People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Biskra University

Faculty of Exact Sciences
Department of Material Sciences

Second Year L.M.D.
Academic
Field: Physics

Course and Exercises

Welcome to P-L-2

Fluid Mechanics

The Modelling Team

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Course Presentation

Introduction

This fluid mechanics course extends classical mechanics and field theory to continuous, deformable media. It transitions students' focus from discrete-particle dynamics to the macroscopic behavior of liquids and gases. The core of the subject lies in applying macroscopic conservation laws to complex, non-linear systems driven by the interplay of internal friction, pressure gradients, and external forces.

This course is designed for second-year physics license students. It is taught in semester 4 and consists of lectures and tutorials (1 h 30 minutes of lecture and 1 h 30 minutes of tutorial per week). The module belongs to the methodology teaching unit, with 4 credits and a coefficient of 2. The fluid mechanics course has been developed in alignment with the official curriculum established by the Ministry of Higher Education and Scientific Research.

Course Objectives

For a first-time, introductory course at the second-year physics level, the goal isn't to have students solve highly complex, real-world flow problems. Instead, the primary objective of this introductory fluid mechanics course for second-year physics students is to fundamentally shift their analytical framework from discrete-particle dynamics to continuous-field theory. By adopting the Eulerian perspective, students learn to recontextualize familiar conservation laws of mass, momentum, and energy for deformable media, thereby providing a rigorous, practical application of their vector calculus toolkit. The course aims to develop a strong physical intuition for how pressure gradients, internal friction, and external forces interact within a continuum. Ultimately, this foundational knowledge serves as a crucial conceptual bridge, preparing students for the mathematical and physical demands of advanced specialized fluid mechanics and other fields such as astrophysics, plasma physics, and electrodynamics.

Course Content

The course is structured into five main chapters. The first chapter builds a comprehensive understanding of continuous media, beginning with foundational continuum definitions and macroscopic properties in Main Concepts in Fluid Mechanics. Students then analyze fluids at equilibrium under pressure and body forces in Fluid Statics before transitioning to the study of motion. This transition begins with Fluid Kinematics, which establishes the Eulerian mathematical framework for describing flow geometries, streamlines, and velocity fields independent of the underlying forces. The analysis of these driving forces is then tackled in two distinct stages: Dynamics of Inviscid Fluid explores idealized, frictionless models to establish baseline conservation principles, while the final chapter, Dynamics of Real Fluids, reintroduces viscosity and internal friction to capture the complex, energy-dissipating behaviors of actual physical systems.

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Chapter II: Fluid Statics

Chapter III: Fluid Kinematics

Chapter I: Basic Concepts in Fluid Mechanics

I. 1 Introduction

I. 1.1 What is fluid mechanics?!

It is a branch of physics that deals with the behavior of fluids at rest and in motion.

I. 1. 2 Why study fluid mechanics?!

Fluid mechanics has many applications in various areas of our lives.

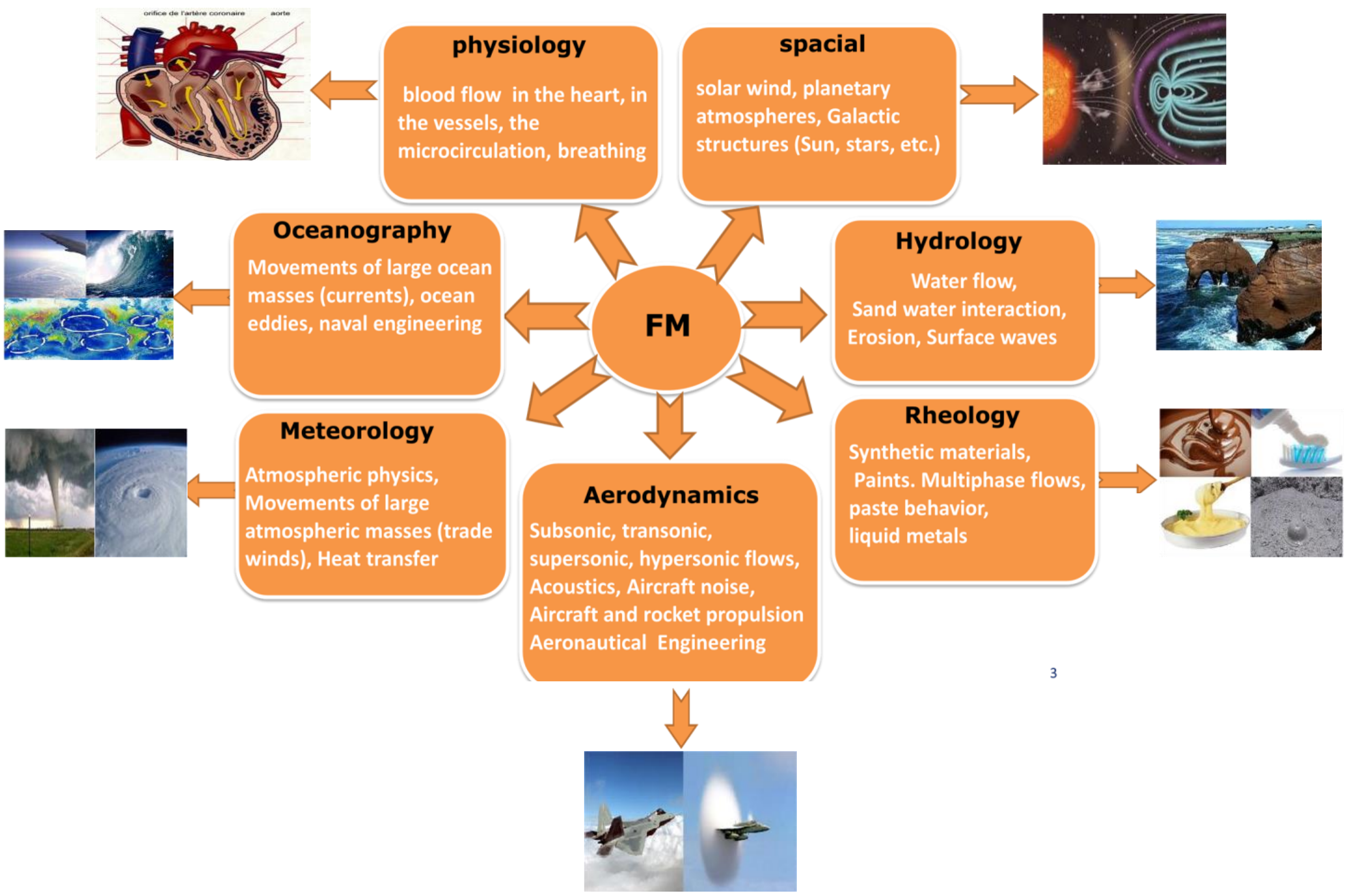


Figure I.1 Fluid mechanics fields

I. 2 Definitions

I. 2.1 The Fluid

A fluid is a continuous medium, made up of very small and very numerous material particles, free to move relative to each other. It is a deformable, non-rigid medium that can flow.

Liquids do not have their own form. They take the form of a container which contains them, but each has its own volume.

Gases have no proper shape or volume.

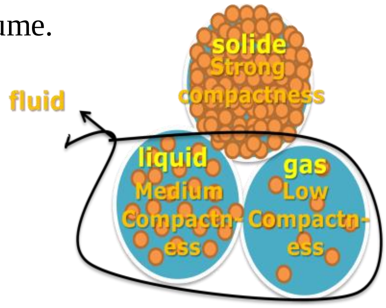


Figure I.2: Matter's States

I. 2.2 Concept of Fluid Particle

The concepts of fluid mechanics are based on the following fundamental assumptions:

- The fluid is a continuous medium.
- The medium is considered continuous when it is possible to define, at any point in this medium (the flow) and at any time, field functions (density, temperature, speed, etc.).
- No matter how small the size element we choose (fog droplet), it will always be much larger than the size of the particles that make it up.
- It is not necessary to follow the evolution of each molecule on a microscopic scale
- The fluid is composed of fluid particles, which comprise a large number of molecules and occupy negligible volume at the macroscopic scale.
- We can assign to each fluid particle a field of velocity V, pressure P, temperature T, mass m…

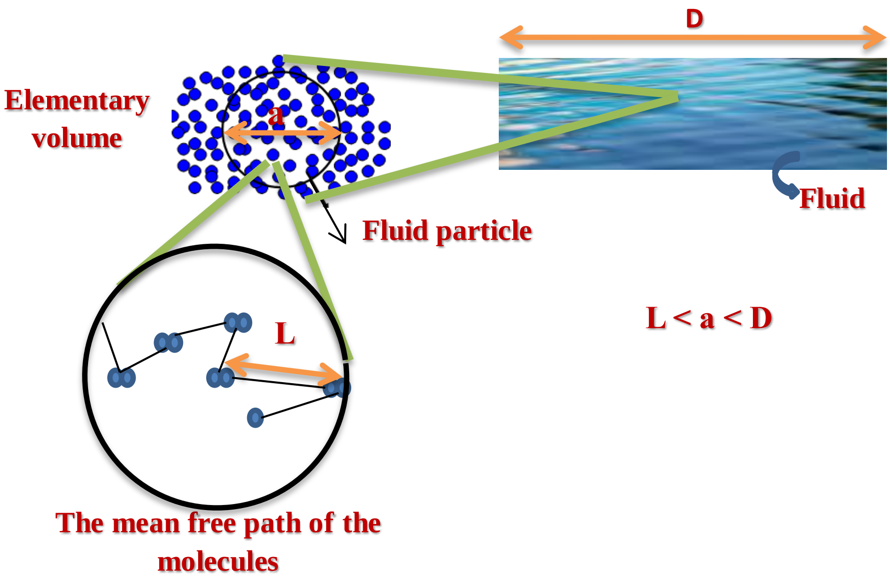


Figure I.3: Notion of fluid particle

D: Characterizes the domain dimension (the macroscopic scale)

L: Characterizes the intermolecular dimension; the mean free path (the microscopic scale)

a: Characterizes the fluid particle dimension (the scale mesoscopic)

I. 2.3 Volume Forces and Surface Forces Applied to a Fluid Domain

I. 2.3 Volume Forces

They are forces acting on the entire volume of the fluid domain, such as gravity, as well as electric, magnetic, and electromagnetic forces.

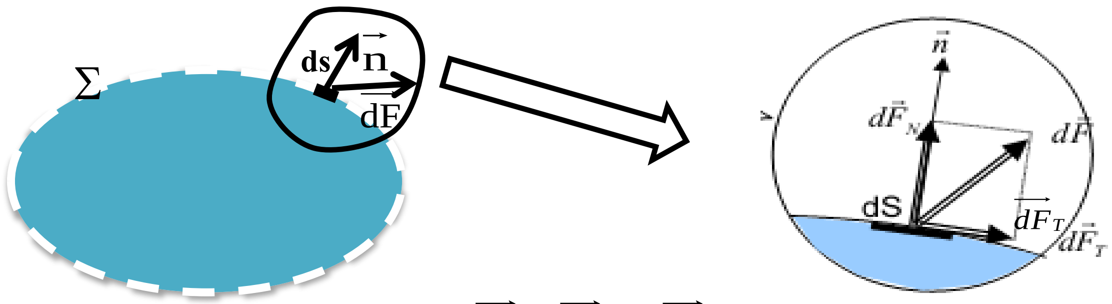
$$\vec{F} = \int_V \vec{f} d\tau$$

$d\tau$: Elementary volume

$\vec{f}$: Force per unit volume

- **Surface Forces:** they are the forces that act on the surface of the fluid domain.

Consider a fluid volume, delimited by a closed surface Σ.



$$\overrightarrow{dF} = \overrightarrow{dF_T} + \overrightarrow{dF_N}$$

Considering the force acting on the elementary surface dS of the normal $\vec{n}$. It is called surface stress.

It is always possible to decompose $\overrightarrow{dF}$ into two components:

1- Tangential stress to dS due to the viscous action called viscous stress or friction.
2- Normal stress to dS due to local pressure, called the force of pressure

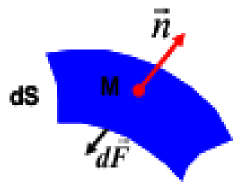
- **Pressure at a point of a fluid**

Pressure is the force per unit area exerted perpendicularly on a surface element dS.

$$\overrightarrow{dF} = -p \cdot \vec{n} \cdot ds \Longrightarrow p = \frac{dF}{ds}$$

dF is the force exerted on the surface element dS.

p is the pressure at point M.



- The pressure force always acts inside the volume delimited by the surface element.
- The pressure is expressed in pascals: Pa = N.m^{-2} = kg.m^{-1}.s^{-2}

We also find: 1 atm = 1,013.10^5 Pa

1bar = 10^5 Pa

1 atm = 760 mm Hg

1. 2.4 Density

The mass density ρ is defined as the mass per unit of volume

$$\rho = \frac{dm}{dV}$$

I. 2.5 Relative Density

The density d is dimensionless and is defined as the mass density of the studied fluid related to the mass density of water (1000 kgm^{-3}) in the case of a liquid, and to the mass density of air (1.29 kgm^{-3}) in the case of a gas

I. 2.6 Incompressible Fluid

A fluid is said to be incompressible when the volume occupied by a given mass does not vary according to external pressure. Liquids can be considered incompressible fluids (e.g., water, oil).

Example: To reduce water volume by 5%, it takes a force of 109 per m^2 of surface

I. 2.7 Compressible Fluid

A fluid is said to be compressible when its volume per unit mass varies with external pressure. Gases are compressible fluids.

Example: air, hydrogen, methane gas,

I. 2.8 Volumetric flow rate and Mass flow rate

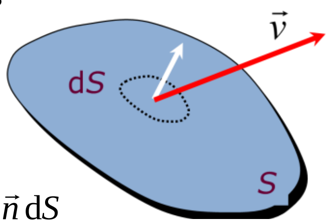
- **Volumetric flow rate**

It is the volume of fluid that passes through a given cross-sectional area per unit time.

The volume flow rate of a fluid is given by:

$$q_v = \int_S \vec{v}.\vec{n}\, dS$$

$q_v = v_n S$; v_n uniform on S



- **Mass flow rate**

The mass flow rate of a fluid is given by :

$$q_m = \int_S \rho \vec{v}.\vec{n}\, dS$$

$q_m = \rho\, v_n\, S$; v_n uniform on the entire surface S

- If the flow is permanent (the flow does not change over time), then the mass flow rate is preserved: $q_m(S_1) = q_m(S_2)$
- If the fluid is incompressible, then the volume flow rate is preserved, $q_v(S_1) = q_v(S_2)$

I.2.9 Inviscid Fluid

A fluid is said to be inviscid or perfect when the resultant of the surface forces exerted on the fluid remains normal to the surface dS. This means that the tangential component FT is zero (friction effects are negligible).

I.2.10 Real Fluid

In a real fluid, the tangential forces of internal friction (viscous friction) which oppose the relative sliding of the fluid layers are taken into consideration.

- This phenomenon of viscous friction appears during fluid movement.
- At rest, the real fluid behaves like an ideal fluid.

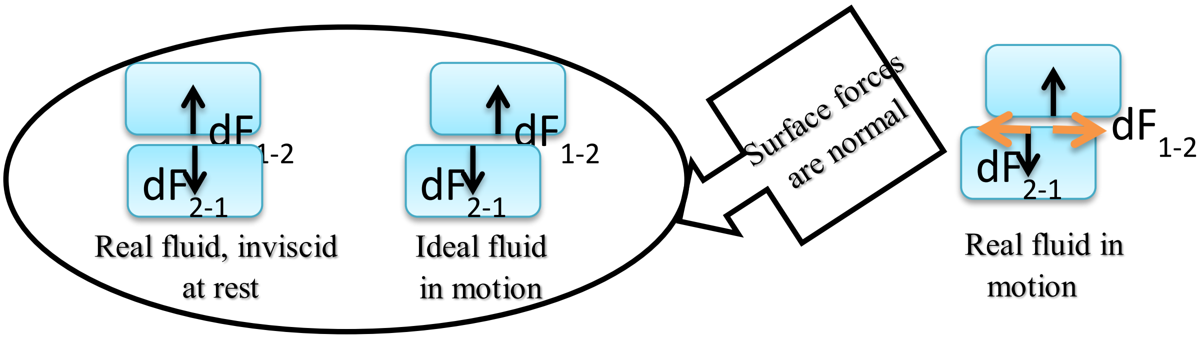
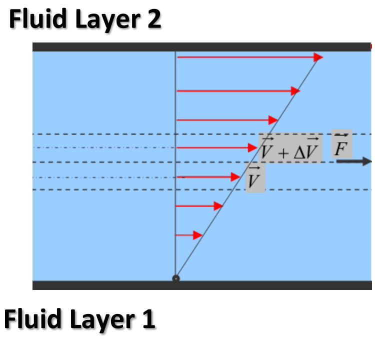


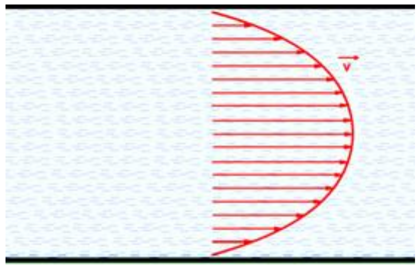
Figure I.4 : Difference Between Real Fluid and Inviscid Fluid

I. 2.11 Viscosity

Viscosity is determined by the training capability (stickiness) of a moving layer over the adjacent layers. In viscous flow, the fluid layers do not flow at the same velocity. It produces a velocity profile.



Linear velocity profile



Parabolic Velocity Profile

Figure I.5: Velocity Profile in a real fluid

We distinguish between dynamic viscosity and kinematic viscosity:

- **Dynamic Viscosity**

Consider two adjacent layers, distant by Δz, of a fluid flowing in straight and parallel streams. The difference in speed between the two fluid layers generates a friction force F, acting on the surface separating them and opposing their relative sliding. It is proportional to the difference in the velocity of the layers Δv, to their surface S, and inversely proportional to Δz.

$$F = \mu \frac{\Delta V}{\Delta Z} S$$

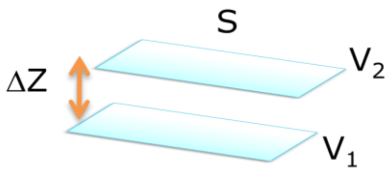


Figure I.6: Newton's law of viscosity for one-dimensional

The proportionality factor **μ** is the coefficient of the dynamic viscosity of the fluid.

$$[\mu] = M \cdot L^{-1} \cdot T^{-1}.$$

Unit: In the international system (SI), the dynamic viscosity unit is the Pascal second (Pa.s) or Poiseuille (Pl)

- **Kinematic viscosity**

In many formulae, the ratio of dynamic viscosity η to density ρ appears. This ratio is called kinematic viscosity.

$$\eta = \frac{\mu}{\rho}$$

$[\eta] = L^2 \cdot T^{-1}$

Unit: In the international system (SI), the viscosity unit has no special name: (m^2/s).
In the CGS system, the unit is the Stokes (St): $1\ m^2/s = 10^4$ St.

I. 2.12 Viscosity Measurement

Viscosity can be measured using several instruments, among these are:

- **Ball Drop Viscometer or Hoepler Viscometer**

A spherical ball falls slowly into a well-calibrated tube containing the viscous liquid. We measure the time t it takes the ball to travel a certain distance.

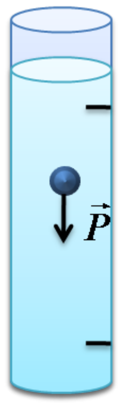


Figure I.7 : Hoepler Viscometer

- **Rotary viscometer or Couette Viscometer**

A solid cylinder (A) rotates at a constant velocity in a liquid contained in a cylindrical container (B), which is trained by the liquid and rotates around its rotation axis. A spring exerting a torsion torque after being turned at an angle α, holds (B) in balance. We show that the dynamic viscosity is proportional to the angle α.

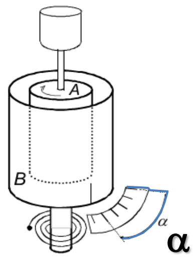


Figure I.8 : Couette Viscometer

I. 3 Flow Regimes

In 1883, British engineer Osborne Reynolds showed the existence of two types of viscous flow in pipes.

I. 3.1 Laminar Regime

It corresponds to regular flows in which the fluid mass is composed of juxtaposed, perfectly individualized threads; the free surfaces are smooth and united; and the threads are subjected only to tangential friction forces due to the fluid's viscosity. These are the forces of viscosity. These flows hardly occur in real life except for some particular problems (at low velocity) or in the case of very viscous fluids

I. 3.2 Turbulent Regime

From a certain velocity, local pressure or velocity fluctuations or external flow disturbances (wall roughness, inertial forces) irreversibly modify the current lines. They do not retain their individualities, which causes pressure gradients on either side of the layer. This leads to the creation of local vortices. When the fluid is filled with these vortices, the regime is called "*turbulent*".

I. 3.3 Transitional Regime

It is an intermediate regime. The flow is said to be transient when small disturbances begin to appear.

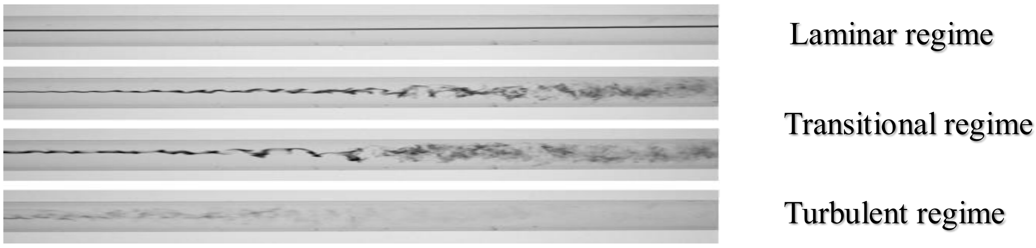


Figure I. 9: Flow Regimes

- **Reynolds Number**

By using fluids of varying viscosity and varying the experimental conditions, Reynolds showed that there is a dimensionless parameter, Re, that classifies these behaviors. This is called the Reynolds number.

$$Re = \frac{\rho V d}{\mu} = \frac{Vd}{\eta}$$

- $Re < 2000$ Laminar regime
- $2000 < Re < 3000$ Transitional regimes
- $Re > 3000$ Turbulent regime

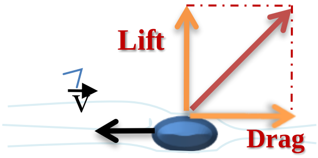
ρ: is the density of the fluid; V: the average velocity; d: a characteristic dimension of the pipe; μ and η: the dynamic and kinematic viscosities of the fluid.

I. 4 drag force

When a fluid passes an object or an object moves through a fluid, the latter exerts a force on the object's surface, which can be split into two components:

One acting in the same direction as the fluid flow, which is called the drag force, T, defined as the component of the force parallel and opposite to the direction of motion and one acting perpendicular to the flow direction, which is called the force of lift.

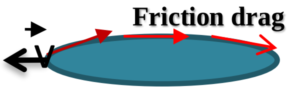
The Drag Force is the result of the internal forces that brake the body in the fluid. It includes two types of drag: friction drag and pressure or form drag. It is distributed on the surface, whose



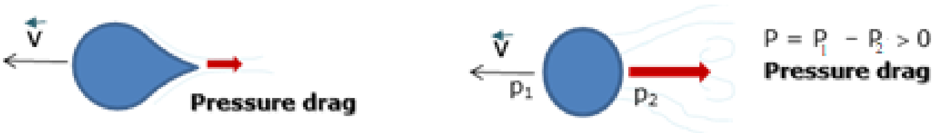
values and distributions depend on:

- ✓ The shape of the body and the roughness of its surface
- ✓ The relative velocity
- ✓ The physical state of the fluid (density, steady, etc.)

➢ Friction drag, also known as skin friction drag, is caused by the friction of fluid. Against the surface of an object that is moving through it. It is directly proportional to the surface area.



➢ Pressure drag, also known as form drag, arises from the shape of the object. It becomes more important for high velocities.



Drag force depends strongly on Reynolds number and the object geometry

❑ Drag force at laminar regime (low Reynolds number):

At low velocity, the fluid, on the surface of an immersed object, has a local velocity equal to that of the object. Hence, the fluid flow over the object is laminar and decreases regularly with increasing distance from the object.

In a laminar regime, it is the viscosity that predominates in the effect of the drag, which is expressed by:

d: the linear dimension of the moving object
K: the numerical coefficient depends on the form and the orientation of the object
μ: the fluid viscosity
V: the velocity of the object

$$T = K V d \mu$$

- **Drag force in the turbulent regime.**

At high speeds, the backflow with low energy separates from the object, creating a relatively low-pressure region, which often leads to pressure drag on the object. This radically changes the overall flow field and, consequently, the inviscid flow region and the pressure distribution on the object.

By the time a turbulent wake develops and grows at the rear of the object, it leads to large pressure drag as the separation point moves from the rear toward the front.

The energy thus communicated from the object to the flow, equal to 1/2 ρV^2 per unit volume, slows the object's motion. Therefore, in a turbulent regime, it is inertia leading to a large pressure drag, which predominates in the drag effect as follows:

$$T = C_x \frac{1}{2} \rho V^2 S$$

The drag coefficient Cx is a dimensionless number. It depends on the object's shape and its orientation relative to the velocity direction. It must be reduced to reduce the drug force.

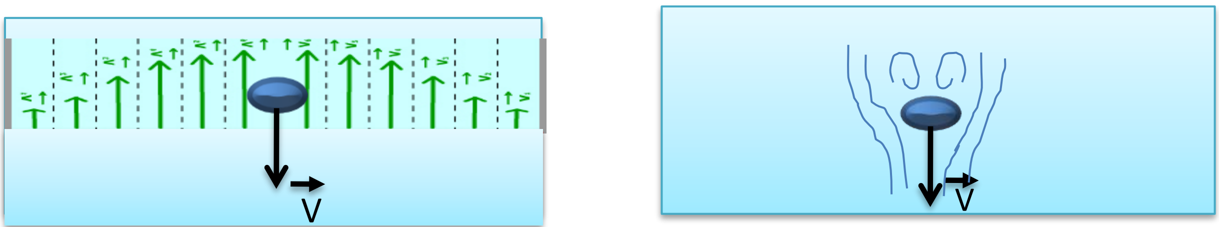


Figure I.10: Drag force in different flow regimes

1.4.1 Drag Coefficient and Reynolds Number

Considering the importance of situations where fluid inertia plays a predominant role (high speed), it is appropriate in all cases to write the drag in the form:

$$T = C_x \frac{1}{2} \rho V^2 S$$

The drag coefficient depends on the velocity. By forming the ratio of the inertia drag and viscous drag, we obtain:

$$\frac{T_{\text{turbulent}}}{T_{\text{laminaire}}} = \frac{\rho d^2 V^2}{\mu dV} = \frac{\rho V d}{\mu} = R_e$$

The Reynolds number, Re, represents the ratio of inertial to viscous forces. Drag force depends on the nature of the flow that determines the value of Cx.

Dimensional analysis identifies the variables involved in Cx and shows:

$$C_x = f(\text{Re})$$

Graphs representing Cx as a function of Re have been compiled by grouping experimental results obtained on very diverse systems (metal alloy balls in oil, wax balls in alcohol, paraffin in aniline, amber in water, air bubbles in water, etc.)

I. 4.2 The Drag Coefficient of a Smooth Sphere

In the case of a sphere of radius r moving at low speeds ($Re \leq 1$) in a fluid, the drag is given by the Stokes formula:

$$T = 6\pi r \mu V$$

In the case of ($1 \leq Re \leq 10^3$), the drag is given by the H.S. Allen formula:

$$C_x = \frac{18.5}{R_e^{0.6}}$$

In the case of high speeds ($1000 \leq Re \leq 5.\ 10^5$) ; where Cx = 0,48, and using $S = \pi r^2$, the drag is given by:

$$T = C_x \frac{1}{2} \rho V^2 \pi r^2$$

In the case of high speeds more than 5.10^5, Cx is approximately equal to 0.2

The drag coefficient of a smooth sphere at low speed is expressed as follows:

With the Reynolds number equal to:

$$R_e = \frac{2r\rho V}{\mu} \quad ; \qquad d = 2r$$

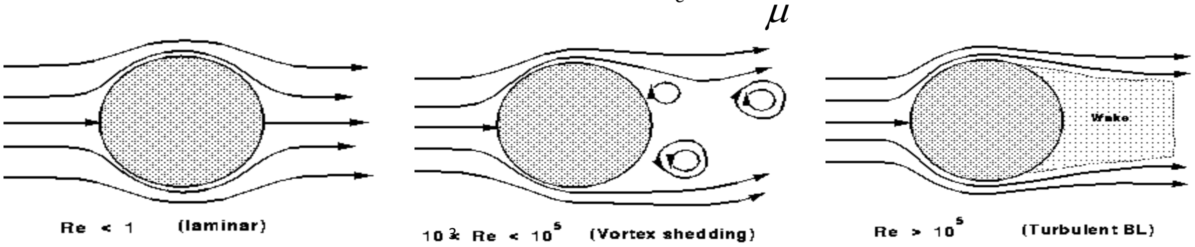
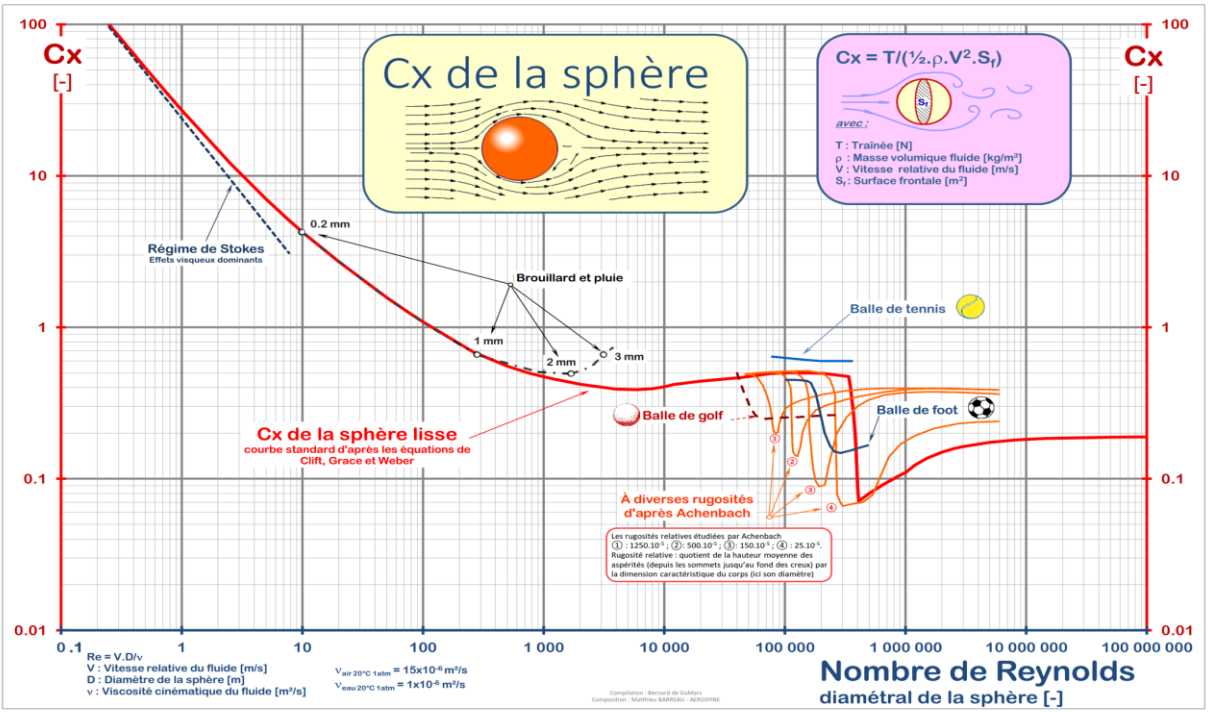


Figure I.11: Reynolds Number at Different Regimes

Graphs representing Cx as a function of Re have been compiled by grouping experimental results obtained on very diverse systems (metal alloy balls in oil, wax balls in alcohol, paraffin in aniline, amber in water, air bubbles in water, etc.)

Different values of Cx based on Re values calculated for different types of spheres

Cx of the smooth sphere according to its diametric Reynolds; Cx of golf balls, tennis balls and footballs; Cx of the rough spheres according to Achenbach; Cx of raindrops and fog according to their equivalent diameter.



I. 5 Exercises

Exercise N°1: Find the height of the free surface if 0.0076 m^3 of water is filled into a cylindrical container with an inner diameter di = 0.162 m, an outer diameter de = 0.20 m, and a height of 0.65 m. The rest of the container is filled with 2.83 kg of oil. Determine the density of this oil. What is the density of the full container (water + oil), knowing that its density when empty is 41 kg m^{-3}?

Exercise N°2: Using the following law, which expresses the variation of the viscosity of liquids as a function of temperature:

$$\mu = \mu_0 \frac{1}{1 + \alpha T + \beta T^2}$$

T is the temperature in degrees C. α and β are constants that depend on the liquid's nature, and μ_0 is the viscosity at 0°C. Taking for water $\alpha = 35{,}6\ 10^{-3}$, $\beta = 0{,}189\ 10^{-3}$ and $\mu_0 = 1{,}787.10^{-3}$ Pl à 0°C.

- Assuming the Sutherland model for gases:

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S}$$

T is the temperature in K, and μ_0 is the viscosity at 0°C, S is the Sutherland temperature. The characteristic constants for air are $\mu_0 = 1{,}711 \cdot 10^{-5}$ Pl, T_0 = 273.15 K, and S =114 K.

1- Calculate the dynamic viscosities of water and air at temperatures of 10°C, 20°C, and 100°C.
2- Same question for the kinematic viscosity, given $\rho_{air} = 1{,}2 \cdot 10^{-3}$ g/cm³ et $\rho_{water} = 1$ g/cm³ and
3- What do you notice?
4- Where does the difference in behavior between the two fluids come from?

Exercise 3: A liquid has a dynamic viscosity $\mu = 95.10^{-3}\ Pa \cdot s$ at T = 20 °C. Calculate its kinematic viscosity η in Stokes given that its specific gravity is d = 0,95.

Exercise 4: air (η_{air} = 0.157 Stokes at 27°C) flows in a pipe with a 15.0 cm diameter at an average velocity v = 4.50 m s^{-1}. What is the flow regime? Calculate the volumetric flow rate relative to the average velocity.

Exercise 6: Consider a hose connected to a water tap, where 53 cm of its length can be filled with 0.006 m^3 of water. When water flows through this hose, 1.19 liters of water can be filled in 10 seconds.

1. Calculate the water flow rate in the hose.

2. What is the flow velocity?

3. Determine the flow regime.

Given: : $\mu_{eau} = 10^{-3}$ *Pa.s*

Exercise 7.

The fall of a plastic ball (d = 2 mm) in a 1 m column of liquid (kinematic viscosity η = 23 st, ρ = 800 kg m^{-3}) took 1.63 s.

1. What is the flow regime?
2. Calculate the drag force.
3. Calculate the friction coefficient.

The experiment is repeated with a steel ball (d = 13 mm), which takes 0.53 s to travel 1 m in the fluid (kinematic viscosity η = 0.23 St, = 800 kg m ^{3}, ρ = 920 kg m^{-3}).

1. What is the flow regime of this object?
2. Calculate the drag force knowing that the drag coefficient is Cx = 0.48.

Exercise 8.

Water flows from a tap with a cross-sectional area of 12.56 cm^2 at a flow rate of 1.4 liters per second. This water was used to fill a cylindrical tube with a 15 cm diameter, and the process took 20 seconds. A steel ball with a diameter of 16 mm was then dropped into the water column, and it took 0.98 seconds to pass through the water.

1- What is the regime of the tap water
2- What is the regime of the ball flow?
3- Calculate the drag force knowing that the drag coefficient is Cx = 0.5.

Given: $\mu eau = 10^{-3}$ Pa · s. ρ_{eau} = 1000 kg/m^3

I.6 Exercise Solutions

Solution 1

1- Height of the Free Surface of Water

The inner cross-sectional area of the cylinder S_i is determined by its inner diameter d_i :

$$S_i = \frac{\pi . d_i^2}{4}$$

Substituting the given value d_i = 0.162m:

$$S_i = \frac{\pi . (0.162)^2}{4} \approx 0.0206\ m^2.$$

The volume of a cylinder is the product of its cross-sectional area and its height. Therefore, the height of the water column, h_w, is expressed as:

$$\mathrm{h_w} = \frac{\mathrm{V_w}}{S_i}$$

Substituting the known values:

$$\mathrm{h_w} = \frac{0.0076}{0.0206} \approx 0.3687m$$

The height of the free surface of the water is approximately 0.369 m.

2- Density of the oil, we must first compute the remaining inner volume of the container V_{oil} occupied by the oil.

The total inner volume $V_{i,total}$ is given by:

$$\mathrm{V_{i,total}} = S_i\,.H$$

$$\mathrm{V_{i,total}} = 0.0206\,(0.65) \approx 0.0134\ \mathrm{m^3}$$

The volume occupied by the oil is the difference between the total inner volume and the water volume:

$$\mathrm{V_{oil}} = \mathrm{V_{i,total}} - \mathrm{V_w}$$

$$\mathrm{V_{oil}} = 0.0134 - 0.0076 = 0.0058\ \mathrm{m^3}$$

The mass density of the oil ρ_{oil} is defined as the ratio of its mass to its volume:

$$\rho_{oil} = \frac{m_{oil}}{V_{oil}}$$

$$\rho_{oil} = \frac{2.83}{0.0058} \approx 488.11 \text{ kg/m}^3.$$

The mass density of the oil is approximately 488.11 kg/m^3

3- <u>Total Density of the Full Container</u>

The density of a complex object or container refers to its bulk or global density, computed relative to its external total volume, V_e , The density when empty is $\rho_{empty} = 41 \text{ kg/m}^3$. The external volume V_e calculated using the outer diameter $d_e = 0.20$ m is:

$$S_e = \frac{\pi . d_e^2}{4}$$

$$S_e = \frac{\pi(0.20)^2}{4} \approx 0.031\text{m}^2$$

$$V_e = S_e . \text{H} \approx 0.02\text{m}^3$$

Using the empty density, the mass of the empty container m_{empty} is:

$$\text{m}_{\text{empty}} = \rho_{empty}.(V_e - V_i) \approx 0.271 \text{ kg}$$

Next, we determine the mass of the water m_w, assuming standard water density $\rho_w = 1000 \text{ kg/m}^3$:

$$m_w = \rho_w . V_w = 1000(0.0076) = 7.60kg.$$

The total mass of the system m_{total} is the sum of the structural mass, water mass, and oil mass:

$$\text{m}_{\text{ total}} = \text{m}_{\text{ empty}} + \text{m}_{\text{w}} + \text{m}_{\text{oil}} = 10.7 \text{ kg}$$

Finally, the total global density ρ_{total} is the total mass divided by the outer volume:

$$\rho_{total} = \frac{\text{m}_{\text{ total}}}{V_e} = \frac{10.7}{0.02} \approx 535.05 \text{ kg/m}^3$$

The global mass density of the full container system is approximately 535 kg/m^3.

<u>Solution 2</u>

1- Dynamic Viscosity of Water

The empirical law governing the temperature dependence of liquid viscosity is given by:

$$\mu = \mu_0 \frac{1}{1 + \alpha T + \beta T^2}$$

Where T: is the temperature in degrees Celsius ℃, and the characteristic constants for water are:

- $\mu_0 = 1.787 \times 10^{-3}$Pl (Poiseuille, or Pa .s), $\alpha = 35.6 \times 10^{-3}$ ℃$^{-1}$, $\beta = 0.189 \times 10^{-3}$ ℃$^{-2}$

At $T = 10$ ℃

We have

$$\mu_{water} = \frac{1.787 \times 10^{-3}}{1 + 35.6 \times 10^{-3}(10) + 0.189 \times 10^{-3}(10)^2} \approx 1.300 \times 10^{-3}\text{PI.}$$

At $T = 20$ ℃

$$\mu_{water} = \frac{1.787 \times 10^{-3}}{1 + 35.6 \times 10^{-3}(20) + 0.189 \times 10^{-3}(20)^2} \approx 1.000 \times 10^{-3}\text{PI.}$$

At $T = 100$ ℃

$$\mu_{water} = 0.277 \times 10^{-3}\text{PI}$$

2- Dynamic Viscosity of Air

The Sutherland model for gases is expressed as:

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S}$$

Where T is the absolute temperature in Kelvin K, and the characteristic constants for air are:

- $\mu_0 = 1.711 \times 10^{-5}$Pl , $T_0 = 273.15\ K$, $S = 114\ K$
- The numerator factor $T_0 + S = 387.15K$

At $T = 10$℃ $= 283.15$K:

$$\mu_{air} = 1.711 \times 10^{-5} \left(\frac{283.15}{273.15}\right)^{3/2} \left(\frac{387.15}{283.15 + 114}\right) \approx 1.760 \times 10^{-5}\text{PI}$$

At $T = 20$℃ $= 293.15$K:

$$\mu_{air} = 1.711 \times 10^{-5} \left(\frac{293.15}{273.15}\right)^{3/2} \left(\frac{387.15}{293.15 + 114}\right) \approx 1.809 \times 10^{-5} \text{PI}$$

At $T = 100°\text{C} = 373.15\text{K}$:

$$\mu_{air} = 2.171 \times 10^{-5} \text{PI}$$

3- Calculation of Kinematic Viscosity

Kinematic viscosity is defined as the ratio of dynamic viscosity to fluid density:

$$\vartheta = \frac{\mu}{\rho}$$

First, the given densities must be converted into the International System of Units (kg/m^3)

- $\rho_{water} = 1 \text{ g/cm}^3 = 1000 \text{ kg/m}^3$
- $\rho_{air} = 1.2 \times 10^{-3} \text{g/cm}^3 = 1.2 \text{ kg/m}^3$

For Water $\vartheta = \left(\frac{\mu}{1000}\right)$:

- $\vartheta(10°\text{C}) = \left(\frac{1.300 \times 10^{-3}}{1000}\right) = 1.300 \times 10^{-6} \text{m}^2/\text{s}$
- $\vartheta(20°\text{C}) = \left(\frac{1.000 \times 10^{-3}}{1000}\right) = 1.000 \times 10^{-6} \text{m}^2/\text{s}$
- $\vartheta(100°\text{C}) = \left(\frac{0.277 \times 10^{-3}}{1000}\right) = 0.277 \times 10^{-6} \text{m}^2/\text{s}$

For Air $\vartheta = \left(\frac{\mu}{1.2}\right)$:

- $\vartheta(10°\text{C}) = \left(\frac{1.760 \times 10^{-5}}{1.2}\right) \approx 14.67 \times 10^{-6} \text{m}^2/\text{s}$
- $\vartheta(20°\text{C}) = \left(\frac{1.809 \times 10^{-5}}{1.2}\right) = 15.08 \times 10^{-6} \text{m}^2/\text{s}$
- $\vartheta(100°\text{C}) = \left(\frac{2.171 \times 10^{-5}}{1.2}\right) = 18.09 \times 10^{-6} \text{m}^2/\text{s}$

Fluid Medium	**Viscous Metric**	**10 °C**	**20 °C**	**100 °C**
Water (Liquid)	Dynamic μ (Pa·s or PI)	1.300×10^{-3}	1.000×10^{-3}	0.277×10^{-3}
	Kinematic ϑ (m²/s)	1.300×10^{-6}	1.000×10^{-6}	0.277×10^{-6}
Air (Gas)	Dynamic μ (Pa·s or PI)	1.760×10^{-5}	1.809×10^{-5}	2.171×10^{-5}
	Kinematic ϑ (m²/s)	14.67×10^{-6}	15.08×10^{-6}	18.09×10^{-6}

Two prominent phenomena can be deduced from the data:

1. Opposing Temperature Trends: As temperature increases, the dynamic and kinematic viscosities of water decrease significantly. Conversely, the dynamic and kinematic viscosities of air increase with rising temperatures.
2. Kinematic Viscosity Inversion: While the dynamic viscosity of water is orders of magnitude larger than that of air ($\mu_{water} \gg \mu_{air}$), the kinematic viscosity of air is substantially greater than that of water ($\vartheta_{water} \gg \vartheta_{air}$). This inversion is due to the dominant effect of the extremely low density of air relative to water.

The fundamental difference in behavior originates from the distinct microscopic structures and intermolecular force dynamics characteristic of the liquid and gaseous phases:

-In liquids, molecules are tightly packed, and viscosity is primarily dictated by intermolecular cohesive forces (such as hydrogen bonds in water). When thermal energy increases (higher temperature), the molecular kinetic energy rises, allowing molecules to overcome these cohesive bonds more easily. Consequently, the internal resistance to shear flow decreases, leading to a drop in viscosity.

-In gases, molecules are widely spaced, rendering intermolecular cohesive forces negligible. Here, viscosity is governed by the momentum transfer via molecular collisions. As the temperature increases, the chaotic thermal velocities of gas molecules increase, leading to a higher frequency of intermolecular collisions. This heightened microscopic collision rate impedes macroscopic fluid layers from sliding past one another, thereby increasing the fluid's internal resistance and viscosity.

Solution 3

Dynamic viscosity $\mu = 95 \times 10^{-3} Pa.s$, Specific gravity d = 0.95, Temperature $T = 20$°C

Calculate the density of the liquid (ρ)

$$\rho = 0.95 \times 1000\ \text{kg/m}^3$$

Calculate kinematic viscosity (η) in SI units

$$\eta = \frac{95 \times 10^{-3} pa.s}{950\ \text{kg/m}^3} = 10^{-4} m^2/s$$

Convert to Stokes (St)

Since $1St = 10^{-4} m^2/s$

Therefore $\eta = 1\text{St}$

The kinematic viscosity of the liquid is 1St.

Solution 4

1. Determination of the Flow Regime

To characterize the fluid flow regime (laminar, transitional, or turbulent), we must evaluate the dimensionless Reynolds number (Re). For internal pipe flow, it is defined as:

$$\text{Re} = \frac{\nu \, . \, \text{D}}{\eta}$$

Where:

- $\nu = 4.50 \, m/s$ (Average flow velocity)
- $\text{D} = 15.0 \, cm$ (Internal diameter of the pipe)
- $\eta = 0.157 \, \text{St}$ (Kinematic viscosity of the air)

$$\eta = 1.57 \times 10^{-5} m^2/s$$

Calculation of Reynolds Number

Substituting the parameters into the Reynolds number formula yields:

$$\text{Re} = \frac{4.50 \times 0.150}{1.57 \times 10^{-5}} \approx 42.993.63$$

Interpretation:

In pipe flow engineering, the standard critical thresholds for the Reynolds number are:

- $Re < 2000$: Laminar flow
- $2000 \leq Re \leq 3000$:Transitional flow
- $Re > 3000$: Turbulent flow

Since the computed value $Re \approx 42994$ significantly exceeds 4000, the fluid flow is firmly within the turbulent regime.

2. Calculation of the Volumetric Flow Rate (Q)

The volumetric flow rate (Q) represents the volume of fluid passing through the cross-sectional area of the pipe per unit time. It is expressed by the continuity equation:

$$Q = s \, . \, \nu$$

Where s is the cross-sectional area of the circular pipe, defined as:

$$S = \frac{\pi . D^2}{4} = \frac{\pi . (0.150)^2}{4} \approx 0.01767 m^2$$

Compute Volumetric Flow Rate (Q):

$$Q = 0.01767 m^2 \times 4.50 \, m/s \approx 0.07952 \, m^3/s$$

Solution 5

1. Calculation of the Water Flow Rate (Q) The volumetric flow rate (Q) characterizes the volume of fluid (ΔV) passing through a given cross-section per unit of time (Δt).

$$Q = \frac{\Delta \mathrm{V}}{\Delta \mathrm{t}} = 1.19 \times 10^{-4} m^3/s$$

2. Determination of the Flow Velocity (ν)

According to the principle of continuity, the volumetric flow rate relates to the average flow velocity (ν) and the cross-sectional area (A) of the conduit is written as follows:

$$Q = A \times \nu$$

Cross-Sectional Area (A)

The problem notes that a section of the hose with a length (L) of 53 cm holds a fixed internal capacity volume ($\mathrm{V}_{\mathrm{internal}}$) of 0.006 m^3. Modeling the hose as a uniform cylinder:

$$\mathrm{V}_{\mathrm{internal}} = A \, . \, L \;\Rightarrow A = \frac{\mathrm{V}_{\mathrm{internal}}}{L}$$

Converting the length metric to meters (L = 53 cm = 0.53 m) :

$$A = \frac{0.006 m^3}{0.53 \text{ m}} \approx 0.011321 m^2$$

Step B: Calculate the Average Velocity ($\boldsymbol{\nu}$)

Substituting the calculated values of Q and A back into the velocity expression:

$$\nu = \frac{1.19 \times 10^{-4} m^3/s}{0.011321 m^2} \approx 0.01051 \; m/s$$

3. Evaluation of the Flow Regime

Reynolds number (Re). For an internal pipe network, it is expressed as:

$$Re = \frac{\rho . \nu . \mathrm{D}}{\mu}$$

Knowing that : $\boldsymbol{\rho} := \mathbf{1000} \boldsymbol{kg/m^3}$, $\mu_{\boldsymbol{eau}} = \mathbf{10^{-3}} \boldsymbol{Pa\ s}$

And : $D = \sqrt{\frac{4.A}{\pi}} = \sqrt{\frac{4 \times 0.011321}{\pi}} \approx 0.12006\text{m}$

Substituting the parameters into the Re formula:

$$Re = \frac{1000 \times 0.01051 \times 0.12006}{10^{-3}} = 1261.8$$

In pipe hydraulics, the flow regime boundaries are defined as:

- $Re < 2300$: Laminar flow
- $2300 \leq Re \leq 4000$: Transitional flow
- $Re > 4000$: Turbulent flow

Because the calculated value (Re≈1262) falls well below the lower critical threshold of 2300, viscous forces dominate over inertial perturbations. Consequently, the fluid stream inside the hose is confirmed to be in a stable, laminar regime.

Solution 6

Part 1: Plastic Ball

Terminal Velocity (ϑ)

Assuming the ball quickly reaches its steady terminal settling velocity during its fall:

$$\vartheta = \frac{\mathrm{L}}{\mathrm{t}} = \frac{1}{1.63} \approx 0.6135\mathrm{m/s}$$

1. Determination of the Flow Regime

The flow regime surrounding a falling sphere is characterized by the particle Reynolds number R_e:

$$R_e = \frac{\nu.\mathrm{d}}{\eta} = \frac{0.6135\mathrm{m/s} \times 2 \times 10^{-3} m}{23 \times 10^{-4}\mathrm{m}^2/\mathrm{s}} \approx 0.533$$

For flow past a sphere, the boundary for the Stokes' flow regime is typically defined as laminar up to 1, so $R_{ep} \approx 0.533$, the flow regime is classified as laminar (Stokesian).

2. Calculation of the Drag Force (F_d)

Because the system operates within the laminar Stokes regime (Base $R_{ep} < 1$), the drag force acting on the sphere can be computed directly using Stokes' Drag Law:

$$F_d = 3.\pi.\mu.d.\nu$$

Where μ is the dynamic viscosity of the fluid, related to kinematic viscosity (η) by

$$\mu = \eta.\rho_f = 23 \times 10^{-4} \times 800 = 1.84\ Pa\ s$$

Substituting μ into Stokes' Law:

$$F_d = 3\pi \times (1.84\ Pa\ s) \times (2 \times 10^{-2} m) \times (0.6135 m\backslash s) \approx 0.0425 N$$

3. Calculation of the Friction (Drag) Coefficient (C_x)

The dimensionless drag coefficient (often denoted as C_x or C_d) represents the ratio of drag force to dynamic pressure forces:

$$C_x = \frac{F_d}{\frac{1}{2}\rho_f . \nu^2 . A}$$

Where A is the projected frontal area of the sphere. Alternatively, for the theoretical Stokes regime, C_x can be calculated directly from the Reynolds number via:

$$C_x = \frac{24}{R_{ep}}$$

Using the Reynolds relation:

$$C_x = \frac{24}{0.5334} \approx 45.0$$

Part 2: Steel Ball

Preliminary Calculation: Terminal Velocity (ν)

$$\nu = \frac{L}{t} = \frac{1}{0.53\ s} \approx 1.8868 m\backslash s$$

1. Determination of the Flow Regime

We recalculate the particle Reynolds number (Base R_{ep}) for the steel ball:

$$R_{ep} = \frac{\nu . d}{\eta} = \frac{1.8868 m/s \times 13 \times 10^{-3} m}{0.23 \times 10^{-4} m^2/s} \approx 1066.4$$

For external flow over spheres, the regime boundaries are categorized as:

- $Re < 1$: Laminar (Stokes) Regime
- $1 \leq Re \leq 1000$: Transitional Regime
- $1000 < Re < 2 \times 10^5$: Turbulent Regime (characterized by a relatively constant drag coefficient)

Since : $Re \approx 1066$, the flow regime has transitioned out of the laminar domain into the turbulent regime.

2. Calculation of the Drag Force (F_d)

Given that the flow is within the Newtonian (external turbulent) regime, we use the generalized quadratic engineering drag equation to calculate the drag force, using the provided drag coefficient $C_x = 0.48$

$$F_d = \frac{1}{2}. \mathrm{Cx}. \rho_f. A. v^2$$

Where:

$$A = \frac{\pi. d^2}{4} \approx 1.3273 \times 10^{-4} m^2$$

Thus;

$$F_d = \frac{1}{2} \times 0.48 \times 800 \times (1.8868)^2 \times (1.3273 \times 10^{-4}) \approx 0.0907N$$

<u>Solution 7</u>

<u>1. What is the regime of the tap water</u>

To determine the flow regime of the tap water, we need to calculate the Reynolds number (Re_{tap}) inside the tap nozzle.

<u>Find the flow velocity</u> (v_{tap})

Using the volumetric flow rate formula ($Q = A \cdot v_{tap}$):

$$v_{tap} = \frac{Q}{A_{tap}} = \frac{1.4 \times 10^{-3}}{12.56 \times 10^{-4}} \approx 1.115 \mathrm{m/s}$$

Find the equivalent diameter of the tap (D_{tap})

$$D_{tap} = \sqrt{\frac{4. A_{tap}}{\pi}} = \sqrt{\frac{4 \times 12.56 \times 10^{-4}}{\pi}} \approx 0.04 \mathrm{~m}$$

Reynolds number

$$Re = \frac{\rho. v. D}{\mu}$$

$$Re = \frac{1000 \times 1.115 \mathrm{~m/s} \times 4 \times 10^{-2} m}{10^{-3}\, Pa\, s} \approx 4.44 \times 10^4$$

Since: Re > 4000, the flow is: Turbulent regime

2. Regime of the ball flow

Height of the water column

The total volume filled is: V= Q t , V= $(1.4\times10^{-3})(20)$, V=0.028 m^3

For the cylindrical tube $V = \frac{\pi\, h\, D^2}{4}$

Thus: $h = \frac{4V}{\pi\, D^2} = 1.58$ m

Velocity of the ball

$$v = \frac{h}{t} = \frac{1.58}{0.98} = 1.61 \text{ m/s}$$

Reynolds number of the ball

$$Re = \frac{\rho.\nu.\mathrm{D}}{\mu}$$

$$Re = \frac{1000 \times 1.61 \times 0.016}{10^{-3}\ Pa\ s} \approx 2.58 \times 10^4$$

Since Re > 1000, the external flow regime is: Turbulent regime

Drag force

The drag force formula is:

$$F_d = \frac{1}{2}.\mathrm{Cx}.\rho_{\mathrm{f}}.A.\nu^2$$

where A is the projected area of the sphere:

:

$$A = \frac{\pi.D^2}{4} \approx 2.01 \times 10^{-4} m^2$$

Now substitute:

$$F_d = \frac{1}{2} \times 0.48 \times 1000 \times (1.61)^2 \times (2.01 \times 10^{-4}) \approx 0.13N$$

Chapter II: Fluid Statics

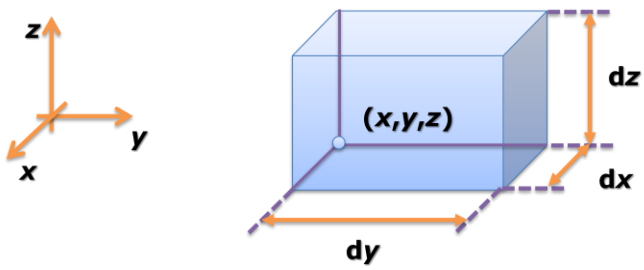
II.1 Definition

Fluid statics is the study of the equilibrium conditions of fluids and their behavior in situations where there is an absence of relative motion between the particles that constitute them:

- Fluids at rest / Zero acceleration
- Fluids in motion without friction: the inviscid fluids
- The surface forces involved are forces due only to pressure.

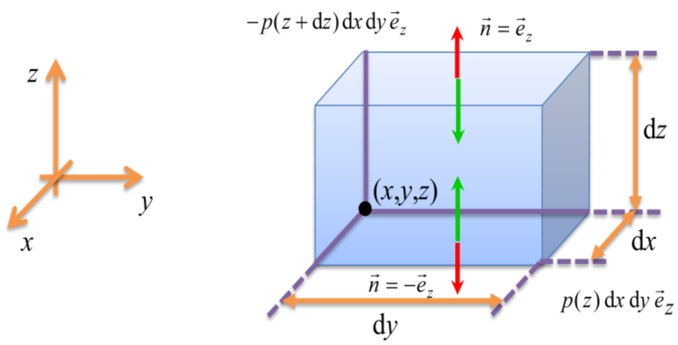
II.2 The fundamental equation of statics

One considers an element of fluid volume of parallelepipedic form, of volume dV = dx.dy.dz, in a Cartesian coordinate system:



The balance of the forces applied to this volume element makes it necessary to distinguish:

- volume forces: weight
- surface forces: pressure forces (stagnant fluid)



volume forces

The expression of the fluid weight is given by:

$$\overrightarrow{dP} = dm\,\vec{g} = \rho\, dV\,\vec{g}$$

Surface forces

This force can be broken down into three components:

$$\overrightarrow{dF} = \overrightarrow{dF_x} + \overrightarrow{dF_y} + \overrightarrow{dF_z} \Longrightarrow \overrightarrow{dF} = dF_x\,\vec{e}_x + dF_y\,\vec{e}_y + dF_z\,\vec{e}_z$$

Let's look at the component $\overrightarrow{dF}_Z$ (the surface forces are necessarily normal), the component along z corresponds to the pressure forces exerted on the surfaces perpendicular to the z-axis.

$$\overrightarrow{dF}_Z = dF_Z\overrightarrow{e_Z} = F_Z(z)\overrightarrow{e_Z} - F_Z(z+dz)\overrightarrow{e_Z} = P(z)dxdy\overrightarrow{e_Z} - P(z+dz)dxdy\overrightarrow{e_Z}$$

So : $dF_z = [p(z) - p(z+dz)]\, dx\, dy$

the development of the first order gives:

$$p(z+dz) = p(z) + \frac{\partial p}{\partial z}dz$$

where : $$dF_z = -\frac{\partial p}{\partial z}dx\,dy\,dz = -\frac{\partial p}{\partial z}dV$$

By analogy, the other two axes give:

$$\begin{cases} dF_x = -\dfrac{\partial p}{\partial x}dV \\ dF_y = -\dfrac{\partial p}{\partial y}dV \end{cases}$$

Hence, the surface force can be written as:

$$\overrightarrow{dF} = -\underbrace{\left(\frac{\partial p}{\partial x}\vec{e}_x + \frac{\partial p}{\partial y}\vec{e}_y + \frac{\partial p}{\partial z}\vec{e}_z\right)}_{\vec{\nabla} p \equiv \overrightarrow{\text{grad}}\, p} dV$$

Then : $\overrightarrow{dF} = -\vec{\nabla} p \, dV$

weight force $\overrightarrow{dP} = \rho \, dV \, \vec{g}$ And $\overrightarrow{dF} = -\vec{\nabla} p \, dV$

According to the fundamental principle of dynamics, the sum of the forces acting on the fluid particle is equivalent to the product of its mass and its acceleration:

$$\overrightarrow{dP} + \overrightarrow{dF} = \rho \, dV \, \vec{a}$$

by substituting, it turns : $\rho \, dV \, \vec{g} - \vec{\nabla} p \, dV = \rho \, dV \, \vec{a}$

From where we get : $\rho \vec{g} - \vec{\nabla} p = \rho \vec{a}$ → The fluid is at rest: $\vec{a} = \vec{0}$

In this case : $\vec{\nabla} p = \rho \vec{g}$ It is the fundamental equation of fluid statics (Local equation)

In the case of a fluid at rest, assuming that $\vec{g} = -g \, \vec{e}_z$ we have :

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \text{ and } \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad p(x, y, z) = p(z)$$

From where: $\frac{dp}{dz} = -\rho g$ It is *the differential equation of fluid statics*

This differential equation should be solved to find the pressure at any point of stagnant fluid

II.2.1 Statics of Incompressible Fluids

A fluid is said to be incompressible if we can consider that its density is at all points the same:

$$\rho = C^{te}$$

In addition, we can consider that the acceleration of gravity is a constant:

$$g = C^{te}$$

As a result :

$$\frac{dp}{dz} = -\rho g = C^{te}$$

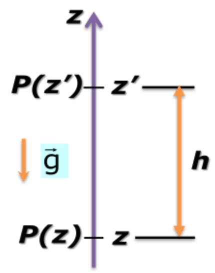
By integration:

$$p(z) = \int \frac{dp}{dz} dz = -\int \rho g\, dz = -\rho g \int dz = -\rho g z + C^{te}$$

then :

$p(z) + \rho g z = C^{te}$ This is the fundamental law of hydrostatics

II.2.2 Hydrostatics theorem



According to the law of hydrostatics, we have:

$$p(z) + \rho g z = C^{te} = p(z') + \rho g z'$$

So :

$$p(z) - p(z') = \rho g (z' - z)$$

$$p(z) - p(z') = \rho g h$$

Generally, the reference level z' = 0 corresponds to the free surface of the fluid, where

$P(0) = P_0 = P_{atm}$

So : $p = p_0 + \rho g h$

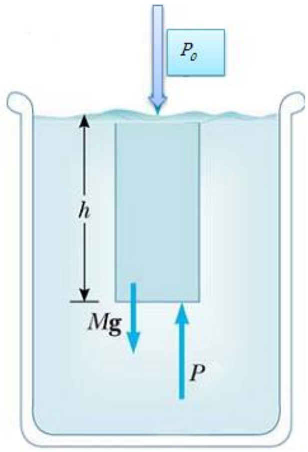
fluid depth below reference level

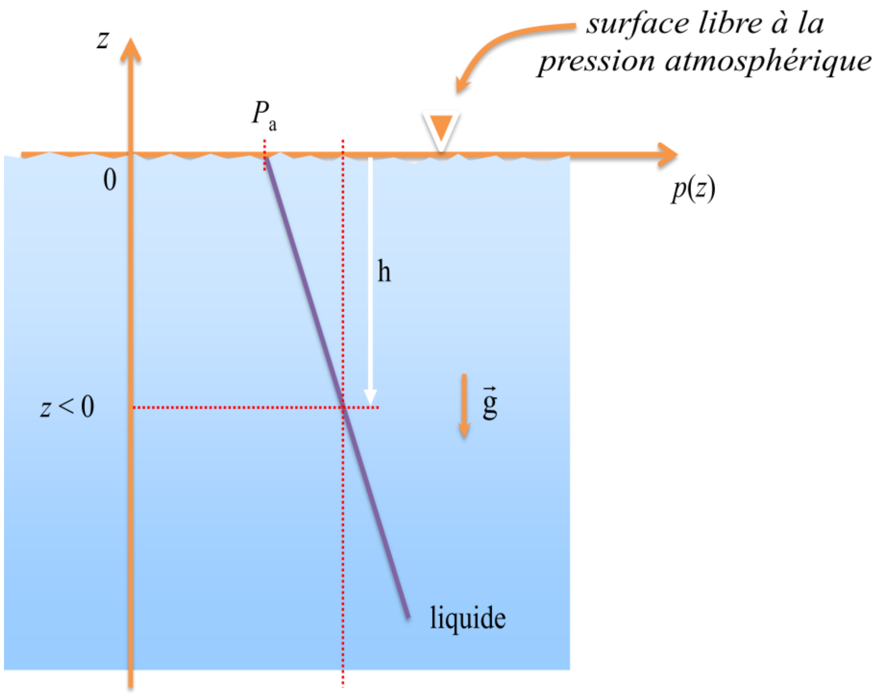
Standard atmospheric pressure: p_{atm} = 1.013.105 Pa

- **Hydrostatics theorem statement**

The difference in pressure between two points of a liquid at equilibrium is equal to the weight of a column of this liquid whose height is the difference in levels between these two points and whose cross-section equals the unit.

$$p - p_0 = \rho g h$$





$$p = p_a - \rho g z$$

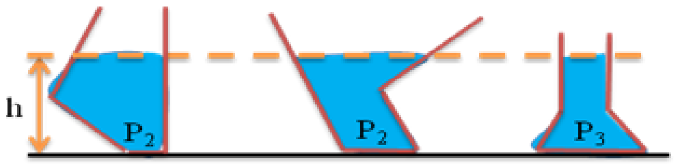
$$p = p_a + \rho g h$$

The pressure increases with the depth in the liquid, and the position of the point in the liquid is identified with respect to the free surface of the liquid

II.3 Consequences of the fundamental theorem of hydrostatics

II.3.1 Equality of pressures at any point of a horizontal plane

- ✓ The pressures at two points M and M' located in the same horizontal plane are equal
- ✓ horizontal planes in a liquid are isobaric planes
- ✓ The pressure at the bottom of a container does not depend on the shape of the container but only on the height of the liquid.

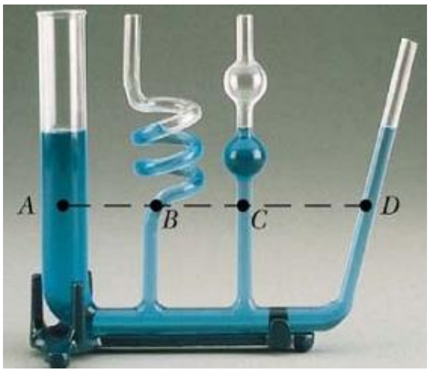


liquid level

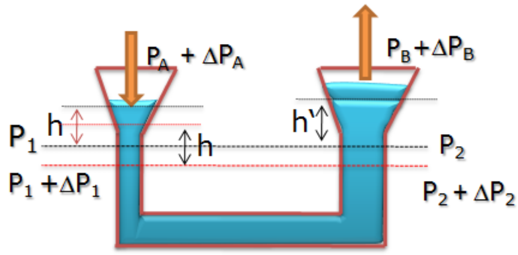
P1 = P2 = P3

- ✓ The free surfaces of communicating vessels are on the same horizontal plane

Horizontal surface ≡ Pressure equality



II.3.2 Pascal's theorem



$$P_1 = P_A+\rho gh \qquad P_2 = P_B+\rho gh' \qquad P_1 = P_2$$

$$P_1+\Delta P_1 = P_A + \Delta P_A + \rho gh \longrightarrow \Delta P_1 = \Delta P_A$$

$$P_1+\Delta P_1 = P_2 + \Delta P_2 \longrightarrow \Delta P_2 = \Delta P_1 = \Delta P_A$$

$$P_2+\Delta P_2 = (P_B+\Delta P_B) + \rho gh' \longrightarrow \Delta P_B = \Delta P_2 = \Delta P_A$$

States :

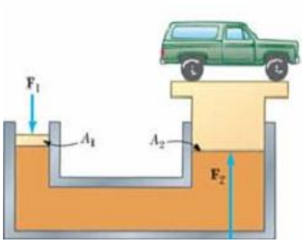
At the equilibrium, any variation of Pressure at one point in an incompressible fluid causes the same pressure change at any other point.

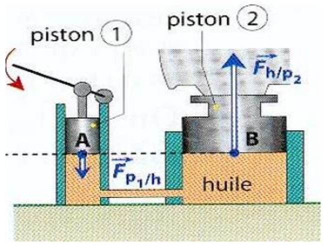
We have $p = \frac{F_A}{S_A} = \frac{F_B}{S_B}$

$$S_B \rangle S_A \Rightarrow F_B = \frac{F_A S_B}{S_A} \Rightarrow F_B \rangle\rangle F_A$$

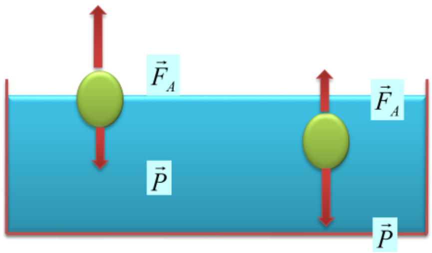
The applied force F_A can be amplified by choosing $S_A << S_B$

<u>Examples of applications of Pascal's theorem</u>: Hydraulic lifter, Hydraulic jack, Hydraulic press, Hydraulic brake, Elevator





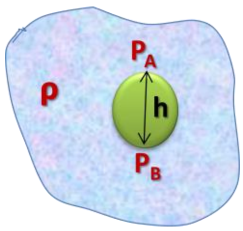
II.3.3 Archimedes principle



$P_B = P_A + \rho gh$ → $P_B - P_A = \rho gh$

→ $(P_B - P_A)\, S = \rho ghS$

→ $F_{ARCH} = \rho gV$



V is the volume of the displaced fluid and equals to the volume of the submerged body, ρ its density.

The pressure difference between the top and the bottom of the solid generates a buoyant force directed towards the decreasing pressures (from the bottom to the top). It is called *Archimedes' thrust*.

States (Archimedes' principle)

Anybody immersed in a fluid receives from this fluid a vertical force (buoyant force), upwards, the intensity of which is equal to the weight of the fluid displaced volume (this volume is, therefore, equal to the submerged volume of the body).

$$F_{ARCH} = \rho_{fluid} . V_{imm} . g$$

Apparent weight

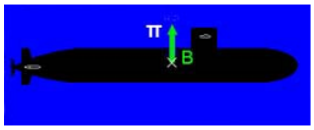
It is the difference between the body's actual weight and the force of Archimedes (the weight of the body inside the fluid).

$$\text{Papp} = \text{P} - \text{F}_{\text{ARCH}} \rightarrow \text{Papp} = \text{Vg}\,(\rho - \rho_{\text{fluid}})$$

$P_{App} > 0$ → Submerged body

$P_{App} < 0$ → Float body

$P_{App} = 0$ → Suspended body



II. 4 Statics of compressible fluids

Generally, this is true of gases, since their density depends on pressure. To simplify the study, we tackle the case of the perfect gases:

$$pV = n\text{R}T$$

Molar mass of the gas

so : $p = \dfrac{n\text{R}T}{V}$, however : $\rho = \dfrac{m}{V} = \dfrac{n\text{M}}{V} \Rightarrow \dfrac{n}{V} = \dfrac{\rho}{\text{M}}$

From where : $p = \dfrac{\rho \text{R}T}{\text{M}} \Rightarrow \rho = \dfrac{\text{M}}{\text{R}T}\,p$ *The density is a function of the pressure ⇒ compressibility*

Starting from the fundamental equation of fluid statics as follows:

$$\frac{\text{d}p}{\text{dz}} = -\rho(p)\,\text{g} \Rightarrow \frac{\text{d}p}{\text{dz}} = -\frac{\text{M}}{\text{R}T}\,p\text{g} \Rightarrow \frac{\text{d}p}{p} = -\frac{\text{M}}{\text{R}T}\,\text{g}\,\text{dz}$$

It is therefore necessary to integrate:

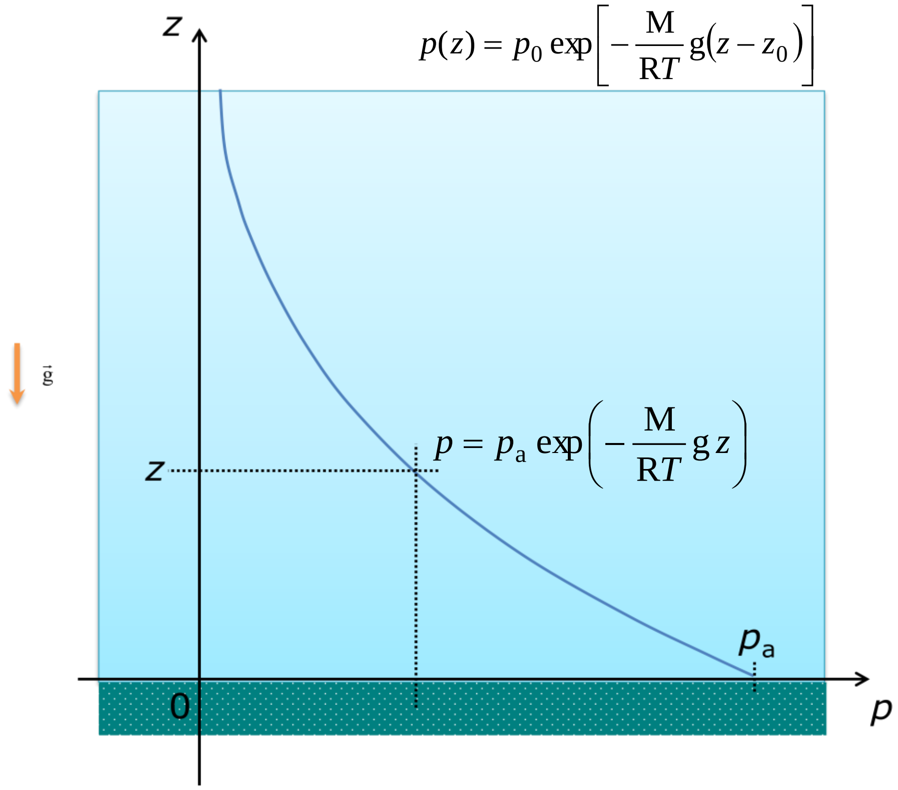
$$\int \frac{dp}{p} = -\int \frac{M}{RT} g\, dz + C^{te}$$ *is constant if the temperature is homogeneous*

As : $\ln p = -\frac{M}{RT} g z + C^{te}$

so: $p(z) = C^{te} \exp\left(-\frac{M}{RT} g z\right)$ where the constant is defined with respect to the pressure at the reference level.

Thus, if p = p_0 at z = z_0, then: $p_0 = C^{te} \exp\left(-\frac{M}{RT} g z_0\right) \Rightarrow C^{te} = p_0 \exp\left(\frac{M}{RT} g z_0\right)$

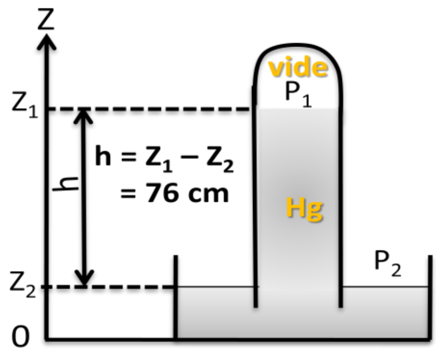
Therefore : $p(z) = p_0 \exp\left[-\frac{M}{RT} g (z - z_0)\right]$



Application: Atmospheric pressure measurement (Torricelli's mercury barometer)

Atmospheric pressure is the force exerted by atmospheric air per unit area.

In this experiment, Torricelli produced a vacuum by filling a glass tube with mercury, inverting it, and then submerging the open end in a vessel of mercury. He observed that the mercury column descended only partially, stopping at a height (independent of the tube's shape and inclination) of about 76 cm.



$P_1 = 0$

P_2 = atmospheric pressure.

$\rho_{mercury} = 13600\ kg.m^{-3}$

$P_2 - P_1 = \rho.gh = 13600 \times 9.81 \times 0.76 = 101396.16$ Pa

II.5 Exercises

Exercise 1

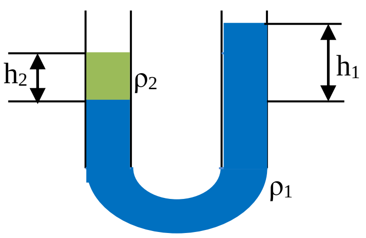
A vertical tube with a 1 cm2 cross-sectional area contains 2 L of water. A piston weighing 200g is placed on this water

- Calculate the pressure at a point located 10 cm from the bottom.
- Calculate the pressure on the bottom of the tube as well as the pressing force.

A mass of 100 g is placed on the piston. Find the new pressures, as well as the pressing force on the bottom.

Exercise 2

Consider the U-tube in the opposite figure containing two liquids of densities ρ_1 and ρ_2. Determine the relationship between h_1, h_2, ρ_1, ρ_2



Exercise 3

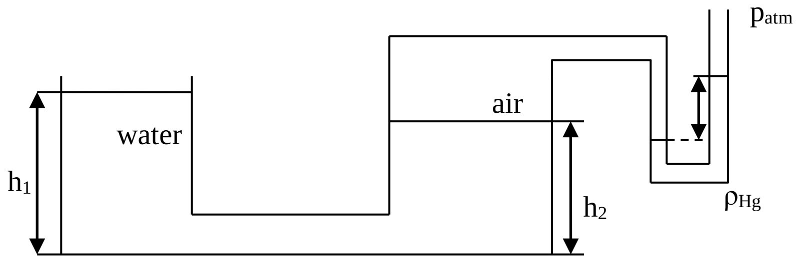
A U-tube of uniform section contains mercury. In branch A, we pour water, and in branch B, we pour alcohol. It can be seen that the free surfaces of the water and the alcohol are in the same horizontal plane and that the mercury has a level difference of 0.5 cm between the two branches. Calculate the heights h and h' of water and alcohol.

We give: $\rho_{mercury}$ =13.6 g.cm^{-3}

$\rho_{alcohol}$ = 0.8 g.cm^{-3}

Exercise 4

Consider the manometer tube shown in the figure below. Determine the height difference h of mercury.



Exercise 5

A hydraulic jack formed by two pistons (1) and (2) of circular section. Under the effect of an action on the lever, the piston (1) acts, at point (A), by a force of pressure F_1 on the oil. The oil acts, at point (B) on the piston (2) by a force F_2 .

We give:

- The diameters of each of the pistons: D_1 = 10 mm; D_2 = 100 mm.

- The pressure force intensity at A: F_1 = 150 N.

1) Determine the oil pressure P_A at point A.

2) What is the P_B pressure?

3) Deduce the intensity of the pressure force F_2

Exercise 6

A block of ice with a volume of 500 cm^3 floats on the surface of the water. Calculate the submerged volume knowing that the density of the ice is 0.92 g.cm^{-3}.

Exercise 8

A gold object weighs 138.5 g; immersed in pure water its apparent weight is 130.5 g.

Show that it is not pure gold, then determine the composition of the alloy of gold and silver which constitutes it

We give: ρ_{or} = 19.25 g.cm^{-3}

ρ_{silver} = 10.5 g.cm^{-3}

Exercise 9

A sphere of radius R = 10 cm floats halfway on the seawater surface (density ρ_{sea}=1025 kg/m3).

1) Determine its weight P.
2) Its density.
3) What will be the fraction of the submerged volume if this sphere floated on the surface of the oil (density ρ_{oil}=800 kg/m^3)?

Exercise 10

The figure below presents a lead solid sphere (1) of radius R_1 = 10 mm suspended, by means of a flexible and light wire (3), to a float (2) in the form of a hollow sphere in plastic material of radius R_2=35 mm and of thickness e = 5 mm.

We give:

- The density of seawater: ρ =1027 kg/m^3

- The density of lead: ρ_1 =11340 kg/m^3,

- The density of the float material: ρ^2 =500 kg/m^3,

- The acceleration due to gravity g = 9.81m.s^{-2}.

Required responses:

1) Calculate the weight P_1 of the sphere (1).

2) Determine the Archimedes thrust $F_{1,}$ which acts on the sphere (1).

3) Write the equilibrium equation of the sphere (1). Deduce the tension T of the thread.

4) Calculate the weight P_2 of the float (2).

5) Write the equilibrium equation of the float. Deduce Archimedes thrust F_2 acting on the sphere (2).

6) Deduce the fraction % of the submerged volume of the float.

II. 6 Exercise Solutions

Solution1

1. Identify the reference interface**:** Let the horizontal line passing through the boundary where liquid 1 and liquid 2 meet (on the left side of the U-tube) be our reference level.
2. Equate the pressures at this level:

Left branch pressure (P_{left}): The pressure is due to the atmospheric pressure (P_{atm}) plus the weight of the column of liquid 2 with height h_2.

$$P_{left} = P_{atm} + \rho 2\ g\ h2$$

Right branch pressure (P_{right}): The pressure at the exact same horizontal level in the right branch is due to (P_{atm}) plus the column of liquid 1 with height h_1.

$$P_{right} = P_{atm} + \rho 1 g\ h1$$

3. Find the relationship**:** Since $P_{left} = P_{right}$:

$$P_{atm} + \rho 2\ g\ h2 = P_{atm} + \rho 1 g\ h1$$

$Patm$ from both sides and divide by g :

$$\rho 1 g\ h1 = \rho 2\ g\ h2$$

The relationship between the variables is:

$$\frac{h1}{h2} = \frac{\rho 2}{\rho 1}$$

Solution2

Given Data

$\rho_{mercury} = 13.6\ g/cm^3$
$\rho_{alcohol} = 0.8\ g/cm^3$
$\rho_{water} = 1.0\ g/cm^3$
$Mercury\ level\ difference\ \Delta\ h = 0.5\ cm$

Let h be the height of the water column in branch A
Let h' be the height of the alcohol column in branch B.

Because the free surfaces (tops) of the water and alcohol are in the same horizontal plane, and the mercury is 0.5 cm higher on the alcohol side (meaning the water pushes the mercury down more because water is denser than alcohol):

$$h = h' + 0.5 \quad implies\, h - h' = 0.5 \qquad (Eqt\ 1)$$

Equating the pressures at the lowest interface (the water-mercury boundary in branch A):

$$\rho_{water}.\text{g}.\text{h} = \rho_{alcohol}.\text{g}.\text{h}' + \rho_{mercury}.\text{g}.\ \Delta\,\text{h}.$$

Cancel out g and substitute the known densities and $\Delta h = 0.5$:

$$1.0.h = 0.8.h' + (13.6.0.5)$$

$$h = 0.8h' + 6.8 \qquad (Eqt\ 2)$$

Substitute Eqt 1 ($h = h' + 0.5$) into Eqt 2:

$$\begin{aligned} h' + 0.5 &= 0.8h' + 6.8 \\ h' - 0.8h' &= 6.8 - 0.5 \\ 0.2h' &= 6.3 \\ h' = \frac{6.3}{0.2} &= 31.5\ cm \end{aligned}$$

Now, find h using Equation 1:

$$h = 31.5 + 0.5 = 32\ cm$$

Solution 3

We use the hydrostatic equation along the continuous fluid paths. Since the pressure of the trapped air is uniform throughout its chamber, we can connect the water side to the mercury side.

The water column has a total height of h_1. The interface with the air is at height h_2.

Therefore, the height of the water column pressing down above that interface is $h_1 - h_2$.

Assuming the open container on the far left is exposed to atmospheric pressure P_{atm}:

$$P_{air} = P_{atm} + \rho_{water}.g.(h_1 - h_2)$$

The mercury column of height h is open to the atmosphere (P_{atm}) at the top and meets the air chamber at the bottom interface.

$$P_{air} = P_{atm} + \rho_{Hg}.g.h$$

Since P_{air} is equal in both expressions, set them equal to each other:

$$P_{atm} + \rho_{water}.g.(h_1 - h_2) = P_{atm} + \rho_{Hg}.g.(h)$$

Subtract P_{atm} and cancel out the gravitational acceleration (g) from both sides:

$$\rho_{water} \, .(h_1 - h_2) = \rho_{Hg} \, .(h)$$

Isolate the height difference of mercury (h):

$$h = \frac{\rho_{water}}{\rho_{Hg}}.(h_1 - h_2)$$

<u>Solution 4</u>

1) Determine the oil pressure P_A at point A

First, calculate the cross-sectional area of piston 1 (A_1):

$$A_1 = \frac{\pi . D_1^2}{4} = \frac{\pi . (0.01)^2}{4} \approx 7.854 \times 10^{-5} \text{ m}^2$$

Now, calculate the pressure P_A:

$$P_A = \frac{F_1}{A_1} = \frac{150}{7.854 \times 10^{-5}} \approx 1.91 \times 10^6 \, Pa = (19.1 \, bar)$$

2) What is the P_B pressure?

According to Pascal's Principle, any change in pressure applied to an enclosed fluid is transmitted undiminished throughout the fluid to all points. Assuming the pistons are at roughly the same height:

$$P_B = P_A \approx 1.91 \times 10^6 \, Pa$$

3) Deduce the intensity of the pressure force F_2

First, calculate the cross-sectional area of piston 2 (A_2):

$$A_2 = \frac{\pi . D_2^2}{4} = \frac{\pi . (0.1)^2}{4} \approx 7.854 \times 10^{-3} \text{ m}^2$$

Using the pressure at point B to find F_2:

$$F_2 = P_B . A_2 = (1.91 \times 10^6).(7.854 \times 10^{-3}) = 15{,}000 \, N$$

<u>Solution 5</u>

The Equilibrium Principle

For any floating object, the upward buoyant force (Archimedes' thrust) equals the downward gravitational force (weight of the object):

$$F_{buoyant} = W_{ice}$$

Expanding this using densities and volumes:

$$\rho_{water} . g . V_{submerged} = \rho_{ice} . g . V_{total}$$

Cancel out the gravitational acceleration (g) from both sides:

$$\rho_{water} . V_{submerged} = \rho_{ice} . V_{total}$$

Isolate the submerged volume ($V_{submerged}$):

$$V_{submerged} = \frac{\rho_{ice}}{\rho_{water}} . V_{total}$$

Substitute the given values:

$$V_{submerged} = \frac{0.92}{1.0} .500 = 460 cm^3$$

The submerged volume of the ice block is $460 cm^3$ (which is 92\% of its total volume).

Solution 6

Showing that it is not pure gold

1. Calculate the buoyant force (Fb): The buoyant force equals the weight of the displaced water, which matches the loss in weight when immersed:

$$F_b = W_{actual} - W_{apparent} = 138.5 \text{ g} - 130.5 \text{ g} = 8.0 \text{ g (mass equivalent)}$$

2. Determine the volume of the object (V_{total}): Since ρ_{water}=1.0 g/cm^3:

$$V_{total} = \frac{\text{Mass of displaced water}}{\rho_{water}} = \frac{8.0}{1.0} = 8.0 \text{ cm}^3$$

3. Calculate the average density (ρ_{avg}):

$$\rho_{avg} = \frac{\text{Mtotal}}{\text{Vtotal}} = \frac{138.5}{8.0} = 17.3125 \text{ g/cm}^3$$

Conclusion: Since 17.3125 g/cm^3 is less than the density of pure gold (ρ_{gold}=19.25 g/cm^3), the object is not pure gold.

Determine the composition of the alloy

Let V_{gold} be the volume of gold and V_{silver} be the volume of silver.

$$V_{gold} + V_{silver} = 8.0 - \text{ (Eq 1)}$$

Total Mass:

$$\rho_{gold}\text{Vgold} + \rho_{silver}\text{Vsilver} = 138.5$$

$$19.25\text{Vgold} + 10.5\text{Vsilver} = 138.5 - \text{ (Eq 2)}$$

$$\text{Vsilver} = 8.0 - \text{Vgold}$$

Substitute into Eq 2:

$$\text{Vgold} = 8.7554.5 \approx 6.23\ cm^3$$

$$\text{Vsilver} = 8.0 - 6.23 = 1.77\ cm^3$$

Calculate the mass composition:

Mass of Gold: Mgold=19.25×6.2286≈119.9 g (or 86.6% of total mass)

Mass of Silver: Msilver=10.5×1.7714≈18.6 g (or 13.4% of total mass)

Solution7

Determine its weight P

The sphere floats halfway on the seawater surface, meaning the submerged volume is exactly half of the total volume:

$$V_{sub} = \frac{V_{total}}{2} \approx 2.094 \times 10^{-3} m^3$$

According to the law of flotation, the weight of the object (P) equals the buoyant force from the seawater (F_b):

$$P = F_b = \rho_{sea} \cdot g \cdot V_{sub}$$
$$P = 1025 \times 9.81 \times (2.094 \times 10^{-3}) \approx 21.06\,N$$

2) Its density

For a floating object completely supported by a single liquid, the ratio of densities equals the ratio of volumes:

$$\frac{\rho_{sphere}}{\rho_{sea}} = \frac{V_{sub}}{V_{total}} = \frac{1}{2}$$

$$\rho_{sphere} = \frac{\rho_{sea}}{2} = \frac{1025}{2} = 512.5Kg\ \text{kg}/m^3$$

3) Fraction of submerged volume if floating on oil

When the sphere floats on oil, its weight (P) is balanced by the buoyant force of the oil:

$$\rho_{sphere}.\,g.\,V_{total} = \rho_{oil} \cdot g.\,V_{sub}$$

Cancel g and arrange to solve for the submerged volume fraction $\frac{V_{sub-oil}}{V_{Total}}$:

$$Fraction = \frac{V_{sub}}{V_{Total}} = \frac{\rho_{sphere}}{\rho_{oil}}$$

Substitute the values:

$$Fraction = \frac{512.5}{800} = 0.6406$$

The sphere will be 64.06% submerged when placed in oil.

Solution10

1) Calculate the weight P_1 of the sphere (1)

$$P_1 = \rho_1 .\, V_1 . g = 11340 \times (4.189 \times 10^{-6}) \times 9.81 = 0.466\ N$$

2) Determine the Archimedes thrust F_1 acting on the sphere (1)

Since the lead sphere is fully submerged in seawater:

$$F_1 = \rho.V1.\ g = 1027 \times (4.189 \times 10^{-6}) \times 9.81 = 0.042N$$

3) Equilibrium equation of sphere (1) and tension T

The forces acting vertically on the sphere are: Upward buoyant force (F_1), upward wire tension (T), and downward weight (P_1).

$$F_1 + T = P_1 \ implies \ T = P_1 - F_1$$
$$T = 0.466 - 0.042 = 0.424 \ N$$

4) Calculate the weight P2 of the float (2)

The mass of the float comes only from its plastic shell layer:

$$P_2 = 2 . V_{shell} . g = 500 \times (6.650 \times 10^{-5}) \times 9.81 = 0.326 \ N$$

5) Equilibrium equation of the float & Archimedes thrust F_2

The forces acting on the float are: Upward buoyant force (F_2), downward weight (P_2), and downward wire tension (T).

$$F_2 = P_2 + T$$
$$F_2 = 0.326 + 0.424 = 0.750 \ N$$

6) Deduce the fraction % of the submerged volume of the float

The Archimedes thrust on the float depends on its partially submerged volume (V_{sub2}):

$$F_2 = \rho . V_{sub2} . g \ \ implies \ V_{sub2} = \frac{F_2}{g}$$

Now, compute the percentage fraction relative to its total outer volume (V2):

$$Fraction \ \% = \frac{V_{sub2}}{v2} \times 100 = \frac{7.444 \times 10^{-5}}{1.796 \times 10^{-4}} \times 100 \approx 41.45\%$$

Chapter III: Fluid Kinematics

The kinematics of fluids concerns the description of the flows without resorting to the calculation forces involved.

III.1 Descriptions of Fluid Motion

➢ Description of Lagrange

This description of the flow consists in following a given particle during its movement within the fluid. With this description, all the unknowns of the problem (position, speed, pressure….etc.) are written according to (x_0; y_0; z_0; t) which are the coordinates of the fluid particle position in its initial configuration t = 0 s, and t represents time.

To know perfectly the evolution of the fluid, it is necessary to determine the 3 following functions:

$$x = x(x_0; y_0; z_0; t)$$

$$y = y(x_0; y_0; z_0; t)$$

$$z = z(x_0; y_0; z_0; t)$$

(x_0; y_0; z_0; t) are variables of Lagrange.

➢ Description of Euler

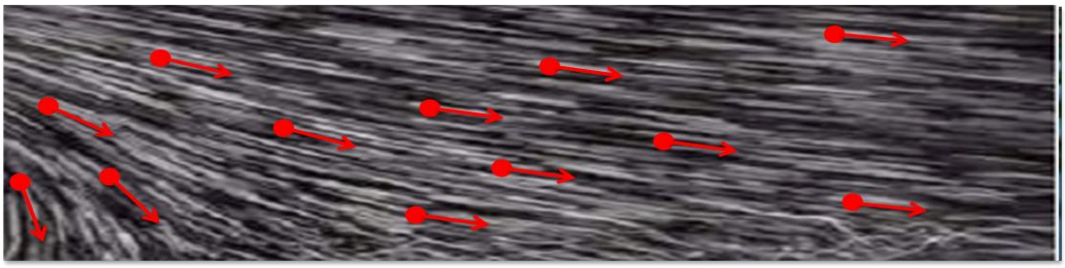
This description consists in knowing the velocity of the particles over time at a given place, determined by its coordinates, for example Cartesian coordinates x, y, z.

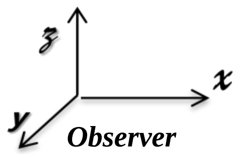
The fluid flow is described by means of a velocity vector field at every time t.

So :

$$\vec{V}(x,y,z,t) = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

$v_x = v_x(x,y,z,t)$, $v_y = v_y(x,y,z,t)$, $v_z = v_z(x,y,z,t)$

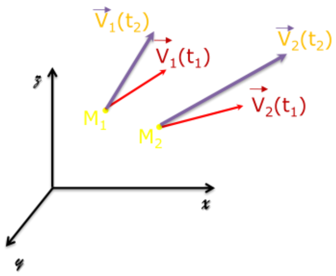




The velocity at all points (x,y,z) of the flow (Velocity field)

"instant photo of the flow"

The velocity $\vec{V}(t)$ associated to a point M evolves over time.



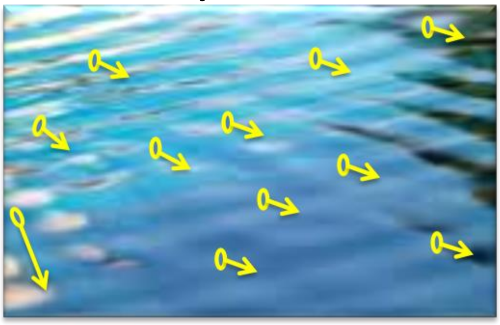
- the physical quantities of the flow are therefore defined according to the coordinates of space X Y Z and time t
- x, y, z coordinates, and time t represent Euler variables
- The Eulerian description is more adapted in practice than the Lagrangian description. the knowledge of the velocity field being sufficient for the description of the fluid in motion.

III.2 Steady flow

A flow is said steady or *permanent* when the velocity $\vec{V}$ vector field does not vary over time. The physical quantities of the flow do not depend explicitly on time.

Steady flow:

Velocity field at time t



Velocity field at time t'

Unsteady flow:

Velocity field at time t

Velocity field at time t'

III. 3 Material Derivative

Consider the scalar function Φ(x,y,z,t), a physical quantity characteristic of the fluid at a point of coordinates x, y, z and time t.

The fluid particle at time t+dt will be at the point of coordinates x+dx, y+dy, z+dz . The variation of the function will therefore be equal to:

$$d\Phi = \Phi(x+dx, y+dy, z+dz, t+dt) - \Phi(x, y, z, t)$$

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz + \frac{\partial\Phi}{\partial t}dt$$

first order development

Since

$$dx = vx\ dt$$

$$dy = vy\ dt$$

$$dz = vz\ dt$$

So :

$$d\Phi = \frac{\partial \Phi}{\partial x} v_x dt + \frac{\partial \Phi}{\partial y} v_y dt + \frac{\partial \Phi}{\partial z} v_z dt + \frac{\partial \Phi}{\partial t} dt$$

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} v_x + \frac{\partial \Phi}{\partial y} v_y + \frac{\partial \Phi}{\partial z} v_z$$

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \vec{v}\,\overrightarrow{grad}\,\Phi$$

Derivative $\frac{d\Phi}{dt}$ that we note $\frac{D\Phi}{Dt}$ is the particle derivative, (related to the particle), is equal to:

$$\frac{D\Phi}{Dt} = \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \vec{v}\,\overrightarrow{grad}\,\Phi = \frac{\partial \Phi}{\partial t} + v_x \frac{\partial \Phi}{\partial x} + v_y \frac{\partial \Phi}{\partial y} + v_z \frac{\partial \Phi}{\partial z}$$

Definition

The particulate derivative of a physical quantity defined by the field Φ(M,t) is the derivative with respect to time of a quantity attached to fluid particles (for example ; the density or the velocity). This derivative appears as the sum of two terms:

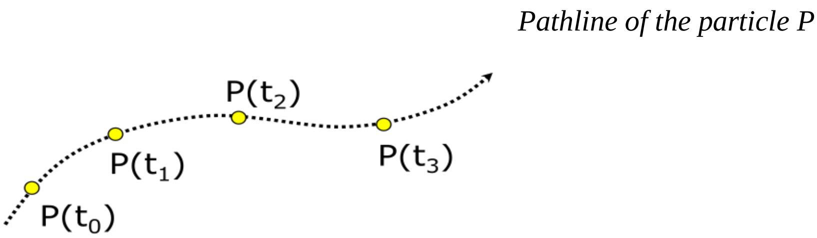
$$\frac{D\Phi}{Dt} = \underbrace{\frac{\partial \Phi}{\partial t}} + \underbrace{v_x \frac{\partial \Phi}{\partial x} + v_y \frac{\partial \Phi}{\partial y} + v_z \frac{\partial \Phi}{\partial z}}$$

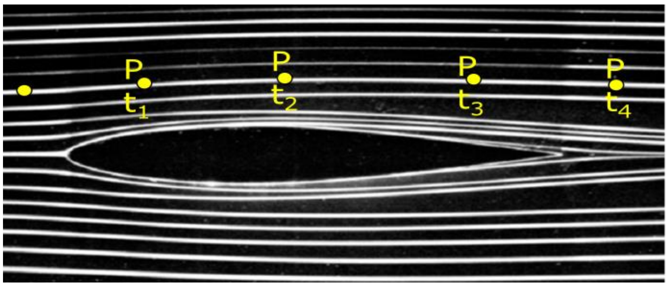
Qualified as temporal, it is due to the unsteady nature of the flow.

Qualified as convective, it is due to the non-uniformity of the flow represents the transport of the quantity Φ(M,t) by the velocity field

III.4 Pathlines

The successive positions occupied by a particle, constitute what is called the pathline of this particle.





Visualization of a pathline using tracers (colorant or smoke) and taking a photo with a long exposure time.

Pathline Equation

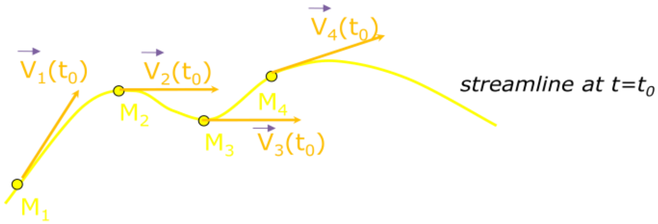
$$\frac{d\vec{r}}{dt} = \vec{V}(\vec{r},t) \quad \Rightarrow \quad \frac{dx}{dt} = V_x(x,y,z,t), \quad \frac{dy}{dt} = V_y(x,y,z,t), \quad \frac{dz}{dt} = V_z(x,y,z,t)$$

$$\frac{dx}{V_x(x,y,z,t)} = \frac{dy}{V_y(x,y,z,t)} = \frac{dz}{V_z(x,y,z,t)} = dt$$

Pathline equation

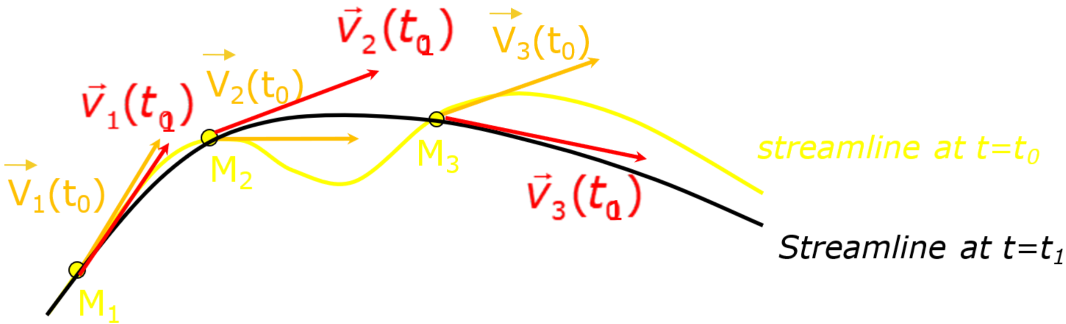
III.5 Streamlines

In Euler's description, the streamline is the curve which, at each of its points, is tangent to the velocity vectors.

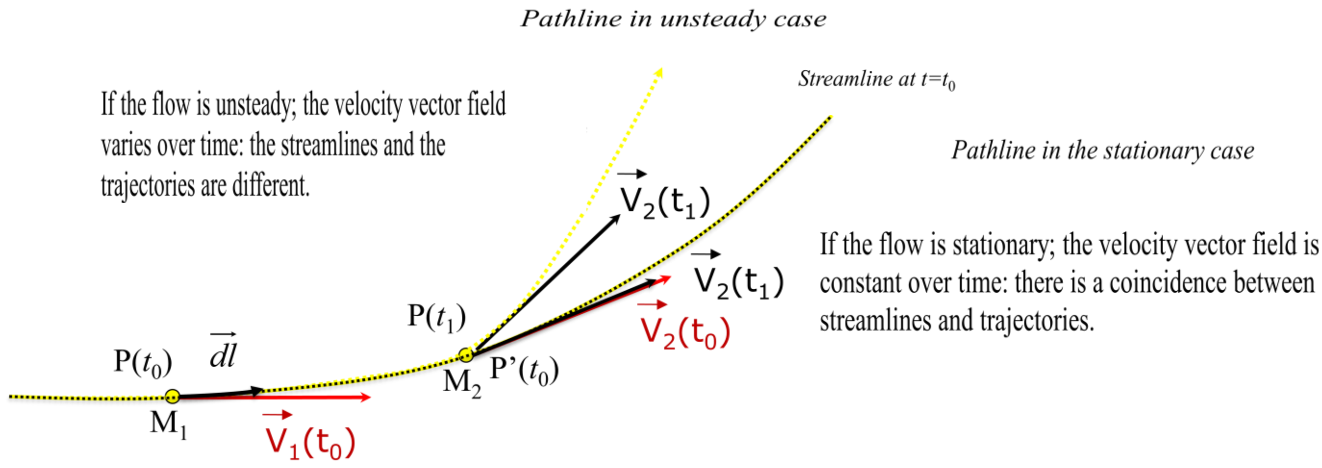


Fumes streamlines

Notice: The streamlines evolve over time, just like the velocity vector field



Notice: it should not confuse streamlines and trajectory. They are two very different concepts.



Equation of streamlines

Let be $\overrightarrow{dl}$ elementary vector of a streamline

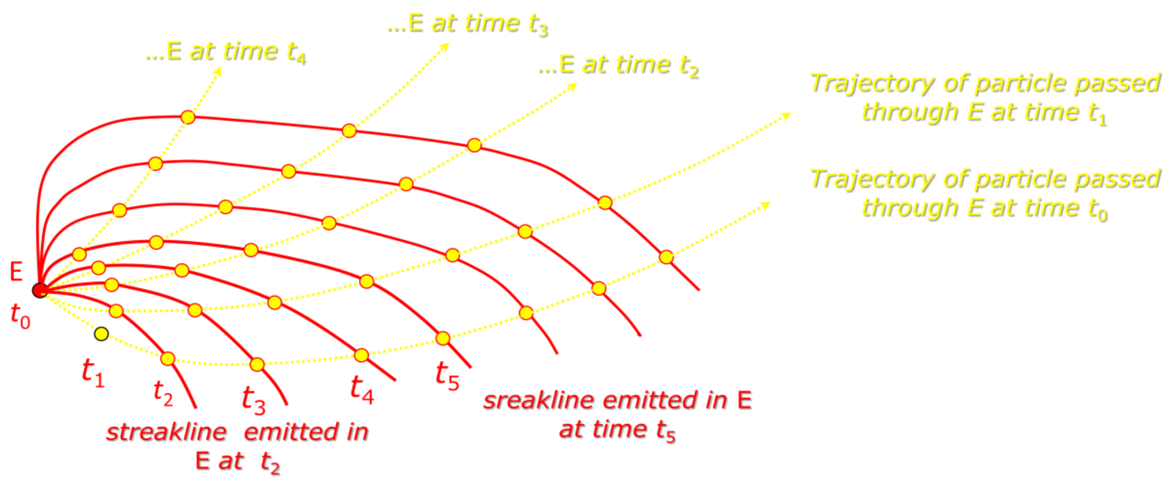
$$\vec{V} \,//\, \overrightarrow{dl} \;\Rightarrow\; \vec{V} \otimes \overrightarrow{dl} = 0 \;\Rightarrow\; \begin{cases} V_y dz - V_z dy = 0 \\ V_z dx - V_x dz = 0 \\ V_x dy - V_y dx = 0 \end{cases}$$

$$\frac{dx}{V_x\,(x,y,z,t)} = \frac{dy}{V_y\,(x,y,z,t)} = \frac{dz}{V_z\,(x,y,z,t)}$$

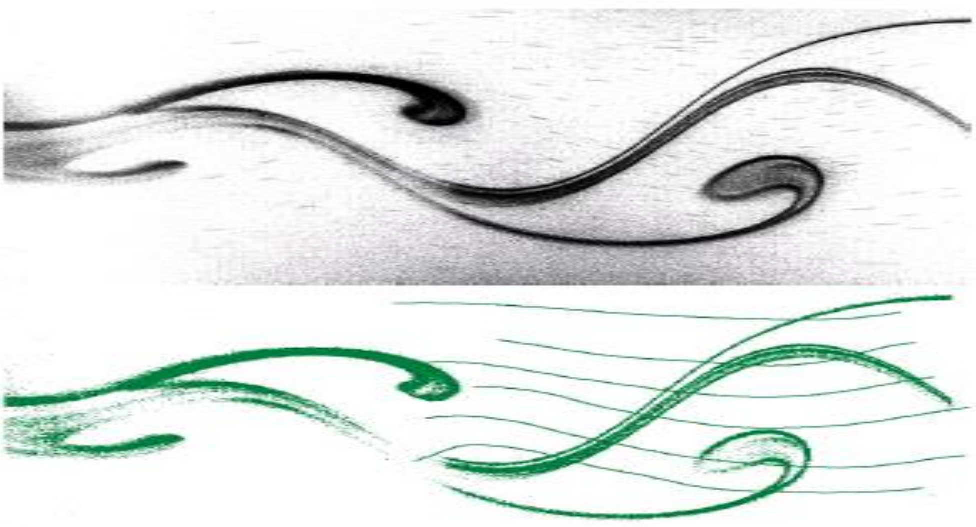
Equation of streamlines

III.6 Streaklines

All fluid particles passing through the same point E (identify by the use of dye or smoke) at different time t_E are joined by a line, at a a given instant t, this line is defined as streakline relaive to the point E.



Practically: a streakline can be visualized by fixing a coloring source at point E: the colored curves then correspond to the streaklines



Unsteady flow behind a cylinder; At the top, simultaneous visualization of streaklines and streamlines; Below, streaklines extracted from the image and some reconstructed streamlines

Streaklines Equation

The parametric equations of the trajectories are:

$$X_i = f(X_{i0}, t)$$

the point $E(x_{iE})$ obeys the parametric equations of the particle pathlines which have passed through this point at a time t_E

So :

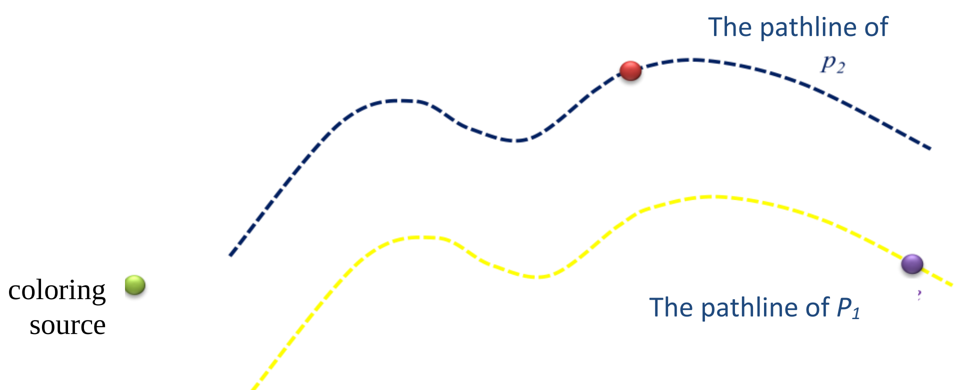
$$X_{iE} = f(X_{i0}, t_E) \rightarrow X_{i0} = f(X_{iE}, t_E)$$

$$X_i = f(X_{iE}, t_E, t)$$

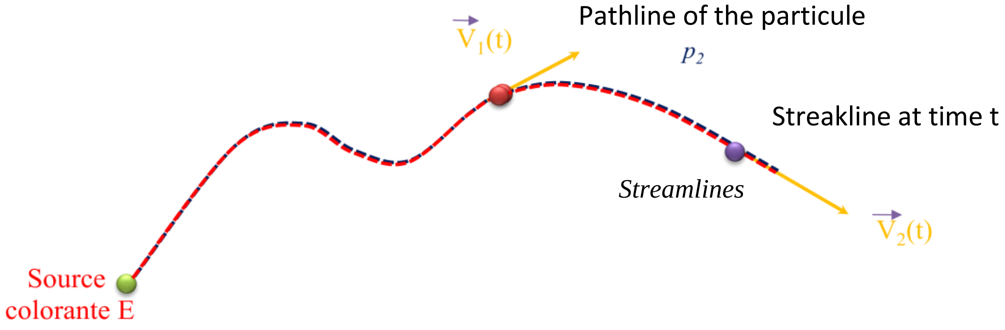
By varying t_E → The streakline at the fixed time t of the particles that have passed through E at different times t_E

By varying t → The pathline of the particle passing through *E* at the fixed time t_E

In the case of a permanent (or stationary) flow, the streaklines coincide with the trajectories and the streamlines.



In the permanent flow The pathlines have the same shape:



Streamlines $\equiv$ Trajectories $\equiv$ Streaklines

III.7 The motion of a fluid element (strain tensor)

Within the flow, each fluid particle undergoes changes of *position*, of *orientation* and of *form*.

to analyze these changes, we consider two adjacent points in the same fluid M(x,y,z) and M'(x+dx,y+dy,z+dz), their corresponding velocities are $\vec{v}_M$ and $\vec{v}_{M'}$, at a time t .

$$\vec{v}_{M'} = \vec{v}(\vec{r} + \overrightarrow{dr}) = \vec{v}(\vec{r}) + \overrightarrow{dv}$$

$\vec{v}_M$ *Increase of the velocity*

$$\vec{v}_{M'} = \vec{v}(\vec{r} + \overrightarrow{dr}) = \vec{v}(\vec{r}) + \overrightarrow{dv}$$

The first-order development of the 3 components of the velocity gives:

$$\begin{cases} u' = u + \dfrac{\partial u}{\partial x}\mathrm{d}x + \dfrac{\partial u}{\partial y}\mathrm{d}y + \dfrac{\partial u}{\partial z}\mathrm{d}z \\ v' = v + \dfrac{\partial v}{\partial x}\mathrm{d}x + \dfrac{\partial v}{\partial y}\mathrm{d}y + \dfrac{\partial v}{\partial z}\mathrm{d}z \\ w' = w + \dfrac{\partial w}{\partial x}\mathrm{d}x + \dfrac{\partial w}{\partial y}\mathrm{d}y + \dfrac{\partial w}{\partial z}\mathrm{d}z \end{cases}$$

$$\Rightarrow \quad \vec{v}(\vec{r} + \overrightarrow{\mathrm{d}r}) = \vec{v}(\vec{r}) + \overline{\overline{G}}\,\overrightarrow{\mathrm{d}r}$$

$$\overline{\overline{G}} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

$\overline{\overline{G}}$ is the strain rate tensor

Meaning of the tensor elements

it is always possible to rewrite any tensor as the sum of one *symmetric tensor* and *antisymmetric tensor*:

$$\overline{\overline{G}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix}}_{\text{Symmetric tensor } \overline{\overline{e}}} + \underbrace{\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(-\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(-\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}}_{\text{Antisymmetric tensor } \overline{\overline{\omega}}}$$

Symmetric part

We can decompose $\overline{\overline{e}}$ as follows : $\overline{\overline{e}} = \overline{\overline{T}} + \overline{\overline{D}}$

$$\overline{\overline{e}} = \begin{pmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & \frac{\partial w}{\partial z} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}$$

Diagonal term tensor $\overline{\overline{T}}$

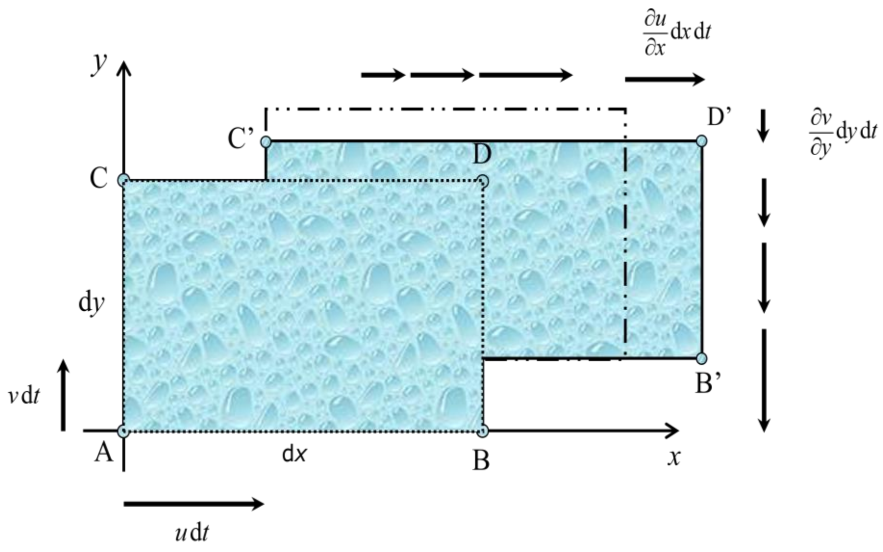
off-diagonal term tensor $\overline{\overline{D}}$

Tensor of the diagonal terms

$$\overline{\overline{T}} = \begin{pmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & \frac{\partial w}{\partial z} \end{pmatrix}$$

Note that in this case, the component u along x of the velocity varies as a function of x, the component v along y of the velocity varies as a function of y, and the component v along z of the velocity varies as a function of z.

The analysis of the particle motion in a plane flow $(\vec{e}_x, \vec{e}_y)$ shows :



$\frac{\partial u}{\partial x}$ elongation rate along x

$\frac{\partial v}{\partial y}$ elongation rate along y

The deformation of the particle is an elongation (or contraction)

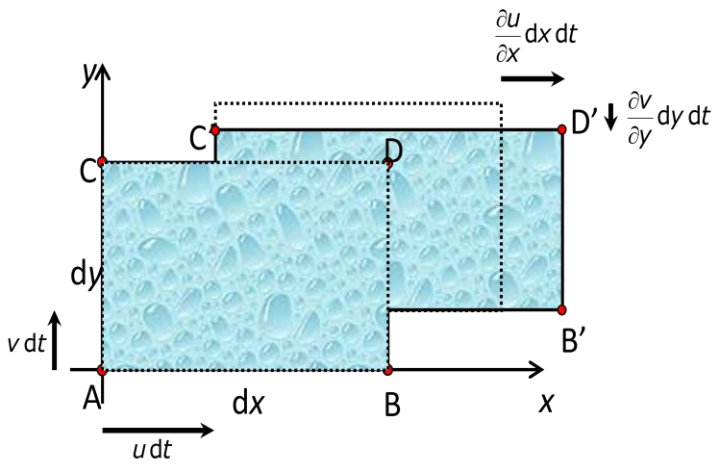
Generalization to 3D

$$\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{\vec{\nabla}\vec{V}} dt = \frac{dV}{V}$$

The relative change in fluid particle volume

$$\vec{\nabla}\vec{V} = \mathrm{Tr}(\overline{\overline{e}}) = \mathrm{Tr}(\overline{\overline{G}})$$

The trace of $\overline{\overline{G}}$ corresponds to the volume expansion rate



Notice

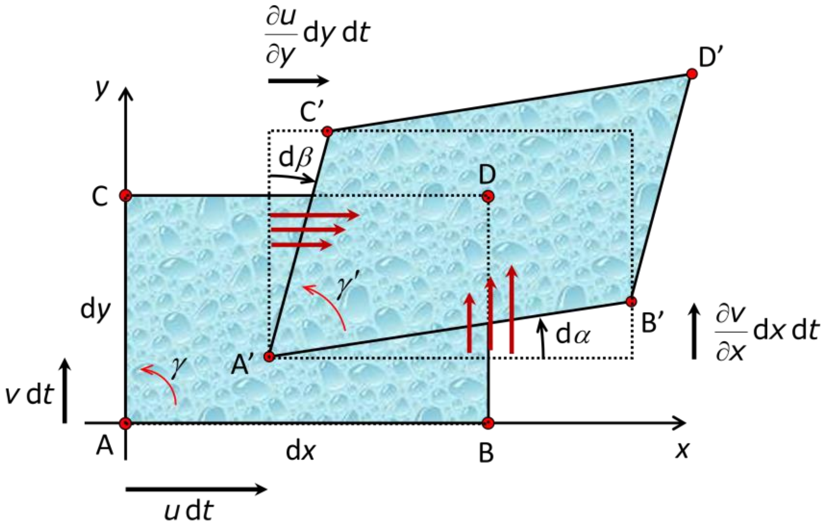
If the fluid is incompressible and the flow is conservative, then $\vec{\nabla}\vec{V} = 0 = \frac{dV}{V}$. In this case, there is no volume variation.

Tensor of off-diagonal terms

$$\overline{\overline{D}} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}$$

Note that in this case, the component u along x of the velocity varies with y, z, the component v along y of the velocity varies with x, z and the component v along z of the velocity varies with x, y

The case of plane flow $(\vec{e_x}, \vec{e_y})$



An Angular deformation of the particle

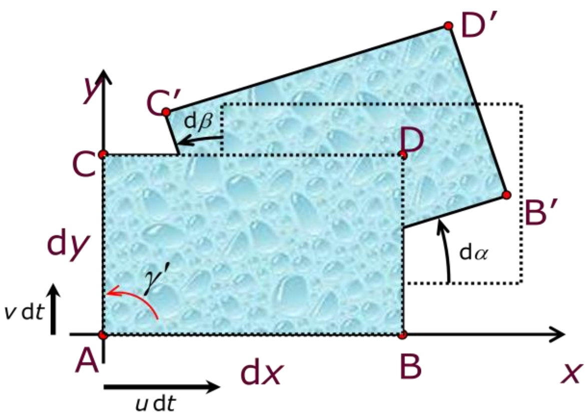
Notice

The symmetric tensor refers to the pure angular deformation

Antisymmetric part

$$\bar{\bar{\omega}} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(-\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(-\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}$$

It is a rotation around the z axis.



- We define the *rotation* of the particle about the z axis as the average of the angular motions of its edges
- Particles will only begin to rotate if they experience a torque caused by only surface shear stresses. In contrast, weight force and normal forces (pressure) accelerate or deform the particle, but cannot generate a torque.
- Rotation of fluid particles will *always* occur for flows in which we have shear stresses (viscous fluid). unless the particles are initially rotating

We can then define an angular velocity of rotation according to z:

$$\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[\vec{\nabla}\wedge\vec{v}\right]_z$$

Generalizing, we define:

$$\vec{\Omega} = \frac{1}{2}\vec{\nabla}\wedge\vec{v}$$

The angular velocity vector of rotation: $\vec{\Omega}$

The rotational velocity is: $\vec{\nabla}\wedge\vec{v}$

$$\vec{\nabla}\wedge\vec{\mathrm{V}} = \begin{cases} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{cases}$$

Recaputilization:

$$\vec{v}(\vec{r}+\overrightarrow{\mathrm{d}r}) = \vec{v}(\vec{r}) + \mathrm{d}\vec{v} = \vec{v}(\vec{r}) + \overline{\overline{G}}\,\overrightarrow{\mathrm{d}r}$$

$\overline{\overline{G}} = \overline{\overline{e}} + \overline{\overline{\omega}}$: Strain rate tensor

$\overline{\overline{e}}$: pure strain rate tensor (elongation + angular deformation)

- elongation → $\overline{\overline{T}}$ Diagonal term tensor
- angular deformation → $\overline{\overline{D}}$ Off-diagonal term tensor

$\overline{\overline{\omega}}$: Pure rotation rate tensor

$$\overline{\overline{G}}\,\overrightarrow{dr} = \overline{\overline{e}}\,\overrightarrow{dr} + \overline{\overline{\omega}}\,\overrightarrow{dr} = \underbrace{\overline{\overline{e}}\,\overrightarrow{dr}}_{\text{Pure deformations}} + \underbrace{\vec{\Omega} \wedge \overrightarrow{dr}}_{\text{Pure Rotation}}$$

III.8 Rotational flow

- It is the flow for which there is a vortex vector field associated with the velocity field;

$$\vec{\Omega} = \frac{1}{2}\vec{\nabla} \wedge \vec{v} = \frac{1}{2}\overrightarrow{rot}\vec{v} \neq 0$$

- We define what is called vector of vorticity ζ to be twice the rotation

$$\vec{\xi} = 2\vec{\Omega} = curl\vec{V} = \overrightarrow{rot}\vec{v}$$

- The vorticity is a measure of the rotation of a fluid particle as it moves in the flow field.

Notice:

we should not confuse rotation of a fluid particle with flow consisting of circular streamlines. rotation of fluid particles will *always* occur for flows in which we have shear stresses it means in viscous flows .

- The *circulation*, Γ, is defined as the line integral of the tangential velocity component about any closed curve fixed in this flow, it is a movement through a circuit.

$$\Gamma = \oint \vec{v}.\overrightarrow{ds}$$

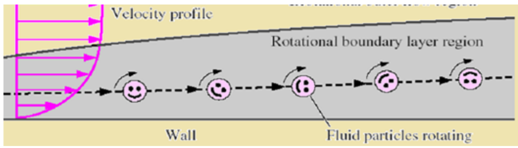
- The circulation around a closed contour is equal to the total vorticity enclosed within it (Stokes Theorem in two dimensions)

$$\Gamma = \oint_C \vec{v}.\overrightarrow{ds} = \oiint_A \xi_Z \, .dA$$

Examples

Rotational Flow $\zeta \neq 0$

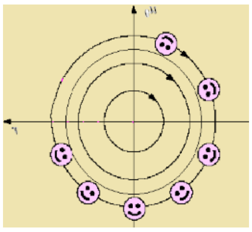
1- Inside a boundary layer, where viscous forces are important, the flow in this region is *rotational* ($\zeta \neq 0$).



$\zeta \neq 0$; $\Gamma = 0$

Forced vortex (rigid-body motion):

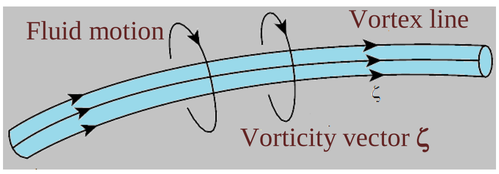
For this flow the vorticity ζ and the rotaion Ω are constant and the circulation depends on the area enclosed by the contour. It is the case for example at the eye of a tornado; the particles rotates as it moves in a circular motion around the center of the flow , creating the "dead" region at the very center and the streamlines are closer together as we move away from the origin.



$\zeta \neq 0$; $\Gamma \neq 0$

Vortex Lines: a line that is everywhere parallel to the local vorticity vector

Vortex Tube: a tube formed by all of the vortex lines that pass through a closed surface S



Vortex tube

Vortex filament: It is a vortex tube of infinitesimal cross-section

Karman vortex: It is the rotation of the particles on theirselves behind an obstacle.

Vortex line equations: The vortex lines are solution of the equations:

$$\frac{dx}{\Omega_x(x,y,z,t)} = \frac{dy}{\Omega_y(x,y,z,t)} = \frac{dz}{\Omega_z(x,y,z,t)}$$

With $\vec{\Omega} = \Omega_x \vec{i} + \Omega_y \vec{j} + \Omega_z \vec{k}$

III.9 Irrotational or potential flow of velocities

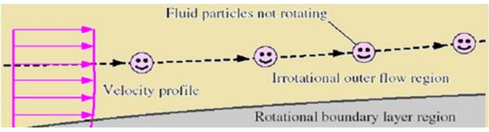
The flow is said to be irrotational or at velocity potential when the fluid particles do not undergo pure rotations:

$$\overline{\overline{\omega}} = \overline{\overline{0}} \quad \overline{\overline{\omega}} = \overline{\overline{0}} \quad \Rightarrow \quad \vec{\Omega} = \tfrac{1}{2}\vec{\nabla} \wedge \vec{v} = \vec{0}$$

So the flow is irrotational when $\overrightarrow{\mathrm{rot}}\,\vec{v} = \vec{0}$ at any point in the fluid.

Examples of Irrotational Flow

1- Flow outside the boundary layer, where viscous forces are not important

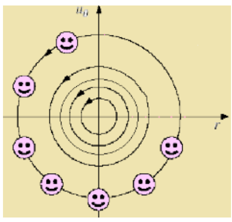


ζ =0 ; Γ =0

2- Free vortex (irrotational motion):

It occurs, for example, outside the eye of a tornado (in such a large region viscous effects are negligible), the particles do not rotate as it moves in a circular motion; also, the streamlines are farther apart as we move away from the origin.

ζ =0 ; $\Gamma \neq 0$



III.10 Potential function

Irrotational flow means that $\overrightarrow{\text{rot}}\,\vec{v} = \vec{0}$ but from a mathematical point of view, the relation

$\vec{\nabla} \wedge (\vec{\nabla}\varphi) = \vec{0} \;\; \forall \varphi$ is always true. so there is a scalar function φ such as:

$$\vec{v} = \overrightarrow{\text{grad}}\, \varphi$$

This means that in an irrotational flow the velocity field is derived from a scalar potential φ .

The flow is called potential flow and the function φ is called the velocity potential function

Properties of the velocity potential

It is then possible to express the components of the velocity vector from the potential of the velocities:

$$\vec{v} = \vec{\nabla}\varphi \quad \Rightarrow \quad v_x = \frac{\partial \varphi}{\partial x} \quad , \quad v_y = \frac{\partial \varphi}{\partial y} \quad \text{et} \quad v_z = \frac{\partial \varphi}{\partial z}.$$

If we assume that the fluid is also incompressible, we must check:

$$\vec{\nabla}\vec{v} = 0 \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

This leads to the relationship:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad \Rightarrow \quad \Delta\varphi = 0 \text{ Laplace's equation}$$

It must be concluded that the potential of the velocities must satisfy Laplace's equation. When a flow is planar, the equation $\varphi(x, y) = C^{te}$ defines, in the plane of the flow, a curve called "equipotential". Along this curve, $\varphi(x, y) = C^{te}$, so: $d\varphi = 0$

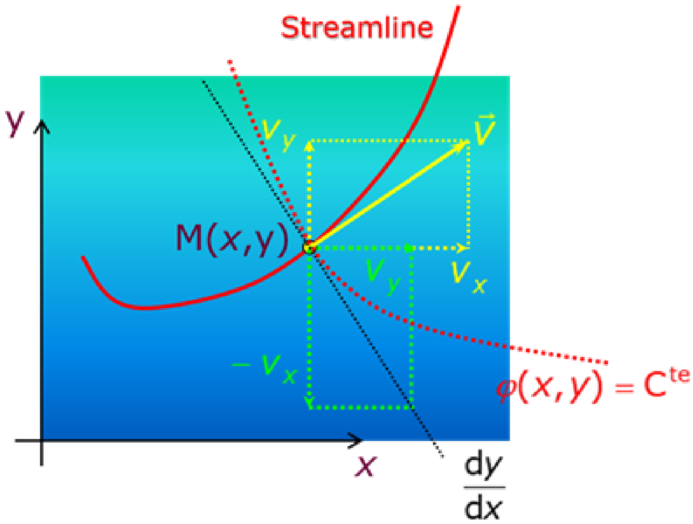
Now, the differential can be written:

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

And as along an equipotential $d\varphi = 0$, so :

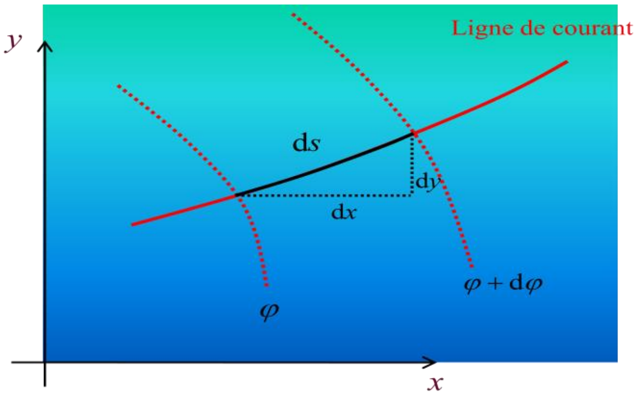
$$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0 \Rightarrow v_x dx + v_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{v_x}{v_y}$$

Relationship which is verified at any point of the equipotential



At any point M(x,y) of the flow plane, the streamline and the equipotential are orthogonal.

Calculation of the length of an arc element along a streamline



$$ds_{\Psi=C^{te}} = \sqrt{dx^2 + dy^2} \ , \vec{v} = \sqrt{v_x{}^2 + v_y{}^2}$$

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = v_x dx + v_y dy$$

Since $\vec{v} // \overrightarrow{ds}$ so :

$$d\varphi = \vec{v}\overrightarrow{ds} = v \ ds$$

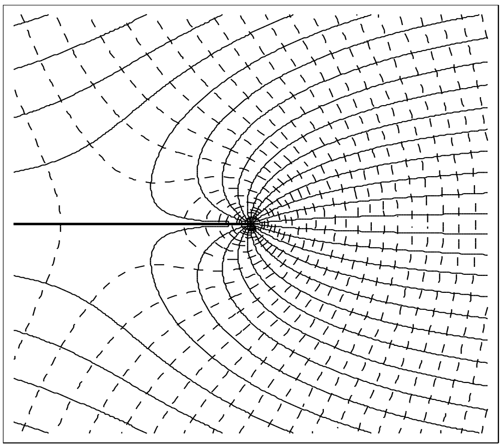
SO :

$$ds_{\Psi=C^{te}} = \sqrt{dx^2 + dy^2} = \frac{d\varphi}{\sqrt{v_x^2 + v_y^2}}$$

Either : $ds_{\Psi=C^{te}} = \dfrac{d\varphi}{v}$

$ds_{\Psi=C^{te}}$ is the distance between two equipotentials, it is inversely proportional to the speed of the flow.

If we choose to represent the equipotentials with a difference $d\varphi = C^{te}$, then the distance between the equipotentials will be all the lower as the speed of the flow is high (and vice versa).



III.11 Stream Function

If the flow of an incompressible fluid is conservative, then the continuity equation is written:

$$\vec{\nabla}\vec{v} = 0$$

However, from a mathematical point of view, the relation $\vec{\nabla}(\vec{\nabla} \wedge \vec{A}) = 0 \;\; \forall \vec{A}$ is always true.

We are then entitled to define a vector $\vec{A}$ such as :

$$\vec{v} = \vec{\nabla} \wedge \vec{A}$$

Or $\vec{A}$ therefore corresponds to a vector potential. It follows:

$$\vec{v} = \vec{\nabla} \wedge \vec{A} = \begin{cases} v_x = \partial A_z / \partial y - \partial A_y / \partial z \\ v_y = \partial A_x / \partial z - \partial A_z / \partial x \\ v_z = \partial A_y / \partial x - \partial A_x / \partial y \end{cases}$$

If we consider a flow in the plane⊥at Oz, and therefore invariant under translation along z, then:

$$v_z = 0 \text{ And } \frac{\partial}{\partial z} = 0$$

From where : $v_x = \frac{\partial A_z}{\partial y}$ And $v_y = -\frac{\partial A_z}{\partial x}$

So in the plan $(\vec{e}_x, \vec{e}_y)$, the speed is at any point defined by means of the single scalar quantity $A_z(x, y)$.

We can then ask: $A_z(x, y) = \Psi(x, y)$ → current function

And $\begin{cases} v_x = \frac{\partial \Psi}{\partial y} \\ v_y = -\frac{\partial \Psi}{\partial x} \end{cases}$ constitutes what will be called the velocity field.

Notice

In cylindrical coordinates, if $v_z = 0$ And $\frac{\partial}{\partial z} = 0$, then we have:

$$\begin{cases} v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \\ v_\theta = -\frac{\partial \Psi}{\partial r} \end{cases} \text{ Or } \Psi = \Psi(r, \theta)$$

Stream function properties

1) $d\Psi$ is an exact total differential

For an incompressible and conservative fluid (conservation of mass) we have $\vec{\nabla}\vec{v} = 0$

SO :

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

and like $v_x = \frac{\partial \Psi}{\partial y}$ And $v_y = -\frac{\partial \Psi}{\partial x}$

thus $\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0$

SO $d\Psi$ is an exact total differential:

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy$$

It means that $d\Psi$ has one and only one primitive:

$$\int_A^B d\Psi = \Psi_B - \Psi_A$$

2) In the plane (x,y), the set of points for which the value of Ψ is constant represents a streamline.

$\Psi(x, y) = C^{te} \quad \Leftrightarrow \quad y(x)$ is a curve along which $d\Psi = 0$

On this curve, we must then verify that: $d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0$, with $\frac{\partial \Psi}{\partial x} = -v_y$ and $\frac{\partial \Psi}{\partial y} = v_x$

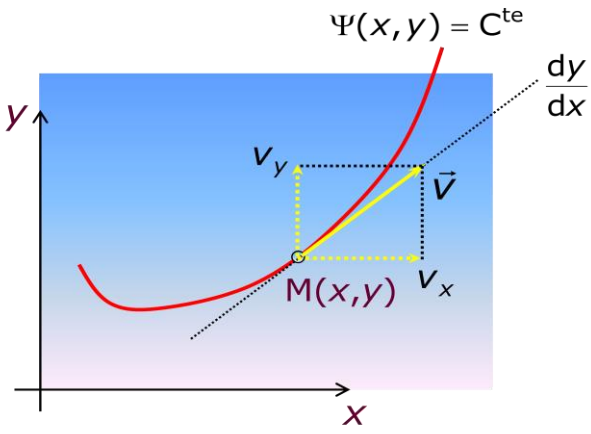
Either :

$$-v_y dx + v_x dy = 0 \Rightarrow \frac{dy}{dx} = \frac{v_y}{v_x}$$

$\frac{dy}{dx}$: slope of the curve y(x) in M(x,y)

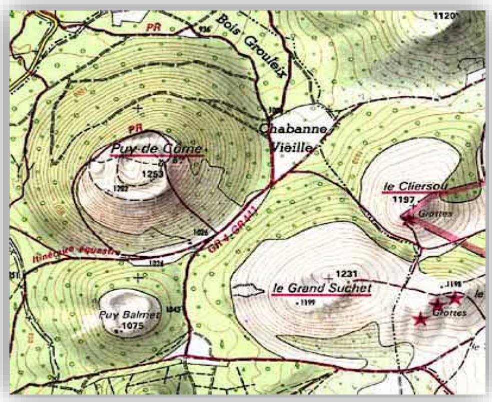
$\frac{v_y}{v_x}$: vector slope $\vec{V}$ in M(x,y)

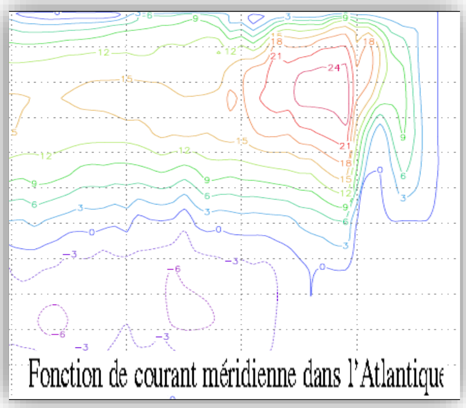
This is the definition of the streamline $\Psi(x, y) = C^{te}$ is therefore a streamline



Noticed :

Each current line is associated with a different constant. The analogy with contour lines in cartography gives:





$$\Psi(x, y) \Leftrightarrow H(x, y) \quad \text{(Altitude)}$$

$$\Psi(x, y) = C^{te} \Leftrightarrow H(x, y) = C^{te}$$

power line *level line*

Noticed:

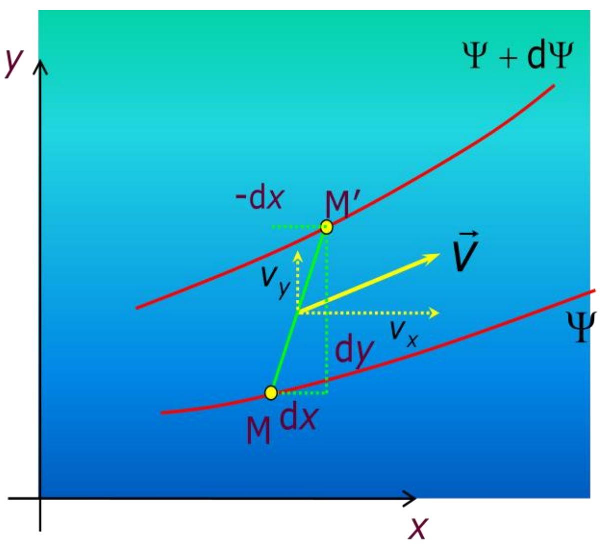
If the flow is irrotational, the current function must also satisfy Laplace's equation:

$$\vec{v} = \begin{cases} \partial\Psi/\partial y \\ -\partial\Psi/\partial x \\ 0 \end{cases} \text{ And } \vec{\nabla} \wedge \vec{v} = \vec{0} \quad \Rightarrow \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \partial\Psi/\partial y \\ -\partial\Psi/\partial x \\ 0 \end{pmatrix} = \vec{0}$$

$$-\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} = 0 \quad \Rightarrow \quad \Delta\Psi = 0$$

III.12 Flow Rates and Streamlines

Let's calculate the volume flow between 2 infinitely close streamlines:



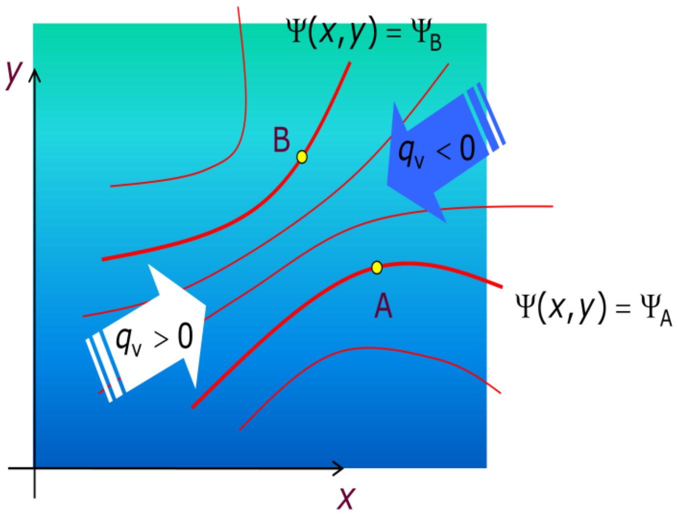
Consider the elementary volume flow between points M and M':

$$dq_v = v_x dy - v_y dx, \quad v_x = \frac{\partial\Psi}{\partial y}, \quad v_y = -\frac{\partial\Psi}{\partial x}$$

$$\Rightarrow dq_v = \frac{\partial\Psi}{\partial y} dy + \frac{\partial\Psi}{\partial x} dx = d\Psi$$

so $dq_v = d\Psi$ therefore, between any 2 current lines, constants Ψ_A And Ψ_B, the volume

flow is given by: $$q_v = \int_A^B dq_v = \int_A^B d\Psi = \Psi_B - \Psi_A$$



III.13 Examples of plane flows

Planar flow can be described by means of a current function $\Psi(x, y)$ and a speed potential $\varphi(x, y)$.

.III 13.1 Uniform flow

Consider the plane flows given by the following functions:

$$\begin{cases} \varphi(x, y) = Ux \\ \Psi(x, y) = Uy \end{cases}$$

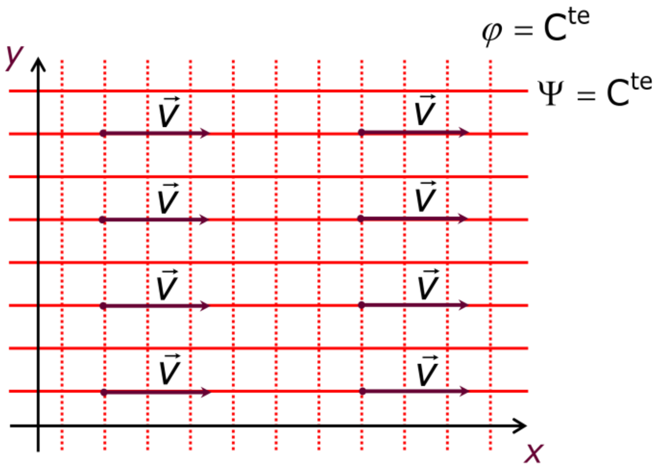
The streamlines are such as: $\Psi(x, y) = Uy = C^{te} \Rightarrow \quad y = C^{te} \ \forall x$ they are horizontal lines.

The equipotentials are such that: $\varphi(x, y) = Ux = C^{te} \Rightarrow \quad x = C^{te} \ \forall y$ they are vertical lines.

Determination of the velocity field:

$$\vec{v} = \begin{cases} v_x = \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \Psi}{\partial y} = U \\ v_y = \dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \Psi}{\partial x} = 0 \end{cases}$$

The speed is uniform: $\vec{v} = U\,\vec{e}_x$



Uniform flow

III. 13.2 Planar flow around a source or sink

Consider the plane flow modeled by the current function and the velocity potential:

$$\begin{cases} \varphi(r,\theta) = \mathrm{C}\ln r \\ \Psi(r,\theta) = \mathrm{C}\,\theta \end{cases}$$ where C is a real constant.

The streamlines are such as: $\Psi(r,\theta) = \mathrm{C}\,\theta = \mathrm{C}^{\mathrm{te}}$

$\Rightarrow \quad \theta = \mathrm{C}^{\mathrm{te}} \ \forall r$ they are straight lines passing through the origin.

The equipotentials are such that: $\varphi(r,\theta) = \mathrm{C}\ln r = \mathrm{C}^{\mathrm{te}}$

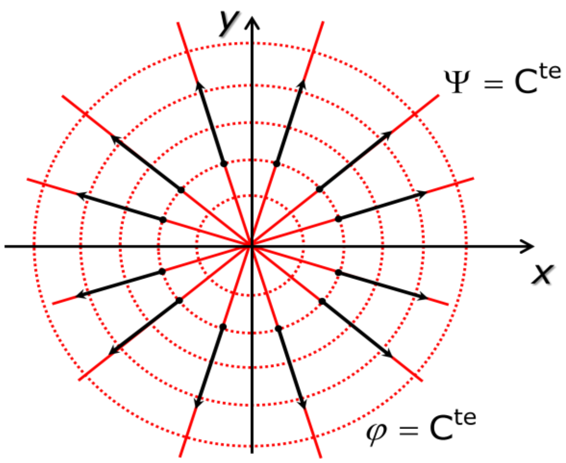
$\Rightarrow \quad r = C^{te} \ \forall\theta$ they are concentric circles centered on the origin.

Determination of the velocity field:

$$\vec{v} = \begin{cases} v_r = \dfrac{\partial\varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta} = -\dfrac{\partial\Psi}{\partial r} \end{cases}$$

Either : $\vec{v} = \begin{cases} v_r = C/r \\ v_\theta = 0 \end{cases} \Rightarrow \vec{v} = \dfrac{C}{r}\vec{e}_r$

The speed is therefore radial and inversely proportional to the distance from the origin.



If C > 0, then the flow is directed outwards⇒divergent flow⇒original source.

If C < 0, then the flow is directed towards the origin⇒convergent flow⇒well at the origin.

Physical meaning of constant C:

Let's calculate the volume flow of this radial flow (source or sink):

$$q_v = \oiint_S \vec{v}.\vec{n}\,\mathrm{d}S$$

where S is a closed surface surrounding the origin.

The flow taking place in a plane⊥at the z axis, we can consider as integration surface a cylinder of heightΔz= 1, and therefore:

$$\oiint_s \ldots \mathrm{d}S = \oint_\ell \ldots \Delta z \mathrm{d}\ell$$

It then remains to integrate on a circle of any radius r, centered on the origin.

$$q_v = \Delta z \oint_\ell \vec{v}.\vec{n}\,r\,\mathrm{d}\theta = \Delta z\, r \int_0^{2\pi} \vec{v}.\vec{n}\mathrm{d}\theta \quad \text{Or} \begin{cases} \vec{v} = \mathrm{C}/r\,\vec{e}_r \\ \vec{n} = \vec{e}_r \end{cases}$$

$$\Rightarrow q_v = \Delta z\, r \int_0^{2\pi} \frac{\mathrm{C}}{r}\mathrm{d}\theta = \Delta z\, r \frac{\mathrm{C}}{r} \int_0^{2\pi} \mathrm{d}\theta = 2\pi \mathrm{C} \underbrace{\Delta z}_{1} \quad \text{(Volume flow per unit height)}$$

$$\Rightarrow \mathrm{C} = \frac{q_v}{2\pi} \quad qv > 0\text{: source flow rate,} < 0\text{: well flow}$$

III. 13.2 Free Swirl

Consider the plane flow modeled by the current function and the velocity potential:

$$\begin{cases} \varphi(r,\theta) = \mathrm{C}\theta \\ \Psi(r,\theta) = -\mathrm{C}\ln r \end{cases} \quad \text{where C is a real constant.}$$

The streamlines are such as: $\Psi(r,\theta) = -C\ln r = C^{te}$

$\Rightarrow \quad r = C^{te} \ \forall\theta$ they are concentric circles centered on the origin.

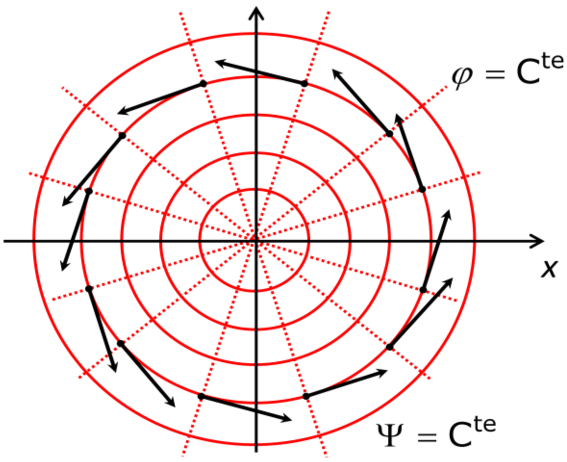
The equipotentials are such that: $\varphi(r,\theta) = C\theta = C^{te} \quad \Rightarrow \quad \theta = C^{te} \ \forall r$ they are straight lines passing through the origin.

Determination of the velocity field:

$$\vec{v} = \begin{cases} v_r = \dfrac{\partial\varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta} = -\dfrac{\partial\Psi}{\partial r} \end{cases}$$

Either : $\vec{v} = \begin{cases} v_r = 0 \\ v_\theta = \dfrac{C}{r} \end{cases} \quad \Rightarrow \quad \vec{v} = \dfrac{C}{r}\vec{e}_\theta$

The speed is therefore ortho radial and inversely proportional to the distance from the origin.



If C > 0, then the flow is around the origin counterclockwise.

If C < 0, then the flow is around the origin in a clockwise direction

Physical meaning of constant C:

Let's calculate the "circulation" of the velocity around the origin:

$$\Gamma = \oint_{\ell} \vec{v}.\mathrm{d}\vec{\ell}$$ Where $\mathrm{d}\vec{\ell}$ is travels any streamline, i.e. a circle of radius r.

$$\vec{v} = \frac{C}{r}\vec{e}_{\theta} \quad \mathrm{d}\vec{\ell} = r\,\mathrm{d}\theta\,\vec{e}_{\theta} \quad \Rightarrow \quad \Gamma = \int_{0}^{2\pi} \frac{C}{r} r\,\mathrm{d}\theta = 2\pi C$$

$$C = \frac{\Gamma}{2\pi}$$ Or$\Gamma$is the circulation of the vortex (free

if $\Gamma > 0$, the vortex rotates counter-clockwise. And if $\Gamma < 0$, the vortex rotates clockwise.

III.14 Exercises

Exercise1

The two-dimensional flow of a fluid is defined by Lagrangian coordinates as follows:

$$x = x_0\, e^{-2t}$$
$$y = y_0\, e^{-t}$$

Where x_0, y_0 are constants.

1. Determine the pathlines of a fluid particle.
2. Find the components of the velocity.
3. Determine the streamlines.
4. Determine the streaklines corresponding to the source S(a. b).
5. flow is it stationary?

Exercise2

The velocity components of a stationary flow are given by:

$$u = 2x^2 y,\ v = -2xy^2,\ w = w_0$$

w_0 is constant.

1. Determine the streamlines, what will be the pathlines?
2. Determine the coordinates x(t), y(t) and z(t) as a function of time of a particle located at $x=x_0$ $y=y_0$ and $z=z_0$ at time t=0.
3. Determine the streamlines relative to the source E (x_s, y_s, z_s).
4. Find the components of the acceleration by deriving twice the functions x(t), y(t), z(t) . Compare this result with the acceleration obtained according to the particle derivative

Exercise 3

We consider the flow whose velocity field is :

$$\vec{\mathbf{v}}(M,t)=(-kx+bt)\,\vec{\mathbf{e_x}} + a\,t^2\,\vec{e_y}$$

1) Characterize this flow.
2) Determine the trajectories of the fluid particles and the streamlines, knowing that at t=0; $x=x_0$, $y=y_0$
3) Calculate the acceleration of a fluid particle in two different ways.

Exercise 4

We have the following two-dimensional flow where the velocity vector of a fluid particle is:

$$\vec{v}(t) = (u_0 + \alpha t)\vec{e_x} + v_0\vec{e_y}$$

u_0, v_0, α constants.

1- Characterize this flow.
2- Determine the streamlines, the streaklines corresponding to the point (a_1, a_2) at time t_0, and the trajectory.

Exercise 5

We are given the following velocity field: $\vec{v} = (x + \alpha t)\,\vec{e}_x + (-y)\vec{e}_y$

1. Characterize the flow.
2. Calculate the velocity potential and determine the equipotential lines
3. Calculate the stream function and determine the streamlines.
4. Calculate the components of the acceleration

Exercise 6

The two-dimensional flow of a fluid can be described using the velocity function:

$$\varphi(x,y) = \frac{x^3}{3} - xy^2$$

1. Determine the equipotential lines.
2. Find the components of the velocity.
3. Is the flow permanent?
4. Is the flow incompressible?
5. Determine the stream function $\Psi(x,y)$ and the streamlines.
6. What is the streamline $\Psi(1,1)$, $\Psi(0,0)$.
7. Calculate the mass flow rate between points (1.1) and (0.0). The density of the fluid is ρ.

Exercise 7

Consider the permanent flow defined with Euler variable by the velocity field:

$$\vec{v} = (2x - 3y)\vec{e}_x + (3x - 2y)\vec{e}_y$$

1. Demonstrate that the fluid is incompressible.
2. Determine the acceleration vector field.
3. Calculate the current function $\Psi(x,y)$. What is the shape of the streamlines?
4. Is the flow rotational?
5. Determine the angular velocity of rotation and vortex lines.
6. Determine the strain rate tensor $\overline{\overline{D}}$.

III. 15 Exercise Solutions

Solution 1.

The two-dimensional flow of a fluid is defined in Lagrangian coordinates by :

$$x = x_0\, e^{-2t}$$
$$y = y_0\, e^{-t}$$

where x_0, y_0 are constants.

1. To determine the trajectory of a fluid particle, it is necessary to eliminate (t)

$$y^2 = \frac{y_0^2}{x_0}\, x$$

2. The components of velocity.

$\vec{V}$ is expressed as a function of the Lagrangian coordinates

$$\vec{V} = \frac{d\vec{r}}{dt} = -2\, x_0\, e^{-2t}\, \vec{\imath} - y_0\, e^{-t}\, \vec{\jmath}$$

3. Is the flow steady?

To determine whether the flow is steady, the velocity vector must be expressed in terms of Eulerian coordinates :

$$\vec{V} = -2\, x_0\, e^{-2t}\, \vec{\imath} - y_0\, e^{-t}\, \vec{\jmath} = -2\, x\, \vec{\imath} - y\, \vec{\jmath}$$

We can observe that $\frac{\partial \vec{V}}{\partial t} = 0$ → The flow is steady

4. The streamlines and the streak lines coincide with the particle trajectories. They are represented by the same equation :

$$y^2 = A\, x$$

Solution 2.

The velocity components of a stationary flow are given by:

$$u = 2x^2y,\ \ v = -2xy^2,\ \ w = w_0$$

w_0 is constant.

1. The streamlines coincide with the pathlines because the flow is stationary (steady). $\frac{\partial \vec{V}}{\partial t} = 0$

To determine the streamlines, it is necessary to solve the following equation at constant t:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}$$

$$\frac{dx}{2x^2y} = \frac{dy}{-2xy^2} = \frac{dz}{w_0}$$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{2Adz}{w_0}$$

$$\begin{cases} xy = A = x_0y_0 \\ y = B\, e^{\frac{-2Az}{w_0}} \\ x = C\, e^{\frac{2Az}{w_0}} \end{cases} \Longrightarrow \begin{cases} xy = A = x_0y_0 \\ y = y_0\, e^{\frac{-2A(z-z0)}{w_0}} \\ x = x_0\, e^{\frac{2A(z-z0)}{w_0}} \end{cases}$$

A, B, and C are determined by the initial conditions, $x=x_0$ $y=y_0$ and $z=z_0$ at time $t = 0$

2. the coordinates x(t), y(t) and z(t) as a function of time of a particle located at $x=x_0$ $y=y_0$ and $z=z_0$ at time $t = 0$.

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = dt$$

$$\frac{dx}{2x^2y} = \frac{dy}{-2xy^2} = \frac{dz}{w_0} = dt$$

$$\begin{cases} \frac{dx}{2x^2y} = \frac{dx}{2Ax} = dt \\ \frac{dy}{-2xy^2} = \frac{dy}{-2Ay} = dt \\ \frac{dz}{w_0} = dt \end{cases}$$

$$\begin{cases} A = x_0y_0 \text{ à } t = 0 \\ x = x_0\, e^{2x_0y_0t} \\ y = y_0\, e^{-2x_0y_0t} \\ z = w_0t + z_0 \end{cases}$$

3. The streaklines coincide with the streamlines in the case of a stationary flow; they have the same profile.

Demonstration:

E(xs, ys, zs) verifies the trajectory equation so:

$$\begin{cases} x_s = x_0\, e^{2x_0y_0\,t_E} \\ y_s = y_0\, e^{-2x_0y_0t_E} \\ z_s = w_0t_E + z_0 \end{cases}$$

$$\begin{cases} x_0 = x_s\, e^{-2x_0 y_0\, t_E} \\ y_0 = y_s\, e^{2x_0 y_0 t_E} \\ \mathrm{z}_0 = -\mathrm{w_s t_E} + \mathrm{z_s} \end{cases}$$

$$\begin{cases} x = x_s\, e^{-2x_0 y_0\, t_E}\, e^{2x_0 y_0 t} \\ y = y_s\, e^{2x_0 y_0 t_E}\, e^{-2x_0 y_0 t} \\ \mathrm{z} = \mathrm{w_0 t} - \mathrm{w_s t_E} + \mathrm{z_s} \end{cases}$$

To determine the streaklines, it is necessary to vary t_E and fix the time t, then we look for a relation between x, y, and z without t_E . We end up with the same profile as the streamlines:

$$\begin{cases} xy = x_s y_s \\ x = x_s\, e^{\frac{2x_0 y_0\,(z-z_s)}{w_0}} \\ y = y_s\, e^{\frac{-2\, x_0 y_0 (z-z_s)}{w_0}} \end{cases}$$

4. The components of the acceleration are obtained by differentiating twice the functions x(t), y(t), and z(t) .

$$\begin{cases} x = x_0\, e^{2x_0 y_0 t} \\ y = y_0\, e^{-2x_0 y_0 t} \\ \mathrm{z} = \mathrm{w_0 t} + \mathrm{z_0} \end{cases}$$

$$\begin{cases} x^{\circ} = 2\, \mathrm{x_0^2} y_0\, e^{\,2x_0 y_0 t} \\ y^{\circ} = -2x_0 \mathrm{y_0^2}\, e^{-2x_0 y_0 t} \\ \mathrm{z}^{\circ} = \mathrm{w_0} \end{cases}$$

$$\begin{cases} a_x = x^{\circ\circ} = 4\, \mathrm{x_0^3 y_0^2}\, e^{\,2x_0 y_0 t} \\ a_y = y^{\circ\circ} = 4\mathrm{x_0^2 y_0^3}\, e^{-2x_0 y_0 t} \\ a_z = \mathrm{z}^{\circ\circ} = 0 \end{cases}$$

$$\begin{cases} a_x = x^{\circ\circ} = 4\mathrm{x\, x_0^2 y_0^2} \\ a_y = y^{\circ\circ} = 4\mathrm{y\, x_0^2 y_0^2} \\ a_z = \mathrm{z}^{\circ\circ} = 0 \end{cases}$$

From the equation of the streamlines, we have $xy = x_0 y_0$, thus

$$\begin{cases} a_x = 4\, \mathrm{x^3 y^2} \\ a_y = 4\, \mathrm{x^2 y^3} \\ a_z = 0 \end{cases}$$

- the acceleration obtained according to the particulate derivative:

$$\frac{\mathrm{D}\vec{V}}{\mathrm{Dt}} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\, \overrightarrow{grad}\, V_i$$

$$\begin{cases} a_x = \dfrac{\partial V_x}{\partial t} + V_x\dfrac{\partial V_x}{\partial x} + V_y\dfrac{\partial V_x}{\partial y} + V_z\dfrac{\partial V_x}{\partial z} \\ a_y = \dfrac{\partial V_y}{\partial t} + V_x\dfrac{\partial V_y}{\partial x} + V_y\dfrac{\partial V_y}{\partial y} + V_z\dfrac{\partial V_y}{\partial z} \\ a_z = \dfrac{\partial V_z}{\partial t} + V_x\dfrac{\partial V_z}{\partial x} + V_y\dfrac{\partial V_z}{\partial y} + V_z\dfrac{\partial V_z}{\partial z} \end{cases}$$

$$\begin{cases} a_x = V_x\dfrac{\partial V_x}{\partial x} + V_y\dfrac{\partial V_x}{\partial y} + V_z\dfrac{\partial V_x}{\partial z} \\ a_y = \dfrac{\partial V_y}{\partial t} + V_x\dfrac{\partial V_y}{\partial x} + V_y\dfrac{\partial V_y}{\partial y} + V_z\dfrac{\partial V_y}{\partial z} \\ a_z = \dfrac{\partial V_z}{\partial t} + V_x\dfrac{\partial V_z}{\partial x} + V_y\dfrac{\partial V_z}{\partial y} + V_z\dfrac{\partial V_z}{\partial z} \end{cases}$$

$$\begin{cases} a_x = 4\,x^3y^2 \\ a_y = 4\,x^2y^3 \\ a_z = 0 \end{cases}$$

Note that both descriptions lead to the same result.

Solution 3.

We consider the flow whose velocity field is :

$$\vec{\mathbf{v}}(M,t)=(-kx+bt)\,\vec{\mathbf{e}}_{\mathbf{x}} + at^2\,\vec{e}_y.$$

1) Characterize this flow.

- Unsteady flow : $\dfrac{\partial \vec{\mathbf{v}}}{\partial t} \neq 0$

- Planar flow (two-dimensional): $v_Z = 0$

- Irrotational flow: $\overrightarrow{\mathrm{rot}}\vec{v} = 0$

- The flow is compressible $\mathrm{div}\vec{v} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \neq 0$

2) The pathlines of the fluid particles in the case of unsteady (non-stationary) flow? knowing that at t =0; $x = x_0$; $y = y_0$

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = dt$$

$$\frac{\mathrm{d}x}{-\mathrm{k}\,\mathrm{x} + \mathrm{b}\,\mathrm{t}} = \frac{\mathrm{d}y}{\mathrm{a}\,\mathrm{t}^2} = dt$$

$$\begin{cases} \dfrac{\mathrm{d}x}{-\mathrm{k}\,\mathrm{x} + \mathrm{b}\,\mathrm{t}} = dt \\ \dfrac{\mathrm{d}y}{\mathrm{a}\,\mathrm{t}^2} = dt \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} + k\,x = b\,t \\ y = \dfrac{a}{3}t^3 + y_0 \end{cases}$$

$$\begin{cases} x = x_0 e^{-kt} + \dfrac{b}{k}t - \dfrac{b}{k^2} \\ y = \dfrac{a}{3}t^3 + y_0 \end{cases}$$

Thus, we obtain the equation of the pathlines as follows :

$$\begin{cases} x = x_0 e^{-k\sqrt[3]{\frac{3}{a}(y_-y_0)}} + \dfrac{b}{k}\sqrt[3]{\dfrac{3}{a}(y_-y_0)} - \dfrac{b}{k^2} \\ t = \sqrt[3]{\dfrac{3}{a}(y_-y_0)} \end{cases}$$

- The streamlines:

$$\frac{dx}{v_x} = \frac{dy}{v_y} \quad \text{à t constant}$$

$$\frac{dx}{-k\,x + b\,t} = \frac{dy}{a\,t^2}$$

Hence, the equation of the streamlines is:

$$x = \frac{bt}{k}\left(1 - e^{\frac{-k}{at^2}y}\right)$$

3) The acceleration of a fluid particle in two different ways.
Acceleration according to Lagrange:

$$\begin{cases} x = x_0 e^{-kt} + \frac{b}{k}t - \frac{b}{k^2} \\ y = \frac{a}{3}t^3 + y_0 \end{cases} \Rightarrow \begin{cases} \overset{\circ}{x} = -kx_0 e^{-kt} + \frac{b}{k} \\ \overset{\circ}{y} = at^2 \end{cases} \Rightarrow \begin{cases} a_x = \overset{\circ\circ}{x} = k^2 x_0 e^{-kt} = k^2 x + b(1 - kt) \\ a_y = \overset{\circ\circ}{y} = 2at \end{cases}$$

Acceleration according to Euler:

$$\begin{cases} a_x = \dfrac{\partial V_x}{\partial t} + V_x \dfrac{\partial V_x}{\partial x} + V_y \dfrac{\partial V_x}{\partial y} \\ a_y = \dfrac{\partial V_y}{\partial t} + V_x \dfrac{\partial V_y}{\partial x} + V_y \dfrac{\partial V_y}{\partial y} \end{cases}$$

So :

$$\begin{cases} a_x = k^2 x + b(1 - kt) \\ a_y = 2at \end{cases}$$

Solution 4.

We have the following two-dimensional flow, so that the velocity vector of a fluid particle is:

$$\vec{v}(t) = (u_0 + \alpha t)\overrightarrow{e_x} + v_0\overrightarrow{e_y}$$

u_0, v_0, α constants.

The streamlines:

$$\frac{dx}{v_x} = \frac{dy}{v_y}$$

$$\frac{dx}{(u_0 + \alpha t_0)} = \frac{dy}{v_0}$$

$$y = \frac{v_0}{u_0 + at_0}x + C$$

The streakline corresponding to the point (a_1, a_2) at an instant ($t = t_0$)

$$\frac{dx}{v_x} = \frac{dy}{v_y} = dt$$

$$\begin{cases} x = \frac{1}{2}\alpha t^2 + u_0\, t + x_0 \\ y = v_0 t + y_0 \end{cases}$$

$$\begin{cases} a_1 = \frac{1}{2}\alpha t_a^2 + u_0\, t_a + x_0 \\ a_2 = v_0 t_a + y_0 \end{cases}$$

$$\begin{cases} x_0 = a_1 - (\frac{1}{2}\alpha t_a^2 + u_0\, t_a) \\ y_0 = a_2 - v_0 t_a \end{cases}$$

$$\begin{cases} x = \frac{1}{2}\alpha t^2 + u_0\, t + a_1 - (\frac{1}{2}\alpha t_a^2 + u_0\, t_a) \quad \ldots .(2) \\ y = v_0 t + a_2 - v_0 t_a \ldots \ldots \quad (1) \end{cases}$$

The streakline at $t = t_0$ is determined by replacing the expression for t_a taken from equation 2 in equation 1, we find:

$$\begin{cases} x = \frac{1}{2}\alpha t_0^2 + (u_0\, t_0 + a_1) - (\frac{1}{2}\alpha \frac{(v_0 t_0 + a_2 - y)^2}{v_0^2} + u_0 \frac{(v_0 t_0 + a_2 - y)}{v_0}) \\ t_a = \frac{(v_0 t_0 + a_2 - y)}{v_0} \end{cases}$$

Returning to the equations: and eliminating the time between the two equations we obtain the equation of the pathlines : $\begin{cases} x = \frac{1}{2}\alpha t^2 + u_0\, t + x_0 \\ y = v_0 t + y_0 \end{cases}$

$$\begin{cases} x = \frac{1}{2}\alpha(\frac{y-y_0}{v_0})^2 + u_0\, \frac{y-y_0}{v_0} + x_0 \\ t = \frac{y-y_0}{v_0} \end{cases}$$

Solution 5.

We are given the following velocity field: $\vec{v} = (x + \alpha t)\,\vec{e}_x + (-y)\vec{e}_y$

1. Characterize the flow.
   - The flow is two-dimensional: $v_z = 0$
   - Unsteady flow (unstationary) : $\frac{\partial v_x}{\partial t} \neq 0$
   - The flow is irrotational: $\overrightarrow{rot}\,\vec{v} = 0$
   - The flow is incompressible: $\text{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

2. Calculate the velocity potential and determine the equipotential lines.

Since the flow is irrotational, a velocity potential exists such that :

$$\vec{v} = \overrightarrow{grad}\varphi$$

$$\begin{cases} v_x = \frac{\partial \varphi}{\partial x} \\ v_y = \frac{\partial \varphi}{\partial y} \end{cases}$$

$$\begin{cases} x + \alpha t = \frac{\partial \varphi}{\partial x} \\ -y = \frac{\partial \varphi}{\partial y} \end{cases}$$

$$\begin{cases} x + \alpha t = \frac{\partial \varphi}{\partial x} \\ -y = \frac{\partial \varphi}{\partial y} \end{cases}$$

$$\begin{cases} \varphi(x,y) = \frac{1}{2}x^2 + \alpha t\, x + C(y) \\ \frac{\partial \varphi(x,y)}{\partial y} = -y = \frac{\partial C(y)}{\partial y} \\ C(y) = -\frac{1}{2}y^2 + C \end{cases}$$

Thus, the expression of φ(x,y) is given as :

$$\varphi(x,y) = \frac{1}{2}x^2 + \alpha t\, x + -\frac{1}{2}y^2 + C$$

The equipotential lines are those that satisfy the equation φ (x,y) = const

$$\frac{1}{2}x^2 - \frac{1}{2}y^2 + \alpha t\, x = C$$

2. The stream function and the streamlines.

$$\begin{cases} v_x = \dfrac{\partial \psi}{\partial y} \\ v_y = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

$$\begin{cases} \dfrac{\partial \psi}{\partial y} = (x + \alpha t) \\ -\dfrac{\partial \psi}{\partial x} = -y \end{cases}$$

$$\begin{cases} \psi(x,y) = (x + \alpha t)y + C(x) \\ \dfrac{\partial \psi}{\partial x} = -y + \dfrac{\partial C(x)}{\partial x} = -y \\ \dfrac{\partial C(x)}{\partial x} = 0 \\ C(x) = \text{cont} \end{cases}$$

Consequently: $\psi(x,y) = (x + \alpha t)y + C$

The equation of the streamlines is given by : $\psi(x,y) = const$ so with:

$$(x + \alpha t)y = \text{Const}$$

3. Calculate the components of the acceleration.

$$\begin{cases} a_x = \dfrac{\partial V_x}{\partial t} + V_x \dfrac{\partial V_x}{\partial x} + V_y \dfrac{\partial V_x}{\partial y} \\ a_y = \dfrac{\partial V_y}{\partial t} + V_x \dfrac{\partial V_y}{\partial x} + V_y \dfrac{\partial V_y}{\partial y} \end{cases}$$

$$\begin{cases} a_x = \alpha + (x + \alpha t) \\ a_y = y \end{cases}$$

Solution 6.

The two-dimensional flow of a fluid can be described using the velocity function:

$$\varphi(x,y) = \frac{x^3}{3} - xy^2$$

1. The equipotential lines $\varphi(x,y) = \text{const} \Longrightarrow \frac{x^3}{3} - xy^2 = c$

2. Velocity components

$$\begin{cases} v_x = \frac{\partial \varphi}{\partial x} \\ v_y = \frac{\partial \varphi}{\partial y} \end{cases}$$

$$\begin{cases} v_x = x^2 - y^2 \\ v_y = -2xy \end{cases}$$

3. Is the flow steady?

$$\frac{\partial \vec{v}}{\partial t} = \vec{0} \rightarrow The\ flow\ is\ steady$$

4. Is the flow incompressible?

$$\text{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow The\ flow\ is\ \text{incompressible}$$

5. The stream function. Ψ(x,y) and the streamlines. .
   We have :

$$\begin{cases} v_x = \frac{\partial \psi}{\partial y} \\ v_y = -\frac{\partial \psi}{\partial x} \end{cases}$$

$$\begin{cases} x^2 - y^2 = \frac{\partial \psi}{\partial y} \\ -2xy = -\frac{\partial \psi}{\partial x} \end{cases}$$

So

$$\begin{cases} \psi(x,y) = x^2 y - \frac{1}{3}y^3 + C(x) \\ -\frac{\partial \psi}{\partial x} = -2xy = -2xy - \frac{\mathrm{d}C(x)}{\mathrm{d}x} \rightarrow C(x) = const \end{cases}$$

$$\psi(x,y) = x^2 y - \frac{1}{3}y^3 + C$$

6. The streamline Ψ(1,1), Ψ(0,0).

$$\psi(1,1) = x^2y - \frac{1}{3}y^3 + C = \frac{2}{3} + C$$

$$\psi(0,0) = x^2y - \frac{1}{3}y^3 + C = C$$

7. The mass flow rate passing between the points (1,1) and (0,0). The fluid density is ρ.

$$q_m = \psi(1,1) - \psi(0,0)$$

$$q_m = \frac{2}{3} + C - C$$

$$q_m = \frac{2}{3}$$

Solution 7.

Consider the steady flow defined in Eulerian variables by the velocity field :

$$\vec{\mathrm{v}} = (2\mathrm{x} - 3\mathrm{y})\vec{\mathrm{e}}_{\mathrm{x}} + (3\mathrm{x} - 2\mathrm{y})\vec{\mathrm{e}}_{\mathrm{y}}$$

1. The fluid is incompressible.

$$\mathrm{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0 \rightarrow \textit{The stream is } \text{incompressible}$$

2. The acceleration vector field.

$$\begin{cases} a_x = \dfrac{\partial V_x}{\partial t} + V_x\dfrac{\partial V_x}{\partial x} + V_y\dfrac{\partial V_x}{\partial y} \\ a_y = \dfrac{\partial V_y}{\partial t} + V_x\dfrac{\partial V_y}{\partial x} + V_y\dfrac{\partial V_y}{\partial y} \end{cases}$$

$$\begin{cases} a_x = -5\,x \\ a_y = -5\,y \end{cases}$$

3. The stream function Ψ(x,y). and the shape of the streamlines?

   In the same way, the stream function is calculated.

$$\begin{cases} v_x = \dfrac{\partial\psi}{\partial y} \\ v_y = -\dfrac{\partial\psi}{\partial x} \end{cases}$$

$$\begin{cases} 2x - 3y = \dfrac{\partial\psi}{\partial y} \\ 3x - 2y = -\dfrac{\partial\psi}{\partial x} \end{cases}$$

So

$$\begin{cases} \psi(x,y) = 2xy - \dfrac{3}{2}y^2 + C(x) \\ -\dfrac{\partial\psi}{\partial x} = 3x - 2y = -2y - \dfrac{dC(x)}{dx} \rightarrow \dfrac{dC(x)}{dx} = -3x \\ C(x) = -\dfrac{3}{2}x^2 + const \end{cases}$$

$$\psi(x,y) = 2xy - \frac{3}{2}y^2 - \frac{3}{2}x^2 + C$$

Streamlines:

$$\psi(x,y) = const \rightarrow 2xy - \frac{3}{2}y^2 - \frac{3}{2}x^2 = C$$

4. Is the flow rotational?

$\overrightarrow{rot}\vec{v} = 6\,\vec{k}$ ⟶ The flow is rotational

5. The vorticity vector field

By definition, the vorticity vector field is:

$$\vec{\omega} = \frac{1}{2}\,\overrightarrow{rot}\vec{v} = 3\,\vec{k}$$

Vortex lines are curves that are tangent at every point to the vorticity vector.

Therefore : $\vec{dl} \,//\, \vec{\omega}$ $\quad (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}) \times (3\,\vec{k}) = \vec{0}$

$$(3\,dx\,\vec{i} - 3\,dy\,\vec{j}) = \vec{0}$$

$$\begin{vmatrix} dx = 0 \\ dy = 0 \\ z = z \end{vmatrix}$$

$$\begin{vmatrix} x = C_1 \;, \quad C_1 \; constant\; for\; the\; same\; line, the\; constant\; changes\; from\; one\; line\; to\; another. \\ y = C_2 \;, \; C_2 \; constant\; for\; the\; same\; line\; if\; we\; change\; the\; line\; the\; constant\; changes \\ z = z \end{vmatrix}$$

6. the strain tensor field $\overline{\overline{G}}$.

with definition :

$$\overline{\overline{G}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \Rightarrow \overline{\overline{G}} = \begin{pmatrix} 2 & -3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Chapter IV: Inviscid Fluid Dynamics

In this chapter, we are interested in establishing the fundamental equations that govern the dynamics of perfect incompressible fluids.

IV.1 Fundamental equation for a perfect fluid - Euler's theorem

IV.1.1 Reynolds transport theorem

Consider a scalar quantity $f(\vec{r},t)$ function of the space and time coordinates:

Over the volume Vs of a system of fluid particles, the integral of $f(\vec{r},t)$ is written:

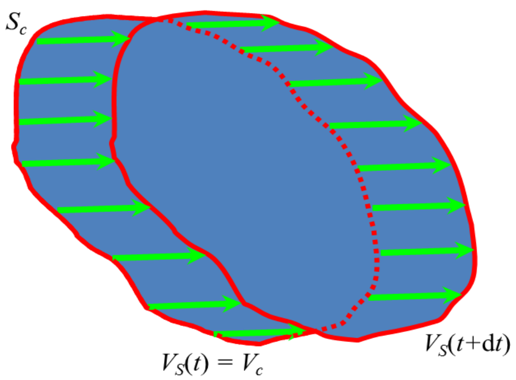
$$F = \iiint_{V_S} f(\vec{r},t)\,\mathrm{d}V$$

If we want to evaluate the variations of F over time, we need to calculate:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_S(t)} f(\vec{r},t)\,\mathrm{d}V$$

The problem is that here Vs is a function of time: it means the system of fluid particles is in motion. The transport theorem consists of using a fixed volume Vc (control volume), delimited by a surface Sc (control surface) through which we can calculate the flow of f:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_S(t)} f(\vec{r},t)\,\mathrm{d}V$$



$$\frac{\mathrm{d}F}{\mathrm{d}t} = \underbrace{\iiint_{V_c} \frac{\partial f}{\partial t}\,\mathrm{d}V}_{\text{local derivative}} + \underbrace{\iint_{S_c} f\,\vec{v}.\vec{n}\,\mathrm{d}S}_{\text{Convective derivative}}$$

local derivative — *Convective derivative*

Instantaneous variations of f over the Vc — *Flux of f through the Sc*

IV.1.2 Transport theorem applied to momentum

The momentum of a fluid system in volume V_S is written:

$$\iiint_{V_S} \rho\vec{v}\,\mathrm{d}V$$

However, the fundamental principle of dynamics states that the derivative with respect to time of the momentum must be equal to the sum of the forces acting on the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_S} \rho\vec{v}\,\mathrm{d}V = \vec{R} + \vec{P}$$

$\vec{R}$: *surface forces*; $\vec{P}$: *volume forces (weight)*

$$\vec{P} = \iiint_{V_S} \rho\,\vec{g}\,\mathrm{d}V$$ ➡ trivial

$$\vec{R} = \iint_{S_S} \overline{\overline{T}}\,\vec{n}\,\mathrm{d}S$$ ➡ difficult access ➡ it is usually more easy to calculate the change in the momentum

So :

$$\vec{R} + \vec{P} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_S} \rho\vec{v}\,\mathrm{d}V = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_S} \rho v_i\,\mathrm{d}V\right]\vec{e}_i$$

$$= \sum_i \left[\iiint_{V_c} \frac{\partial(\rho v_i)}{\partial t}\mathrm{d}V + \iint_{S_c} (\rho v_i)\,\vec{v}\vec{n}\,\mathrm{d}S\right]\vec{e}_i$$

$$= \iiint_{V_c}\left[\sum_i \frac{\partial(\rho v_i)}{\partial t}\vec{e}_i\right]\mathrm{d}V + \iint_{S_c}\left[\sum_i (\rho v_i)\vec{e}_i\right]\vec{v}\vec{n}\,\mathrm{d}S$$

$$= \underbrace{\iiint_{V_c} \frac{\partial(\rho\vec{v})}{\partial t}\mathrm{d}V}_{\text{Instantaneous derivative momentum}} + \underbrace{\iint_{S_c} (\rho\vec{v})\,\vec{v}\vec{n}\,\mathrm{d}S}_{\text{Momentum flux through the control surface}}$$

Instantaneous derivative momentum

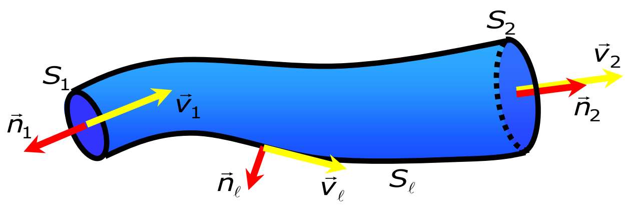
Momentum flux through the control surface

For a stationary flow, the instantaneous derivative is zero, so we will have:

$$\vec{R} + \vec{P} = \iint_{S_c} (\rho \vec{v}) \vec{v}\vec{n} \, dS$$

IV.1.3 Euler's theorem

Let's apply the previous result to the case of a stream tube:



We will assume that the speed is constant at any point of the same section (the average speed), and the flow is stationary.

$$\vec{R} + \vec{P} = \iint_{S_1+S_2+S_\ell} (\rho \vec{v}) \vec{v}\vec{n} \, dS$$

$$\vec{R} + \vec{P} = \underbrace{\iint_{S_1} (\rho \vec{v}_1) \underbrace{\vec{v}_1\vec{n}_1}_{-V1} dS}_{-\rho v_1 \vec{v}_1 S_1} + \underbrace{\iint_{S_2} (\rho \vec{v}_2) \underbrace{\vec{v}_2\vec{n}_2}_{-V2} dS}_{\rho v_2 \vec{v}_2 S_2} + \iint_{S_\ell} (\rho \vec{v}_\ell) \underbrace{\vec{v}_\ell\vec{n}_\ell}_{0} dS$$

Where : $\vec{R} + \vec{P} = -\rho v_1 \vec{v}_1 S_1 + \rho v_2 \vec{v}_2 S_2$

However, we know that in a current tube the mass flow is conserved:

$$q_m = \rho v_1 S_1 = \rho v_2 S_2$$

Hence we obtain the following simple result:

$$\vec{R} + \vec{P} = q_m (\vec{v}_2 - \vec{v}_1) \qquad \textit{Euler's theorem}$$

Theorem statement:

The resultant (ΣFext) of the external mechanical actions exerted on an isolated fluid (fluid contained in the envelope limited by S1 and S2) is equal to the variation of the quantity of movement of the fluid which enters S1 with a speed V1 and leaves through S2 with a speed V2.

Noticed

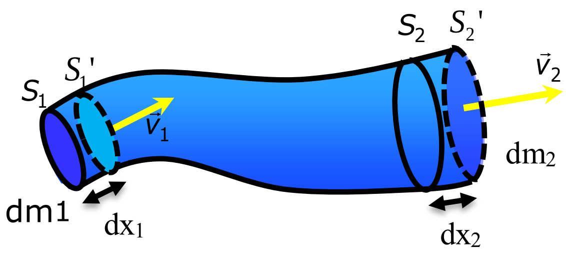
This theorem makes it possible to determine the forces exerted by the moving fluid on the objects which surround them. A direct application of Euler's theorem is the evaluation of the forces exerted by water jets. These are exploited in various fields: production of electrical energy from hydraulic energy using turbines. On the other hand, it makes it possible to simply obtain the resultant of the surface forces (in particular the friction forces) without having to calculate the tensor of the constraints

IV.2 Bernoulli's Equation

In this paragraph we will determine the fundamental equation of the flow of a fluid which will verify the following hypotheses:

- The fluid is perfect and incompressible.
- The flow is permanent.
- The flow is in a perfectly smooth pipe.

Let’s consider a stream tube of an incompressible fluid of density ρ animated by a permanent flow



We denote by:

- S_1 and S_2 respectively the inlet section and the outlet section of the fluid at time t.
- S'_1 and S'_2 respectively the inlet and outlet sections of the fluid at time t'=(t+dt).

- V_1 and V_2 the flow velocity vectors respectively through the sections S_1 and S_2 of the tube.

- dx_1 and dx_2 respectively the displacements of sections S_1 and S_2 during the time interval dt,

- dm_1: incoming elementary mass between sections S_1 and S'_1,

- dm_2: outgoing elementary mass between sections S_2 and S'_2,

- dV_1: incoming elementary volume between sections S_1 and S'_1,

- dV_2: outgoing elementary volume between sections S_2 and S'_2,

By conservation of mass:

$dm_1 = dm_2$

So $\rho_1 .dV_1 = \rho_2 .dV_2$

Or $\rho_1 .S_1 .dx_1 = \rho_2 .S_2 .dx_2$

By dividing by dt we ended up with:

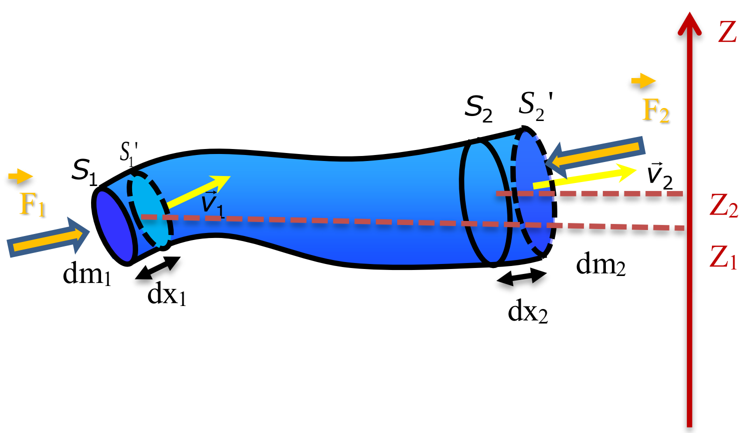
$$\rho_1 s_1 \frac{dx_1}{dt} = \rho_2 s_2 \frac{dx_2}{dt}$$

$$\rho_1 s_1 v_1 = \rho_2 s_2 v_2$$

Since the fluid is incompressible: $\rho = \rho_1 = \rho_2$, , We can simplify and arrive at the following continuity equation:

$$s_1 v_1 = s_2 v_2$$

We consider a vertical axis OZ directed upwards.



We note Z_1, Z_2 respectively the altitudes of the centers of gravity of the masses dm_1, dm_2
And we designate by F_1 and F_2 respectively the forces of the fluid pressure acting on the sections S_1 and S_2.
At time t the fluid of mass dm_1 is between S_1 and S'_1. Its mechanical energy is:

$$E_m = E_{pot} + E_{cin} = dm_1 g\, z_1 + \tfrac{1}{2} dm_1 v_1^2$$

At time t+dt the fluid of mass dm_1 is between S_2 and S'_2. Its mechanical energy is:

$$E'_m = E'_{pot} + E'_{cin} = dm_2 g\, z_2 + \tfrac{1}{2} dm_2 v_2^2$$

We apply the theorem of mechanical energy (The variation of mechanical energy is equal to the sum of the works of the external forces) to the fluid between t and t':

$$E'_m - E_m = w_{forcesdepression} = F_1 dx_1 - F_2 dx_2 = P_1 S_1 dx_1 - P_2 S_2 dx_2$$

$$= P_1 dV_1 - P_2 dV_2 = \frac{P_1}{\rho_1} dm_1 - \frac{P_2}{\rho_2} dm_2$$

By conservation of mass: dm1 = dm2 and since the fluid is incompressible: ρ1 = ρ2 = ρ, We arrive at Bernoulli's equation:

$$\frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(Z_2 - Z_1) = 0$$

$$\Downarrow$$

$$P_1 + \rho g Z_1 + \tfrac{1}{2}\rho V_1^2 = P_2 + \rho g Z_2 + \tfrac{1}{2}\rho V_2^2$$

Bernoulli's equation along the same streamline:

$$P + \rho g Z + \tfrac{1}{2}\rho V^2 = cont$$

IV.3 Interpretation of Bernoulli's equation

IV.3.1 Interpretation in terms of energy

Let's multiply all the terms of Bernoulli's equation by a volume V:

$$p.V + \rho gz.V + \tfrac{1}{2}\rho v^2.V = \mathrm{C}^{\mathrm{te}} \times V$$

$p.V$: Work of pressure forces: potential energy due to pressure forces.

$\rho gz.V = mgz$: Potential energy due to the forces of gravity.

$\tfrac{1}{2}\rho v^2.V = \tfrac{1}{2}mv^2$: Kinetic energy

$\mathrm{C}^{\mathrm{te}} \times V = E_m$: Total energy: mechanical energy.

- Consequently : $p + \rho gz + \tfrac{1}{2}\rho v^2 = \dfrac{E_m}{V}$ corresponds to mechanical energy per unit of volume.
- The mechanical energy then remains constant along a streamline (there is no energy dissipation).

IV.3.2 Interpretation in terms of pressure

$$p + \rho gz + \tfrac{1}{2}\rho v^2 = \mathrm{C}^{\mathrm{te}}$$

P: Static pressure (it exists even if there is no movement)

$p + \rho gz = p^*$: Driving pressure (it generates the movement)

$\tfrac{1}{2}\rho v^2$: Dynamic pressure (it results from the movement)

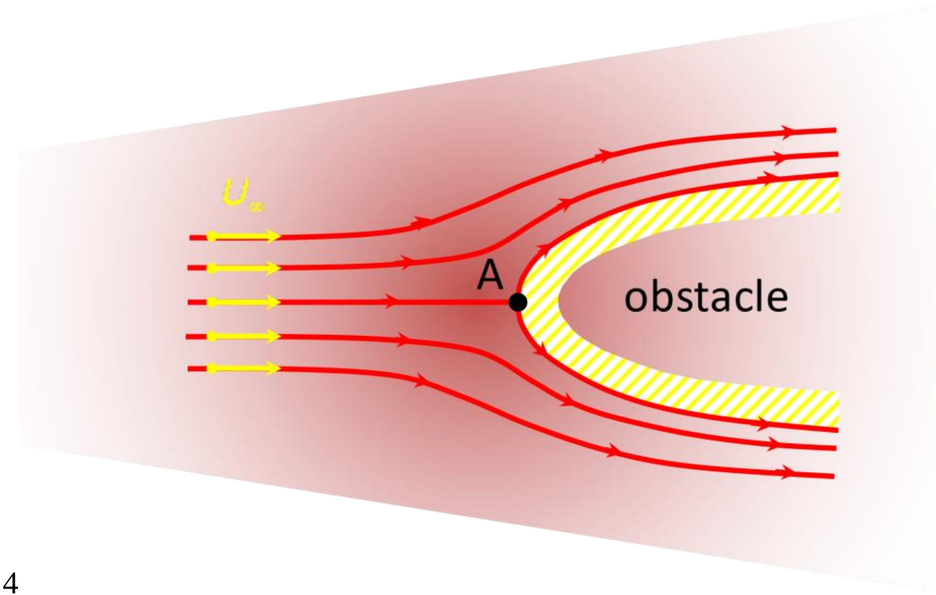
$p + \rho gz + \tfrac{1}{2}\rho v^2 = P_t$: Total pressure (or load)

Bernoulli's equation then shows that the charge remains constant along the same streamline (no charge loss in the flow of an ideal fluid).

IV. Applications 4

IV.4.1 The pressure upstream of an obstacle

The pressure upstream of an obstacle (aircraft) can be calculated using Bernoulli's theorem. Point A on the attack front of the object is a stationary point.



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Consider the streamline passing through the stopping point and apply Bernoulli between point A and a point located far upstream. Along the same streamline, we check:

$$p + \rho g z + \tfrac{1}{2}\rho v^2 = C^{te}$$

$$p_\infty + \rho g z_\infty + \tfrac{1}{2}\rho \underbrace{v_\infty^2}_{U_\infty^2} = p_A + \rho g z_A + \tfrac{1}{2}\rho \underbrace{v_A^2}_{0} \Rightarrow p_\infty + \rho g z_\infty + \tfrac{1}{2}\rho U_\infty^2 = p_A + \rho g z_A$$

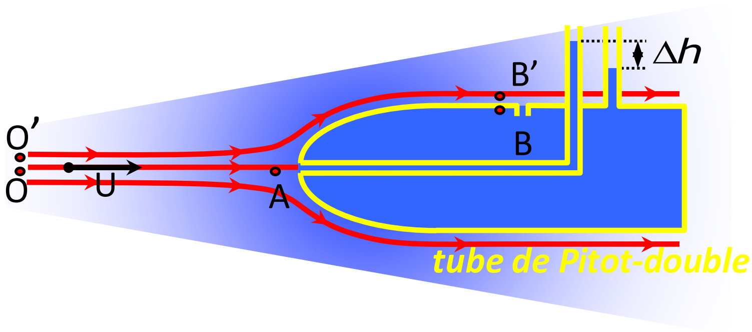
In the case of plane flow : $z = Cte$, so:

$$p_\infty + \tfrac{1}{2}\rho U_\infty^2 = p_A$$

The pressure P_A is called stagnation pressure.

IV.4.2 Pitot tube

Different Types of Prandtl Probes
(pitot-double tubes)



At O and O', the flow is supposed to be uniform, of speed U. Streamlines are supposed to be rectilinear and parallel, so the pressure is the same at O and O' ⇒ $p_0 = p_0'$

For the same reasons, the pressure is the same in B and B'⇒pB = pB'

The fluid is immobile inside the probe; thereby, the pressure is uniform and equal to the pressure at B.

- The first manometer gives the pressure at A
- The second manometer gives the pressure at B

$$p_A - p_B = \rho g \Delta h$$

By applying Bernoulli's equation between O and A, we have:

$$p_0 + \tfrac{1}{2}\rho U^2 = p_A$$

Then between O' and B':

$$p_{0'} + \tfrac{1}{2}\rho U^2 = p_{B'} + \tfrac{1}{2}\rho v_{B'}^2$$

We can then assume that the flow has become uniform again far after the front of the object:

$$v_{B'} = U$$

where: $p_{0'} = p_{B'}$ However, we have seen that: $p_{0'} = p_0$, $p_{B'} = p_B \Rightarrow p_O = p_B$

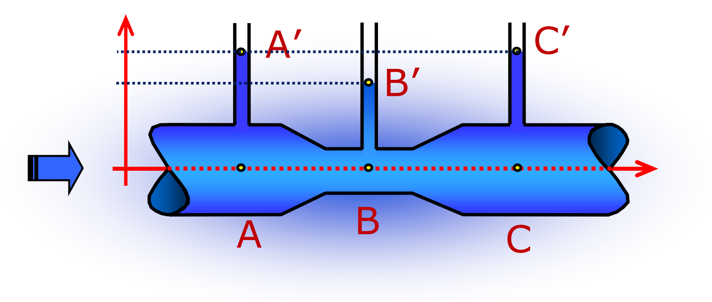
As a result: $p_B + \frac{1}{2}\rho U^2 = p_A$

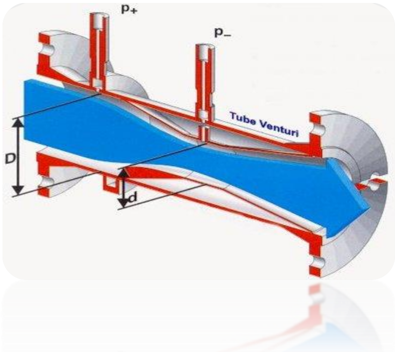
$$\Rightarrow \quad \tfrac{1}{2}\rho U^2 = p_A - p_B = \rho g \Delta h$$

$$\Rightarrow \quad U = \sqrt{2g\Delta h}$$

IV.4.3 Venturi tube - Flow rate measurement

It is an instrument for measuring with accuracy the flow rate of fluids in pipes. It consists of a tube with a constricted center section and widened ends





There are 3 pressure probes (manometers) placed:

- at the Upstream of the constriction $\Rightarrow P_A$
- At the constriction $\Rightarrow P_B$
- At the Downstream of the constriction $\Rightarrow P_C$ (optional probe)

Under each pressure probe, the streamlines can be considered rectilinear and parallel in the perpendicular direction (along z). The hydrostatic law can be used to calculate the pressure at points A, B, and C as follows.

$$\begin{cases} p_A = p_{A'} + \rho g z_{A'} \\ p_B = p_{B'} + \rho g z_{B'} \\ p_C = p_{C'} + \rho g z_{C'} \end{cases} \quad \text{where} \quad p_{A'} = p_{B'} = p_{C'} = p_{atm}$$

Applying Bernoulli's equation to the streamline passing through A, B, and C:

$$p_A + \rho g z_A + \tfrac{1}{2}\rho v_A^2 = p_B + \rho g z_B + \tfrac{1}{2}\rho v_B^2 = p_C + \rho g z_C + \tfrac{1}{2}\rho v_C^2$$

$$z_A = z_B = z_C = 0$$

$$\Rightarrow \overbrace{p_{atm} + \rho g z_{A'}} + \tfrac{1}{2}\rho v_A^2 = p_{atm} + \rho g z_{B'} + \tfrac{1}{2}\rho v_B^2 = p_{atm} + \rho g z_{C'} + \tfrac{1}{2}\rho v_C^2$$

$$\Rightarrow z_{A'} + \frac{1}{2}\frac{v_A^2}{g} = z_{B'} + \frac{1}{2}\frac{v_B^2}{g} = z_{C'} + \frac{1}{2}\frac{v_C^2}{g}$$

knowing that the volume flow rate is conserved and assuming that the velocity is uniform over the same section:

$$q_v = S_A v_A = S_B v_B = S_C v_C$$

Note that: $S_A > S_B \Rightarrow v_A < v_B \Rightarrow z_{A'} > z_{B'}$

(constriction) (acceleration) (depression)

1. And if $S_A = S_C$ then $v_A = v_C$ and, $z_{A'} = z_{C'}$: the 3rd probe will only be used for a study of pressure drops (loss of charge)

$$z_{A'} + \frac{1}{2}\frac{v_A^2}{g} = z_{B'} + \frac{1}{2}\frac{v_B^2}{g} \Rightarrow \Delta z = z_{A'} - z_{B'} = \frac{1}{2g}(v_B^2 - v_A^2)$$

$$S_A v_A = S_B v_B \Rightarrow v_B = v_A \frac{S_A}{S_B}$$

So : $\Delta z = \frac{1}{2g} v_A^2 (S_A^2 / S_B^2 - 1)$ thereby : $v_A = \sqrt{\frac{2g\Delta z}{(S_A/S_B)^2 - 1}}$

The flow rate in the pipe is obtained by:

$$q_v = S_A \sqrt{\frac{2g\Delta z}{(S_A/S_B)^2 - 1}}$$

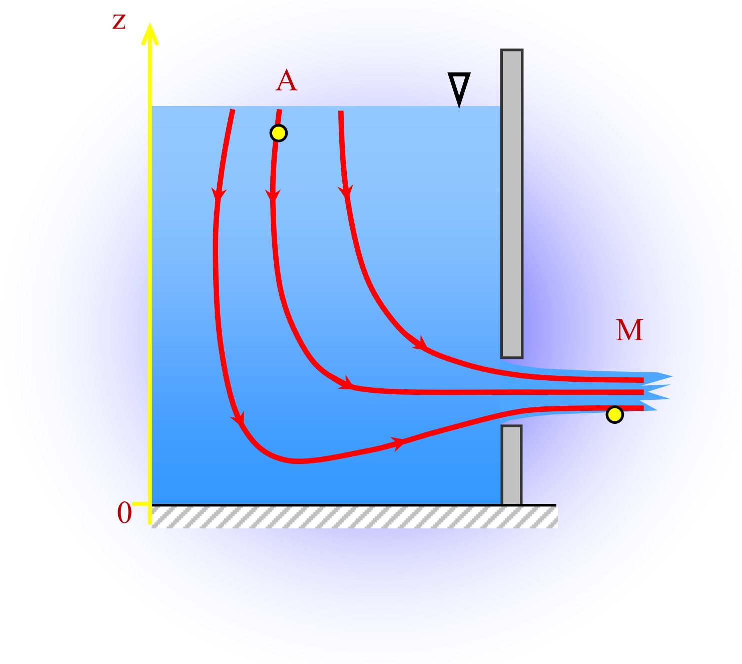
As a function of the diameter D of the pipe and the diameter d of the constriction, the flow rate is expressed by:

$$q_v = \frac{\pi D^2}{4}\sqrt{\frac{2g\Delta z}{(D/d)^4 - 1}}$$

IV.4.4 Flow through an orifice (draining of a tank)

Torricelli formula

Consider the draining of a tank through an orifice placed under the free surface:



Let's apply Bernoulli's equation between a point A at the free surface and a point M at the jet:

$$p_A + \rho g z_A + \tfrac{1}{2}\rho v_A^2 = p_M + \rho g z_M + \tfrac{1}{2}\rho v_M^2$$

Assumption 1: In the jet, the streamlines are rectilinear and parallel, the altitude variations being negligible, the static pressure can then be considered uniform throughout the jet.

Since there is no pressure discontinuity at the jet-atmosphere interface, the static pressure in the jet is equal to atmospheric pressure.

As a result : $p_A = p_M = p_{atm}$

$$p_A + \rho g z_A + \tfrac{1}{2}\rho v_A^2 = p_M + \rho g z_M + \tfrac{1}{2}\rho v_M^2$$

($p_A \to$ pat, $p_M \to$ pat)

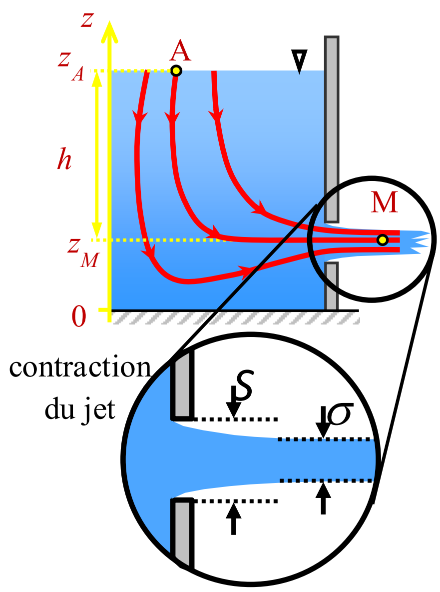
$$\Downarrow$$

$$\rho g z_A + \tfrac{1}{2}\rho v_A^2 = \rho g z_M + \tfrac{1}{2}\rho v_M^2$$

<u>Assumption 2</u>: The speed of the descent level of the free surface can be considered negligible compared to that of the fluid flowing in the jet: $v_A << v_M$

therefore, we get the Torricelli velocity:

$$\rho g \underbrace{(z_A - z_M)}_{h} = \tfrac{1}{2}\rho(v_M^2 - v_A^2) \approx \tfrac{1}{2}\rho v_M^2 \Rightarrow v_M = \sqrt{2gh}$$

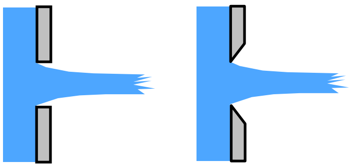


Flow rate calculation:

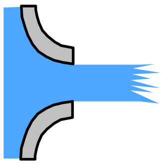
$$q_v = \sigma v_M = \sigma\sqrt{2gh} \text{ Or } \sigma = C_c S$$

C_C is the coefficient of contraction.

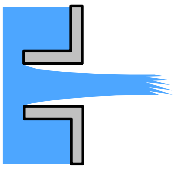
The coefficient of contraction depends on the geometry of the orifice. In general, Cc is determined experimentally :



Thin walls Cc = 0.61



Orifice with profiled edges Cc = 1.00



Re-entrant orifice CC = 0.50

IV.4.5 Cavitation phenomenon

The phenomenon of cavitation corresponds to the formation of bubbles of vapor within a moving liquid.

As a consequence of Bernoulli's equation, as the speed increases, the pressure decreases. If the pressure falls below the saturation vapor pressure, then the liquid evaporates, which involves the formation of bubbles. In practice, and most cases, this phenomenon is troublesome. For example:

- Cavitation consumes energy: the energy consumed for the formation of bubbles (phase transition) + stresses
- Cavitation is the cause of the premature deterioration of ship propellers.

The bubbles created by cavitation migrate spontaneously to areas where the fluid pressure is higher: they burst, and the mechanical shock causes damage.

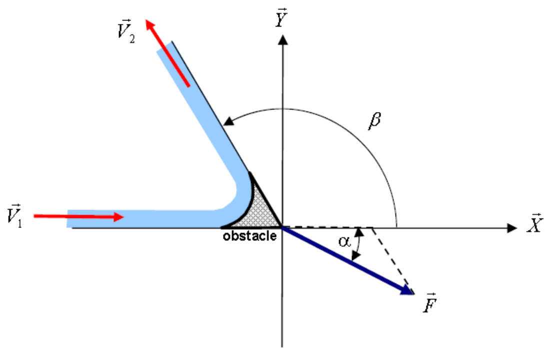
IV. 5 Exercises

Exercise 1

The figure below represents a horizontal jet of water striking an obstacle with a mass flow rate of

q_m=2 kg/s. The obstacle deflects the jet by an angle β = 120 °.
$\vec{v_1}$ denotes the water flow velocity at the inlet of the obstacle. It is directed along the X-axis.
$\vec{v_2}$ denotes the water flow velocity at the outlet of the obstacle. It is directed along a line making an angle β=120° with respect to the X-axis.
Assume that: $\|\vec{v_1}\| = \|\vec{v_2}\|$ = 3 m/s

1. By applying Euler's theorem, give the vector expression of the force $\vec{F}$ exerted by the liquid on the obstacle as a function of q_m, $\vec{v_2}$, and $\vec{v_1}$. Then calculate its components F_x and F_y.
2. What is its inclination angle α?



Exercise 2

Water (assumed to be a perfect fluid) flows from point A to point B with a volume flow rate of 350 L/s. The pressure at A is 0.70 bar. Calculate the pressure at B (detail the calculations, then the numerical applications).

Data: Diameters at points A and B: DA = 35.0 cm, DB = 64.0 cm.

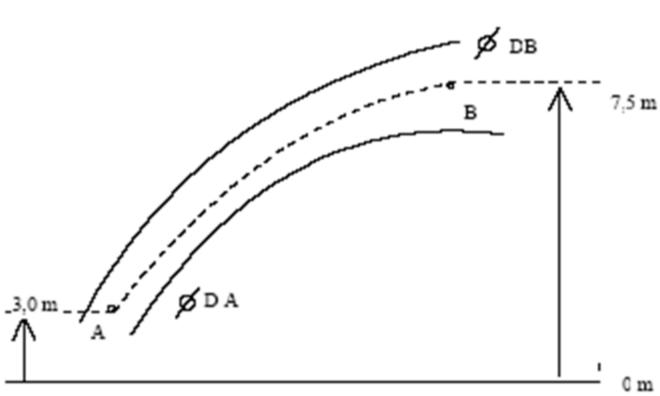


Figure.2

Exercise 3

We consider the horizontal convergent in Figure 3 in which air (assumed perfect incompressible fluid) circulates. The volume flow rate is: $q_v = 220\ ls^{-1}$. $S_1 = 6.5\times10^{-2}\ m^2$ and
$S_2 = 2.0\times10^{-2}\ m^2$.

1- Calculate the mass flow rate q_m . Assume the density of the air is constant ρ (air) = 3.20 kg.m-3.

2- Calculate the average speeds V_1 and V_2.

3- Calculate the pressure difference $\Delta P = P_1 - P_2$ at the terminals of the convergent. Give its value in Pascal and mbar.

4- Calculate the difference in level h of a water differential manometer connected between points 1 and 2.

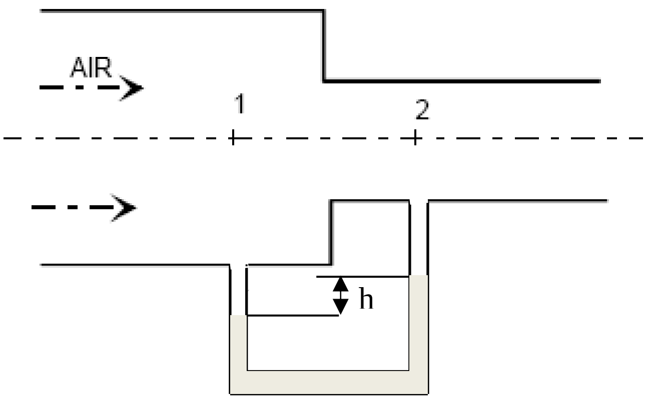


Figure.3

Exercise 4

We want to accelerate the circulation of a perfect fluid in a pipe so that its speed is multiplied by 4. For this, the pipe has a convergent characterized by the angle α (Figure.4).

1- Calculate the radius ratio R1/R2. Numerical application.

2- Calculate (R1- R2) as a function of L and α. Deduce the length L. (R1 = 50 mm, α = 15°)

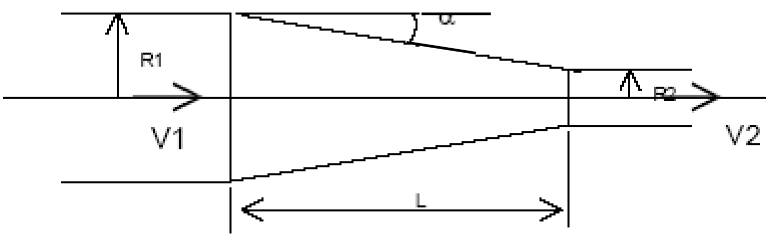


Figure. 4

Exercise 5

A vertical tank (see figure.5) filled with water; we assumed that the level A in the tank is constant. The fluid flows through an orifice of diameter D located at the bottom of the tank. Water is considered an incompressible perfect fluid.

1- Apply Bernoulli's relation between points A and B and determine the literal expression of the velocity V_B at the level of the orifice
2- Numerically calculate the velocity V_B and the volume flow rate q_v at point B.
3- In fact, the actual flow rate is 0.92 L/s. Compare to the value found in question 2. Justify?

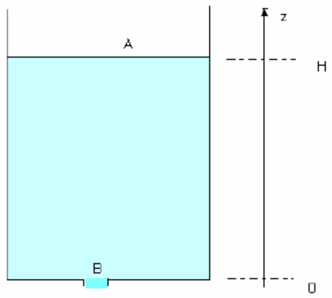


Figure 5

4- This difference is partly explained by a contraction of the liquid vein at the outlet of the orifice. Deduce the diameter D' of the liquid stream at the outlet of the tank.
Numerical values:
H=0.82 m, D=2.0 cm.
ρ(water) = 1000 kg.m^{-3}.
g = 9.81 ms^{-2}.

IV. 6 Exercise Solutions

Solution 1

Given Data :

Mass flow rate: Q_m=2 kg/s

Speed magnitude: $\|\vec{v_1}\| = \|\vec{v_2}\|$ = 3 m/s

Deflection angle: β=120°

Applying Euler's theorem (momentum theorem) for a steady flow gives:

$$\sum\vec{F} = Q_m(\vec{v_2} - \vec{v_1})$$

This equation gives the force exerted by the obstacle on the fluid. But the problem asks for the force exerted by the liquid on the obstacle.

According to Newton's third law:

$$\vec{F} = -Q_m(\vec{v_2} - \vec{v_1})$$

Therefore:

$$\vec{F} = Q_m(\overrightarrow{v_1} - \overrightarrow{v_2})$$

Velocity vectors

Inlet velocity: The inlet velocity is along the positive X-axis:

$\overrightarrow{v_1} = 3\vec{\imath}$

Outlet velocity: The outlet velocity makes an angle of 120° with the X-axis:

$\overrightarrow{v_2} = 3\cos 120\ \vec{\imath} + 3\sin 120\ \vec{\jmath}$

Thus : $\overrightarrow{v_2} = -\frac{3}{2}\vec{\imath} + \frac{3\sqrt{3}}{2}\vec{\jmath}$

Calculate the force vector

$\vec{F} = Q_m(\overrightarrow{v_1} - \overrightarrow{v_2}) = 2\ (3\vec{\imath} - \frac{3}{2}\vec{\imath} + \frac{3\sqrt{3}}{2}\vec{\jmath})$

Simplify:

$\vec{F} = (9\vec{\imath} - 3\sqrt{3}\ \vec{\jmath})$ N

Components F_x and F_y

$F_x = 9$ N, $Fy = -5.20$ N

Inclination angle α

The angle is measured below the positive X-axis. Using:

$$\tan\alpha = \frac{\|F_y\|}{F_x} = \frac{1}{\sqrt{3}}$$

Therefore: α=30°

Solution 2

Given data :

Volume flow rate: Q=350 L/s = 0.350 m3/s
Pressure at A: PA=0.70 bar $=7.0\times10^4$ Pa
Diameters: D_A= 35.0 cm = 0.35 m
D_B= 64.0 cm = 0.64 m
Density of water: $\rho = 1000$ kg/m^3
We assume points A and B are at the same height: $z_A=z_B$

1) The velocities at A and B

Using the continuity equation: Q = S v

With $S_A = \frac{\pi\, DA^2}{4} = 0.0962\ m^2$ and $S_B = \frac{\pi\, DB^2}{4} = \ = 0.3217\ m^2$

Thus $v_A = \frac{Q}{S_A} = \approx 3.64\ \mathrm{m/s}$ and $v_A = \frac{Q}{S_A} = \approx 1.09\ \mathrm{m/s}$

2) Apply Bernoulli's equation for an inviscid fluid:

$$P_A + \frac{1}{2}\rho v_A^2 + \rho g z_A = P_B + \frac{1}{2}\rho v_B^2 + \rho g z_B$$

Since $z_A = z_B$

$$P_A + \frac{1}{2}\rho v_A^2 = P_B + \frac{1}{2}\rho v_B^2$$

$$P_B = P_A + \frac{1}{2}\rho\,(v_A^2 - v_B^2)$$

$$P_B \approx 76030\ \mathrm{Pa} = 0.76\ \mathrm{bar}$$

Because the diameter at B is larger, the velocity decreases, and therefore the pressure increases according to Bernoulli's principle.

Solution 3

1. Mass flow rate Q_m

$$Q_m = \rho\, q_v = 3.20 \times 0.220 = 0.704\ \mathrm{kg/s}$$

2. Average velocities V_1, V_2

Applying the continuity equation: $q_v = S \cdot V$

At section 1:

$$v_1 = \frac{Qv}{S1} = \frac{0.220}{6.5\times 10^{-2}} = 3.38\ m/s$$

At section 2:

$$v_2 = \frac{0.220}{2 \times 10^{-2}} = 11\ m/s$$

3. Pressure difference $\Delta P = P1 - P2$

Bernoulli equation (horizontal flow):

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Delta P = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

$$\Delta P = 0.5 \times 3.2 \times (11^2 - 3.38^2)$$
$$\Delta P = 1.6 \times (109.58) \approx 175.3\ Pa$$

Thus: $\Delta P \approx 175.3\ Pa = 1.75$ mbar

$$\Delta P \approx 1.75 \text{ mbar}$$

4. Water manometer height difference h

$$\Delta P = \rho_{wate}\ g\ h$$

Take:
ρ_{water}= 1000 kg/m^3 and g = 9.81m/s^2

$$h = \frac{175.3}{1000 \times 9.81} \approx 0.0179\ m$$

$$h \approx 1.79\ cm$$

Solution 4

The fluid velocity is multiplied by 4: $V_2 = 4V_1$
The pipe contains a convergent section.
Given: R_1 = 50 mm , α=15∘

Calculate the radius ratio $\frac{R_1}{R_2}$:

For an incompressible perfect fluid, the continuity equation gives:

$$Q = S_1 v_1 = S_2 v_2$$

Here $S = \pi R^2$

Thus: Simplify:

$$\pi R_2^2 v_1 = \pi R_1^2 v_1$$

Since:
$v_2 = 4 v_1$

We get:

$$R_1^2 = 4R_2^2$$

Take the square root:

$$\frac{R_1}{R_2} = 2$$

Numerical Application

Given: R_1=50 mm , then: R2 = 25 mm

<u>Calculate (R1−R2) as a function of L and α</u>

from the geometry of the convergent:

$$\tan\alpha = \frac{R1-R2}{L}$$

Therefore:

$$R1-R2 = L\tan\alpha$$

<u>Deduce the length L</u>

$$L = \frac{R1-R2}{\tan\alpha}$$

Substitute the values:

$$R1-R2=25\text{ mm}\quad\text{then}\quad L=\frac{25}{\tan 15}$$

we obtain: L ≈ 93.3 mm = 9.3 cm

<u>Solution 5</u>

Given data:

Height of water: H=0.82 m Orifice diameter:

D = 2.0 cm=0.020 m

Density of water: ρ =1000 kg/m^3

g = 9.81 m/s^2

Flow rate: qv_{real}=0.92 L/s

<u>1) Apply Bernoulli's equation between A and B</u>

We apply Bernoulli's theorem between:

- Point A is in the free surface of the tank

- Point B is at the outlet orifice

For a perfect incompressible fluid:

$$P_A + \frac{1}{2}\rho v_A^2 + \rho g z_A = P_B + \frac{1}{2}\rho v_B^2 + \rho g z_B$$

We notice that both points are exposed to atmospheric pressure:

$P_A = P_B = P_{atm}$

The tank is large compared with the orifice, so: $v_A \approx 0$

Height difference : $z_A - z_B = H$

Therefore:

$$g\,H = \frac{1}{2} v_B^2$$

Hence:

$$v_B = \sqrt{2gH}$$

This is Torricelli's formula.

2) Numerical calculation of V_B and q_v

Velocity at B: $v_B = 4.01$m/s

Volume flow rate : $qv = S_B\, v_B$ where: $S_B = \frac{\pi D^2}{4} = 3.14\times10^{-4}$ m^2

Thus: $qv \approx 1.26\times10^{-3}$ m^3/s = 1.26 L/s

3) Compare with the actual flow rate

The theoretical value: qv_{th}=1.26 L/s

Actual value: $qv_{,real}$=0.92 L/s

We observe: $q_{v,real} < q_{v,thq}$

4) Determine the diameter D′ of the contracted jet

The actual flow rate is:

$$q_{v,real} = S' v_B$$

where:

$$S'_B = \frac{\pi D'^2}{4}$$

Thus: $q_{v,real} = \frac{\pi D'^2}{4} v_B$

$$D' = \sqrt{\frac{4q_{vreal}}{\pi v_B}} \qquad D' \approx 1.71 \text{ cm}$$

Comparison

$q_{v,real} < q_{v,th}$ because of viscosity, turbulence, and contraction of the jet.

Chapter V: Real Fluid Dynamics

This approach considers the various forces acting on fluid particles in motion, from which a quantitative description of the flow can be derived using local fundamental equations.

V.1 Application of the Fundamental Principle of Dynamics

To establish the fundamental equation of the dynamics, we apply the fundamental principle of the dynamics to an element of fluid volume in motion. We must therefore take the balance of the forces acting on the surface and in volume.

$$d\vec{F} = \boxed{d\vec{F}_S} + \boxed{d\vec{F}_V} = \rho dV \frac{d\vec{v}}{dt}$$

Surface forces   Volume forces (weight forces)

$$d\vec{F}_V = \rho dV\vec{g}$$

- If $\vec{v}$ is a vector field in Euler description, we should use the particle derivative:

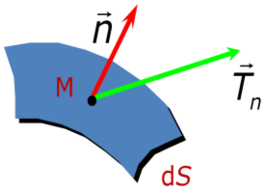
$$\text{So; } \frac{d\vec{v}}{dt} = \frac{D\vec{v}}{Dt}$$

V.1.1 Surface Forces - Stress Tensor

For a real (viscous) fluid in motion, surface forces are no longer just normal to the surface: there are tangential stresses due to viscosity (friction).

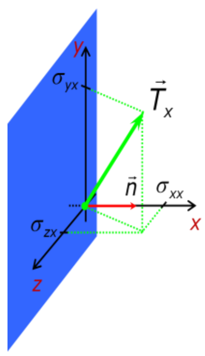
At a point M of a surface dS, the surface force is expressed as:

$$d\vec{F} = \vec{T}_n\, dS$$



Yet, $\vec{T}_n$ is the stress acting on the normal surface $\vec{n}$.

Consider a surface ⊥ to the x-axis. The normal to this surface is : $\vec{n} = \vec{e}_x$



The stress exerted on this surface is then noted $\vec{T}_x$ and can be broken down as: $\vec{T}_x = \sigma_{xx}\vec{e}_x + \sigma_{yx}\vec{e}_y + \sigma_{zx}\vec{e}_z$

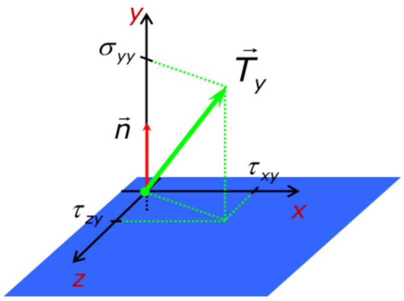
We notice that the components σ_{yx} And σ_{zx} are tangential components: we will rather denote them τ_{yx} And τ_{zx} to distinguish them from the normal component σ_{xx} .

We can also consider the surface ⊥ to the y axis. we thereby have the stress force:

$$\vec{T}_y = \tau_{xy}\vec{e}_x + \sigma_{yy}\vec{e}_y + \tau_{zy}\vec{e}_z$$

And for the surface ⊥ at the z axis the stress is expressed:

$$\vec{T}_z = \tau_{xz}\vec{e}_x + \tau_{yz}\vec{e}_y + \sigma_{zz}\vec{e}_z$$

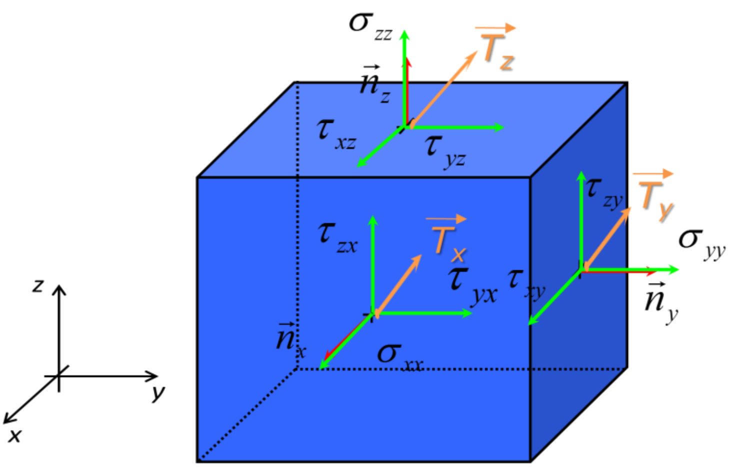


Let's now consider a surface whose orientation is arbitrary. In the Cartesian coordinate system, its normal can be broken down into:

$$\vec{n} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$$

In this case, the stress exerted on this surface is expressed as:

$$\vec{T}_n = n_x\vec{T}_x + n_y\vec{T}_y + n_z\vec{T}_z$$



Expanding, we get:

$$\vec{T}_n = \left| \begin{array}{l} n_x(\sigma_{xx}\vec{e}_x + \tau_{yx}\vec{e}_y + \tau_{zx}\vec{e}_z) \\ + n_y(\tau_{xy}\vec{e}_x + \sigma_{yy}\vec{e}_y + \tau_{zy}\vec{e}_z) \\ + n_z(\tau_{xz}\vec{e}_x + \tau_{yz}\vec{e}_y + \sigma_{zz}\vec{e}_z) \end{array} \right. = \begin{cases} (n_x\sigma_{xx} + n_y\tau_{xy} + n_z\tau_{xz})\vec{e}_x \\ (n_x\tau_{yx} + n_y\sigma_{yy} + n_z\tau_{yz})\vec{e}_y \\ (n_x\tau_{zx} + n_y\tau_{zy} + n_z\sigma_{zz})\vec{e}_z \end{cases}$$

Which refers to the product of a matrix by the normal:

$$\vec{T}_n = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \Rightarrow \vec{T}_n = \overline{\overline{T}}\,\vec{n}$$

Stress Tensor

Using the divergence theorem (the surface integral of a vecteur field over a closed suface, which is called the flux through the surface, is equal to the volume integral of the divergence over the region inside the surface) the surface forces dF are written

$$\mathrm{d}\vec{F} = \vec{T}_n\,\mathrm{d}S \Longrightarrow \mathrm{d}\vec{F} = \overline{\overline{T}}\,\vec{n}\,\mathrm{d}S \Longrightarrow \mathrm{d}\vec{F} = \vec{\nabla}\overline{\overline{T}}\,\mathrm{d}V$$

Substitute in the forces balance, we get :

$$\mathrm{d}\vec{F} = \mathrm{d}\vec{F}_S + \mathrm{d}\vec{F}_V = \rho\,\mathrm{d}V\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

$$\Rightarrow \vec{\nabla}\overline{\overline{T}}\,\cancel{\mathrm{d}V} + \rho\,\vec{g}\,\cancel{\mathrm{d}V} = \rho\,\cancel{\mathrm{d}V}\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

$$\Rightarrow \vec{\nabla}\overline{\overline{T}} + \rho\,\vec{g} = \rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

We can then show the two parts of the stress tensor:

$$\overline{\overline{T}} = -p\overline{\overline{I}} + \overline{\overline{T}}'$$

$$\vec{\nabla}\overline{\overline{T}} = -\vec{\nabla}\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \vec{\nabla}\overline{\overline{T'}}$$

$$= -\vec{\nabla}p + \vec{\nabla}\overline{\overline{T'}}$$

Finally the fundamental equation of the dynamics (local equation):

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla}p + \vec{\nabla}\overline{\overline{T'}} + \rho\,\vec{g}$$

V.2 Newtonian fluid and Navier-Stokes equation

By definition, “Newtonian” fluids are those for which the components of the viscosity stress tensor $\overline{\overline{T'}}$ depend linearly on the tensor components of the pure strain rates $\overline{\overline{e}}$.

Notice

- A pure rotation generates no deformation: consequently there is no constraint (stress). That is why $\overline{\overline{T'}}$ And $\overline{\overline{\omega}}$ are not related.
- All the fluids which will be studied can be considered Newtonian.

Consider the strain tensor elements of $\overline{\overline{e}}$:

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

this tensor is symmetric, because $e_{ij} = e_{ji}$

We then admit that for an isotropic fluid, the tensorial elements of $\overline{\overline{T'}}$ And $\overline{\overline{e}}$ are linked by the following relationship:

$$\sigma'_{ij} = 2\mu e_{ij} + \mu'(e_{xx} + e_{yy} + e_{zz})\delta_{ij}$$

viscosity (μ) — Viscosity of dilation (μ') — Kronecker symbol (δ_{ij})

Note that : $$e_{xx}+e_{yy}+e_{zz}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=\vec{\nabla}\vec{v}=\frac{\Delta \mathrm{V}}{\mathrm{V}}$$

So, if the fluid is incompressible, we have $\vec{\nabla}\vec{v}=0$ and in this case :

$$\sigma'_{ij}=2\mu e_{ij}\ \Rightarrow\ \overline{\overline{T'}}=2\mu\,\overline{\overline{e}}$$

Let’s take the fundamental equation of the dynamics:

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=-\vec{\nabla}p+\vec{\nabla}\overline{\overline{T'}}+\rho\,\vec{g}$$

Case of a Newtonian incompressible fluid

For a Newtonian incompressible fluid, this equation therefore becomes:

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=-\vec{\nabla}p+2\mu\vec{\nabla}\overline{\overline{e}}+\rho\,\vec{g}$$

Meaning of the term $\vec{\nabla}\overline{\overline{e}}$

$$\vec{\nabla}\overline{\overline{e}}=\sum_i\left(\sum_j\frac{\partial e_{ij}}{\partial x_j}\right)\vec{e}_i \quad \text{où}\quad e_{ij}=\tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$$

$$\Rightarrow\quad \vec{\nabla}\overline{\overline{e}}=\sum_i\sum_j\left(\tfrac{1}{2}\frac{\partial^2 v_i}{\partial x_j^2}+\tfrac{1}{2}\frac{\partial^2 v_j}{\partial x_j\partial x_i}\right)\vec{e}_i$$

$$=\tfrac{1}{2}\underbrace{\sum_i\Big(\underbrace{\sum_j\frac{\partial^2 v_i}{\partial x_j^2}}_{\Delta v_i}\Big)\vec{e}_i}_{\Delta\vec{v}}+\tfrac{1}{2}\underbrace{\sum_i\frac{\partial}{\partial x_i}\Big(\underbrace{\sum_j\frac{\partial v_j}{\partial x_j}}_{\vec{\nabla}\vec{v}}\Big)\vec{e}_i}_{\vec{\nabla}(\vec{\nabla}\vec{v})}=\tfrac{1}{2}\Delta\vec{v}+\tfrac{1}{2}\vec{\nabla}(\cancel{\vec{\nabla}\vec{v}})$$

Incompressible fluid

Thus, It remains: $\vec{\nabla}\overset{=}{e} = \frac{1}{2}\Delta\vec{v}$ Which leads to :

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p + 2\mu\vec{\nabla}\overset{=}{e} + \rho\vec{g} \Rightarrow \rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \mu\Delta\vec{v} + \rho\vec{g}$$

Fundamental equation of dynamics for an incompressible Newtonian fluid

Navier–Stokes equation (local equation)

The particle derivative of the velocity is as follows:

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\,\vec{\nabla})\,\vec{v}$$

The instantaneous

The convective derivative

$$\Rightarrow \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\,\vec{\nabla})\,\vec{v}\right) = -\vec{\nabla}p + \mu\Delta\vec{v} + \rho\vec{g}$$

By putting $\vec{g} = -g\vec{e}_z$, the projection of the Navier-Stokes equation on the 3 axes of the Cartesian coordinate system gives:

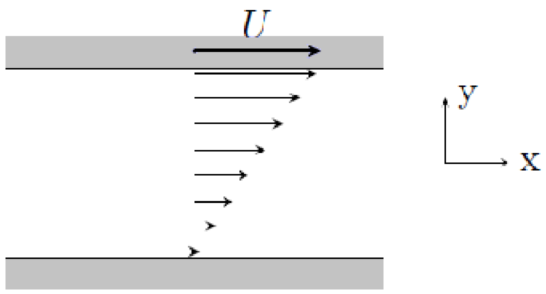
$$\begin{cases} \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \\ \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) \\ \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \rho g \end{cases}$$

V.3 Application on particular real flows

Solving complex Navier-Stokes equations is difficult and most of the time requires the adoption of simplifying assumptions. In the following real flows, we will present some assumptions that will allow us to simplify the Navier-Stokes equations and determine the velocity field.

V.3.1 Couette flow

A stationary flow of an incompressible viscous fluid occurs between 2 horizontal plates separated by a distance h, one of which is moving tangentially relative to the other with a constant speed U. The relative motion of the surfaces imposes a shear stress on the fluid and induces flow in the direction x without pressure gradient The effects of the gravitational field are neglected.



Two-dimensional flow $\Longrightarrow$ $\vec{v} = v(x,y)\,\overrightarrow{e_x}$

Incompressible flow $\Longrightarrow$ $\mathrm{div}\vec{v} = 0$ in the present case $\frac{\partial v}{\partial x} = 0$

v does not depend on x so $\vec{v} = v(y)\,\overrightarrow{e_x}$

$$\begin{cases}
\rho\left(\underbrace{\frac{\partial v_x}{\partial t}}_{0} + v_x\underbrace{\frac{\partial v_x}{\partial x}}_{0} + \underbrace{v_y}_{0}\frac{\partial v_x}{\partial y} + \underbrace{v_z}_{0}\underbrace{\frac{\partial v_x}{\partial z}}_{0}\right) = -\underbrace{\frac{\partial p}{\partial x}}_{0} + \mu\left(\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{0} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{0}\right) \\
\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{0} + v_x\underbrace{\frac{\partial v_y}{\partial x}}_{0} + \underbrace{v_y}_{0}\underbrace{\frac{\partial v_y}{\partial y}}_{0} + \underbrace{v_z}_{0}\underbrace{\frac{\partial v_y}{\partial z}}_{0}\right) = -\underbrace{\frac{\partial p}{\partial y}}_{0} + \mu\left(\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{0}\right) \\
\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{0} + v_x\underbrace{\frac{\partial v_z}{\partial x}}_{0} + \underbrace{v_y}_{0}\underbrace{\frac{\partial v_z}{\partial y}}_{0} + \underbrace{v_z}_{0}\underbrace{\frac{\partial v_z}{\partial z}}_{0}\right) = -\underbrace{\frac{\partial p}{\partial z}}_{0} + \mu\left(\underbrace{\frac{\partial^2 v_z}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{0}\right)
\end{cases}$$

The Navier Stokes equations reduce to:

$$\mu\left(\frac{\partial^2 v_x}{\partial y^2}\right) = 0$$

After integration we obtain the velocity field:

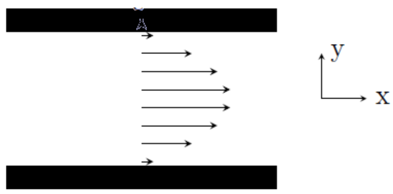
$$\vec{v} = (Ay + B)\overrightarrow{e_x}$$

Using boundary conditions $v(0) = 0$ and $v(\mathrm{h}) = \mathrm{U}$, we obtain the following linear velocity profile:

$$v(y) = \frac{\mathrm{U}}{\mathrm{h}}y$$

V.3.2 Poiseuille plane flow

Stationary flow of an incompressible fluid between two fixed plates of length L, separated by a width h, and oriented along the direction x. The flow is done by a pressure gradient along the direction x



The Navier-Stokes equations reduce in

$$\begin{cases} 0 = -\dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 v_x}{\partial y^2}\right) \Rightarrow \dfrac{\partial p}{\partial x} = \dfrac{dp}{dx} = \mu\left(\dfrac{d^2 v_x}{dy^2}\right) \\ 0 = -\dfrac{\partial p}{\partial y} \\ 0 = -\dfrac{\partial p}{\partial z} \end{cases} \Longrightarrow \quad p = p(x)$$

After integration we will have:

$$\frac{dv}{dy} = y\frac{1}{\mu}\frac{dp}{dx} + A \quad \Rightarrow v(y) = \frac{y^2}{2\mu}\frac{dp}{dx} + Ay + B$$

Non-slip conditions on the plates, $v(0) = v(h) = 0$, determine the integration constants to give:

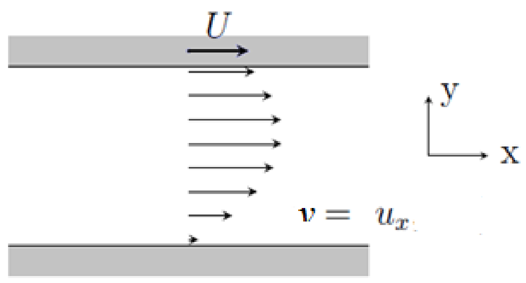
$$\begin{cases} v(0) = 0 = B \\ v(h) = 0 = \dfrac{h^2}{2\mu}\dfrac{dp}{dx} + Ah \end{cases} \Longrightarrow \begin{cases} B = 0 \\ A = -\dfrac{h}{2\mu}\dfrac{dp}{dx} \end{cases}$$

The velocity field of Poiseuille plane flow is:

$$v(y) = \frac{y}{2\mu}\frac{dp}{dx}(y-h)$$

V.3.3 Couette-Poiseuille plane shear flow

We consider a stationary flow between two infinite plates, one of which is immobile while the other is driven by a constant speed U. The flow undergoes a pressure gradient along the direction x



Same equation of motion as that of the plane Poiseuille flow is obtained after simplification of the Navier-Stokes equations:

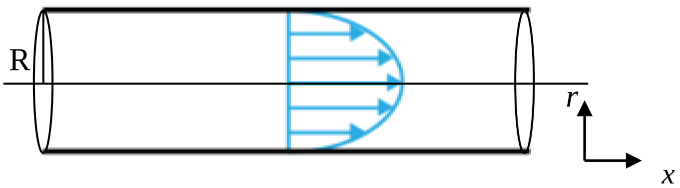
$$\frac{dv}{dy} = y\frac{1}{\mu}\frac{dp}{dx} + A \quad \Rightarrow \quad v(y) = \frac{y^2}{2\mu}\frac{dp}{dx} + Ay + B$$

The velocities at the two plates, u_x (y = 0) = 0 and u_x (y = h) = U, determine the integration constants to give the velocity profile:

$$v(y) = \frac{dp}{2\mu\, dx} y(y-h) + U\frac{y}{h}$$

V.3.4 Laminar flow in a cylindrical conduit: Poiseuille's law

The resolution of the Navier-Stokes equations for the flow of an incompressible Newtonian fluid in the laminar regime leads to the Poiseuille law, which describes the pressure drop due to fluid viscosity.



The Navier-Stokes equations in cylindrical coordinates (x, r,θ):

$$\frac{\partial u}{\partial t} + v_r\frac{\partial u}{\partial r} + \frac{v_\theta}{r}\frac{\partial u}{\partial \theta} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + u\frac{\partial v_r}{\partial x} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + u\frac{\partial v_\theta}{\partial x} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta v_\theta - \frac{v_\theta}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

The continuity equation in cylindrical coordinates:

$$\frac{\partial(ru_r)}{r\partial r} + \frac{\partial u_\theta}{r\partial \theta} + \frac{\partial u_x}{\partial x} = 0$$

Simplification of the Navier-Stokes equations according to the hypotheses of this flow:

- Steady flow $\longrightarrow$ $\frac{\partial}{\partial t} = 0$
- The radial and azimuthal components of the fluid velocity are zero ($v_r = v_\theta = 0$). (streamlines are parallel to the x axe ; there is no rotation around the axis)
- The flow is axisymmetric $\longrightarrow$ $\frac{\partial}{\partial \theta} = 0$
- Incompressible flow $\longrightarrow$ $\frac{\partial v}{\partial x} = 0$

We obtain :

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) = 0$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \Rightarrow P(x,r,\theta) = p(x)$$

Knowing that:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

Transforming the resulting equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{1}{\mu}\frac{dp}{dx}$$

After integration it comes:

$$v(r) = \frac{r^2}{4\mu}\frac{dp}{dx} + A\ln r + B$$

v (r=0)$\rightarrow \infty$; Knowing that the velocity has a finite value at r = 0, so A= 0

The non-slip boundary condition at the pipe wall requires that v (r = R) = 0, which gives

$B = -\frac{R^2}{4\mu}\frac{dp}{dx}$:

Thus the parabolic velocity profile is written:

$$v(r) = \frac{1}{4\mu}\frac{dp}{dx}(r^2 - R^2)$$

The maximum velocity occurs at the pipe centerline (r = 0), $u_{max} = \frac{1}{4\mu}\frac{dp}{dx}R^2$. This is average velocity can be obtained by integrating over the pipe cross-section,

$$u_{avg} = \frac{1}{\pi R^2}\int_0^R 2\pi r u \, dr = \frac{1}{2}u_{max}$$

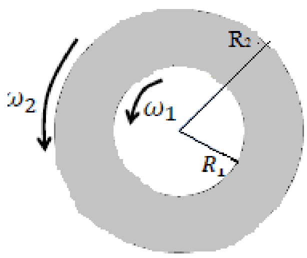
The volumetric flow rate $Q = \pi R^2 u_{avg}$.

The rearrangement of the equation gives the Hagen–Poiseuille equation

$$p = \frac{8\mu Q L}{\pi R^4}$$

V.3.5 Cylindrical Couette flow

This is the axisymmetric version of plane Couette flow; the viscous fluid is contained between two concentric cylinders of radius $R_1$and R_2 and angular velocity ω_1 and ω_2



The Navier-Stokes equations in cylindrical coordinates, neglecting the effects of gravity, are written:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial}{\partial \theta}\frac{u_\theta}{} - \frac{u_r}{r^2}\right]$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} - \frac{u_r u_\theta}{r} =$$
$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right]$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} - \frac{u_r u_\theta}{r} =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r}\right]$$

The continuity equation in cylindrical coordinates:

$$\frac{\partial(ru_r)}{r\partial r}+\frac{\partial u_\theta}{r\partial\theta}+\frac{\partial u_z}{\partial z}=0$$

According to the hypotheses of the flow, it comes:

The movement is permanent $\longrightarrow$ $\frac{\partial}{\partial t}=0$

Axisymmetrical flow $\longrightarrow$ $\frac{\partial}{\partial\theta}=0$

The direction along z is assumed to be infinite and speedless $\longrightarrow$ $v_z = 0$

The flow is incompressible $\longrightarrow$ $v_r = 0$

The Navier-Stokes equations give, according to the three projections:

$$-\rho\frac{u_\theta^2}{r}=-\frac{\partial p}{\partial r}$$

$$0=\mu\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$0=-\frac{\partial p}{\partial z}$$

Which give :

$$p(r,\theta,z)=p(r)$$

$$u_\theta^2=\frac{r}{\rho}\frac{\partial p}{\partial r}\Rightarrow u_\theta(r,\theta,z)=u_\theta(r)$$

Integrating the second equation gives:

$$\frac{d^2u_\theta}{\partial r^2}+\frac{1}{r}\frac{du_\theta}{dr}-\frac{u_\theta}{r^2}=0\Leftrightarrow\frac{d}{dr}\left(\frac{du_\theta}{dr}+\frac{u_\theta}{r}\right)=0,\quad \text{so}\quad \frac{du_\theta}{dr}+\frac{u_\theta}{r}=A \longrightarrow \frac{1}{r}\frac{d(ru_\theta)}{dr}=A$$

$$\text{soit}: u_\theta(r)=\frac{A}{2}r+\frac{B}{r}\quad and\quad u_\theta(R_1)=\omega_1 R_1\quad and\quad u_\theta(R_2)=\omega_2 R_2$$

The velocity profile of this type of flow is written as follows:

$$u_\theta(r) = \frac{(\omega_1 - \omega_2)R_2^2 R_1^2}{R_2^2 - R_1^2}\frac{1}{r} + \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^2 - R_1^2} r, \; avec, \;\; B = \frac{(\omega_1 - \omega_2)R_2^2 R_1^2}{R_2^2 - R_1^2} \;\; and \;\; A = 2\frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^2 - R_1^2}$$

And the expression of the pressure field is:

$$P(r) = \rho(\frac{A^8}{8} r^2 + AB\ln(r) - \frac{B^2}{2r^2}) + p_0$$

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