2].

The interaction of a charged particle with an electric dipole is a fundamental problem, which received a lot of attention since the early days of nuclear and molecular physics. This received a lot of attention in the physics literature for more than 60 years [3].

In Appendix A, we address the issue of the Coulomb problem in any dimension and calculate the potentials of a point electric dipole and quadrupole in two dimensions. The problem of a point charge in the field of an electric quadrupole in 2D (Fig. 1) is equivalent to the problem of a point charge in 3D moving in a plane normal to four parallel uniform line charges [3].



**Fig. 1:** An electric quadrupole configuration in 2D with  $d \ll r$ .

In 2007 A.D. Alhaidari obtains analytic solution of the time-independent Schrödinger equation in two dimensions for a charged particle moving in field of an electric quadrupole [3]. There is an other people who had studied the bound states of the Schrödinger equation (SE) for an electron confined to two dimensions and subject to a potential  $V(r, \theta) = D \cos \theta / r$  where  $D$  is the strength of the "dipole" potential (2012) [4]. And in 2013 C R Handy and D Vrinceanu examine the effectiveness of a new spectral method in solving the two dimensional dipole problem (DP). This deceptively simple problem has a long history of poorly approximated energy values, particularly for the ground state, until the recent work by Amore and Fernandez (AF, 2012 Phys. Rev. B: At. Mol. Opt. Phys. **45** 235004) [5].

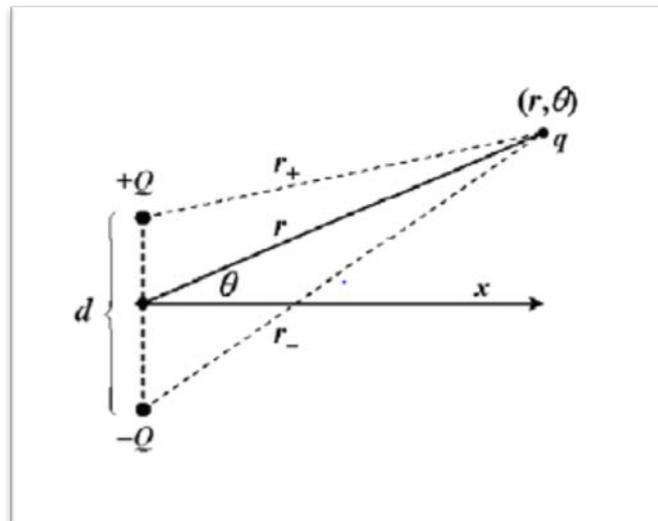
## Introduction

In this work we will calculate the exact potential: unperturbed potential (Coulomb potential) and perturbed potential (dipole and quadrupole parts). The corrections induced by multipolar terms that add to the monopole term (Coulomb energies) are calculated using stationary perturbation theory.

In chapter one, we will see multipole expansion to calculate the potentials of a point electric charge, a dipole and a quadrupole in two dimensions.

In the second chapter we write the solution of Schrödinger equation in 2D for the Coulomb potential. First of all, we will discuss, in a general view, on the Schrödinger equation in one-dimensional hydrogen atom and in two-dimensional hydrogen atom.

In the third chapter we will talk about perturbation theory exactly the time-independent perturbation theory.



**Fig. 2:** The electric dipole configuration in 2D with  $d \ll r$ .

**I-1- I.1.Introduction**

Recently, multipole expansion technique was applied in modern fields such as nanostructures, near-field diffraction, highly directional antennas. The special properties of toroid moments make them very interesting in constructing meta-materials with particular characteristics [6-7].

In this chapter we study the multipole expansion for calculate the potentials of a point electric monopole, dipole and quadrupole in two dimensions  $V(r, \theta)$  where  $\theta$  is the azimuthally angle in the  $x - y$  plane

**I.2.Definition of Multipole Expansion**

“...A multipole expansion is a series expansion of the effect produced by a given system in terms of an expansion parameter which becomes small as the distance away from the system increases”[8].

The subject of multipole expansion of the electromagnetic field is treated in many textbooks on classical electrodynamics. Nevertheless, the correct relation between the radiation source and the radiation field was explained only together with the introduction of the class of toroid moments and distributions [9]. Previously, we have come to a conclusion that very far from a charge distribution, this charge distribution “looked” like a point charge (e.g. the field and the potential) with charge magnitude equal to the sum of charges in the original charge distribution [10]. Consider an extended charge distribution  $q(r')$ . We wish to find the electrostatic potential due to this charge distribution at a given point  $r$ , we assume that this point is at a large distance from the charge distribution, that is if  $r'$  varies over the charge distribution then  $r \gg r'$  the coulomb potential for a charge distribution is given by[11]:

$$V(r) = k \sum \frac{q}{r_i} \quad \text{where} \quad k = \frac{1}{4\pi\epsilon r} \tag{I.1}$$

In cases where there is no  $\phi$  dependence the Laplace solution in polar coordinates is:

$$\sum_l \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \tag{I.2}$$

The multipole expansion of  $\|\vec{r} - \vec{r}'\|^{-1}$  shows the relation and demonstrates that at long distances ( $r \gg r'$ ), we can expand the potential as a multipole, i.e. Eq. (2), with  $A_l = 0$ . More than that, we can actually get general expressions for the coefficients  $B_l$  in terms of  $(r')$ . First let see that Eq. (1) and (2) are related, but doing a systematic expansion of  $\|\vec{r} - \vec{r}'\|^{-1}$ , in the case where  $r'/r < 1$  [12].

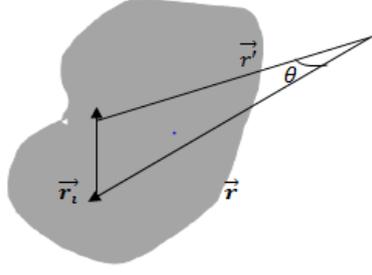


Fig.3. Extended distribution of charges

We write [12]:

$$\frac{1}{|r-r'|} = \frac{1}{[r^2+r'^2-2\vec{r}\cdot\vec{r}']^{1/2}} = \frac{1}{r} \left[ 1 + \left(\frac{r'}{r}\right)^2 - \frac{2rr'}{r^2} \right]^{-1/2} \quad (\text{I.3})$$

We use  $x = \left(\frac{r'}{r}\right)^2 - \frac{2rr'}{r^2}$ , and make a Taylor expansion of  $(1+x)^{-n}$ , i.e. use:

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \dots \quad (\text{I.4})$$

We have [13]:

$$\vec{r}_i = \sqrt{(\vec{r} - \vec{r}')^2} = \sqrt{r^2 \left( \vec{u}_r - \frac{r'}{r} \vec{u}' \right)^2} \quad (\text{I.5})$$

$$\vec{r}_i = r \left( 1 - 2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^{1/2} \quad (\text{I.6})$$

$$\frac{1}{\vec{r}_i} = \frac{1}{r} \left( 1 - 2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^{-1/2} \quad (\text{I.7})$$

We use expansion of Taylor:

$$(1+x)^{-1/2} = 1 + \frac{-1/2 x}{1!} + \frac{-1/2(-1/2-1)x^2}{2!} + \dots \quad (\text{I.8})$$

$$(1+x)^{-1/2} = 1 + \frac{-1}{2} x + \frac{3}{8} x^2 + \dots \quad (\text{I.9})$$

$$\left( 1 - 2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^{-1/2} = 1 - \frac{1}{2} \left( -2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right) + \frac{3}{8} \left( -2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^2 + \dots \quad (\text{I.10})$$

$$\left( 1 - 2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^{-1/2} = 1 + \frac{r'}{r} \cos \theta - \frac{1}{2} \frac{r'^2}{r^2} + \frac{3}{8} \left( 4 \frac{r'^3}{r^3} \cos^2 \theta + \frac{r'^4}{r^4} - 4 \frac{r'}{r} \cos \theta \frac{r'^2}{r^2} \right) \quad (\text{I.11})$$

$$r' \ll r \Rightarrow \frac{r'^3}{r^3} \approx 0 \Rightarrow \left( 1 - 2 \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \right)^{-1/2} = 1 + \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \left( \frac{3 \cos^2 \theta - 1}{2} \right) \quad (\text{I.12})$$

$$\frac{1}{\|\vec{r}_i\|} = \frac{1}{r} \left[ \underbrace{1}_{\text{monopole}} + \underbrace{\frac{r'}{r} \cos \theta}_{\text{dipole}} + \underbrace{\frac{r'^2}{r^2} \left( \frac{3 \cos^2 \theta - 1}{2} \right)}_{\text{quadrupole}} \right] \quad (\text{I.13})$$

$$V(r) = k \sum \frac{q}{r_i} = k \int_{v'} \frac{q(r')}{r_i} dv' = \frac{k}{r} \int_{v'} q(r') \left( 1 + \frac{r'}{r} \cos \theta + \frac{r'^2}{r^2} \left( \frac{3 \cos^2 \theta - 1}{2} \right) \right) dv' \quad (\text{I.14})$$

$$V_{mono} = k \frac{1}{r} \int_{v'} q(r') dv' \quad (\text{I.15})$$

$$V_{dip} = k \frac{1}{r^2} \int_{v'} q(r') r' \cos \theta dv' \quad (\text{I.16})$$

$$V_{qua} = k \frac{1}{r^3} \int_{v'} q(r') r'^2 \left( \frac{3 \cos^2 \theta - 1}{2} \right) dv' [14]. \quad (\text{I.17})$$

### I.2.1. Definition of Monopole:

A point charge is a hypothetical charge located at a Single point in space. While an electron can for many purposes be considered a point charge, its size can be characterized by length scale known as the electron radius [15].

The electric potential of an electric charge monopole is given by:

$$V_{mono} = k \frac{1}{r} \int_{v'} q(r') dv' [14]. \quad (\text{I.18})$$

### II.3. Definition of Dipole:

A physical dipole consists of two equal and opposite point charges. Its field at large distances (distances large in comparison to the separation of the poles) depends almost entirely on the dipole moment. A point (electric) dipole is the limit obtained by letting the separation tends to 0 while keeping the dipole moment fixed. The field of a point dipole has a particularly simple form, and the 1<sup>st</sup> order term in the multipole expansion is precisely the point dipole field [16]. The electric potential of an electric charge dipole is given by[14]:

$$V_{dip} = k \frac{1}{r^2} \int_{v'} q(r') r' \cos \theta dv'. \quad (\text{I.19})$$

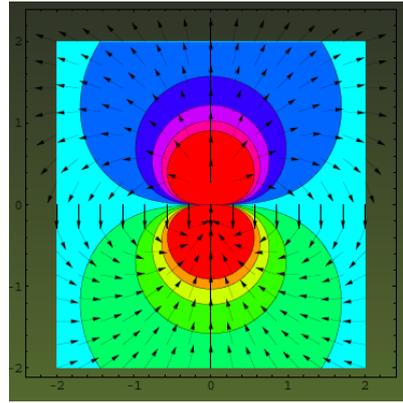


Fig.4. Dipole Potential and Field

**II.4. Definition of Quadruple:**

A quadrupole is one of a sequence of configurations of electric charge or current or gravitational mass that can exist in ideal form, but it is usually just part of a multipole expansion of a more complex structure reflecting various orders of complexity [17].

The simplest example of an electric quadrupole consists of alternating positive and negative charges, arranged on the corners of a square. The monopole moment (just the total charge) of this arrangement is zero. Similarly, the dipole moment is zero, but the quadrupole moment of the arrangement in the diagram cannot be reduced to zero [16]. The electric potential of an electric charge quadrupole is given by [14]:

$$V_{qua} = k \frac{1}{r^3} \int_{v'} q(r') r'^2 \left( \frac{3 \cos^2 \theta - 1}{2} \right) dv' \tag{I.20}$$

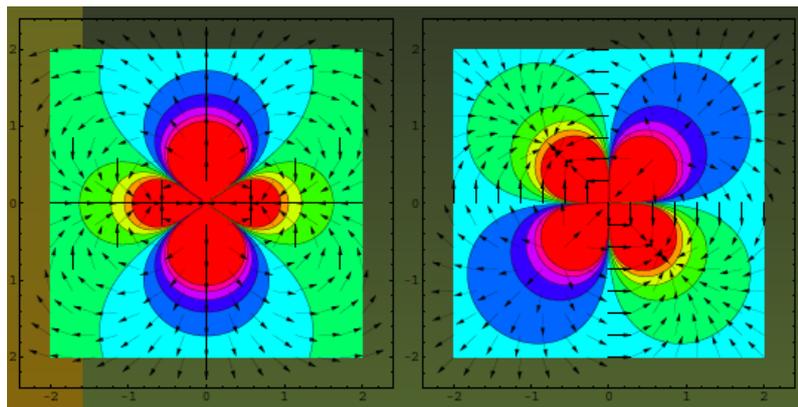


Fig.5. Quadrupole Potential and Field

## II.1.Introduction

One and two dimensional reductions of the three-dimensional Schrödinger equation of the hydrogen atom are considered. These reductions are carried out from the point of view of the two Common sets of space coordinates: Cartesian and spherical. The resulting systems have features that relate more readily to the old quantum theory models of Bohr and Summerfield than the general three-dimensional hydrogen atom. Furthermore, the consideration yield interesting insights into the quantum mechanics of the hydrogen atom and may serve as helpful intermediary preparation, in introductory presentation of the subject, for the unreduced three-dimensional case [19].

## II.2.Generality of Schrödinger equation

It is convenient for our purposes to begin with the time-independent Schrödinger equation of a particle in a potential. This equation, when expressed in standard notation, reads as follows [19]:

$$\frac{-\hbar^2}{2m} \nabla^2 \psi(x, y, z) + V(x, y, z) \psi(x, y, z) = E \psi(x, y, z) \quad \text{II.1}$$

One usually illustrates this equation, in an introductory presentation of the subject, in terms of Cartesian one-dimensional systems have simple potentials, e.g. the square well and the rectangular potential barrier. The important case of hydrogen atom, however, is dealt with from the outset in three dimensions. Yet, one- and two-dimensional reductions of the hydrogen atom system can be studied towards illustration of Schrödinger's Eq. (1), and towards building up to the three-dimensional case. Additionally, the study of the resulting reduced systems leads to a better appreciation of the relation between the old quantum theory and Schrödinger's theory[19].

### II.2.1.One-Dimensional Hydrogen Atom

The dimensional reduction of Schrödinger's equation depends on the set of coordinates in which it is expressed. Attention will be confined to the hydrogen atom in the two Common sets: Cartesian and spherical coordinates, the potential  $V$  in Eq. (1) is then the electrostatic Coulomb potential of the electron in the field of the proton. For the one-dimensional reduction from the proton of view of Cartesian coordinates, there is only one distinct reduction. This corresponds to any one of  $x, y$ , and  $z$  being variable while the remaining two are held constant. Let the chosen variable be denoted by  $x$ . The resulting reduced equation from Eq. (1) [19].

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} - \frac{e^2}{4\pi\epsilon_0 |x|} \psi(x) = E \psi(x) \quad \text{II.1}$$

### II.2.2. Two-Dimensional Hydrogen Atom

As in the one-dimensional case, the reduction of the hydrogen atom Schrödinger equation to two dimensions will be carried out from the point of view of the two Common sets of coordinates: Cartesian set of coordinates  $x, y$ , and  $z$ , there is only one distinct reduction. This corresponds to any pair of the three coordinates being variable while the third is held constant. Choosing  $x$  and  $y$  to be the pair, Eq. (II.1) becomes [19].

$$\frac{-\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y, z) - \frac{e^2}{4\pi\epsilon_0 \sqrt{x^2 + y^2}} \psi(x, y) = E \psi(x, y) \quad \text{II.2}$$

Equation (3), when expressed in terms of the transformed coordinates  $r$  and  $\theta$ , has the form

$$\left[ -\frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) - \frac{e^2}{4\pi\epsilon_0 \sqrt{x^2 + y^2}} \right] \psi(r, \theta) = E \psi(r, \theta) \quad \text{II.3}$$

### II.3. Analytic Solution of a Two-Dimensional Hydrogen Atom

The hydrogen atom is the name given to the system composed of an electron with mass  $m$ , and charge  $-e$ —and a positively charged nucleus ( $Ze$ ) located at the origin of the coordinate system. The central force between the electron and the nucleus is determined by the attractive Coulomb potential function [20].

$$V(r) = -\frac{Ze^2}{r} \quad \text{II.4}$$

The three-dimensional (3D) hydrogen atom played a central role in the early formulation and development of quantum mechanics and is now part of the standard curriculum in modern undergraduate physics. If the motion of the electron around the nucleus is constrained in a plane by certain boundary conditions, then such a system is called the two-dimensional hydrogen atom. We would like to point out that "2D" in the name "2D hydrogen atom" only emphasizes that the motion of the electron around a positive point charge (not a line charge) is constrained in a plane. This system is not 2D in a strict sense that all fields including electromagnetic fields, photon emission, angular momentum, and spin are not confined to a plane [20].

The eigenstates of a 2D hydrogen atom are described by the Schrödinger equation, in polar coordinates [20].

$$\left[ -\frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + V(r) \right] \psi(r, \theta) = E \psi(r, \theta) \quad \text{II.5}$$

$$\left[ -\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{2m}{\hbar^2} V(r) \right] \psi(r, \theta) = \frac{2m}{\hbar^2} E \psi(r, \theta) \quad \text{II.6}$$

We multiply this equation by  $r^2$ :

$$\left[ -\left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{2m}{\hbar^2} r^2 V(r) \right] \psi(r, \theta) = r^2 \frac{2m}{\hbar^2} E \psi \quad \text{II.7}$$

$$\left[ \left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{2m}{\hbar^2} r^2 (E - V(r)) \right] \psi(r, \theta) = 0 \quad \text{II.8}$$

Using separation of variables  $\psi(r, \theta) = R(r)\Phi(\theta)$ .

We have:

$$\left[ \left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{2m}{\hbar^2} r^2 (E - V(r)) \right] R(r)\Phi(\theta) = 0 \quad \text{II.9}$$

$$\left[ \left( r^2 \Phi(\theta) \frac{\partial^2 R(r)}{\partial r^2} + r \Phi(\theta) \frac{\partial R(r)}{\partial r} + R(r) \frac{\partial^2 \Phi(\theta)}{\partial \theta^2} \right) + \frac{2m r^2}{\hbar^2} (E - V(r)) \right] R(r)\Phi(\theta) = 0 \quad \text{II.10}$$

We obtain two separable equations called radial part and angular part of the total solution.

For the angular solution, we have:

$$\frac{\partial^2 \Phi(\theta)}{\partial \theta^2} + \frac{2mr^2}{\hbar^2} (E - V(r)) \Phi(\theta) = 0 \quad \text{II.11}$$

The solution is:

$$\Phi(\theta) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{im\theta} ; m = 0, \pm 1, \pm 2 \quad \text{II.12}$$

And the radial part writes:

$$\frac{d^2}{dr^2} R(r) + \frac{1}{r} \frac{d}{dr} R(r) + \left[ \frac{2m_e}{\hbar^2} (E + v(r)) - \frac{m^2}{r^2} \right] R(r) = 0 \quad \text{II.13}$$

For simplicity, we choose the Gaussian units  $\hbar = 2m = e^2/2$  and for negative energy, we use  $E = -q_0^2$ ,

$$\frac{d^2}{dr^2} R(r) + \frac{1}{r} \frac{d}{dr} R(r) + \left( \frac{2}{r} - q_0^2 - \frac{m^2}{r^2} \right) R(r) = 0 \quad \text{II.14}$$

The solution is:

$$R(x) = x^{|m|} e^{-x/2} G(x) \quad \text{II.15}$$

The equation then becomes:

$$x \frac{d^2 G}{dx^2} + [(2|m| + 1) - x] \frac{dG}{dx} - \left(-N + |m| + \frac{1}{2}\right) G = 0 \quad \text{II.16}$$

The solution of Eq which is regular at  $x = 0$ , is the confluent hypergeometric function

$$G(x) = {}_1F_1\left(-N + |m| + \frac{1}{2}, 2|m| + 1, x\right) \quad \text{II.17}$$

A solution which satisfies the condition at infinity is obtained only for negative integral (or zero) values of  $-N + |m| + \frac{1}{2}$ , then the function reduces to a finite polynomial; otherwise it diverges at  $r \rightarrow \infty$ . Thus we arrive

We have:  $N = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ , and let  $n = N + \frac{1}{2} = 1, 2, 3, \dots$

Where  $n$ , is an integer called the principal quantum number. For a given  $n$ ,  $|m|$  can take the values :

$$|m| = 0, 1, 2, 3, \dots, n - 1 \quad \text{II.18}$$

$$R(x) = x^{|m|} e^{-x/2} G(x) \quad \text{II.19}$$

$$R(x) = x^{|m|} e^{-x/2} {}_1F_1\left(-N + |m| + \frac{1}{2}, 2|m| + 1, x\right) \quad \text{II.20}$$

The normalized radial eigen function is given by [20]:

$$R_{nl} = \frac{\beta_n}{(2|m|)!} \left[ \frac{(n+|m|-1)!}{(2n-1)(n-|m|-1)!} \right]^{1/2} (\beta_n r)^{|m|} \exp(-\beta_n r/2) {}_1F_1(-n+|m|+1, 2|m|+1, \beta_n r) \quad \text{II.21}$$

Where  $\beta_n = \frac{2Z}{n-\frac{1}{2}} \frac{m_e e^2}{\hbar^2}$

So we give the first few function  $R_{nl}$  explicitly:

$$R_{10} = \beta_1 e^{-\frac{\beta_1 r}{2}}, \quad \text{II.22}$$

$$R_{20} = \left( \frac{\beta_2}{3^{1/2}} \right) (1 - \beta_2 r) e^{-\beta_2 r/2}, \quad \text{II.23}$$

$$R_{21} = \left( \beta_2^2 / 6^{1/2} \right) r e^{-\beta_2 r/2}, \dots \quad \text{II.24}$$

We have  $\psi(r, \theta) = R(r) \Phi(\theta)$

$$\psi(r, \theta) = \frac{\beta_n}{(2|m|)!} \left[ \frac{(n+|m|-1)!}{(2n-1)(n-|m|-1)!} \right]^{1/2} (\beta_n r)^{|m|} \exp\left(\frac{-\beta_n r}{2}\right) {}_1F_1(-n+|m|, -1, 2|m|+1, \beta_n r) \frac{1}{(2\pi)^{1/2}} e^{im\theta} \quad \text{II.25}$$

This ends this section where we have writing the global expression of the Schrödinger equation for 2D hydrogen atom.

Perturbation theory is an important tool for describing real quantum systems. As it turns out to be very difficult to find exact solutions to Schrödinger for Hamiltonians of even moderate complexity [21].

### III.1. Perturbations Theory

Generally finding the exact solution of most interesting problems is impossible or at least so difficult that it is not practical to obtain it. Sometimes it is possible with a bit of work to obtain a so-called asymptotic series approximation of the solution that gives a good approximation to the solution. In what follows we hope to provide, mostly by way of numerous examples, some insight into this important branch of mathematics which we refer to as perturbation theory. Our goal is to examine several examples including the asymptotic analysis of solutions of algebraic, transcendental, differential equations and the evaluation of integrals.

In quantum mechanics, perturbation theory is a set of approximation schemes directly related to mathematical perturbation for describing a complicated quantum system in terms of a simpler one. The idea is to start with a simple system for which a mathematical solution is known, and add an additional "perturbing" Hamiltonian representing a weak disturbance to the system. If the disturbance is not too large, the various physical quantities associated with the perturbed system (e.g. its energy levels and eigenstates) can, from considerations of continuity, be expressed as 'corrections' to those of the simple system. These corrections, being 'small' compared to the size of the quantities themselves, can be calculated using approximate methods such as asymptotic series. The complicated system can therefore be studied based on knowledge of the simpler one [22].

### III.2. Time-Independent Perturbation Theory

Time-independent perturbation theory is one of two categories of perturbation theory. In time-independent perturbation theory the perturbation Hamiltonian is static (i.e., possesses no time dependence). Time-independent perturbation theory was presented by Erwin Schrödinger in a 1926 paper [23]. Shortly after he produced his theories in wave mechanics. In this paper Schrödinger referred to earlier work of Lord Rayleigh [24], who investigated harmonic vibrations of a string perturbed by small inhomogeneities. This is why this perturbation theory is often referred to as Rayleigh Schrödinger perturbation theory.

$$H = H_0 + V \tag{II.26}$$

And now we go to first order corrections.

### III.2.1. First Order Corrections

We begin with an unperturbed Hamiltonian  $H_0$ , which is also assumed to have no time dependence. It has known energy levels and eigenstates, arising from the time-independent Schrödinger equation [24]:

$$H_0|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle, n=1, 2, 3, \dots \tag{III.1}$$

For simplicity, we have assumed that the energies are discrete. The (0) superscripts denote that these quantities are associated with the unperturbed system. Note the use of Bra-ket notation. We now introduce a perturbation to the Hamiltonian. Let  $V$  be a Hamiltonian representing a weak physical disturbance, such as a potential energy produced by an external field. (Thus,  $V$  is formally a Hermitian operator) Let  $\lambda$  be a dimensionless parameter that can take on values ranging continuously from 0 (no perturbation) to 1 (the full perturbation). The perturbed Hamiltonian is [24] :

$$H = H_0 + \lambda V \text{ where } \lambda = 0 \text{ or } 1 \tag{III.2}$$

The energy levels and eigenstates of the perturbed Hamiltonian are again given by the Schrödinger equation:

$$(H_0 + \lambda V)|n\rangle = E_n|n\rangle. \tag{III.3}$$

Our goal is to express  $E_n$  and  $|n\rangle$  in terms of the energy levels and eigenstates of the old Hamiltonian. If the perturbation is sufficiently weak, we can write them as power series in  $\lambda$ :

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \tag{III.4}$$

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \dots \tag{III.5}$$

Where  $E_n^{(k)} = \frac{1}{k!} \frac{d^k E_n}{d\lambda^k}$  and  $|n^{(k)}\rangle = \frac{1}{k!} \frac{d^k |n\rangle}{d\lambda^k}$  (III.6)

When  $\lambda = 0$ , these reduce to the unperturbed values, which are the first term in each series. Since the perturbation is weak, the energy levels and eigenstates should not deviate too much

from their unperturbed values, and the terms should rapidly become smaller as we go to higher order.

Substituting the power series expansion into the Schrödinger equation, we obtain:

$$(H_0 + \lambda V)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \dots) = (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \dots) \quad (\text{III.7})$$

Expanding this equation and comparing coefficients of each power of  $\lambda$  results in an infinite series of simultaneous equations. The zeroth-order equation is simply the Schrödinger equation for the unperturbed system. The first-order equation is

$$H_0|n^{(1)}\rangle + V|n^{(0)}\rangle = E_n^{(0)}|n^{(1)}\rangle + E_n^{(1)}|n^{(0)}\rangle \quad (\text{III.8})$$

Operating through by  $\langle n^{(0)}|$ , the first term on the left-hand side cancels with the first term on the right-hand side. (Recall, the unperturbed Hamiltonian is Hermitian). This leads to the first-order energy shift [24]:

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle \quad (\text{III.9})$$

### III.2.2. Second-Order Correction

We can find the higher-order deviations by a similar procedure, though the calculations become quite tedious with our current formulation. Our normalization prescription gives that

$$2\langle n^{(0)}|n^{(2)}\rangle + \langle n^{(1)}|n^{(1)}\rangle = 0 \quad (\text{III.10})$$

Up to second order, the expressions for the energies and (normalized) eigenstates are [24]:

$$E_n = E_n^{(0)} + \lambda \langle n^{(0)}|V|n^{(0)}\rangle + \lambda^2 \sum_{k \neq n} \frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} + O(\lambda^3) \quad (\text{III.11})$$

$|n\rangle =$  (III.12)

$$\begin{aligned}
 & |n^{(0)}\rangle + \lambda \sum_{k \neq n} |k^{(0)}\rangle \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}} + \lambda^2 \sum_{k \neq n} \sum_{l \neq n} \frac{\langle k^{(0)}|V|l^{(0)}\rangle \langle l^{(0)}|V|n^{(0)}\rangle}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_l^{(0)})} - \\
 & \lambda^2 \sum_{k \neq n} |k^{(0)}\rangle \frac{\langle n^{(0)}|V|n^{(0)}\rangle \langle k^{(0)}|V|n^{(0)}\rangle}{(E_n^{(0)} - E_k^{(0)})^2} - \frac{1}{2} \lambda^2 |n^{(0)}\rangle \sum_{k \neq n} \frac{\langle n^{(0)}|V|k^{(0)}\rangle \langle k^{(0)}|V|n^{(0)}\rangle}{(E_n^{(0)} - E_k^{(0)})^2} + O(\lambda^3)
 \end{aligned}$$

**Results and Discussion:**

We present in this section the results of studying the multipole potential (monopole, dipole and quadrupole contributions). We have calculated the exact potential: unperturbed potential (Coulomb potential) and perturbed potential (dipole and quadrupole parts). The corrections added to Coulomb energies are calculated using stationary perturbation theory.

We use the software Mathematica for the calculation.

➤ **For The First Level :**

We have:  $z = 1 ; n = 1 ; m = 0 ; a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ .

- **The Monopole Potential:** is given with the following equation

$$V_c = -\frac{Zq^2}{4\pi\epsilon_0 r} \tag{IV.1}$$

So the energy of Coulomb potential is ( $z = 1 ; n = 1$ )

$$E_C = -\frac{z^2 q^4 m_e}{2(4\pi\epsilon_0)^2 \hbar^2 n^2} = -2.180 \times 10^{-18} \frac{\text{Kilogram Meter}^2}{\text{Second}^2} \tag{IV.2}$$

We plot the monopole potential

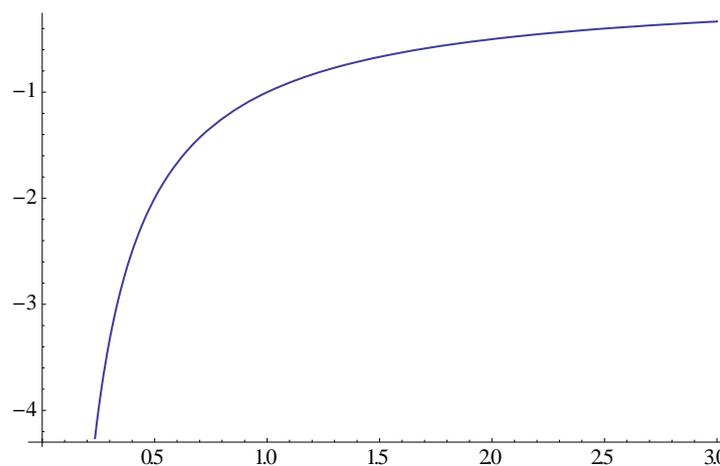


Fig.6. Monopole (Coulomb) potential

• **The Dipole Potential:**

The Confirmation of  $\int_0^{2\pi} \int_0^\infty \psi \frac{\cos[\theta]}{r^2} \psi^* r dr d\theta = 0$  in all situations:

$$\begin{aligned}
 \langle \psi | V_D | \psi^* \rangle &= \int_0^{2\pi} \int_0^\infty r dr d\theta \left( \frac{\left(\frac{2z}{n-\frac{1}{2}}\right)^{\frac{1}{a}}}{(2|m|)!} \sqrt{\frac{(n+|m|-1)}{((2n-1)(n-|m|-1))!}} \frac{1}{\sqrt{2\pi}} \left(|m| \frac{r}{a}\right)^{|m|} e^{-\frac{2z}{2} \frac{r}{\left(n-\frac{1}{2}\right)^a}} \times \quad \text{IV.3} \\
 &\quad {}_1F_1 \left[ -n + |m| + 1, 2|m| + 1, \frac{2z}{\left(n-\frac{1}{2}\right)^a} \frac{r}{a} \right] e^{im\theta} \times \frac{\cos[\theta]}{r^2} \times \\
 &\quad \frac{\left(\frac{2z}{n'-\frac{1}{2}}\right)^{\frac{1}{a}}}{(2|m'|)!} \sqrt{\frac{(n'+|m'|-1)}{((2n'-1)(n'-|m'|-1))!}} \frac{1}{\sqrt{2\pi}} \left(|m'| \frac{r}{a}\right)^{|m'|} e^{-\frac{2z}{2} \frac{r}{\left(n'-\frac{1}{2}\right)^a}} \times {}_1F_1 \left[ -n' + |m'| + 1, 2|m'| + \right. \\
 &\quad \left. 1, \frac{2z}{\left(n'-\frac{1}{2}\right)^a} \frac{r}{a} \right] e^{-im'\theta} \Big) \\
 &= \frac{\left(\frac{2z}{n-\frac{1}{2}}\right)^{\frac{1}{a}}}{(2|m|)!} \sqrt{\frac{(n+|m|-1)}{((2n-1)(n-|m|-1))!}} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{\left(\frac{2z}{n'-\frac{1}{2}}\right)^{\frac{1}{a}}}{(2|m'|)!} \sqrt{\frac{(n'+|m'|-1)}{((2n'-1)(n'-|m'|-1))!}} \int_0^\infty \left(|m| \frac{r}{a}\right)^{|m|} e^{-\frac{2z}{2} \frac{r}{\left(n-\frac{1}{2}\right)^a}} \times \\
 &\quad {}_1F_1 \left[ -n + |m| + 1, 2|m| + 1, \frac{2z}{\left(n-\frac{1}{2}\right)^a} \frac{r}{a} \right] \left(|m'| \frac{r}{a}\right)^{|m'|} e^{-\frac{2z}{2} \frac{r}{\left(n'-\frac{1}{2}\right)^a}} \times {}_1F_1 \left[ -n' + |m'| + 1, 2|m'| + \right. \\
 &\quad \left. 1, \frac{2z}{\left(n'-\frac{1}{2}\right)^a} \frac{r}{a} \right] \frac{1}{r^2} r dr \int_0^{2\pi} \cos[\theta] \underbrace{e^{im\theta} \times e^{-im'\theta}}_{=1} d\theta
 \end{aligned}$$

Because  $m = m'$  so  $e^{im\theta} \times e^{-im'\theta} = 1$  and we have  $\int_0^{2\pi} \cos[\theta] d\theta = 0$  so

$$\int_0^{2\pi} \int_0^\infty \psi \frac{\cos[\theta]}{r^2} \psi^* r dr d\theta = 0 \text{ in all situations}$$

The first order correction is always zero in all situations  $\langle \psi | V_D | \psi^* \rangle = 0$ ; so we go to second order correction  $\sum_{n' \neq n} \frac{|\langle \psi | V_D | \psi^* \rangle|^2}{E_{n'} - E_n}$ .

We compute the corrections for different levels and we use these abbreviations:

$$E_n = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2 n^2 h^2}; E_{n'} = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2 n'^2 h^2}$$

We have  $D = qd$  where  $d$  is the distance between the charges.

We have the following abbreviation for the integration:

$$\begin{aligned} \langle \psi_{n,l,m} | V | \psi_{n',l',m'}^* \rangle &= \int_0^{2\pi} \int_0^\infty \psi_{n,l,m} \times V \times \psi_{n',l',m'}^* dr d\theta \quad \text{IV.4} \\ &= \int_0^{2\pi} \int_0^\infty r dr d\theta \left( \frac{\frac{2z}{(n-\frac{1}{2})a}}{(2|m|)!} \sqrt{\frac{(n+|m|-1)}{((2n-1)(n-|m|-1)!)} \frac{1}{\sqrt{2\pi}}} \left( |m| \frac{r}{a} \right)^{|m|} e^{-\frac{2z}{2} \frac{r}{(n-\frac{1}{2})a}} \right. \\ &\quad \times {}_1F_1 \left[ -n+|m|+1, 2|m|+1, \frac{2z}{(n-\frac{1}{2})a} r \right] e^{im\theta} \times V \\ &\quad \times \frac{\frac{2z}{(n'-\frac{1}{2})a}}{(2|m'|)!} \sqrt{\frac{(n'+|m'|-1)}{((2n'-1)(n'-|m'|-1)!)} \frac{1}{\sqrt{2\pi}}} \left( |m'| \frac{r}{a} \right)^{|m'|} e^{-\frac{2z}{2} \frac{r}{(n'-\frac{1}{2})a}} \\ &\quad \left. \times {}_1F_1 \left[ -n'+|m'|+1, 2|m'|+1, \frac{2z}{(n'-\frac{1}{2})a} r \right] e^{-im'\theta} \right) \end{aligned}$$

We have for the first correction the following value:

$$\sum_{n=2}^{10} \sum_{m=1}^{n-1} \left( \left| \langle \psi_{1,0,0} | \frac{D}{4\pi\epsilon_0} \frac{\cos[\theta]}{r^2} | \psi_{n,l,m}^* \rangle \right|^2 / (E_1 - E_n) \right) = 0 \quad \text{IV.5}$$

- **The Quadrupole Potential:**

The quadrupole potential is given by following relation:

$$\frac{D^2}{4\pi\epsilon_0} \frac{3\cos^2[\theta]-1}{r^3} \quad \text{IV.6}$$

And we have for its correction the following value:

$$\left\langle \psi_{1,0,0} \left| \frac{D^2 \cos^2[\theta] - 1}{4\pi\epsilon_0 r^3} \right| \psi_{1,0,0}^* \right\rangle = 0 \quad \text{IV.7}$$

We have found from the fact that  $\langle \psi | V_D | \psi^* \rangle = 0$  for all states and from IV-4 and IV-6, that for the first level, there is no corrections coming from the dipole and quadrupole contributions of the potential; so the total energy is:

$$E_T = E_C + E_D + E_Q = E_C = -2.180 \times 10^{-18} \frac{\text{Kilogram Meter}^2}{\text{Second}^2} \quad \text{IV.8}$$

➤ **For The Second Level :**  $z = 1$  ;  $n = 2$  we will follow the same steps:

- **The Monopole Potential:** Does not change and is given by equation IV-1; So the energy of the second level is given by:

$$E_C = -\frac{Z^2 q^4 m_e}{2(4\pi\epsilon_0)^2 \hbar^2 n^2} = -5.450 \times 10^{-19} \frac{\text{Kilogram Meter}^2}{\text{Second}^2} \quad \text{IV.9}$$

- **The Dipole Potential:** The first order always equals zero so we have calculated the second order which gives us a value

$$\begin{aligned} & \left( \left| \left\langle \psi_{2,0,0} \left| \frac{D \cos[\theta]}{4\pi\epsilon_0 r^2} \right| \psi_{1,0,0}^* \right\rangle \right|^2 / (E_2 - E_1) \right) \\ & + \sum_{n=3}^{10} \sum_{m=1}^{n-1} \left( \left| \left\langle \psi_{2,0,0} \left| \frac{D \cos[\theta]}{4\pi\epsilon_0 r^2} \right| \psi_{n,l,m}^* \right\rangle \right|^2 / (E_2 - E_n) \right) \\ & = 3.748 \times 10^{40} D^2 \frac{\text{Kilogram}}{\text{Coulomb}^2 \text{Second}^2} \quad \text{IV.10} \end{aligned}$$

We plot the dipole potential

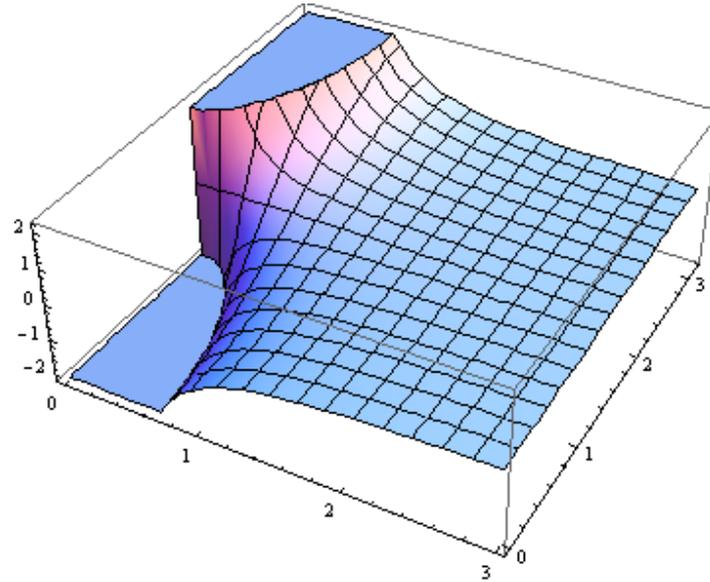


Fig.7. Dipole potential  $V(r, \theta) = \cos \theta / r^2$

- **The Quadrupole Potential :** the first correction is given with  $\langle \psi | V | \psi \rangle$

$$\left\langle \psi_{2,0,0} \left| \frac{D^2}{4\pi\epsilon_0} \frac{\cos^2[\theta]-1}{r^3} \right| \psi_{2,0,0}^* \right\rangle = 2.022 \times 10^{40} D^2 \frac{\text{Kilogram}}{\text{Coulomb}^2 \text{Second}^2} \quad \text{IV.11}$$

We plot the quadrupole potential.

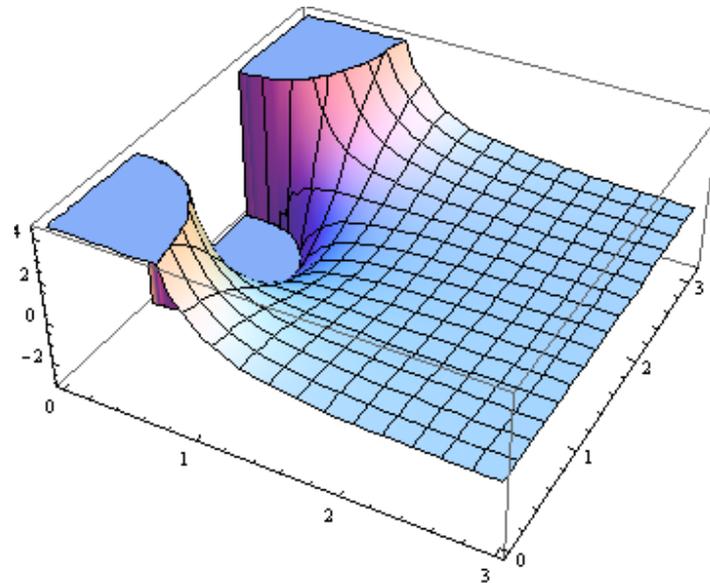


Fig.8. Quadrupole potential  $V(r, \theta) = (3 \cos^2 \theta - 1) / r^3$

The Total Energy: We take the values computed in IV-8, IV-9 and IV-10 to write:

$$E_T = E_c - E_d + E_Q = \quad \text{IV.12}$$

$$\left(-5.450 \times 10^{-19} + 5.770 \times 10^{40} \frac{D^2}{\text{Coulomb}^2 \text{Meter}^2}\right) \frac{\text{KilogramMeter}^2}{\text{Second}^2}$$

We plot the total potential for the second level.

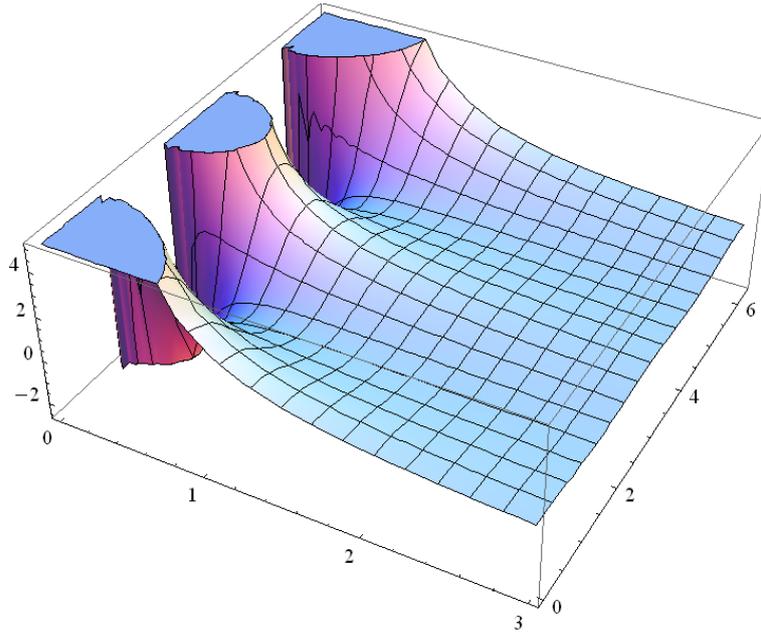


Fig.9. Total potential  $V(r, \theta) = -1/r + \cos \theta/r^2 + (3 \cos^2 \theta - 1)/r^3$

➤ **For The Third Level :z = 1 ; n = 3** We will follow the same steps that we have followed in the first level and second level

- **The Monopole Potential:** Does not change and is given by equation IV-1; So the energy of this level is given by:

$$E_C = - \frac{Z^2 q^4 m_e}{2(4\pi\epsilon_0)^2 \hbar^2 n^2} = -6.975 \times 10^{-19} \frac{\text{Kilogram Meter}^2}{\text{Second}^2} \quad \text{IV.13}$$

- **The Dipole Potential:** Like the second level, the first order always equals zero so we calculate the second order from the expression  $\sum_{n' \neq n} \frac{|\langle \psi | V_D | \psi^* \rangle|^2}{E_{n'} - E_n}$  which gives us a value

$$\begin{aligned}
 & \sum_{n=1}^2 \sum_{m=1}^{n-1} \left( \left| \left\langle \psi_{3,0,0} \left| \frac{D}{4\pi\epsilon_0} \frac{\cos[\theta]}{r^2} \right| \psi_{n,l,m}^* \right\rangle \right|^2 / (E_3 - E_n) \right) \\
 & + \sum_{n=4}^{10} \sum_{m=1}^{n-1} \left( \left| \left\langle \psi_{3,0,0} \left| \frac{D}{4\pi\epsilon_0} \frac{\cos[\theta]}{r^2} \right| \psi_{n,l,m}^* \right\rangle \right|^2 / (E_3 - E_n) \right) \\
 & = 1.066 \times 10^{34} D^2 \frac{\text{Kilogram}}{\text{Coulomb}^2 \text{Second}^2} \quad \text{IV.14}
 \end{aligned}$$

- **The Quadrupole Potential :** the first correction had given a value

$$\int_0^{2\pi} \int_0^\infty \left\langle \psi_{3,0,0} \left| \frac{D^2}{4\pi\epsilon_0} \frac{\cos^2[\theta]-1}{r^3} \right| \psi_{3,0,0}^* \right\rangle = 8.424 \times 10^{37} \times D^2 \frac{\text{Kilogram}}{\text{Coulomb}^2 \text{Second}^2} \quad \text{IV.15}$$

The Total Energy: Comes from the values computed in IV-13, IV-14 and IV-15:

$$\begin{aligned}
 E_T &= E_C - E_D + E_Q = \quad \text{IV.16} \\
 & \left( -6.975 \times 10^{-19} + 8.425 \times 10^{37} \frac{D^2}{\text{Coulomb}^2 \text{Meter}^2} \right) \frac{\text{KilogramMeter}^2}{\text{Second}^2}
 \end{aligned}$$

So we have

$$E_T(32) = 8.425 \times 10^{37} D^2 \frac{\text{Kilogram}}{\text{Coulomb}^2 \text{Second}^2} - 6.975 \times 10^{-19} \frac{\text{KilogramMeter}^2}{\text{Second}^2}$$

We summarize all the results in the following table ( $[D] = \text{Coulomb Meter}$ ):

|       | $E_M$ in Joules          | $E_D$ in Joules            | $E_Q$ in Joules            |
|-------|--------------------------|----------------------------|----------------------------|
| n = 1 | $-2.180 \times 10^{-18}$ | 0                          | 0                          |
| n = 2 | $-5.450 \times 10^{-19}$ | $3.748 \times 10^{40} D^2$ | $2.022 \times 10^{40} D^2$ |
| n = 3 | $-6.975 \times 10^{-19}$ | $1.067 \times 10^{34} D^2$ | $8.424 \times 10^{37} D^2$ |

It is interesting to see from the precedent results, that we can have a total energy that vanishes and thus the corresponding bound state disappears.

For the second level we put  $E_T = 0$  in IV-11 to get:

$$D_{crit} = 5.795 \times 10^{-30} \text{Coulomb Meter} \quad \text{IV.17}$$

And this gives us the following value for the dimension of the dipole which characterizes the extended system considered ( $q$  is the elementary charge):

$$d_{crit} = D_{crit}/q = 0.683 a \quad \text{IV.18}$$

For the third level we put  $E_T = 0$  in IV-15 to get:

$$D_{crit} = 9.099 \times 10^{-29} \text{Coulomb Meter} \quad \text{IV.19}$$

And this gives us the following characteristic dimension of the extended system considered ( $q$  is the elementary charge):

$$d_{crit} = D_{crit}/q = 10.732 a \quad \text{IV.20}$$

We see that the extended nature of the charge. has a role which is opposed to the attractive nature of the potential of the point charge, if the characteristics dimensions of the system exceeds a certain critical value  $d_{crit}$ . The higher the level considered is, the greater our critical value is. But if this dimensions are bellow this critical value, it is the attractive nature of the system that prevails and we have bound states.

We can explain this behaviour by the fact that the dipole and quadrupole terms decrease the depth of the potential well that results from the Coulomb term. This makes the bound states of the potential well disappearing one after the other with the diminution of the depth or with the increase of the dipole moment from IV-16 and IV-18.

This effect is easier to see if we represent the behaviour of the potential as a function of the distance  $r$  and giving a constant value to the angle  $\theta$ : we will consider the limit cases  $\theta = \pm 1$  where the angular effect is most pronounced.

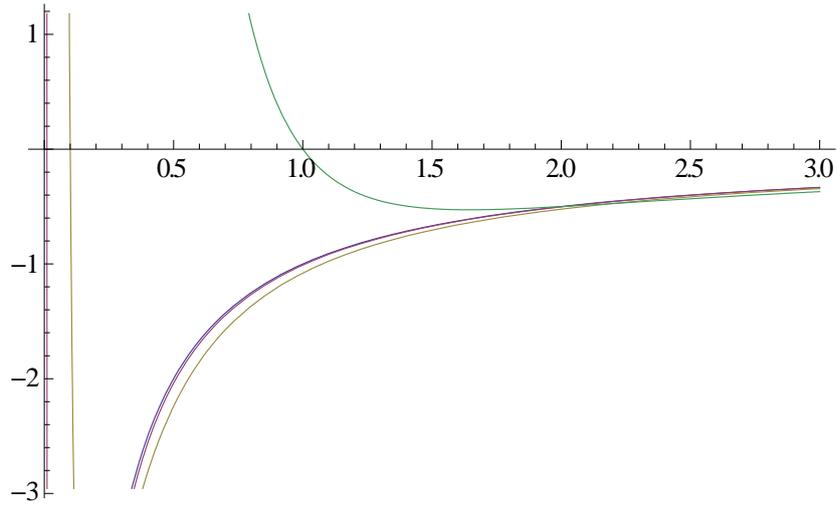


Fig.10. Representation of potential  $\frac{-1}{r} + D \frac{\cos\theta}{r^2} + D^2 \frac{3\cos^2\theta-1}{r^3}$  with  $D = 0,0.01; 0.1,1$  and  $\cos\theta = +1$  from bottom to top

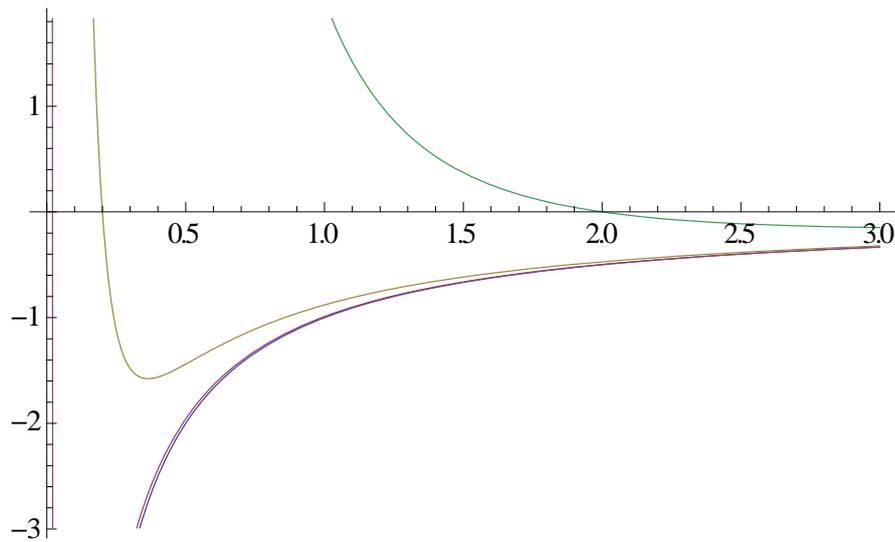


Fig.11. Representation of potential  $\frac{-1}{r} + D \frac{\cos\theta}{r^2} + D^2 \frac{3\cos^2\theta-1}{r^3}$  with  $D = 0,0.01; 0.1,1$  and  $\cos\theta = -1$  from bottom to top

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# Conclusion

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## Conclusion

The Coulomb problem in any dimension is defined as that of an electrostatic point charge at the origin of free space. This problem in 2D is one of the most familiar of all problems in Physics. So in our work, we have studied the spectrum of multipolar potential by perturbation theory in 2D system. We have calculated the exact potential: unperturbed potential (Coulomb potential) and perturbed potential (dipole and quadrupole parts).

In chapter one, we talk about multipole expansion to calculate the potentials of a point electric charge, a dipole and a quadrupole in two dimensions. In the second chapter we write the solutions of Schrödinger equation in 2D for Coulomb potential (monopole part). In the last chapter, we use perturbation theory to calculate the corrections induced by perturbed potential (dipole and quadrupole parts).

For the first level, we have found that there are no contributions from dipole and quadrupole potentials; so the total energy is the energy of the monopole (coulomb) potential and is equal to  $[-2.180 \times 10^{-18} \text{Joules}]$ . But for the second and the third level, we have found that there is a value for the contributions of dipole and quadrupole potential and the total energy is the sum of monopole, dipole and quadrupole potential. For the second level the total energy equal to  $(-5.450 \times 10^{-19} + 5.770 \times 10^{40} D^2) \text{Joules}$ . The total energy for the third level is equal to  $(-6.975 \times 10^{-19} + 8.425 \times 10^{37} D^2) \text{Joules}$ .

There is a special case where the Coulomb potential is suppressed by multipole corrections and thus the total energy becomes zero. So for the second and third levels we put  $E_T = 0$  to get  $d_{crit}$ , where  $d$  is the distance between the charges. For the second level we found that  $d_{crit} = D_{crit} / q = 0.68 a$  ( $a$  is the first Bohr radius). But for the third level we found that  $d_{crit} = D_{crit} / q = 10.732 a$ .

Finally we conclude by saying that the dipole and quadrupole terms decrease the depth of the potential well that results from the Coulomb term. This makes the bound states of the potential well disappearing one after the other with the diminution of the depth or with the increase of the dipole moment.

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## Abstract

We discuss in this work a very important topic in the field of quantum physics; which refers to quantum spectrum of the multipole expansion (monopole, dipole and quadrupole potential). We write the solutions for solutions of Schrödinger equation in two-dimensional system for the Coulomb potential. We talked about the theory of perturbation theory use this theory. to calculate the corrections induced by dipole and quadrupole potentials. to the energies of the monopole term by Schrödinger theory.

Finally, we discuss the results. We have found that there is a special case where the dipole and quadrupole corrections equal the Coulomb energy and thus the total energy goes to zero and there is no bound state for our system in this case.

*Key words:* two-dimensional system, Schrödinger equation, perturbation theory, monopole, dipole, quadrupole.

## ملخص

تطرقنا في هذا العمل إلى موضوع هام جدا في مجال فيزياء الكم، وهو حساب طيف كمون متعددات الأقطاب، حيث تحدثنا في الفصل الأول عن متعددات الأقطاب (أحادي وثنائي ورباعي الأقطاب). أما الفصل الثاني فكان عن حلول معادلة شرودنجر في نظام ذو بعدين. الفصل الثالث تكلمنا فيه عن نظرية الاضطرابات المستقرة، بعد ذلك قمنا بحساب الكمون الكولومبي وكمون ثنائي الأقطاب ورباعي الأقطاب، كما اعتبرنا كمون ثنائي الأقطاب ورباعي الأقطاب كتصحيحات للكمون الكولومبي. وفي الأخير قمنا بمناقشة النتائج التي تحصلنا عليها. لقد استخدمنا برنامج (Mathematica) لحساب هذه النتائج. ومن بين الحالات التي مررنا بها وجدنا انه توجد حالة خاصة يكون فيها الكمون الكولومبي مساويا التصحيحات وبالتالي تصبح الطاقة الكلية معدومة و تزول الحالات المستقرة عندها.