Linear state feedback stabilisation of non linear systems via fuzzy models

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Abstract : A Lyapunov based linear stabilising control design method for non linear systems using fuzzy model is proposed. The linear stabiliser is constructed using a fuzzy design model of the dynamical system to be controlled. The fuzzy design model is represented as an uncertain linear system and then sufficiency conditions for the uncertain system to be globally stabilisable by a linear controller are given. The proposed design method is illustrated on a problem of balancing an inverted pendulum on a cart.

Keywords : Fuzzy models, uncertain system, stabilisation.

1. Introduction

The tasks of stabilization and tracking are two typical control problems. In the stabilization problem of a non linear plant, we are concerned with constructing a controller so that starting from an arbitrary point in some neighborhood of the operating point, the controller forces the closed-loop system trajectory to converge to the operating point. On the other hand, if the starting point coincides with the operating point, the closed-loop system trajectory is expected to stay at this point for all subsequent time.

Stabilization of non linear systems is difficult because no systematic mathematical tools exist to help find necessary and sufficiant conditions to garantee stability and performance. By using a Takagi-Sugeno fuzzy model, a non linear system can be expressed as a weighted sum of simple subsystems. This model gives a fixed structure to some non linear systems and thus facilate their analysis. There are two ways to obtain the fuzzy model: 1)- by applying identification methods with input-output data from the plant[1][2], 2)-or directly from the mathematical model of the non linear plant[3].

In this paper, we prpose a Lyapunov based design of linear stabilizing controllers for a class of fuzzy system models. We provide a sufficiency condition for a fuzzy model to be globally stabilizable by a linear controller.

The rest of the paper is organized as follows. Section 2 reviews the continuous T-S fuzzy models. The controller design method is proposed in section 3. Then we provide an application example of an inverted pendulum on a cart. Finally, we present our conclusions.

2. Fuzzy model

Many physical systems are very complex in practice so that rigorous mathematical models can be very difficult to obtain, if not impossible. However, many of these systems can be expressed in some form of mathematical models. Takagi and Sugeno have proposed a fuzzy model to describe the complex systems[1]. The continuous-time Takagi-Sugeno fuzzy dynamic model is a piecewise interpolation of several linear models through membership functions. The fuzzy model is described by fuzzy if-then rules and will be employed here to deal with the control design problem for the non linear system. The ith rule of the fuzzy model for the non linear system is of the form:

Plant rule i:

$$\textit{if}\ \ z_1(t)\ \textit{is}\ F_{i1}\ \textit{and}\ \ldots\ z_g(t)\ \textit{is}\ F_{ig}\ \textit{then}$$

$$\dot{x}(t) = A_i x(t) + B_i u(t) \qquad \textit{for} \quad i = 1,2,\ldots,r \tag{1}$$

where $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^T \in R^n$ denotes the state vector, $u(t)=[u_1(t),\ldots,u_m(t)]^T \in R^m$ is the input vector. F_{ij} is the fuzzy set, $A_i \in R^{nxn}$, $B_i \in R^{nxm}$, r is the number of if-then rules, and $z_1(t),z_2(t),\ldots,z_g(t)$ are some measurable system variables, i.e. the premise variables.

The output of the fuzzy model with central average deffuzifier product inference and singleton fuzzifier can be expressed as:

$$\dot{x}(t)=\frac{\sum_{i=1}^{r}\omega_i(z(t))[A_i x(t)+B_i u(t)]}{\sum_{i=1}^{r}\omega_i(z(t))}$$

$$=\sum_{i=1}^{r}\alpha_i(z(t))[A_i x(t)+B_i u(t)] \quad (2)$$

where

$$\omega_i(z(t))=\prod_{j=1}^{g}F_{ij}(z(t))$$

$$\alpha_i(z(t))=\frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))}$$

$$z(t)=[z_1(t),z_2(t),\ldots,z_g(t)] \quad (3)$$

$F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in F_{ij}.

We assume $\omega_i(z(t))\geq 0$

and $\sum_{i=1}^{r}\omega_i(z(t))>0 \quad for \quad i=1,2,\ldots,r$

for all t.

Therefore we get :

$$\alpha_i(z(t))\geq 0 \quad for \quad i=1,2,\ldots,r \quad (4)$$

and

$$\sum_{i=1}^{r}\alpha_i(z(t))=1 \quad (5)$$

The TS fuzzy model (2) is a general non linear time varying equation and has been used to model the behaviour of complex non linear dynamic systems.

The state of the fuzzy system can be represented as:

$$\dot{x}(t)=\sum_{i=1}^{r}\alpha_i(t)A_i x(t)+\sum_{i=1}^{r}\alpha_i(t)B_i(t)$$

$$=\left(\sum_{i=1}^{r}\alpha_i(t)A_i\right)x(t)+\left(\sum_{i=1}^{r}\alpha_i(t)B_i\right)u(t) \quad (6)$$

Let $\overline{A}$, $\underline{A}$, $\overline{B}$ and $\underline{B}$ four matrices defined as:

$$\overline{A}(i,j)=\max_{k=1,r} A_k(i,j)$$
$$\underline{A}(i,j)=\min_{k=1,r} A_k(i,j) \quad (7)$$

for i=1,..,n and j=1,..,n

$$\overline{B}(i,j)=\max_{k=1,r} B_k(i,j)$$
$$\underline{B}(i,j)=\min_{k=1,r} B_k(i,j) \quad (8)$$

for i=1,..,n and j=1,..,m.

Let (A,B) a completely controllable pair chosen so that :

$$\underline{A}(i,j)\leq A(i,j)\leq \overline{A}(i,j)$$
$$\underline{B}(i,j)\leq B(i,j)\leq \overline{B}(i,j) \quad (9)$$

The model (6) can be represented as:

$$\dot{x}(t)=\left(A+\sum_{i=1}^{r}\alpha_i(t)(A_i-A)\right)x(t)+\left(B+\sum_{i=1}^{r}\alpha_i(t)(B_i-B)\right)u(t)$$

$$\dot{x}(t)=(A+\Delta A)x(t)+(B+\Delta B)u(t) \quad (10)$$

where

$$\Delta A(\alpha(t))=\sum_{i=1}^{r}\alpha_i(t)(A_i-A)$$
$$\Delta B(\alpha(t))=\sum_{i=1}^{r}\alpha_i(t)(B_i-B) \quad (11)$$

let :

$$\alpha(t)=[\alpha_1(t),\alpha_2(t),\ldots,\alpha_r(t)]^T$$
$$\alpha(t)\in[0,1]^r \quad (12)$$

model (10) can be viewed as an uncertain linear dynamical system model where $\alpha(t)$ is called the vector of uncertain parameters[4]. The uncertain system modelled has linear uncertainty structure because the uncertain elements ΔA and ΔB are linear matrices functions[4].

We assume that the matrices ΔA and ΔB can be written as:

$$\Delta A(\alpha(t))=B.D(\alpha(t))+E(\alpha(t))$$
$$\Delta B(\alpha(t))=B.F(\alpha(t)) \quad (13)$$

and

$$F(\alpha(t)) + F^T(\alpha(t)) + 2I_m > 0 \quad (14)$$

The matrices $B.D(\alpha(t))$ and $B.F(\alpha(t))$ model the matched uncertainties whereas the term $E(\alpha(t))$ represent the mismatched uncertainties.
We assume also that the matrix functions $D(\alpha(t))$ and $E(\alpha(t))$ are bounded by:

$$\|D(\alpha(t))\| \le \overline{D}$$

$$\|E(\alpha(t))\| \le \overline{E}$$

$$\sigma = \min_{\alpha(t)} \left[\frac{1}{2} \lambda_{\min} (F(\alpha(t)) + F^T(\alpha(t)) \right] \quad (15)$$

for all $\alpha(t) \in [0,1]^r$.

3. Controller design

The structure of the controller to be designed in this paper is:

$$u(t) = -\frac{\gamma}{2} B^T P x(t) \qquad (16)$$

The parameter $\gamma > 0$ is a scalar and the matrix P is the positive definite solution of the following algebraic Ricatti equation :

$$A^T P + PA - \eta PBB^T P = -2Q \qquad (17)$$

where $Q \in R^{n \times n}$ is a symmetric positive definite matrix and η is any given positive constant.

Theorem :
The state feedback controller given by (16) where :

$$\gamma = \frac{\eta + \delta^2 \overline{D}^2}{1 + \sigma} \qquad (18)$$

globally asymptotically stabilise the uncertain system for arbitrary $D(\alpha(t))$, $E(\alpha(t))$ and $F(\alpha(t))$ that satisfy the norm bounds (15) and the conditions (13) and (14) if :

$$\frac{1}{\delta^2} < 2(\lambda_{\min}(Q) - \overline{E}\lambda_{\max}(P)) \qquad (19)$$

and

$$\overline{E} < \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \qquad (20)$$

Proof : Let define the positive definite function V as :

$$V(t) = x^T P x \qquad (21)$$

where $P \in R^{n \times n}$ is the solution of algebraic Ricatti equation (17).

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\
&= \left[(A + \Delta A)x + (B + \Delta B)u\right]^T P x + \\
&\quad x^T P[(A + \Delta A)x + (B + \Delta B)u] \\
&= x^T (A^T P + PA)x + x^T (\Delta A^T P + P\Delta A)x \\
&\quad + u^T B^T P x + x^T P B u \\
&\quad + u^T \Delta B^T P x + x^T P \Delta B u \\
&= x^T (A^T P + PA)x + x^T (\Delta A^T P + P\Delta A)x \\
&\quad - \gamma x^T PB \left[I_m + \frac{1}{2}(F^T + F) \right] B^T P x \\
&= x^T (A^T P + PA)x + 2x^T PBDx \\
&\quad + 2x^T PEx - \gamma x^T PB \left[I_m + \frac{1}{2}(F^T + F) \right] B^T P x \\
&\le x^T (A^T P + PA)x + 2x^T PBDx \\
&\quad + 2x^T PEx - \gamma x^T PB(1 + \sigma) B^T P x \\
&\le x^T (A^T P + PA)x + 2x^T PBDx \\
&\quad + 2x^T PEx - \gamma x^T PB(1 + \sigma) B^T P x \\
&\le x^T (A^T P + PA - \eta PBB^T P)x + 2x^T PBDx \\
&\quad + 2x^T PEx - \delta^2 \overline{D}^2 x^T PBB^T P x \\
&\le -2x^T Qx + 2x^T PBDx \\
&\quad + 2x^T PEx - \delta^2 \overline{D}^2 x^T PBB^T P x \\
&\le -2x^T Qx + 2\|B^T P x\|\|D\|\|x\| + 2\|PEx\|\|x\| \\
&\quad - \delta^2 \overline{D}^2 \|B^T P x\|^2 \\
&\le -2x^T Qx + 2\overline{D}\|B^T P x\|\|x\| + 2\|PEx\|\|x\| \\
&\quad - \delta^2 \overline{D}^2 \|B^T P x\|^2 \\
&\le -2x^T Qx + 2\|PEx\|\|x\| - \left[\delta \overline{D} \|B^T P x\| - \frac{1}{\delta}\|x\| \right]^2 \\
&\qquad + \frac{1}{\delta^2}\|x\|^2 \\
&\le -2x^T Qx + 2\|PEx\|\|x\| + \frac{1}{\delta^2}\|x\|^2 \\
&\le -2\lambda_{\min}(Q)\|x\|^2 + 2\overline{E}\lambda_{\max}(P)\|x\|^2 + \frac{1}{\delta^2}\|x\|^2 \\
&\le \left[-2\lambda_{\min}(Q) + 2\overline{E}\lambda_{\max}(P) + \frac{1}{\delta^2} \right] \|x\|^2
\end{aligned}$$

(22)

$$\dot{V} < 0 \quad \Rightarrow \quad -2\lambda_{\min}(Q) + 2\bar{E}\lambda_{\max}(P) + \frac{1}{\delta^2} < 0$$

$$\dot{V} < 0 \quad \Rightarrow \quad \frac{1}{\delta^2} < 2\lambda_{\min}(Q) - 2\bar{E}\lambda_{\max}(P)$$

$$2\lambda_{\min}(Q) - 2\bar{E}\lambda_{\max}(P) > 0 \Rightarrow 2\bar{E}\lambda_{\max}(P) < 2\lambda_{\min}(Q)$$

$$\bar{E} < \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \qquad (23)$$

A similar method was proposed by Zak[4] but the conditions of stability in our method are weaker then those in [4].

4. Simulation example

An application example will be given here to show the design procedure of the linear stabilizer. A cart-pole inverted pendulum is shown in Fig. 1.

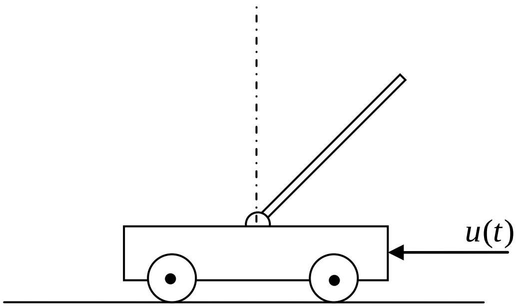

Fig. 1. The inverted pendulum system

The motion of the pendulum can be described by the following differential equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{g\sin(x_1) - \frac{1}{2} aml x_2^2 \sin(2x_1) - a\cos(x_1)u}{\frac{4l}{3} - a\cos^2(x_1)} \end{cases}$$

$$a = \frac{1}{M+m}$$

x_1 is the angle of the pendulum from the vertical line, x_2 is the angular velocity of the pendulum, u is the control force applied to the cart, $g = 9.8\, m/s^2$ is the gravity, $m = 2.0\, kg$ is the mass of the pendulum, $M = 8.0\, kg$ is the mass of the cart and $l = 0.5\, m$ is the half length of the pendulum.

The inverted pendulum can be modeled by a fuzzy plant model having the following two rules:

$Rule1$: *if* $x_1(t)$ *is close to* 0 *then*

$$\dot{x}(t) = A_1 x(t) + B_1 u(t)$$

and

$Rule2$: *if* $x_1(t)$ *is close to* $\pm\pi/2$ *then*

$$\dot{x}(t) = A_2 x(t) + B_2 u(t)$$

(25)

where:

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{4l/3 - aml} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\dfrac{a}{4l/3 - aml} \end{bmatrix}$$

(26)

and

$$A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi\left(4l/3 - amlb^2\right)} & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\dfrac{ab}{4l/3 - amlb^2} \end{bmatrix}$$

(27)

where $a = 1/(M+m)$ and $b = \cos(88°)$ and the membership functions are:

$$\omega_1(x_1) = 1 - \frac{2}{\pi}|x_1(t)|$$
$$\omega_2(x_1) = \frac{2}{\pi}|x_1(t)| \qquad (28)$$

The corresponding fuzzy model is :

$$\dot{x} = (\alpha_1 A_1 + \alpha_2 A_2)x + (\alpha_1 B_1 + \alpha_2 B_2)u \quad (29)$$

where :

$$\alpha_1(t) = \frac{\omega_1(x_1(t))}{\omega_1(x_1(t)) + \omega_2(x_1(t))} = \omega_1(x_1(t))$$

$$\alpha_2(t) = \frac{\omega_2(x_1(t))}{\omega_1(x_1(t)) + \omega_2(x_{2(t))}} = \omega_2(x_1(t))$$

(30)

$$\dot{x} = \left[\alpha_1 \begin{bmatrix} 0 & 1 \\ 17.3118 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 9.3696 & 0 \end{bmatrix}\right] x + \left[\alpha_1 \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ -0.0052 \end{bmatrix}\right] u$$

(31)

$\bar{A} = A_1$, $\underline{A} = A_2$, $\bar{B} = B_2$ and $\underline{B} = B_1$. We choose :

$$A = \begin{bmatrix} 0 & 1 \\ 15 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ -0.15 \end{bmatrix} \tag{32}$$

The fuzzy model can be written as :

$$\dot{x} = \left[\begin{bmatrix} 0 & 1 \\ 15 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2.3118 & 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 & 0 \\ -5.6304 & 0 \end{bmatrix} \alpha_2 \right] x$$
$$+ \left[\begin{bmatrix} 0 \\ -0.15 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ -0.02647 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0.14476 \end{bmatrix} \right] u$$
$$\Delta A = \begin{bmatrix} 0 & 0 \\ 2.3115 & 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 & 0 \\ -5.6304 & 0 \end{bmatrix} \alpha_2$$
$$\Delta B = \begin{bmatrix} 0 \\ -0.02647 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 \\ 0.14476 \end{bmatrix} \alpha_2 \tag{33}$$

$$\Delta A = B.D(\alpha) = \begin{bmatrix} 0 \\ -0.15 \end{bmatrix} [-15.412\alpha_1 + 37.536\alpha_2 \quad 0]$$
$$\Delta B = B.F(\alpha) = \begin{bmatrix} 0 \\ -0.15 \end{bmatrix} (0.17647\alpha_1 - 0.96507\alpha_2)$$
$$D(\alpha) = [-15.412\alpha_1 + 37.536\alpha_2 \quad 0]$$
$$\overline{D} = \max_{\alpha} \|D(\alpha)\| = 37.536$$
$$F(\alpha) = 0.17647\alpha_1 - 0.96507\alpha_2$$
$$F(\alpha) + F^T(\alpha) = 0.35294\alpha_1 - 1.93014\alpha_2$$
$$F(\alpha) + F^T(\alpha) + 2I \geq 0.0699$$
$$\sigma = \min_{\alpha(t)} \left[\frac{1}{2} \lambda_{\min} (0.35294\alpha_1 - 1.93014\alpha_2) \right]$$
$$\sigma = -0.9651 \tag{34}$$

$$\text{let } Q = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} \text{ and } \eta = 8.0 \tag{35}$$

Solving the Recatti equation yields

$$P = \begin{bmatrix} 770.6016 & 169.9359 \\ 169.9359 & 49.4341 \end{bmatrix} \tag{36}$$
$$\lambda_{\min}(Q) = 50 .$$
$$\frac{1}{\delta^2} < 2\lambda_{\min}(Q) - 2\overline{E}\lambda_{\max}(P)$$
$$\overline{E} = 0 \Rightarrow \frac{1}{\delta^2} < 100 \Rightarrow \delta > 0.1 \tag{37}$$

we choose $\delta = 0.12$

Then the controller is :

$$u(t) = -\frac{1}{2}\gamma B^T P x(t) = [10301.35 \quad 2996.65] x(t) \tag{38}$$

The performance of the controller is illustrated in Fig. 2, where plots of x_1 versus time are shown for different initial angular posditions. The controller was tested on the thruth model, the fuzzy model was used only to determine the parameters of th linear controller. As can be seen from Fig. 2 the linear controller can stabilize the pendulum in less then 2 seconds.

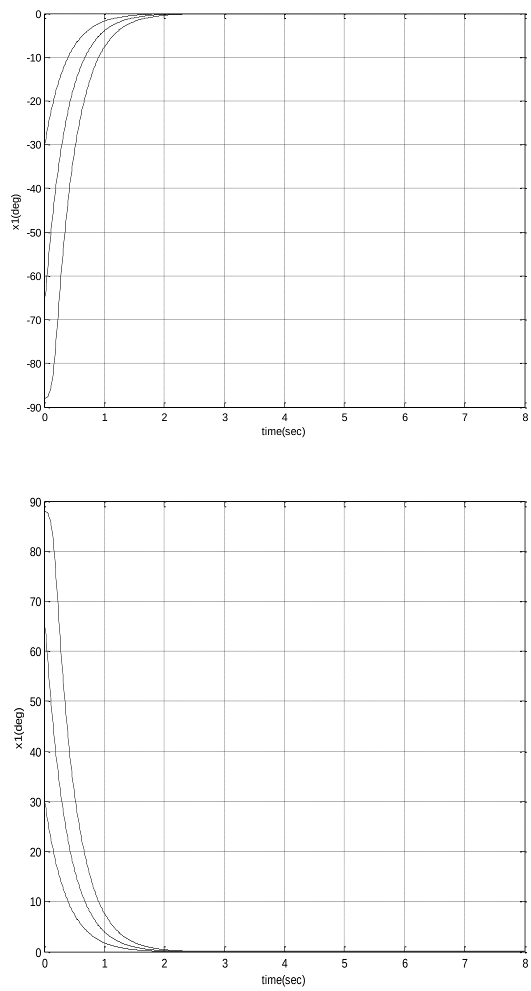

Fig. 2 Plots of x_1 for different initial values

5. Conclusions

A Lyapunov based method has been proposed to design a stabilizing controller for non linear systems via fuzzy model.

Under some conditions this linear controller has the ability to stabilize the non linear system. A simulation example of an inverted pendulum on a cart has been given to demonstrate the effectiveness of this approach.

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