# A Switching Controller for Nonlinear Systems via Fuzzy Models

M. Boumehraz[1], K. Benmahammed[2]

[1]Department of Automatic, University of Biskra, Algeria
[2]Department of Electronics, University of Setif, Algeria
E-mail: medboumehraz@netcourrier.com, Khierben@ieee.org

## Abstract

A Lyapunov based switching control design method for non linear systems using fuzzy models is proposed. The switching controller consists of several linear state feedback controllers; only one of the linear controllers is employed at each moment according to a switching scheme. The gains of the linear state feedback controllers are derived based on Lyapunov stability theory. The fuzzy design model is represented as a set of uncertain linear subsystems and then sufficiency conditions for the system to be globally stabilisable by the switching controller are given. The proposed design method is illustrated trough numerical simulations on the chaotic Lorenz system.

## 1 Introduction

Fuzzy techniques have been widely adapted to model complex non linear plants. By using a Takagi-Sugeno fuzzy model, a non linear system can be expressed as a weighted sum of simple subsystems. This model gives a fixed structure to some non linear systems and thus facilitates their analysis. There are two ways to obtain the fuzzy model: 1) - by applying identification methods with input-output data from the plant [1] [2], 2)-or directly from the mathematical model of the non linear plant [3].
In this paper, we propose a Lyapunov based design of a switching linear controller for a class of fuzzy models.
The rest of the paper is organized as follows. Section 2 reviews the continuous T-S fuzzy models. Section 3 gives the structure of the switching controller and the controller design method is proposed in section 4. Then we provide an application example; the control problem of the chaotic Lorenz system. Finally, we present our conclusions.

## 2 Fuzzy model

The continuous-time Takagi-Sugeno fuzzy dynamic model is a piecewise interpolation of several linear models through membership functions. The fuzzy model is described by fuzzy if-then rules. The ith rule of the fuzzy model for the non linear system is of the form:
***Plant rule i:***

$$\textit{if} \;\; z_1(t) \;\; \textit{is} \;\; F_{i1} \;\; \textit{and} \;\; \ldots \;\; z_g(t) \;\; \textit{is} \;\; F_{ig} \;\; \textit{then}$$
$$\dot{x}(t) = A_i x(t) + B_i u(t) \qquad \textit{for} \;\; i = 1, 2, \ldots, r \qquad (1)$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ denotes the state vector, $u(t) = [u_1(t), \ldots, u_m(t)]^T \in R^m$ is the input vector. $F_{ij}$ is the fuzzy set, $A_i \in R^{nxn}$, $B_i \in R^{nxm}$, r is the number of if-then rules, and $z_1(t), z_2(t), \ldots, z_g(t)$ are some measurable system variables, i.e. the premise variables.
The output of the fuzzy model can be expressed as:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} \omega_i(z(t))[A_i x(t) + B_i u(t)]}{\sum_{i=1}^{r} \omega_i(z(t))} = \sum_{i=1}^{r} \alpha_i(z(t))[A_i x(t) + B_i u(t)] \qquad (2)$$

where $\omega_i(z(t)) = \prod_{j=1}^{g} F_{ij}(z(t))$

$F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in $F_{ij}$.
The TS fuzzy model (2) is a general non linear time varying equation and has been used to model the behaviour of complex non linear dynamic systems.
The TS fuzzy model (2) has strong nonlinear interactions among the fuzzy rules, which complicates the analysis and the control of the system. In order to overcome these difficulties, the TS fuzzy model (2) is represented as a set of uncertain linear systems. Each sub-space is defined as:

$$\mathbb{S}_l = \left\{ X \,\middle|\, \alpha_l(x) \geq \alpha_l^{min} \right\}, \quad l = 1, 2, \ldots, r \qquad (3)$$

where $0 \leq \alpha_i^{min} \leq 1$ is a scalar to be determined.
The characteristic function of $\mathbb{S}_l$ is defined by:

$$\zeta_i = \begin{cases} 1 & x \in \mathbb{S}_l \\ 0 & x \notin \mathbb{S}_l \end{cases} \qquad \sum_{i=1}^{r} \zeta_i = 1 \qquad (4)$$

In each subspace $\mathbb{S}_l$, the fuzzy model (2) can be represented as:

$$\dot{x}(t) = \left( A_l + \sum_{\substack{i=1 \\ i \neq l}}^{r_l} \alpha_i(t)(A_i - A_l) \right) x(t) +$$

$$\left( B_l + \sum_{\substack{i=1 \\ i \neq l}}^{r_l} \alpha_i(t)(B_i - B_l) \right) u(t) \qquad (5)$$

$$\dot{x}(t) = \left( A_l + (1-\alpha_l(t)) \sum_{\substack{i=1 \\ i \neq l}}^{r} \alpha_i'(t)(A_i - A_l) \right) x(t) +$$

$$\left( B_l + (1-\alpha_l(t)) \sum_{\substack{i=1 \\ i \neq l}}^{r} \alpha_i'(t)(B_i - B_l) \right) u(t)$$

$$\sum_{\substack{i=1 \\ i \neq l}}^{r} \alpha_i'(t) = \sum_{\substack{i=1 \\ i \neq l}}^{r} \frac{\alpha_i(t)}{1-\alpha_l(t)} = 1$$

$$\begin{aligned} \dot{x}(t) = (A_l + (1-\alpha_l(t))\Delta A_l)\, x(t) + \\ (B_l + (1-\alpha_l(t))\Delta B_l)\, u(t) \end{aligned} \qquad (6)$$

where

$$\Delta A_l(\alpha'(t)) = \sum_{\substack{i=1 \\ i \neq l}}^{r} \alpha_i'(t)(A_i - A_l)$$

$$\Delta B_l(\alpha'(t)) = \sum_{\substack{i=1 \\ i \neq l}}^{r} \alpha_i'(t)(B_i - B_l) \qquad (7)$$

The fuzzy system in subspace $\mathbb{S}_l$ consists of a dominant nominal system with matrix $A_l$ and a set of interacting systems determining the effect of the control law on the other non-dominant systems. The fuzzy model (6) can be viewed as an uncertain linear dynamical system model [4].

We assume that the matrices $\Delta A_l$ and $\Delta B_l$ can be written as:

$$\begin{aligned} \Delta A_l(\alpha'(t)) &= B_l . D_l(\alpha'(t)) + E_l(\alpha'(t)) \\ \Delta B_l(\alpha'(t)) &= B_l . F_l(\alpha'(t)) \end{aligned} \qquad (8)$$

The matrices $B_l.D_l(\alpha'(t))$ and $B_l.F_l(\alpha'(t))$ model the matched uncertainties whereas the term $E_l(\alpha'(t))$ represent the mismatched uncertainties.

We assume that:

$$F_l(\alpha'(t)) + F_l^T(\alpha'(t)) + \frac{2}{1-\alpha_l(t)} I_m > 0 \qquad (9)$$

We assume also that the matrix functions $D_l(\alpha'(t))$ and $E_l(\alpha'(t))$ are bounded by:

$$\|D_l(\alpha'(t))\| \leq \overline{D_l}, \quad \|E_l(\alpha'(t))\| \leq \overline{E_l}$$

$$\sigma_l = \min_{\alpha(t)} \left[ \frac{1}{2} \lambda_{min} \left( F_l(\alpha'(t)) + F_l^T(\alpha'(t)) \right) \right] \qquad (10)$$

for all $\alpha'(t) \in [0,1]^{r-1}$.

## 3 Switching controller approach

A switching controller is employed to control the fuzzy system (2). The switching controller consists of some linear state feedback controllers that will be switched from one to another to control the system. The switching controller is described by:

$$u(t) = -\sum_{l=1}^{r} \zeta_l K_l x(t) \qquad (11)$$

with $$K_l = \frac{\gamma_l}{2} B_l^T P_l \qquad (12)$$

and $$\sum_{l=1}^{r} \zeta_l = 1, \quad \zeta_l \in \{0,1\} \qquad (13)$$

$K_l$ is the local state feedback gain in subspace $\mathbb{S}_l$ to be designed. The parameter $\gamma_l > 0$ is a scalar and the matrix $P_l$ is the positive definite solution of the following algebraic Ricatti equation:

$$A_l^T P_l + P_l A_l - \eta_l P_l B_l B_l^T P_l = -2Q_l \qquad (14)$$

where $Q_l \in R^{n \times n}$ is a symmetric positive definite matrix and $\eta_l$ is any given positive constant.

It can be seen that (11) is a linear combination of $r$ linear state-feedback controllers.

## 4 Controller design

In this section, the switching controller will be designed to guarantee the system stability.

**Theorem:**

The state feedback controller given by (12) where:

$$\gamma_l = \frac{\eta_l + \delta_l^2 \overline{D_l}^2}{1+\sigma_l} \qquad (15)$$

globally asymptotically stabilise the uncertain sub-system (6) for arbitrary $D_l(\alpha'(t))$, $E_l(\alpha'(t))$ and $F_l(\alpha'(t))$ that satisfy the norm bounds (10) and the conditions (8) and (9) if :

$$\frac{1}{\delta_l^2} < 2\left(\frac{1}{1-\alpha_l(t)}\lambda_{min}(Q_l) - \overline{E}_l\lambda_{max}(P_l)\right) \quad (16)$$

and

$$\alpha_l(t) > 1 - \frac{\lambda_{min}(Q_l)}{\overline{E}_l\lambda_{max}(P_l)} \quad (17)$$

***Proof:*** Let define the positive definite function $V_l$ as:

$$V_l(t) = x^T P_l x \quad (18)$$

where $P_l \in R^{n\times n}$ is the solution of algebraic Ricatti equation (14).

$$\begin{aligned}
\dot{V}_l &= \dot{x}^T P_l x + x^T P_l \dot{x} \\
&= x^T\left(A_l^T P_l + P_l A_l\right)x + 2(1-\alpha_l)x^T P_l B_l D_l x + 2(1-\alpha_l)x^T P_l E_l x \\
&\quad -\gamma_l x^T P_l B_l\left[I_m + \frac{1}{2}(1-\alpha_l)\left(F_l^T + F_l\right)\right]B_l^T P_l x \\
&\le x^T\left(A_l^T P_l + P_l A_l\right)x + 2(1-\alpha_l)x^T P_l B_l D_l x \\
&\quad + 2(1-\alpha_l)x^T P_l E_l x - \gamma_l x^T P_l B_l(1+\sigma_l)B_l^T P_l x \\
&\le x^T\left(A_l^T P_l + P_l A_l - \eta_l P_l B_l B_l^T P_l\right)x + 2(1-\alpha_l)x^T P_l B_l D_l x \\
&\quad + 2(1-\alpha_l)x^T P_l E_l x - \delta_l^2\overline{D}_l^2 x^T P_l B_l B_l^T P_l x \\
&\le -2\lambda_{min}(Q_l)\|x\|^2 - \delta_l^2\overline{D}_l^2\left\|B_l^T P_l x\right\|^2 + \\
&\quad 2(1-\alpha_l)\left\|B_l^T P_l x\right\|\|D_l\|\|x\| + 2(1-\alpha_l)\|P_l E_l x\|\|x\| \\
&\le -2\lambda_{min}(Q_l)\|x\|^2 + 2(1-\alpha_l)\|P_l E_l x\|\|x\| + \frac{1}{\delta_l^2}(1-\alpha_l)\|x\|^2 \\
&\le \left[-2\lambda_{min}(Q_l) + 2(1-\alpha_l)\overline{E}_l\lambda_{max}(P_l) + \frac{1}{\delta_l^2}(1-\alpha_l)\right]\|x\|^2
\end{aligned} \quad (19)$$

$$\dot{V}_l < 0 \Rightarrow -2\lambda_{min}(Q_l) + 2(1-\alpha_l)\overline{E}_l\lambda_{max}(P_l) + \frac{1}{\delta_l^2}(1-\alpha_l) < 0$$

$$\dot{V}_l < 0 \Rightarrow \frac{1}{\delta_l^2} < \frac{2}{1-\alpha_l}\lambda_{min}(Q_l) - 2\overline{E}_l\lambda_{max}(P_l)$$

$$\frac{2}{1-\alpha_l}\lambda_{min}(Q_l) - 2\overline{E}_l\lambda_{max}(P_l) > 0 \Rightarrow$$

$$2\overline{E}_l\lambda_{max}(P_l) < \frac{2}{1-\alpha_l}\lambda_{min}(Q_l)$$

$$\alpha_l > 1 - \frac{\lambda_{min}(Q_l)}{\overline{E}_l\lambda_{max}(P_l)} \quad (20)$$

We define:

$$\alpha_l^{min} = max\left(0, 1 - \frac{\lambda_{min}(Q_l)}{\overline{E}_l\lambda_{max}(P_l)}\right) \quad (21)$$

$\alpha_l^{min}$ is the minimum value of $\alpha_l$ that guarantee the stability of the global system using the local subsystem $\mathbb{S}_l$ state-feedback gain $K_l$.

In each subspace, the command is given by:

$$u(t) = -K_l x(t) \quad (22)$$

The boundary of the sub-region $\mathbb{S}_l$ is determined by the minimal value that guarantees its stability $\alpha_l^{min}$.

***Lemma:***

The global system is asymptotically stable if there exists, at each moment *t*, at least one value $\alpha_k(t)$ satisfying:

$$\alpha_k(t) \ge \alpha_k^{min}, k = 1,2,\ldots,r \quad (23)$$

or $\bigcup_{l=1}^{r}\mathbb{S}_l = \mathbb{S}$ where $\mathbb{S}$ is the global state space.

In overlapping regions many subsystems may satisfy this condition. In this case the control is inferred by selecting the control of the dominant system whose membership function is of maximum distance from the boundary of its stability region determined by $\alpha_l^{min}$:

$$u(t) = -K_l x(t) \quad (24)$$

$$l = arg\ \max_{i=1,r}\left\{\alpha_i(t) - \alpha_i^{min}, i = 1,2,\ldots,r\right\} \quad (25)$$

The design procedure of the switching controller is summarized in the following steps:

- *Step 1:* Obtain the fuzzy plant model of the non linear plant by means of the methods in [1],[2], [4], or other suitable ways.
- *Step 2:* Determine the subsystems $\mathbb{S}_l$ matrices $A_l, B_l$, $\Delta A_l$ and $\Delta B_l$ for $l = 1,\ldots,r$ and check if condition (9) is verified for each subsystem.
- *Step 3:* Design the state-feedback gain $K_l$ for each subsystem $\mathbb{S}_l$ according to (14), (15) and (16). And determine the value of $\alpha_l^{min}$ for $l = 1,\ldots,r$.
- *Step 4:* Check if the condition (23) is satisfied, otherwise go to *Step 3* and choose other values for the free design parameters.

## 5 Simulation example

To show the effectiveness of the proposed method, we simulate the control of the chaotic Lorenz system. The control objective is to drive its chaotic trajectory to the origin. The Lorenz equations are as follows [5]:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\sigma x_1(t) + \sigma x_2(t) \\ r x_1(t) - x_2(t) - x_1(t)x_3(t) \\ x_1(t)x_2(t) - b x_3(t) \end{bmatrix} \quad (26)$$

The nominal values of $(\sigma, r, b)$ are $(10, 28, 8/3)$ for chaos to emerge. The system can be described by the following T-S fuzzy model [5]:

Rule 1: If $x_1(t)$ is about $M_1$ then $\dot{x}(t) = A_1 x(t)$

Rule 2: If $x_1(t)$ is about $M_2$ then $\dot{x}(t) = A_2 x(t)$

Where

$$A_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_1 \\ 0 & M_1 & -b \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_2 \\ 0 & M_2 & -b \end{bmatrix}$$

$$M_1 = -20 \text{ and } M_2 = 30 \qquad (27)$$

The membership functions are given by:

$$\mu_1(x(t)) = \frac{-x_1(t) + M_2}{M_2 - M_1}, \quad \mu_2(x(t)) = \frac{x_1(t) - M_1}{M_2 - M_1} \qquad (28)$$

The input matrices $B_1$ and $B_2$ are chosen as:

$$B_1 = B_2 = I_3 \qquad (29)$$

The fuzzy model can be decomposed into two subsystems:

Subsystem 1: $\dot{x}(t) = (A_1 + (1 - \mu_1)\Delta A_1)\, x(t) + B_1 u(t)$

Subsystem 2: $\dot{x}(t) = (A_2 + (1 - \mu_2)\Delta A_2)\, x(t) + B_2 u(t)$

Where: $\Delta A_1 = A_2 - A_1$, $\Delta A_2 = A_1 - A_2$, $\Delta B_1 = \Delta B_2 = 0$

For $Q_1 = Q_2 = 50 I_3$ and $\eta_1 = \eta_2 = 0.1$ the resolution of the Ricatti equation (14) gives:

$$P_1 = \begin{bmatrix} 3.1211 & 1.5040 & 0.4201 \\ 1.5040 & 3.1526 & 0.0437 \\ 0.4201 & 0.0437 & 2.9063 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.0589 & 1.3771 & -0.5838 \\ 1.3771 & 3.1339 & -0.0675 \\ -0.5838 & -0.0675 & 2.9162 \end{bmatrix}$$

and

$$\alpha_1^{\min} = \max(0, -0.4181) = 0, \alpha_2^{\min} = \max(0, -0.4490) = 0$$

The state feedback gains:

$$K_1 = \begin{bmatrix} 96.433 & 46.467 & 12.981 \\ 46.467 & 97.404 & 1.350 \\ 12.981 & 1.350 & 89.794 \end{bmatrix} \quad K_2 = \begin{bmatrix} 91.425 & 41.160 & -17.447 \\ 41.160 & 93.668 & -2.016 \\ -17.447 & -2.016 & 87.160 \end{bmatrix}$$

The initial values of states are $x(0)^T = [20, 20, 20]$. The simulation time is 40 s. The control input is activated at t=20s. Before the activation of the control the phase trajectory of the Lorenz system was chaotic. However, after the activation of the command the phase trajectory is quickly directed to the origin as shown in figures 1 and 2. In this example the boundary of the two sub-spaces are determined by $\alpha_1^{\min} = 0$ and $\alpha_2^{\min} = 0$ which means that the two sub-spaces are equal to the global state space and the chaotic system can be controlled using only one controller.

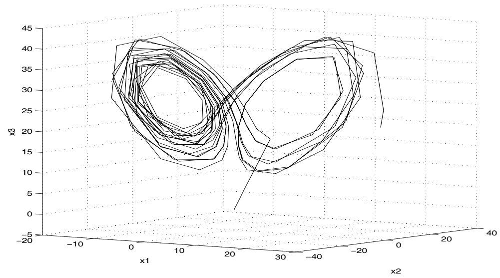


Fig. 1. The phase trajectory of the controlled Lorenz system.

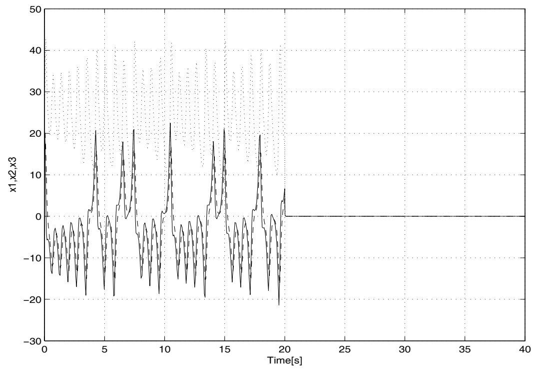


Fig. 2. States of the Lorenz system.

## 6 Conclusion

In this paper a Lyapunov based method has been proposed to design a fuzzy model based switching controller for non linear systems. Under some conditions this switching controller has the ability to stabilize the non linear system. The control of the chaotic Lorenz system has been used demonstrate the effectiveness of this approach.

## References

[1] Takagi, T., Sugeno M., (1985) Fuzzy identification of systems an its application to modelling and control, IEEE Transactions on Systems, men and Cybernitics 15: 116-132.

[2] Sugeno M., Kang, G. T. (1988) Structure identification of fuzzy models, Fuzzy Sets and Systems, 28:15-33.

[3] Kim E., Park M., Ji S., Park M. (1999) A new approach to fuzzy modeling, IEEE Transactions on fuzzy systems, 7: 120-132.

[4]- Zak S. H. (1999) Stabilizing fuzzy system models using linear controllers, IEEE Transactions on Fuzzy Systems, 7: 236-240.

[5] Lee H. J., Park J. B., Chen G. (2001) Robust fuzzy control of nonlinear systems with parametric uncertainties, IEEE Transactions on fuzzy systems, 9: 369-380.