PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
Ministry of Higher Education and Scientific Research

University of Biskra
Faculty of Exact Sciences
Department of Mathematics

MATHEMATICS 1

Course and Corrected Exercises

For First Year Students – Common Core
Science and Technology Domain

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Semester: **1**

Teaching Unit: **UEF 1.1.1**

Subject: **Mathematics 1**

Total Hours: **67 hours 30 minutes** (Lectures: 3h00, Tutorials: 1h30)

Credits: **6**   Coefficient: **3**

Objectives

By the end of this course, students will have a solid understanding of mathematical reasoning methods, set theory, relations and functions, limits and continuity of real functions, differentiation, elementary functions (power, exponential, logarithmic, trigonometric, hyperbolic, inverse), Taylor expansions, and basic concepts of linear algebra (vector spaces, linear transformations, kernel, image and rank).

Prerequisites

Secondary school mathematics.

Course Content Overview

Chapter	**Weeks**
Chapter 1: Methods of Mathematical Reasoning 1.1 Direct reasoning 1.2 Reasoning by contraposition 1.3 Reasoning by contradiction (reductio ad absurdum) 1.4 Reasoning by counterexample 1.5 Reasoning by mathematical induction	01
Chapter 2: Sets, Relations and Functions 2.1 Set theory 2.2 Order relations and equivalence relations 2.3 Injective, surjective, bijective functions: definition, direct image, inverse image, characteristics of a mapping	02
Chapter 3: Real Functions of One Real Variable 3.1 Limits and continuity of a function 3.2 Derivative and differentiability of a function	03
Chapter 4: Applications to Elementary Functions 4.1 Power functions 4.2 Logarithmic function 4.3 Exponential function 4.4 Hyperbolic functions 4.5 Trigonometric functions 4.6 Inverse functions	03
Chapter 5: Taylor Expansions 5.1 Taylor's formula 5.2 Taylor expansion 5.3 Applications	02
Chapter 6: Linear Algebra 6.1 Internal laws and composition 6.2 Vector spaces, basis, dimension (definitions and elementary properties) 6.3 Linear transformations, kernel, image, rank	04

Assessment

Continuous assessment: **40%**
Final examination: **60%**

Contents

Foreword

This document presents the Mathematics 1 course. It is intended for first-year students of the Science and Technology domain, and is designed to serve as a concise instructional manual aligned with the official curriculum and the allotted teaching schedule.

The content is structured into six main chapters. The first half establishes the fundamental tools of mathematical reasoning and analysis: Chapter 1 introduces the methods of mathematical logic and proof, including direct proof, contradiction, contrapositive, and induction; Chapter 2 develops the theory of sets, relations, and functions, covering equivalence and order relations, images, and bijectivity; Chapter 3 studies real functions of one real variable, with a thorough treatment of limits, continuity, and differentiability.

The second half of the course builds on these foundations through the study of classical functions and algebraic structures. Chapter 4 is devoted to elementary functions, including logarithmic, exponential, trigonometric, inverse trigonometric, and hyperbolic functions. Chapter 5 introduces Taylor expansions (Taylor series) and their applications to the computation of limits and local behaviour of functions. Chapter 6 provides an introduction to linear algebra, covering groups, rings, fields, vector spaces, and linear maps. Each chapter ends with fully solved exercises to reinforce understanding and promote deeper assimilation of the concepts introduced.

Any remarks or suggestions are welcome to help improve the content of this work.

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CHAPTER 1

Methods of Mathematical Reasoning

1.1 Mathematical Logic

Propositions

Definition 1.1 (Proposition).

A **proposition** (or statement) is a sentence that is either true or false, but not both.

Example 1.1. The following sentences are propositions:

1. $2 + 3 = 5$ (True)
2. $7 > 10$ (False)
3. $\forall x \in \mathbb{R}, x^2 \geq 0$ (True)
4. The number 17 is prime. (True)
5. There are infinitely many prime numbers. (True)

The following sentences are **not** propositions:

1. $x + 1 = 5$ (Depends on x)
2. What time is it? (Question)
3. This sentence is false. (Paradox)
4. n is an even number. (Depends on n)

Logical Connectives

Definition 1.2 (Negation).

The **negation** of a proposition P, denoted $\neg P$ or $\overline{P}$, is true when P is false, and false when P is true.

P	$\neg P$
T	F
F	T

Example 1.2.

1. P: "It is raining." $\neg P$: "It is not raining."
2. Q: "$2+2=4$." $\neg Q$: "$2+2\neq 4$."
3. R: "All cats are black." $\neg R$: "There exists a cat that is not black."

Definition 1.3 (Conjunction - "and").

The statement "P and Q" (denoted $P \wedge Q$) is true if P is true and Q is true, and false otherwise.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.3.

1. $(3 < 5) \wedge (2 \text{ divides } 4)$ is true.
2. $(2+2=4) \wedge (2\times 3 = 7)$ is false.

Definition 1.4 (Disjunction - "or").

The statement "P or Q" (denoted $P \vee Q$) is true if at least one of the propositions P or Q is true. It is false if both P and Q are false.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.4.

1. $(2+2=4) \vee (3\times 2 = 6)$ is true.
2. $(2=4) \vee (4\times 3 = 7)$ is false.

Definition 1.5 (Implication).

The statement "P implies Q" (denoted $P \Rightarrow Q$) is false only when P is true and Q is false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.5.

1. P: "It is raining." Q: "The streets are wet."
   $P \Rightarrow Q$: "If it is raining, then the streets are wet."
2. "$2+2=5 \Rightarrow \sqrt{2}=2$" is true! (If P is false then the statement $P \Rightarrow Q$ is always true.)

Definition 1.6 (Equivalence).

The statement "P is equivalent to Q" (denoted $P \Leftrightarrow Q$) is true when P and Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.6. For $x, y \in \mathbb{R}$, the equivalence "$xy=0 \Leftrightarrow x=0$ or $y=0$" is true.

Theorem 1.1 (Important Logical Equivalences).

Let P, Q, and R be three propositions. We have the following equivalences:

1. *Double Negation:* $\neg(\neg P) \equiv P$
2. *De Morgan's Laws:*
   - $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
   - $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
3. *Commutativity:* $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$
4. *Associativity:* $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$, $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
5. *Distributivity:* $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
6. *Implication Equivalences:*
   - $P \Rightarrow Q \equiv \neg P \vee Q$
   - $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$ *(Contrapositive)*

Example 1.7. Show that $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$.

Solution:

$$\begin{aligned}
\neg(P \Rightarrow Q) &\equiv \neg(\neg P \vee Q) \quad (\text{since } P \Rightarrow Q \equiv \neg P \vee Q) \\
&\equiv \neg(\neg P) \wedge \neg Q \quad (\text{De Morgan's Law}) \\
&\equiv P \wedge \neg Q \quad (\text{Double Negation})
\end{aligned}$$

Quantifiers

Definition 1.7 (Universal Quantifier).

The quantifier $\forall$ means "for all". The statement

$$\forall x \in E, P(x)$$

is true when the statements $P(x)$ are true for all elements x of the set E.

Example 1.8.

1. $\forall x \in \mathbb{R}, x^2 \geq 0$ is true.
2. $\forall x \in \mathbb{R}, x^2 \geq 1$ is false (take $x = 0$).

Definition 1.8 (Existential Quantifier).

The quantifier $\exists$ means "there exists". The statement

$$\exists x \in E, P(x)$$

is true when we can find at least one element x of E for which $P(x)$ is true.

Example 1.9.

1. $\exists x \in \mathbb{R}, x^2 \leq 0$ is true (for example $x = 0$).
2. $\exists x \in \mathbb{R}, x^2 < 0$ is false.

Negation of Quantifiers

Theorem 1.2 (Negation of Quantifiers).

1. $\neg(\forall x \in E, P(x)) \equiv \exists x \in E, \neg P(x)$
2. $\neg(\exists x \in E, P(x)) \equiv \forall x \in E, \neg P(x)$

Example 1.10.

1. The negation of $\forall x \in \mathbb{R}, x^2 \geq 0$ is $\exists x \in \mathbb{R}, x^2 < 0$.
2. The negation of $\exists x \in \mathbb{R}, x < 0$ is $\forall x \in \mathbb{R}, x \geq 0$.

Multiple Quantifiers

We can combine several quantifiers in a quantified proposition, but we must not change their order if they are of different types.

Example 1.11.

1. $(\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y = 2)$ and $(\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : 2x + y = 2)$ are two different quantified propositions.
2. $(\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y = 2)$ and $(\exists y \in \mathbb{R}, \exists x \in \mathbb{R} : 2x + y = 2)$ are two equivalent quantified propositions.

Negation of a Quantified Proposition

When forming the negation of a quantified proposition, we replace the universal quantifier $\forall$ with the existential quantifier $\exists$ and vice versa, and the property $P(x)$ with its negation $\neg P(x)$.

Example 1.12.

1. The negation of $(\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y = 2)$ is $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y \neq 2$.
2. The negation of $(\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : (x+y = 1) \wedge (2xy \leq 1))$ is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (x+y \neq 1) \vee (2xy > 1)$.

1.2 Reasoning Methods

Direct Proof

We want to show that the statement $P \Rightarrow Q$ is true. We assume that P is true and then we show that Q is true.

Example 1.13. Let $a, b \in \mathbb{R}$. Show that $a = b \Rightarrow \frac{a+b}{2} = b$.

Solution: Take $a = b$, then $\frac{a}{2} = \frac{b}{2}$, so

$$\frac{a}{2} + \frac{b}{2} = \frac{b}{2} + \frac{b}{2} \Rightarrow \frac{a+b}{2} = b.$$

Example 1.14. Show that the sum of two even integers is even.

Solution: Let a and b be even integers. Then there exist integers k and l such that $a = 2k$ and $b = 2l$. Then:

$$a + b = 2k + 2l = 2(k + l)$$

Since $k + l$ is an integer, $a + b$ is even.

Proof by Contrapositive

Proof by contrapositive is based on the equivalence: $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$. So if we want to show the statement $P \Rightarrow Q$, we actually show that if $\neg Q$ is true then $\neg P$ is true.

Example 1.15. Let $x \in \mathbb{R}$. Show that $(x \neq 2$ and $x \neq -2) \Rightarrow (x^2 \neq 4)$.

Solution: By contrapositive, this is equivalent to $(x^2 = 4) \Rightarrow (x = 2$ or $x = -2)$. Indeed, take $x^2 = 4$, then $(x - 2)(x + 2) = 0$, so $x = 2$ or $x = -2$.

Example 1.16. Show: If n^2 is even, then n is even.

Solution by contrapositive: We show: If n is odd, then n^2 is odd. Let n be odd. Then $n = 2k + 1$ for some integer k. Then:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which is odd.

Proof by Contradiction

Proof by contradiction to show $P \Rightarrow Q$ relies on the following principle: We assume both that P is true and that Q is false and we look for a contradiction. Thus if P is true then Q must be true, so $P \Rightarrow Q$ is true.

Example 1.17. Let $a, b > 0$. Show that if $\frac{a}{1+b} = \frac{b}{1+a}$ then $a = b$.

Solution: We reason by contradiction by assuming $\frac{a}{1+b} = \frac{b}{1+a}$ and $a \neq b$. We have

$$\frac{a}{1+b} = \frac{b}{1+a} \Leftrightarrow a(a+1) = b(b+1) \Leftrightarrow a^2 - b^2 = -(a-b) \Leftrightarrow (a-b)(a+b) = -(a-b)$$

Since $a - b \neq 0$, we can divide by $a - b$, obtaining $a + b = -1$. The sum of two positive numbers cannot be negative. We obtain a contradiction.

Example 1.18. Show that $\sqrt{2}$ is irrational.

Solution: Assume for contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ where a and b are coprime integers and $b \neq 0$.

Squaring: $2 = \frac{a^2}{b^2}$ so $a^2 = 2b^2$. Thus a^2 is even, so a is even. Write $a = 2k$. Then:

$$(2k)^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$$

So b^2 is even, hence b is even. But then a and b are both even, contradicting the fact that they are coprime. Therefore, $\sqrt{2}$ is irrational.

Proof by Cases

When we want to verify a property $P(x)$ for all elements x of a set E, we can divide this set into n (non-empty) subsets $A_1, A_2, \ldots, A_n$ (according to the number of cases to treat) and show that the property holds on each of them without exception, then conclude that it holds everywhere on E.

Example 1.19. Let $n \in \mathbb{N}$. Show that $a_n = \frac{1}{2}n(n+1) \in \mathbb{N}$.

Solution: Let $n \in \mathbb{N}$. We distinguish two cases.

- First case: $n = 2k$, $k \in \mathbb{N}$, then $a_n = \frac{1}{2}(2k)(2k+1) = k(2k+1) \in \mathbb{N}$.
- Second case: $n = 2k+1$, $k \in \mathbb{N}$, then $a_n = \frac{1}{2}(2k+1)(2k+2) = (2k+1)(k+1) \in \mathbb{N}$.

In all cases, $a_n = \frac{1}{2}n(n+1) \in \mathbb{N}$.

Proof by Counterexample

If we want to show that a statement of the type $(\forall x \in E : P(x))$ is true, then for each x in E we must show that $P(x)$ is true. On the other hand, to show that this statement is false, it suffices to find $x \in E$ such that $P(x)$ is false.

Example 1.20. Show that the statement $(\forall x \in \mathbb{R}, x^2 - 1 > 1)$ is false.

Solution: A counterexample is $x = 0 \in \mathbb{R}$, because $0^2 - 1 = -1 > 1$ is false.

Proof by Induction

The principle of induction allows us to show that a statement $P(n)$, depending on n, is true for all $n \in \mathbb{N}$.

A proof by induction proceeds in two steps:

1. We prove that $P(0)$ is true (initialization).

2. We assume $n \geq 0$ given with $P(n)$ true (induction hypothesis), and we then show that $P(n+1)$ is true (heredity).

Finally, in the conclusion, we recall that by the principle of induction $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.21. Show that for all $n \in \mathbb{N} : 2^n > n$.

Solution: Let $P(n) : 2^n > n$, for all $n \in \mathbb{N}$.

1. For $n = 0$ we have $2^0 = 1 > 0$, so $P(0)$ is true.
2. Let $n \in \mathbb{N}$, assume $P(n)$ is true. We will show that $P(n+1)$ is true.

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ &> n + 2^n \quad \text{(since by } P(n) \text{ we know that } 2^n > n) \\ &\geq n + 1 \quad \text{(since } 2^n \geq 1) \end{aligned}$$

So $P(n+1)$ is true.

By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.22. Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}^*$.

Solution:

1. **Initialization:** For $n = 1$: $1 = \frac{1(2)}{2} = 1$. So $P(1)$ is true.
2. **Induction step:** Assume $P(k)$ is true for some $k \geq 1$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Then for $n = k + 1$:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

which is the formula for $n = k + 1$. So $P(k+1)$ is true.

By induction, the formula holds for all $n \in \mathbb{N}^*$.

1.3 Exercises and Solutions

Exercise 1.1 — Truth Tables and Logical Equivalences.

Construct the truth tables for the following statements and determine which pairs are logically equivalent:

(a) $P \vee Q \Rightarrow P \wedge Q$
(b) $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$
(c) $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$
(d) $P \Rightarrow Q$ and $\neg P \vee Q$
(e) $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$
(f) $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$

Exercise 1.2 — Negations and Truth Values.

For each of the following statements, give its negation and determine its truth value:

(1) $\forall x \in \mathbb{R}, \quad \dfrac{x}{1+x^2} \leq \dfrac{1}{2}$

(2) $\forall x \in \mathbb{R}^*,\ \exists y \in \mathbb{R},\quad x^2 - xy + y^2 = 0$
(3) $\forall x \in \mathbb{R}_+,\ \forall y \in \mathbb{R}_+,\quad x + y > xy$

Exercise 1.3 — Proofs using Direct Methods and Counterexamples.

Prove or disprove the following statements:

(1) Let n and m be integers. If n and m are both even, then $n+m$ is even.
(2) Let $x, y, z \in \mathbb{Z}$. If $x + y = x + z$, then $y = z$.
(3) For each $n \in \mathbb{N}$, if n is prime, then $2^n - 1$ is prime.

Exercise 1.4 — Proofs by Contrapositive, Contradiction, and Counterexample.

Prove or disprove the following statements:

(1) If n is a positive integer with $n^2 > 100$, then $n > 10$.
(2) $\sqrt{3}$ is irrational.
(3) Let $x \in \mathbb{R}$. If $x^3 > 8$, then $x > 2$.
(4) The statement $\forall x \in \mathbb{R},\ (x^2 - 2x \neq 1) \Rightarrow (x \neq 1)$ is **false**.

Exercise 1.5 — Proofs by Mathematical Induction.

Prove the following statements by mathematical induction:

(1) For all $n \geq 1$:
$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}.$$
(2) For all $n \geq 1$:
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$
(3) For every positive integer n, the number $6^n - 1$ is divisible by 5.

Exercise 1.6 — Further Proofs by Induction.

Prove the following statements by mathematical induction:

(1) For all $n \geq 1$:
$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$
(2) For every integer $n \geq 0$, the number $n^3 + 2n$ is divisible by 3.
(3) For all $n \geq 1$ and all $x > -1$:
$$(1+x)^n \geq 1 + nx.$$
(This is known as **Bernoulli's inequality**.)

Solution — Exercise 1.1 — Truth Tables and Logical Equivalences.

Step 1 — Recall the method.
Two propositions are logically equivalent when their truth tables produce identical columns.

Step 2 — Check each statement.

- **(a)** $P \Rightarrow Q \equiv P \wedge Q$: **False.** When P is true and Q is false, $P \Rightarrow Q$ is false but $P \wedge Q$ is also false, yet when P is false, $P \Rightarrow Q$ is true but $P \wedge Q$ is false. Not equivalent.
- **(b)** De Morgan's Law: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$. ✓ Equivalent
- **(c)** De Morgan's Law: $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$. ✓ Equivalent
- **(d)** $P \Rightarrow Q \equiv \neg P \vee Q$. ✓ Equivalent
- **(e)** $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$. ✓ Equivalent
- **(f)** Contrapositive law: $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$. ✓ Equivalent

Conclusion. Statement **(a)** is **not** a logical equivalence; all others **(b)**–**(f)** are valid equivalences. □

Solution — Exercise 1.2 — Negations and Truth Values.

Statement (1): $\forall x \in \mathbb{R}, \ \dfrac{x}{1+x^2} \leq \dfrac{1}{2}$.

Step 1 — Truth value.
Since $(1-x)^2 \geq 0$, expanding gives $1 - 2x + x^2 \geq 0$, so $1 + x^2 \geq 2x$. Dividing both sides by $1 + x^2 > 0$:

$$\frac{x}{1+x^2} \leq \frac{1}{2}.$$

True. ✓

Step 2 — Negation.

$$\exists x \in \mathbb{R}, \quad \frac{x}{1+x^2} > \frac{1}{2}.$$

Statement (2): $\forall x \in \mathbb{R}^*, \ \exists y \in \mathbb{R} : x^2 - xy + y^2 = 0$.

Step 3 — Truth value.
Viewing $x^2 - xy + y^2 = 0$ as a quadratic in y: discriminant $\Delta = x^2 - 4x^2 = -3x^2 < 0$ for all $x \neq 0$. No real solution y exists. **False.**

Step 4 — Negation.

$$\exists x \in \mathbb{R}^*, \ \forall y \in \mathbb{R}, \quad x^2 - xy + y^2 \neq 0.$$

Statement (3): $\forall x, y \in \mathbb{R}_+, \ x + y > xy$.

Step 5 — Truth value.
Counterexample: $x = 2$, $y = 2$ gives $x + y = 4 = xy$. So $x + y > xy$ fails. **False.**

Step 6 — Negation.

$$\exists x, y \in \mathbb{R}_+, \quad x + y \leq xy.$$

Solution — Exercise 1.3 — Direct Methods and Counterexamples.

Statement (1): The sum of two even integers is even.

Step 1 — Direct proof.
Let $n = 2k$ and $m = 2\ell$ with $k, \ell \in \mathbb{Z}$. Then $n+m = 2k+2\ell = 2(k+\ell)$, which is even.
✓ True

Statement (2): $x + y = x + z \Rightarrow y = z$ in $\mathbb{R}$.

Step 2 — Direct proof.
$x + y = x + z \Rightarrow (x + y) - (x + z) = 0 \Rightarrow y - z = 0 \Rightarrow y = z.$ ✓ True

Statement (3): If n is prime, then $2^n - 1$ is prime.

Step 3 — Counterexample.
Take $n = 11$ (prime). Then $2^{11} - 1 = 2047 = 23 \times 89$, which is composite. **False.**

Solution — Exercise 1.4 — Contrapositive, Contradiction and Counterexample.

Statement (1): $n > 0$ and $n^2 > 100 \Rightarrow n > 10$.

Step 1 — Direct/contrapositive proof.
Since $n > 0$, the function $x \mapsto x^2$ is strictly increasing on $\mathbb{R}^+$. Thus:

$$n^2 > 100 = 10^2 \Rightarrow n > 10.$$

✓ True

Statement (2): $\sqrt{3}$ is irrational.

Step 2 — Proof by contradiction.
Assume $\sqrt{3} = \dfrac{p}{q}$ in lowest terms with $p, q \in \mathbb{N}^*$. Then $3q^2 = p^2$, so $3 \mid p^2$, hence $3 \mid p$.
Write $p = 3k$:

$$3q^2 = 9k^2 \Rightarrow q^2 = 3k^2 \Rightarrow 3 \mid q.$$

Both p and q are divisible by 3, contradicting $\gcd(p, q) = 1$. ✓ Irrational

Statement (3): $x^3 > 8 \Rightarrow x > 2$.

Step 3 — Proof using monotonicity.
The function $x \mapsto x^3$ is strictly increasing on $\mathbb{R}$. So:

$$x^3 > 8 = 2^3 \Rightarrow x > 2.$$

✓ True

Statement (4): $x^2 - 2x = 1 \Rightarrow x \neq 1$.

Step 4 — Counterexample.
At $x = 1$: premise $x^2 - 2x = 1 - 2 = -1 \neq 1$, so the premise is *false*. The implication is vacuously true. However, recheck: the claim says the implication itself is a valid result to prove. Testing $x^2 - 2x = 1 \Rightarrow x = 1 \pm \sqrt{2}$, so indeed $x \neq 1$ when the premise holds.
✓ True

Solution — Exercise 1.5 — Mathematical Induction.

Statement (1): $\sum_{k=1}^{n}(3k-2) = \frac{n(3n-1)}{2}$.

Step 1 — Base case $(n = 1)$:

$$3(1) - 2 = 1 \quad \text{and} \quad \frac{1 \cdot 2}{2} = 1. \quad \checkmark$$

Step 2 — Induction step.
Assume the formula holds for some $k \geq 1$. Then:

$$\begin{aligned}\sum_{j=1}^{k+1}(3j-2) &= \frac{k(3k-1)}{2} + \big(3(k+1) - 2\big) \\ &= \frac{k(3k-1)}{2} + (3k+1) \\ &= \frac{k(3k-1) + 2(3k+1)}{2} = \frac{3k^2 + 5k + 2}{2} = \frac{(k+1)(3k+2)}{2}.\end{aligned}$$

This is the formula for $n = k + 1$. Hence the result holds for all $n \geq 1$. □

Statement (2): $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 3 — Base case $(n = 1)$:

$$1^2 = 1 \quad \text{and} \quad \frac{1 \cdot 2 \cdot 3}{6} = 1. \quad \checkmark$$

Step 4 — Induction step.
Assume the formula holds for k. Then:

$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\ &= (k+1)\cdot\frac{k(2k+1)+6(k+1)}{6} \\ &= (k+1)\cdot\frac{2k^2+7k+6}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

Hence the result holds for all $n \geq 1$. □

Statement (3): $5 \mid 6^n - 1$ for all $n \geq 1$.

Step 5 — Base case $(n = 1)$:

$$6^1 - 1 = 5. \quad 5 \mid 5. \quad \checkmark$$

Step 6 — Induction step.
Assume $5 \mid 6^k - 1$, i.e., $6^k = 5m + 1$ for some $m \in \mathbb{Z}$. Then:

$$6^{k+1} - 1 = 6 \cdot 6^k - 1 = 6(5m+1) - 1 = 30m + 5 = 5(6m+1).$$

So $5 \mid 6^{k+1} - 1$. Hence the result holds for all $n \geq 1$. □

Solution — Exercise 1.6 — Further Proofs by Induction.

Statement (1): $\displaystyle\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Step 1 — Base case $(n = 1)$:

$$1 \cdot 2 = 2 \quad \text{and} \quad \frac{1\cdot 2\cdot 3}{3} = 2. \quad \checkmark$$

Step 2 — Induction step.
Assume the formula holds for some $k \geq 1$. For $n = k+1$:

$$\begin{aligned}\sum_{j=1}^{k+1} j(j+1) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)\left[\frac{k}{3}+1\right] \\ &= (k+1)(k+2)\cdot\frac{k+3}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

This is the formula for $n = k+1$. Hence the result holds for all $n \geq 1$. □

Statement (2): $3 \mid n^3 + 2n$ for all $n \geq 0$.

Step 3 — Base case $(n = 0)$:

$$0^3 + 2\cdot 0 = 0 = 3\cdot 0. \quad 3 \mid 0. \quad \checkmark$$

Step 4 — Induction step.
Assume $3 \mid k^3 + 2k$ for some $k \geq 0$, i.e., $k^3 + 2k = 3m$ for some $m \in \mathbb{Z}$. Then:

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + 3k^2 + 3k + 3 \\ &= 3m + 3(k^2 + k + 1) \\ &= 3\big(m + k^2 + k + 1\big).\end{aligned}$$

Since $m + k^2 + k + 1 \in \mathbb{Z}$, we have $3 \mid (k+1)^3 + 2(k+1)$. Hence the result holds for all $n \geq 0$. □

Statement (3): Bernoulli's Inequality. For all $n \geq 1$ and $x > -1$: $(1+x)^n \geq 1+nx$.

Step 5 — Base case $(n = 1)$:

$$(1+x)^1 = 1 + x = 1 + 1\cdot x. \quad \checkmark$$

Step 6 — Induction step.
Assume $(1+x)^k \geq 1 + kx$ for some $k \geq 1$ (induction hypothesis). Since $x > -1$, we have $1 + x > 0$. Multiplying both sides by $(1+x) > 0$:

$$(1+x)^{k+1} = (1+x)^k \cdot (1+x) \geq (1+kx)(1+x).$$

Step 7 — Expand.

$$(1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x,$$

since $kx^2 \geq 0$.

Conclusion. $(1 + x)^{k+1} \geq 1 + (k + 1)x$. By induction, Bernoulli's inequality holds for all $n \geq 1$ and all $x > -1$. □

CHAPTER 2

Sets, Relations, and Functions

2.1 Set Theory

Definition 2.1 (Set).

A **set** is a collection of elements.

Among sets, one is special: the empty set, denoted $\emptyset$.

Let E be a set. We write $x \in E$ if x is an element of E, and $x \notin E$ otherwise.

Example 2.1. We have $\{0, 1\}$, $\{\text{red}, \text{black}\}$, and $\{0, 1, 2, \dots\} = \mathbb{N}$ are sets. Then $0 \in \{0, 1\}$ and $2 \notin \{0, 1\}$.

Inclusion, Union, Intersection, Complement

Definition 2.2 (Subset).

A set E is included in a set F, if every element of E is also an element of F, and we write $E \subset F$. In other words:

$$\forall x, \quad x \in E \Rightarrow x \in F.$$

We then say that E is a subset of F or a part of F.

Example 2.2. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Definition 2.3 (Set Equality).

Two sets E and F are equal if and only if each is included in the other, i.e.,

$$E = F \Leftrightarrow E \subset F \text{ and } F \subset E.$$

Example 2.3. If $E = \mathbb{R}$, then

$$\begin{aligned} A &= \{x \in \mathbb{R} : |x - 1| \le 1\} \\ &= \{x \in \mathbb{R} : -1 \le x - 1 \le 1\} \\ &= \{x \in \mathbb{R} : 0 \le x \le 2\} = [0, 2]. \end{aligned}$$

Definition 2.4 (Power Set).

Let E be a set. The set of all subsets of E, denoted $\mathcal{P}(E)$, is characterized by the following relation:

$$\mathcal{P}(E) = \{A : A \subset E\}.$$

Example 2.4. If $E = \{1, 2, 3\}$, then

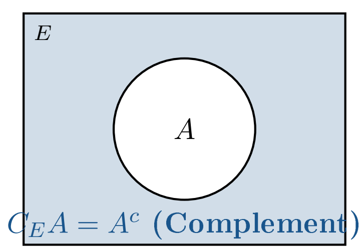
$$\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

so $\{1\} \in \mathcal{P}(E)$ and $E \in \mathcal{P}(E)$.

Definition 2.5 (Complement).

Let E be a set. The complement of $A \subset E$, denoted $C_E A$ or A^c, is the set of elements of E that are not in A, i.e.,

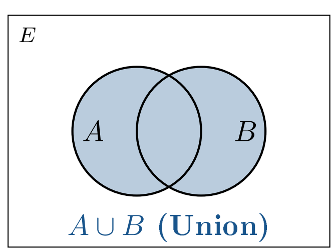
$$C_E A = \{x \in E : x \notin A\}.$$



Definition 2.6 (Union).

The union of two sets A and B, denoted $A \cup B$, is the set of elements x that belong to A or belong to B, i.e.,

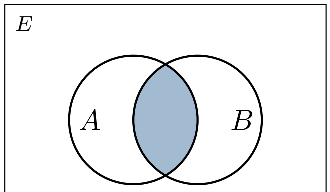
$$A \cup B = \{x \in E : x \in A \text{ or } x \in B\}.$$



Definition 2.7 (Intersection).

The intersection of two sets A and B, denoted $A \cap B$, is the set of elements x that belong to A and belong to B, i.e.,

$$A \cap B = \{x \in E : x \in A \text{ and } x \in B\}.$$



$A \cap B$ **(Intersection)**

Example 2.5. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap B = \{2, 3\}$.

Definition 2.8 (Finite Set, Cardinal).

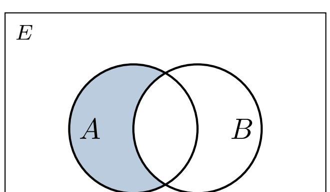
A set E is said to be finite if the number of elements of E is finite. The number of elements of E is called the cardinal of E, denoted $\text{Card}(E)$.

Example 2.6. If $E = \{0, 1, 2, 3\}$, then $\text{Card}(E) = 4$. The set $\mathbb{N}$ is not finite. $\text{Card}(\emptyset) = 0$.

Definition 2.9 (Difference).

Let E be a set. The difference of A and B, denoted $A \setminus B$, is the set of elements x that belong to A and do not belong to B, i.e.,

$$A \setminus B = \{x \in A : x \notin B\}.$$

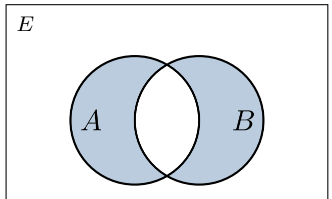


$A \setminus B$ **(Difference)**

Definition 2.10 (Symmetric Difference).

The symmetric difference of A and B, denoted $A \triangle B$, is the set of elements x that belong to $A \cup B$ and do not belong to $A \cap B$, i.e.,

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$



$A \triangle B$ **(Symmetric Difference)**

Example 2.7. If $E = \mathbb{R}$, $A = [0, 1]$ and $B =]0, +\infty[$, then

$$A \setminus B = \{0\}, \quad B \setminus A =]1, +\infty[\quad \text{and} \quad A \triangle B = \{0\} \cup]1, +\infty[.$$

Remark.

Let A, B, C be subsets of a set E. It is clear that:

- Commutativity: $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
- Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$.

Proposition 2.1 (Distributive Laws).

Let A, B, C be subsets of a set E. Then:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

We have

$$\begin{aligned}
(x \in A \cap (B \cup C)) &\Leftrightarrow (x \in A \text{ and } x \in B \cup C) \\
&\Leftrightarrow (x \in A \text{ and } (x \in B \text{ or } x \in C)) \\
&\Leftrightarrow ((x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)) \\
&\Leftrightarrow (x \in A \cap B \text{ or } x \in A \cap C) \\
&\Leftrightarrow x \in (A \cap B) \cup (A \cap C),
\end{aligned}$$

so $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$$\begin{aligned}
(x \in A \cup (B \cap C)) &\Leftrightarrow (x \in A \text{ or } x \in B \cap C) \\
&\Leftrightarrow (x \in A \text{ or } (x \in B \text{ and } x \in C)) \\
&\Leftrightarrow ((x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)) \\
&\Leftrightarrow (x \in A \cup B \text{ and } x \in A \cup C) \\
&\Leftrightarrow x \in (A \cup B) \cap (A \cup C),
\end{aligned}$$

so $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proposition 2.2 (De Morgan's Laws).

Let A and B be subsets of a set E. Then:

$$C_E(A \cap B) = C_E(A) \cup C_E(B) \quad \text{and} \quad C_E(A \cup B) = C_E(A) \cap C_E(B).$$

Proof. We have

$$\begin{aligned}(x \in C_E(A \cap B)) &\Leftrightarrow (x \notin A \cap B) \Leftrightarrow (x \notin A \text{ or } x \notin B) \\ &\Leftrightarrow (x \in C_E(A) \text{ or } x \in C_E(B)) \Leftrightarrow x \in C_E(A) \cup C_E(B),\end{aligned}$$

so $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

$$\begin{aligned}(x \in C_E(A \cup B)) &\Leftrightarrow (x \notin A \cup B) \Leftrightarrow x \notin A \text{ and } x \notin B \\ &\Leftrightarrow (x \in C_E(A) \text{ and } x \in C_E(B)) \Leftrightarrow x \in C_E(A) \cap C_E(B),\end{aligned}$$

so $C_E(A \cup B) = C_E(A) \cap C_E(B)$. □

Remark.

Let A be a subset of E. Then $C_E(C_E(A)) = A$. Indeed,

$$(x \in C_E(C_E(A))) \Leftrightarrow (x \notin C_E(A)) \Leftrightarrow (x \in A).$$

Cartesian Product

Definition 2.11 (Cartesian Product).

The Cartesian product of two sets E and F, denoted $E \times F$, is the set of ordered pairs (x, y) where $x \in E$ and $y \in F$.

$$E \times F = \{(x, y) : x \in E \text{ and } y \in F\}.$$

Example 2.8. If $E = \{1, 2\}$ and $F = \{3, 5\}$, then

$$E \times F = \{(1, 3), (1, 5), (2, 3), (2, 5)\},$$

$$F \times E = \{(3, 1), (3, 2), (5, 1), (5, 2)\} \neq E \times F.$$

We denote E^2 as the Cartesian square $E \times E$. More generally, we define the Cartesian product of n sets $E_1, E_2, \ldots, E_n$ by

$$E_1 \times E_2 \times \cdots \times E_n = \{(x_1, x_2, \ldots, x_n) : x_i \in E_i, \text{ for } i = 1, \ldots, n\}.$$

Example 2.9. If $E = \{1, 2\}$, then

$$E^2 = E \times E = \{(1,1),(1,2),(2,1),(2,2)\}$$

$$E^3 = E \times E \times E = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,2,1),(2,1,2),(2,2,2)\}.$$

2.2 Order Relations, Equivalence Relations

Binary Relations

Definition 2.12 (Binary Relation).

A binary relation on a set E is any statement between two objects, which can be verified or not, denoted $x\mathcal{R}y$ and read as "x is related to y".

Example 2.10. On $\mathbb{R}$, we define the relation $\mathcal{R}$ by:

$$x\mathcal{R}y \Leftrightarrow x - y \geq 0.$$

Definition 2.13 (Properties of Relations).

Let $\mathcal{R}$ be a binary relation on a set E. For all $x, y, z \in E$, we say that $\mathcal{R}$ is:

1. **Reflexive** if each element is related to itself, i.e.,
$$x\mathcal{R}x, \quad \forall x \in E.$$
2. **Symmetric** if for all $x, y \in E$, if x is related to y then y is related to x, i.e.,
$$x\mathcal{R}y \Rightarrow y\mathcal{R}x, \quad \forall x, y \in E.$$
3. **Transitive** if for all $x, y, z \in E$, if x is related to y and y is related to z, then x is related to z, i.e.,
$$(x\mathcal{R}y \text{ and } y\mathcal{R}z) \Rightarrow x\mathcal{R}z, \quad \forall x, y, z \in E.$$
4. **Antisymmetric** if whenever two elements are related to each other, they are equal, i.e.,
$$(x\mathcal{R}y \text{ and } y\mathcal{R}x) \Rightarrow x = y, \quad \forall x, y \in E.$$

Equivalence Relations

Definition 2.14 (Equivalence Relation).

A binary relation $\mathcal{R}$ on E is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2.15 (Equivalence Class).

Let $\mathcal{R}$ be an equivalence relation on E. The **equivalence class** of $x \in E$ is the set of elements of E related to x by $\mathcal{R}$, denoted $\dot{x}$ or $cl(x)$ or $\mathcal{C}(x)$:

$$\mathcal{C}(x) = \{y \in E : y\mathcal{R}x\}.$$

The equivalence class $\mathcal{C}(x)$ is non-empty because $\mathcal{R}$ is reflexive and thus contains at least x. We denote by

$$E/\mathcal{R} = \{\mathcal{C}(x) : x \in E\}$$

the set of equivalence classes of E under the relation $\mathcal{R}$.

Example 2.11. On $\mathbb{R}$, define the relation $\mathcal{R}$ by:

$$x\mathcal{R}y \Leftrightarrow x - y \in \mathbb{Z}.$$

This relation is indeed an equivalence relation.

- **Reflexive:** For $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$, so $x\mathcal{R}x$.
- **Symmetric:** For $x, y \in \mathbb{R}$, $x\mathcal{R}y \Rightarrow x - y \in \mathbb{Z} \Rightarrow y - x = -(x - y) \in \mathbb{Z} \Rightarrow y\mathcal{R}x$.
- **Transitive:** For $x, y, z \in \mathbb{R}$,

$$\begin{aligned}(x\mathcal{R}y \text{ and } y\mathcal{R}z) &\Rightarrow (x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}) \\ &\Rightarrow (x - y + y - z \in \mathbb{Z}) \\ &\Rightarrow (x - z \in \mathbb{Z}) \Rightarrow x\mathcal{R}z.\end{aligned}$$

The equivalence class of x is:

$$\begin{aligned}\mathcal{C}(x) &= \{y \in \mathbb{R} : y - x \in \mathbb{Z}\} \\ &= \{y \in \mathbb{R} : y = x + k, k \in \mathbb{Z}\}.\end{aligned}$$

If $x \in \mathbb{Z}$, then $\mathcal{C}(x) = \mathbb{Z}$.

Order Relations

Definition 2.16 (Order Relation).

A binary relation $\mathcal{R}$ on E is said to be an **order relation** if it is reflexive, antisymmetric, and transitive.

Example 2.12. Let $\mathcal{R}$ be the relation defined on $\mathbb{N}^*$ by: x divides y, i.e.,

$$x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{N}^* : y = kx.$$

- **Reflexive:** For $x \in \mathbb{N}^*$, $x = 1 \cdot x$, so $x\mathcal{R}x$.
- **Transitive:** If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $y = k_1x$ and $z = k_2y$, so $z = (k_1k_2)x$, thus $x\mathcal{R}z$.
- **Antisymmetric:** If $x\mathcal{R}y$ and $y\mathcal{R}x$, then $y = k_1x$ and $x = k_2y$. Substituting, $x = k_2(k_1x) = (k_1k_2)x$, so $k_1k_2 = 1$. Since $k_1, k_2 \in \mathbb{N}^*$, we have $k_1 = k_2 = 1$, hence $x = y$.

Thus $\mathcal{R}$ is an order relation.

Total Order and Partial Order

Definition 2.17 (Total and Partial Order).

Let $\mathcal{R}$ be an order relation defined on a set E. We say that $\mathcal{R}$ is a **total order** if for all $x, y \in E$,

$$x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Otherwise, we say $\mathcal{R}$ is a **partial order**, i.e.,

$$\exists x, y \in E : \text{neither } x\mathcal{R}y \text{ nor } y\mathcal{R}x.$$

Example 2.13. Let $\mathcal{R}$ be the order relation defined on $\mathbb{N}^*$ by:

$$x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{N} : y = kx.$$

For $x = 2$ and $y = 3$, we have neither $x\mathcal{R}y$ nor $y\mathcal{R}x$, so $\mathcal{R}$ is a partial order.

2.3 Functions (Applications)

Definition of a Function

Definition 2.18 (Function).

Let E and F be given sets. A **function** (or map) from E to F is a correspondence f between the elements of E and those of F that associates to every element of E one and only one element of F. We write

$$\begin{aligned} f : E &\to F \\ x &\mapsto f(x). \end{aligned}$$

Example 2.14. Let $f : \mathbb{N} \to \mathbb{C}$ be defined by $f(n) = n + ie^n$. Then f is a function, with $E = \mathbb{N}$ and $F = \mathbb{C}$.

Definition 2.19 (Graph).

Let E and F be given sets. The **graph** of a function $f : E \to F$ is:

$$\Gamma_f := \{(x, f(x)) : x \in E\} \subset E \times F.$$

Definition 2.20 (Equality of Functions).

Let $f, g : E \to F$ be functions. f and g are said to be equal if and only if

$$\text{for all } x \in E : f(x) = g(x).$$

We then write $f = g$.

Definition 2.21 (Composition).

Let E, F, and G be three sets, and f and g two functions such that

$$E \xrightarrow{f} F \xrightarrow{g} G.$$

We can deduce a function from E to G denoted $g \circ f$ and called the **composite function** of f and g, defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all } x \in E.$$

Definition 2.22 (Identity Function).

Let E be a set. The **identity function**, denoted $id_E : E \to E$, is the function that satisfies $id_E(x) = x$ for all $x \in E$.

Example 2.15. Let $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [1, +\infty[$ be defined by:

$$f(x) = x^2 \quad \forall x \in \mathbb{R}, \quad g(x) = 2x + 1 \quad \forall x \in \mathbb{R}^+.$$

Then $g \circ f : \mathbb{R} \to [1, +\infty[$ is given by

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1 \quad \forall x \in \mathbb{R}.$$

Definition 2.23 (Domain of Definition).

Let $f : E \to F$ be a function. The **domain of definition** of f, denoted D_f, is the set of elements $x \in E$ for which there exists a unique element $y \in F$ such that $y = f(x)$.

Example 2.16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sqrt{x+1}$. Then

$$D_f = \{x \in \mathbb{R} : x + 1 \geq 0\} = [-1, +\infty[.$$

Definition 2.24 (Restriction).

Let $A \subset E$ and $f : E \to F$ be a function. The **restriction** of f to A is the function $f|_A : E \to F$ defined by

$$f|_A(x) = f(x), \quad \text{for all } x \in A.$$

Definition 2.25 (Extension).

Let $E \subset G$ and $f : E \to F$ be a function. An **extension** of f to G is any function $g : G \to F$ whose restriction to E is f.

Direct Image, Inverse Image

Definition 2.26 (Direct Image).

Let $A \subset E$ and $f : E \to F$. The **direct image** of A under f is the set

$$f(A) = \{f(x) : x \in A\} \subset F.$$

Example 2.17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$. If $A = [0, 1]$, then

$$f([0, 1]) = \{f(x) : x \in [0, 1]\} = \{2x + 1 : x \in [0, 1]\}.$$

We have $x \in [0, 1] \Leftrightarrow 0 \leq x \leq 1 \Leftrightarrow 1 \leq 2x + 1 \leq 3$, so $f([0, 1]) = [1, 3]$.

Definition 2.27 (Inverse Image).

Let $B \subset F$ and $f : E \to F$. The **inverse image** of B under f is the set

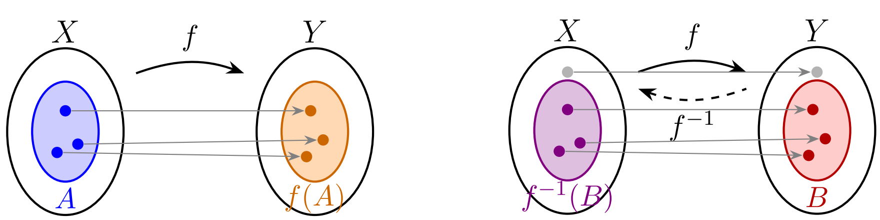
$$f^{-1}(B) = \{x \in E : f(x) \in B\} \subset E.$$

Example 2.18. Let f be the function defined by $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}^+$. Then

$$\begin{aligned} f^{-1}([0, 1]) &= \{x \in \mathbb{R} : 0 \leq x^2 \leq 1\} \\ &= \{x \in \mathbb{R} : 0 \leq |x| \leq 1\} = [-1, 1]. \end{aligned}$$

Let g be defined by $g(x) = \sin(\pi x)$ from $\mathbb{R}$ to $\mathbb{R}$. Then

$$g^{-1}(\{0\}) = \{x \in \mathbb{R} : \sin(\pi x) = 0\} = \{x : x = k, \text{ with } k \in \mathbb{Z}\} = \mathbb{Z}.$$



Direct Image: $f(A)$ **Inverse Image:** $f^{-1}(B)$

Proposition 2.3 (Properties of Images).

Let E and F be any sets and let $f : E \to F$ be a function. For all $A, B \subset E$ and $X, Y \subset F$, we have the following properties:

(1) $A \subset B \Rightarrow f(A) \subset f(B)$ *and* $X \subset Y \Rightarrow f^{-1}(X) \subset f^{-1}(Y)$

(2) $f(A \cap B) \subset f(A) \cap f(B)$ *and* $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

(3) $f(A \cup B) = f(A) \cup f(B)$ *and* $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

(4) $A \subset f^{-1}(f(A))$ *and* $f(f^{-1}(X)) \subset X$.

Injective, Surjective, Bijective Functions

Definition 2.28 (Injective Function).

Let $f : E \to F$. f is said to be **injective** if and only if:

$$\forall x_1, x_2 \in E : f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Definition 2.29 (Surjective Function).

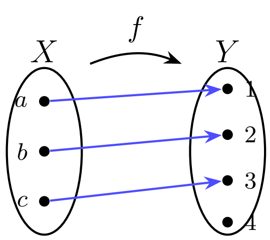
Let $f : E \to F$. f is said to be **surjective** if and only if every element of F has at least one preimage in E:

$$\forall y \in F,\ \exists x \in E : f(x) = y.$$

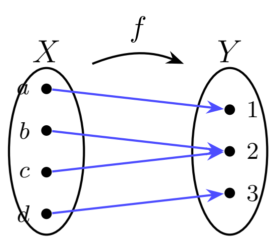
Definition 2.30 (Bijective Function).

Let $f : E \to F$. f is said to be **bijective** if and only if f is both injective and surjective, i.e. every element of F has exactly one preimage in E:

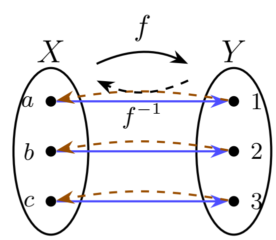
$$\forall y \in F,\ \exists!\, x \in E : f(x) = y.$$



Injective (one-to-one)



Surjective (onto)



Bijective (one-to-one & onto)

Example 2.19. Let f be the function defined by $f(x) = x - 7$ from $\mathbb{Z}$ to $\mathbb{Z}$. Then f is bijective. Indeed, let $y \in \mathbb{Z}$ such that $f(x) = y$, then $x = y + 7$, so there exists a unique x in $\mathbb{Z}$ such that $y = f(x)$.

Remark.

If the function f is bijective, and only in this case, to every $y \in F$ we can associate a unique $x \in E$.

Definition 2.31 (Inverse Function).

Let $f : E \to F$ be a bijective function. We define the function $f^{-1} : F \to E$, called the **inverse function** of f, by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Example 2.20. Let f be the function defined by $f(x) = x^2 + 1$ from $\mathbb{R}^+$ to $[1, +\infty[$. Then f is bijective, because for every $y \in [1, \infty[$, the equation $y = f(x)$ has a unique solution $x = \sqrt{y-1}$. The inverse bijection is $f^{-1} : [1, +\infty[\to \mathbb{R}^+$ defined by:

$$f^{-1}(x) = \sqrt{x-1}, \quad \text{for all } x \in [1, +\infty[.$$

Proposition 2.4 (Characterization of Bijections).

Let E and F be sets and $f : E \to F$ be a function. The function f is bijective if and only if there exists a function $g : F \to E$ such that

$$f \circ g = id_F \quad \textit{and} \quad g \circ f = id_E.$$

Example 2.21. Let $f : \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = e^x$ for all $x \in \mathbb{R}$. f is bijective, and its inverse bijection is $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \ln x$. We have $f \circ g : \mathbb{R}^+ \to \mathbb{R}^+$ and $g \circ f : \mathbb{R} \to \mathbb{R}$, with

$$(f \circ g)(x) = e^{\ln x} = x = id_{\mathbb{R}^+}(x) \quad \text{and} \quad (g \circ f)(x) = \ln e^x = x = id_{\mathbb{R}}(x).$$

Proposition 2.5 (Inverse of a Composition).

Let $f : E \to F$ and $g : F \to G$ be bijective functions. Then $g \circ f$ is bijective and its inverse bijection is

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

2.4 Exercises and Solutions

Exercise 2.1.

Let A, B and C be three parts of a set E . Give a simplified writing of the following subsets:

$$[A \cup (A \cap B)] \cup B$$

$$(A \cap B) \cup (A \cap B^c)$$

$$(A \cup B) \cap (B \cap C) \cap (A \cup C)$$

$$(A \cup B)^c \cup (C \cup A^c)$$

Exercise 2.2.

1. What is the power set of $\{a, b, c, d\}$?
2. Prove that $(A \cap B = A \cup B) \Longrightarrow A = B$

Exercise 2.3.

We define on $\mathbb{R}$ the relation $\mathcal{R}$ by:

$$x\mathcal{R}y \iff \sin^2 x + \cos^2 y = 1.$$

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Give the equivalence class of 0 and $\frac{\pi}{2}$

Exercise 2.4.

We define on $\mathbb{R}^2$ the relation T by:

$$(x,y)T(x',y') \iff |x-x'| \leq y'-y.$$

1. Show that T is an order relation.
2. Is T a total order relation?

Exercise 2.5.

Let A, B be two sets and f an application. Show that:

$$A \subset B \Longrightarrow f(A) \subset f(B).$$

$$f(A \cap B) \subset f(A) \cap f(B).$$
$$f(A \cup B) = f(A) \cup f(B).$$

Exercise 2.6.

Let A, B be two sets and f a bijective application. Show that:

$$A \subset B \Longrightarrow f^{-1}(A) \subset f^{-1}(B).$$

$$f(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B).$$
$$f(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

Exercise 2.7.

Let the application $f : \mathbb{R} \longrightarrow \mathbb{R}_+^*$ defined by

$$f(x) = \frac{2+e^x}{e^{-x}}.$$

Show that f is bijective and find its reciprocal bijection f^{-1} .

Solution — Exercise 2.1 — Simplifying Set Expressions.

Let A, B, C be subsets of a set E.

Expression 1: $[A \cup (A \cap B)] \cup B$
Step 1 — Since $A \cap B \subseteq A$, the absorption law gives $A \cup (A \cap B) = A$.
Step 2 — Therefore:

$$[A \cup (A \cap B)] \cup B = A \cup B.$$

$$\boxed{A \cup B}$$

Expression 2: $(A \cap B) \cup (A \cap B^c)$

Step 3 — Factor out A using the distributive law:

$$(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c).$$

Step 4 — Since $B \cup B^c = E$ and $A \cap E = A$:

$$\boxed{A}$$

Expression 3: $(A \cup B) \cap (B \cap C) \cap (A \cup C)$

Step 5 — Note that $B \cap C \subseteq B \subseteq A \cup B$ and $B \cap C \subseteq C \subseteq A \cup C$. So any element of $B \cap C$ automatically belongs to all three sets. Thus the intersection reduces to $B \cap C$.

Verification: $[(A \cup B) \cap (A \cup C)] \cap (B \cap C) = [A \cup (B \cap C)] \cap (B \cap C) = B \cap C$ (absorption).

$$\boxed{B \cap C}$$

Expression 4: $(A \cup B)^c \cup (C \cup A^c)$

Step 6 — Apply De Morgan's law: $(A \cup B)^c = A^c \cap B^c$. So:

$$(A^c \cap B^c) \cup C \cup A^c.$$

Step 7 — Since $A^c \cap B^c \subseteq A^c$, by absorption: $A^c \cup (A^c \cap B^c) = A^c$. Thus:

$$\boxed{A^c \cup C}$$

Solution — Exercise 2.2 — Power Set and Equality Proof.

Part 1 — Power set of $\{a, b, c, d\}$.

Step 1 — A set with 4 elements has $2^4 = 16$ subsets:

$$\emptyset; \quad \{a\}, \{b\}, \{c\}, \{d\};$$

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\};$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}; \quad \{a, b, c, d\}.$$

Part 2 — Prove: $A \cap B = A \cup B \Rightarrow A = B$.

Step 2 — Set up.
Let $S = A \cap B = A \cup B$.

Step 3 — Show $A \subseteq B$.
Let $x \in A$. Then $x \in A \cup B = S = A \cap B$, so $x \in B$.

Step 4 — Show $B \subseteq A$.
Let $x \in B$. Then $x \in A \cup B = S = A \cap B$, so $x \in A$.

Conclusion. $A \subseteq B$ and $B \subseteq A$, therefore $A = B$. $\square$

Solution — Exercise 2.3 — Equivalence Relation on $\mathbb{R}$.

The relation is $x\mathcal{R}y \Leftrightarrow \sin^2 x + \cos^2 y = 1$.

Step 1 — Simplify the condition.
$\sin^2 x + \cos^2 y = 1 \Leftrightarrow \cos^2 y = 1 - \sin^2 x = \cos^2 x \Leftrightarrow |\cos y| = |\cos x|$.

Step 2 — Reflexivity.
$|\cos x| = |\cos x|$ is always true, so $x\mathcal{R}x$ for all $x \in \mathbb{R}$. ✓

Step 3 — Symmetry.
If $x\mathcal{R}y$ then $|\cos y| = |\cos x|$, hence $|\cos x| = |\cos y|$, so $y\mathcal{R}x$. ✓

Step 4 — Transitivity.
If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $|\cos y| = |\cos x|$ and $|\cos z| = |\cos y|$, hence $|\cos z| = |\cos x|$, so $x\mathcal{R}z$. ✓

Step 5 — Equivalence classes.
Class of 0: $x\mathcal{R}0 \Leftrightarrow \sin^2 x + \cos^2 0 = 1 \Leftrightarrow \sin^2 x = 0 \Leftrightarrow x = k\pi$, $k \in \mathbb{Z}$.

$$\mathcal{C}(0) = \{k\pi : k \in \mathbb{Z}\}.$$

Class of $\frac{\pi}{2}$: $x\mathcal{R}\frac{\pi}{2} \Leftrightarrow \sin^2 x + \cos^2 \frac{\pi}{2} = 1 \Leftrightarrow \sin^2 x = 1 \Leftrightarrow x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

$$\mathcal{C}\left(\tfrac{\pi}{2}\right) = \left\{\tfrac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}.$$

Solution — Exercise 2.4 — Order Relation on $\mathbb{R}^2$.

Recall the relation: $(x, y)\, T\, (x', y') \Leftrightarrow |x - x'| \le y' - y$.

Part 1 — T is an order relation.

Step 1 — Reflexivity.
Set $(x', y') = (x, y)$: $|x - x| = 0 \le y - y = 0$. ✓

Step 2 — Antisymmetry.
Suppose $(x, y)\, T\, (x', y')$ and $(x', y')\, T\, (x, y)$. Then:

$$|x - x'| \le y' - y \quad \text{and} \quad |x' - x| \le y - y'.$$

Since $|x - x'| = |x' - x| \ge 0$, adding both inequalities: $2|x - x'| \le 0$, so $|x - x'| = 0$, giving $x = x'$. Then $0 \le y' - y$ and $0 \le y - y'$, so $y = y'$.

Step 3 — Transitivity.
Suppose $(x, y)\, T\, (x', y')$ and $(x', y')\, T\, (x'', y'')$. Then by the triangle inequality:

$$|x - x''| \le |x - x'| + |x' - x''| \le (y' - y) + (y'' - y') = y'' - y.$$

So $(x, y)\, T\, (x'', y'')$. ✓

Part 2 — T is not a total order.

Step 4 — Find incomparable elements.
Take $(0,0)$ and $(1,0)$:

$$(0,0)\,T\,(1,0): \quad |0-1| = 1 \leq 0-0 = 0? \quad \text{False.}$$

$$(1,0)\,T\,(0,0): \quad |1-0| = 1 \leq 0-0 = 0? \quad \text{False.}$$

These two elements are incomparable, so T is only a **partial order**. □

Solution — Exercise 2.5 — Images of Sets under a Function.

Let $f : E \to F$ and $A, B \subseteq E$.

Property 1: $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

Step 1 — Let $y \in f(A)$. Then $\exists x \in A$ with $y = f(x)$. Since $A \subseteq B$, $x \in B$, hence $y = f(x) \in f(B)$. ✓

Property 2: $f(A \cap B) \subseteq f(A) \cap f(B)$.

Step 2 — Let $y \in f(A \cap B)$. Then $\exists x \in A \cap B$ with $y = f(x)$. Since $x \in A$, $y \in f(A)$; since $x \in B$, $y \in f(B)$. Hence $y \in f(A) \cap f(B)$. ✓

Note: The reverse inclusion fails in general; equality holds if and only if f is injective.

Property 3: $f(A \cup B) = f(A) \cup f(B)$.

Step 3 — (⊆). Let $y \in f(A \cup B)$. Then $\exists x \in A \cup B$ with $y = f(x)$. Either $x \in A$ (so $y \in f(A)$) or $x \in B$ (so $y \in f(B)$). Thus $y \in f(A) \cup f(B)$.

Step 4 — (⊇). Let $y \in f(A) \cup f(B)$, say $y \in f(A)$. Then $\exists x \in A \subseteq A \cup B$ with $f(x) = y$, so $y \in f(A \cup B)$. ✓

Solution — Exercise 2.6 — Preimages under a Bijective Function.

Let $f : E \to F$ be bijective and $A, B \subseteq F$.

Property 1: $A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$.

Step 1 — Let $x \in f^{-1}(A)$. Then $f(x) \in A \subseteq B$, so $x \in f^{-1}(B)$. ✓

Property 2: $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Step 2 — (⊆). Let $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B$, so $f(x) \in A$ gives $x \in f^{-1}(A)$ and $f(x) \in B$ gives $x \in f^{-1}(B)$.

Step 3 — (⊇). Let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, hence $x \in f^{-1}(A \cap B)$.

Note: This equality holds for *any* function (not just bijections).

Property 3: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Step 4 — (⊆). Let $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. If $f(x) \in A$, then $x \in f^{-1}(A)$; if $f(x) \in B$, then $x \in f^{-1}(B)$.

Step 5 — (⊇). Let $x \in f^{-1}(A) \cup f^{-1}(B)$, say $x \in f^{-1}(A)$. Then $f(x) \in A \subseteq A \cup B$, so $x \in f^{-1}(A \cup B)$. ✓

Solution — Exercise 2.7 — Bijectivity and Inverse of f.

Note: $f(x) = \dfrac{2+e^x}{e^{-x}} = (2+e^x) \cdot e^x = e^{2x} + 2e^x$.

Step 1 — Injectivity.
$f'(x) = 2e^{2x} + 2e^x = 2e^x(e^x+1) > 0$ for all $x \in \mathbb{R}$, so f is strictly increasing, hence injective. ✓

Step 2 — Surjectivity onto $(0, +\infty)$.
As $x \to -\infty$: $e^{2x} \to 0$ and $2e^x \to 0$, so $f(x) \to 0^+$.
As $x \to +\infty$: $f(x) \to +\infty$.
By continuity, f takes every value in $(0, +\infty)$. ✓

Step 3 — Computing f^{-1}.
Set $y = e^{2x} + 2e^x > 0$. Let $u = e^x > 0$, so $y = u^2 + 2u$:

$$u^2 + 2u - y = 0 \Rightarrow u = \frac{-2 + \sqrt{4+4y}}{2} = \sqrt{1+y} - 1.$$

Since $u = e^x > 0$ we take the positive root: $e^x = \sqrt{1+y} - 1$, hence $x = \ln(\sqrt{1+y} - 1)$.

$$\boxed{f^{-1}(y) = \ln\left(\sqrt{1+y} - 1\right), \quad y > 0.}$$

CHAPTER 3

Real Functions of One Real Variable

3.1 Basic Concepts of Functions

General Definitions

Definition 3.1 (Function).

A **numerical function** on a set D is any process that, for every element x of D, associates at most one element of $\mathbb{R}$, called the image of x and denoted $f(x)$. The elements of D that have an image under f form the **domain of definition** of f, denoted D_f.

Example 3.1. The function $f : x \mapsto \sqrt{x-1}$ is defined for all $x \in \mathbb{R}$ such that $x - 1 \geq 0$. Thus $D_f = [1, +\infty[$.

Graph of a Function

Definition 3.2 (Graph).

The **graph**, or representative curve, of a function f defined on an interval $D_f \subset \mathbb{R}$ is the set

$$C_f = \{(x, f(x)) : x \in D_f\}$$

of points $(x, f(x)) \in \mathbb{R}^2$ in the plane with an orthonormal coordinate system $(O, \vec{i}, \vec{j})$.

Bounded Functions, Monotonic Functions

Definition 3.3 (Bounded Functions).

Let $f : D \to \mathbb{R}$ be a function. We say that:

- f is **bounded above** on D if $\exists M \in \mathbb{R}, \forall x \in D : f(x) \leq M$.
- f is **bounded below** on D if $\exists m \in \mathbb{R}, \forall x \in D : f(x) \geq m$.

- f is **bounded** on D if it is both bounded above and below, i.e., $\exists M > 0, \forall x \in D : |f(x)| \leq M$.

Definition 3.4 (Monotonic Functions).

Let $f : D \to \mathbb{R}$ be a function. We say that:

- f is **increasing** on D if $\forall x, y \in D, x > y \Rightarrow f(x) \geq f(y)$.
- f is **strictly increasing** on D if $\forall x, y \in D, x > y \Rightarrow f(x) > f(y)$.
- f is **decreasing** on D if $\forall x, y \in D, x > y \Rightarrow f(x) \leq f(y)$.
- f is **strictly decreasing** on D if $\forall x, y \in D, x > y \Rightarrow f(x) < f(y)$.
- f is **monotonic** (resp. **strictly monotonic**) on D if f is increasing or decreasing (resp. strictly increasing or strictly decreasing) on D.

Example 3.2.

- The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing.
- The absolute value function $x \mapsto |x|$ defined on $\mathbb{R}$ is not monotonic.

Even, Odd, Periodic Functions

Definition 3.5 (Even and Odd Functions).

Let I be an interval symmetric about 0 (i.e., if $x \in I$, then $-x \in I$). Let $f : I \to \mathbb{R}$ be a function. We say that:

- f is **even** if $\forall x \in I : f(-x) = f(x)$.
- f is **odd** if $\forall x \in I : f(-x) = -f(x)$.

Example 3.3.

- The function $x \mapsto x^{2n}$ ($n \in \mathbb{N}$) defined on $\mathbb{R}$ is even.
- The function $x \mapsto x^{2n+1}$ ($n \in \mathbb{N}$) defined on $\mathbb{R}$ is odd.

Definition 3.6 (Periodic Function).

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and T a positive real number. The function f is said to be **periodic** with period T if

$$\forall x \in \mathbb{R} : f(x+T) = f(x).$$

Example 3.4. The functions sin and cos are 2π-periodic. The tangent function is π-periodic.

Algebraic Operations on Functions

The set of functions from $D \subset \mathbb{R}$ to $\mathbb{R}$ is denoted $\mathcal{F}(D, \mathbb{R})$.

Definition 3.7 (Operations on Functions).

Let $f, g \in \mathcal{F}(D, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We define:

- **Sum** of two functions: $f + g : x \mapsto (f + g)(x) = f(x) + g(x)$.
- **Scalar multiplication**: For $\lambda \in \mathbb{R}$, $\lambda f : x \mapsto (\lambda f)(x) = \lambda f(x)$.
- **Product** of two functions: $fg : x \mapsto (fg)(x) = f(x)g(x)$.

Remark.

The functions $f + g$, λf, and fg are all functions belonging to $\mathcal{F}(D, \mathbb{R})$.

Definition 3.8 (Order on Functions).

Let $f, g \in \mathcal{F}(D, \mathbb{R})$. We say that:

- $f \leq g$ if $\forall x \in D,\ f(x) \leq g(x)$.
- $f < g$ if $\forall x \in D,\ f(x) < g(x)$.

Example 3.5. Let f and g be two functions defined on $]0, 1[$ by $f(x) = x$, $g(x) = x^2$. We have $g < f$, because for all $x \in]0, 1[$, $x^2 < x$.

3.2 Limit of a Function

General Definitions

Let $f : I \to \mathbb{R}$ be a function defined on an interval I of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a point in I or an endpoint of I.

Definition 3.9 (Limit of a Function).

Let $\ell \in \mathbb{R}$. We say that f has limit ℓ at x_0 if

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I, \quad |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

In this case, we write $\lim\limits_{x \to x_0} f(x) = \ell$.

Example 3.6. Consider the function $f(x) = 2x - 1$, defined on $\mathbb{R}$. At $x = 1$, we have $\lim_{x \to 1} f(x) = 1$. Indeed, for any $\varepsilon > 0$, we have $|f(x) - 1| = 2|x - 1| < \varepsilon$ if $|x - 1| < \varepsilon/2$. So we can choose $\delta = \varepsilon/2$.

Uniqueness of the Limit

Proposition 3.1 (Uniqueness of the Limit).

If f has a limit at x_0, this limit is unique.

Proof. If f has two limits ℓ_1 and ℓ_2 at x_0, then for any $\varepsilon > 0$,

$$\exists \delta_1 > 0, \forall x \in I, |x - x_0| < \delta_1 \Rightarrow |f(x) - \ell_1| < \frac{\varepsilon}{2},$$
$$\exists \delta_2 > 0, \forall x \in I, |x - x_0| < \delta_2 \Rightarrow |f(x) - \ell_2| < \frac{\varepsilon}{2}.$$

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then for x with $|x - x_0| < \delta$, we have

$$|\ell_1 - \ell_2| \leq |f(x) - \ell_1| + |f(x) - \ell_2| \leq \varepsilon.$$

Since ε is arbitrary, taking $\varepsilon = \frac{|\ell_1 - \ell_2|}{2}$ forces $\ell_1 = \ell_2$. □

Right-hand Limit, Left-hand Limit

Definition 3.10 (Right-hand Limit).

We say that f has right-hand limit ℓ at x_0, or that $f(x)$ tends to ℓ as x approaches x_0 from the right, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0 < x < x_0 + \delta$ implies $|f(x) - \ell| \leq \varepsilon$. We denote this by:

$$\lim_{x \to x_0^+} f(x) = \ell.$$

Definition 3.11 (Left-hand Limit).

We say that f has left-hand limit ℓ at x_0, or that $f(x)$ tends to ℓ as x approaches x_0 from the left, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0 - \delta < x < x_0$ implies $|f(x) - \ell| \leq \varepsilon$. We denote this by:

$$\lim_{x \to x_0^-} f(x) = \ell.$$

Example 3.7. The function $x \in \mathbb{R}^+ \mapsto \sqrt{x}$ tends to 0 as $x \to 0^+$.

Remark.

If f has a left-hand limit ℓ and a right-hand limit ℓ' at x_0, for f to have a limit at x_0, it is necessary and sufficient that $\ell = \ell'$.

Example 3.8. Consider the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

It has right-hand limit 1 at 0 and left-hand limit -1 at 0. But it has no limit at 0.

Limits at Infinity

We define by convention:

- $\lim_{x\to+\infty} f(x) = \ell$ if $\forall\varepsilon > 0, \exists A > 0$ such that $x > A \Rightarrow |f(x) - \ell| < \varepsilon$.
- $\lim_{x\to-\infty} f(x) = \ell$ if $\forall\varepsilon > 0, \exists A > 0$ such that $x < -A \Rightarrow |f(x) - \ell| < \varepsilon$.

Infinite Limits

Let $x_0 \in \mathbb{R}$. We define by convention:

- $\lim_{x\to x_0} f(x) = +\infty$ if $\forall A > 0, \exists\delta > 0$ such that $|x - x_0| < \delta \Rightarrow f(x) > A$.
- $\lim_{x\to x_0} f(x) = -\infty$ if $\forall A > 0, \exists\delta > 0$ such that $|x - x_0| < \delta \Rightarrow f(x) < -A$.

If $x_0 = +\infty$ or $x_0 = -\infty$, we define:

- $\lim_{x\to+\infty} f(x) = +\infty$ if $\forall A > 0, \exists B > 0$ such that $x > B \Rightarrow f(x) > A$.
- $\lim_{x\to-\infty} f(x) = +\infty$ if $\forall A > 0, \exists B > 0$ such that $x < -B \Rightarrow f(x) > A$.
- $\lim_{x\to+\infty} f(x) = -\infty$ if $\forall A > 0, \exists B > 0$ such that $x > B \Rightarrow f(x) < -A$.
- $\lim_{x\to-\infty} f(x) = -\infty$ if $\forall A > 0, \exists B > 0$ such that $x < -B \Rightarrow f(x) < -A$.

Theorems on Limits

Theorem 3.2 (Sequential Characterization of Limits).

Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$. The following two properties are equivalent:

$$(1)\ \lim_{x\to x_0} f(x) = \ell,$$

$$(2)\quad \textit{For every sequence } (x_n)_{n\in\mathbb{N}} \subset]a, b[\textit{ with } \lim_{n\to+\infty} x_n = x_0,$$
$$\textit{we have } \lim_{n\to+\infty} f(x_n) = \ell.$$

Example 3.9. The function $f : x \in \mathbb{R}^* \to \sin(1/x)$ has no limit at 0. Consider the sequences

$$x_n = \frac{1}{(n+1)\pi} \quad \text{and} \quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}, \quad n \in \mathbb{N}.$$

We have $x_n \to 0$ and $y_n \to 0$ as $n \to +\infty$, but

$$\sin(1/x_n) = \sin((n+1)\pi) = 0, \quad \sin(1/y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1.$$

Thus the limits are different, so f has no limit at 0.

Operations on Limits

Theorem 3.3 (Limit Laws).

Let $f, g : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$, with $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} g(x) = \ell'$. Then:

- $\lim_{x\to x_0} [f(x)+g(x)] = \ell + \ell'$.
- $\lim_{x\to x_0} (\lambda f(x)) = \lambda\ell$, *for any* $\lambda \in \mathbb{R}$.
- $\lim_{x\to x_0} f(x)g(x) = \ell\ell'$.
- $\lim_{x\to x_0} |f(x)| = |\ell|$.
- $\lim_{x\to x_0} |f(x)-\ell| = 0$.
- $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{\ell'}$, *if* $\ell' \neq 0$.

Theorem 3.4 (Limit of a Composition).

Let $f : [a,b] \to [c,d]$, $g : [c,d] \to \mathbb{R}$, *and* $x_0 \in]a,b[$, $y_0 \in [c,d]$, *such that*

$$\lim_{x\to x_0} f(x) = y_0 \quad \text{and} \quad \lim_{y\to y_0} g(y) = \ell.$$

Then $\lim_{x\to x_0}(g\circ f)(x) = \ell$.

Proposition 3.5 (Properties of Infinite Limits).

Let $f, g : [a,b] \to \mathbb{R}$ *and* $x_0 \in]a,b[$. *Then:*

- *If* $\lim_{x\to x_0} f(x) = +\infty$, *then* $\lim_{x\to x_0} \frac{1}{f(x)} = 0$.
- *If* $\lim_{x\to x_0} f(x) = -\infty$, *then* $\lim_{x\to x_0} \frac{1}{f(x)} = 0$.
- *If* $f \leq g$, *and* $\lim_{x\to x_0} f(x) = \ell$, $\lim_{x\to x_0} g(x) = \ell'$, *then* $\ell \leq \ell'$.
- *If* $f \leq g$ *and* $\lim_{x\to x_0} f(x) = +\infty$, *then* $\lim_{x\to x_0} g(x) = +\infty$.

Theorem 3.6 (Squeeze Theorem).

Let $f, g, h : [a,b] \to \mathbb{R}$ *and* $x_0 \in]a,b[$, *with:*

- $f(x) \leq g(x) \leq h(x)$ *for all* $x \in]a,b[$,
- $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \ell \in \mathbb{R}$.

Then $\lim_{x\to x_0} g(x) = \ell$.

Remark (Indeterminate Forms).

Here is a list of indeterminate forms:

$$+\infty - \infty, \quad 0\times\infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 1^{\infty}, \quad \infty^0, \quad 0^0.$$

3.3 Continuity of a Function

General Definitions

Definition 3.12 (Continuity at a Point).

Let $f : I \to \mathbb{R}$ be a function, where I is an interval of $\mathbb{R}$. f is said to be **continuous** at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

i.e., if

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I, \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Example 3.10. Consider the real function f defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

At $x_0 = 0$, we have $|f(x) - f(0)| = |x \sin(1/x)| \leq |x|$. For any $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$. Hence f is continuous at 0.

Definition 3.13 (Continuity on an Interval).

A function defined on an interval I is **continuous on** I if it is continuous at every point of I. The set of continuous functions on I is denoted $C(I)$.

Left and Right Continuity

Definition 3.14 (Left and Right Continuity).

Let $f : I \to \mathbb{R}$ be a function.

1. f is **left continuous** at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$.
2. f is **right continuous** at x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$.

Remark.

f is continuous at x_0 if and only if f is both left continuous and right continuous at x_0:

$$f \text{ is continuous at } x_0 \Leftrightarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0).$$

Example 3.11. The function defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x \leq 0, \end{cases}$$

is continuous on $\mathbb{R}^*$. At $x_0 = 0$, f is left continuous, but not right continuous because $\lim_{x \to 0^+} f(x) = 1 \neq f(0) = -1$.

Continuity Extension

Definition 3.15 (Extension by Continuity).

Let I be an interval and $x_0 \in I$. If f is not defined at x_0 but has a finite limit ℓ at x_0, the function defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ \ell & \text{if } x = x_0 \end{cases}$$

is called the **extension by continuity** of f at x_0.

Example 3.12. The function $f(x) = x\sin(1/x)$ is defined and continuous on $\mathbb{R}^*$. Since $\lim_{x\to 0} f(x) = 0$, the extension by continuity at 0 is

$$\tilde{f}(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Operations on Continuous Functions

Theorem 3.7 (Algebra of Continuous Functions).

Let I be an interval, and f and g functions defined on I and continuous at $x_0 \in I$. Then:

1. *λf is continuous at x_0, for $\lambda \in \mathbb{R}$.*
2. *$f + g$ is continuous at x_0.*
3. *fg is continuous at x_0.*
4. *$\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.*

Continuity of Composite Functions

Theorem 3.8 (Continuity of Composite Functions).

Let $f : I \to J$ and $g : J \to \mathbb{R}$ be functions, where I and J are intervals of $\mathbb{R}$. If f is continuous at $x_0 \in I$ and g is continuous at $y_0 = f(x_0) \in J$, then the composite function $g \circ f : I \to \mathbb{R}$ is continuous at x_0.

3.4 Differentiable Functions

Definition and Properties

Definition 3.16 (Derivative).

Let I be an open interval of $\mathbb{R}$, $x_0 \in I$, and $f : I \to \mathbb{R}$ a function. f is said to be **differentiable** at x_0 if the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the **derivative** of f at x_0 and is denoted $f'(x_0)$.

Remark.

Setting $x = x_0 + h$, we have

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

We can also write

$$f(x_0+h) = f(x_0) + hf'(x_0) + h\varepsilon(h), \quad \lim_{h\to 0} \varepsilon(h) = 0.$$

Example 3.13. Let f be the real function defined on $\mathbb{R}$ by $f(x) = x^2$. The derivative of f at $x_0 \in \mathbb{R}$ is

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2x_0 h}{h} = \lim_{h\to 0}(h + 2x_0) = 2x_0. \end{aligned}$$

Definition 3.17 (Derivative Function).

The function that assigns to every x in I the derivative $f'(x)$ is called the **derivative function** of f and is denoted f' or $\frac{df}{dx}$.

Proposition 3.9 (Differentiability Implies Continuity).

Every function differentiable at a point is continuous at that point.

Proof. If f is differentiable at x_0, then for h sufficiently small,

$$f(x_0+h) = f(x_0) + hf'(x_0) + h\varepsilon(h), \quad \lim_{h\to 0} \varepsilon(h) = 0.$$

Hence $\lim_{h\to 0} f(x_0+h) = f(x_0)$. □

Higher Order Derivatives

Definition 3.18 (Higher Order Derivatives).

The derivative f' of $f : I \to \mathbb{R}$ is a function on I. If f' is itself differentiable, its derivative, denoted $f'' = (f')'$, is called the **second derivative** of f. This notion can be generalized to order n. The n-th order derivative of f is defined by

$$f^{(n)}(x) = (f^{(n-1)})'(x).$$

Example 3.14. Let $f(x) = \sin x$ defined on $\mathbb{R}$. The first and second derivatives are

$$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right), \quad f''(x) = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right).$$

By induction, the n-th derivative is

$$\sin^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right).$$

Class C^n Functions

Definition 3.19 (C^n Functions).

Let $n \in \mathbb{N}^*$. A function defined on an interval I is said to be of **class** C^n, or n times continuously differentiable, if it is n times differentiable and $f^{(n)}$ is continuous on I. We denote $C^n(I)$ the set of functions of class C^n.

Definition 3.20 (C^0 and C^∞ Functions).

A function is said to be of class C^0 if it is continuous on I, and of class C^∞ if it is infinitely differentiable on I (i.e., $f^{(n)}$ exists for all n).

Example 3.15. The function $x \mapsto |x|$ defined on $\mathbb{R}$ is of class $C^0(\mathbb{R})$, but not of class $C^1(\mathbb{R})$, because it is not differentiable at the origin.

Left and Right Derivatives

Definition 3.21 (Left and Right Derivatives).

f is said to be **right differentiable** at x_0 if the limit

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. Similarly, f is **left differentiable** at x_0 if the corresponding left-hand limit exists and is finite. We denote:

$$f'_d(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \quad f'_g(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Remark.

The derivative of f at x_0 exists if and only if both $f'_d(x_0)$ and $f'_g(x_0)$ exist and are equal:

$$f \text{ is differentiable at } x_0 \Leftrightarrow f'_d(x_0) = f'_g(x_0) = f'(x_0).$$

Definition 3.22 (Angular Point).

If the left and right derivatives exist but are different, there exist two half-tangents to the curve C_f at the point $(x_0, f(x_0))$, called an **angular point**.

Example 3.16. Consider the function $f(x) = |x^2 - x|$, defined on $\mathbb{R}$. It has two angular points: $(0,0)$ and $(1,0)$.

- At $(0,0)$:

$$f'_g(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} (x-1) = -1,$$

$$f'_d(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} (1-x) = 1.$$

- At $(1,0)$:

$$f'_g(1) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} (-x) = -1,$$

$$f'_d(1) = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} x = 1.$$

At the origin, there are two half-tangents: $y = x$ and $y = -x$. At $(1,0)$, the half-tangents are $y = x - 1$ and $y = -x + 1$.

Differentiation Rules

Theorem 3.10 (Differentiation Rules).

Let f and g be two functions defined on an interval $I \subset \mathbb{R}$, differentiable at $x_0 \in I$. Then:

1. $\forall \alpha \in \mathbb{R}$, αf *is differentiable at* x_0 *and* $(\alpha f)'(x_0) = \alpha f'(x_0)$.
2. $f + g$ *is differentiable at* x_0 *and* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.
3. fg *is differentiable at* x_0 *and* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ *(Product Rule).*
4. *If* $g(x_0) \neq 0$, *then* $\frac{f}{g}$ *is differentiable at* x_0 *and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

In particular, $\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$.

Derivative of Composite Functions

Theorem 3.11 (Chain Rule).

Let I and J be intervals of $\mathbb{R}$, $f : I \to J$ and $g : J \to \mathbb{R}$. If f is differentiable at $x_0 \in I$ and g is differentiable at $f(x_0) \in J$, then $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = f'(x_0)\, g'(f(x_0)).$$

Derivative of Inverse Functions

Theorem 3.12 (Derivative of an Inverse Function).

Let J be an interval of $\mathbb{R}$ and $f : I \to J$ a continuous bijection. The inverse function $f^{-1} : J \to I$ is also continuous on J. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{(f' \circ f^{-1})(y_0)}.$$

Rolle's Theorem

Theorem 3.13 (Rolle's Theorem).

Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on $]a, b[$, with $f(a) = f(b)$. Then there exists $c \in]a, b[$ such that $f'(c) = 0$.

Mean Value Theorem

Theorem 3.14 (Mean Value Theorem).

Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists $c \in]a, b[$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Define

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

φ is continuous on $[a, b]$, differentiable on $]a, b[$, and $\varphi(a) = \varphi(b) = 0$. By Rolle's theorem, there exists $c \in]a, b[$ such that $\varphi'(c) = 0$, i.e.,

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f(b) - f(a) = f'(c)(b - a).$$

□

Example 3.17. Show that $\sqrt{1 + x} < 1 + \frac{x}{2}$ for $x > 0$. Let $f(t) = \sqrt{1 + t}$. Then $f'(t) = \frac{1}{2\sqrt{t+1}}$ and $f(0) = 1$. For $x > 0$, applying the Mean Value Theorem on $[0, x]$, there exists $c \in]0, x[$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{1}{2\sqrt{c + 1}} \leq \frac{1}{2}.$$

Thus $f(x) - 1 \leq \frac{x}{2}$, i.e., $\sqrt{1 + x} \leq 1 + \frac{x}{2}$. For $x > 0$, the inequality is strict.

Corollary 3.15 (Monotonicity and Derivative).

Let $f :]a, b[\to \mathbb{R}$ be differentiable on $]a, b[$. Then:

1. *$f'(x) = 0$ for all $x \in]a, b[$ if and only if f is constant on $]a, b[$.*
2. *If $f'(x) \geq 0$ (resp. $f'(x) > 0$) for all $x \in]a, b[$, then f is increasing (resp. strictly*

increasing) on $]a, b[$.

3. *If* $f'(x) \leq 0$ *(resp.* $f'(x) < 0$*) for all* $x \in]a, b[$*, then* f *is decreasing (resp. strictly decreasing) on* $]a, b[$.

Generalized Mean Value Theorem (Cauchy's Theorem)

Theorem 3.16 (Cauchy's Mean Value Theorem).

Let f *and* g *be two functions continuous on* $[a, b]$ *and differentiable on* $]a, b[$ *such that* $g'(x) \neq 0$ *on this interval and* $g(a) \neq g(b)$*. Then there exists* $c \in]a, b[$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Define

$$\varphi(x) = [f(b) - f(a)][g(x) - g(a)] - [g(b) - g(a)][f(x) - f(a)].$$

φ is continuous on $[a, b]$, differentiable on $]a, b[$, and $\varphi(a) = \varphi(b) = 0$. By Rolle's theorem, there exists $c \in]a, b[$ such that $\varphi'(c) = 0$, i.e.,

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0 \Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

□

L'Hôpital's Rule

Theorem 3.17 (L'Hôpital's Rule (0/0 case)).

Let f *and* g *be two differentiable functions on* $]a, b[$*, both tending to* 0 *as* $x \to a^+$*. Assume* $g'(x) \neq 0$ *in a neighborhood of* a *and that* $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \ell$*. Then*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \ell.$$

This result holds whether ℓ *is finite,* $+\infty$*, or* $-\infty$*.*

Remark.

The theorem also holds when f and g tend to 0 as $x \to b^-$.

Theorem 3.18 (L'Hôpital's Rule (Guillaume de l'Hôpital)).

Let f *and* g *be differentiable functions on* $]a, b[$*. Assume* $g'(x) \neq 0$ *on* $]a, b[$ *and that*

$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell$ *at* $x_0 \in]a,b[$. *Then*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell.$$

Proof. Apply Cauchy's Mean Value Theorem to the interval $[x_0, x]$:

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c_x)}{g'(c_x)}, \quad \text{with } c_x \in]x_0, x[.$$

As $x \to x_0$, $c_x \to x_0$, so $\frac{f'(c_x)}{g'(c_x)} \to \ell$. □

Example 3.18. Consider $\varphi(x) = \dfrac{\ln(x^2+x+1)-\ln 3}{\ln x}$ for $x \in]0,2[$. Let $f(x) = \ln(x^2+x+1)$ and $g(x) = \ln x$. Then $f(1) = \ln 3$, $g(1) = 0$, and

$$f'(x) = \frac{2x+1}{x^2+x+1}, \quad g'(x) = \frac{1}{x}.$$

Since $f(x) - f(1) = \ln(x^2+x+1) - \ln 3 \to 0$ and $g(x) - g(1) = \ln x \to 0$ as $x \to 1$, we have a $\frac{0}{0}$ form and Theorem 3.18 applies:

$$\lim_{x\to 1} \frac{f(x)-f(1)}{g(x)-g(1)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{2x+1}{x^2+x+1} \cdot x = \frac{3}{3} \cdot 1 = 1.$$

3.5 Equivalent Functions

Definition 3.23 (Equivalent Functions).

Let f and g be two functions defined near $x_0 \in \mathbb{R}$, except possibly at x_0. f is said to be **equivalent** to g as $x \to x_0$ (or near x_0), denoted $f \sim g$, if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

Example 3.19. Let P be a polynomial of degree n written as

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Then $P(x) \sim a_n x^n$ as $x \to +\infty$. Indeed,

$$\lim_{x\to+\infty} \frac{P(x)}{a_n x^n} = \lim_{x\to+\infty} \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}\right) = 1.$$

Proposition 3.19 (Properties of Equivalence).

Let f_1, g_1, f_2, g_2 be functions from I to $\mathbb{R}$.

1. If $f_1 \sim g_1$ and $f_2 \sim g_2$, then $f_1 f_2 \sim g_1 g_2$ and $\frac{f_1}{f_2} \sim \frac{g_1}{g_2}$.
2. If $f_1 \sim g_1$ and $\lim_{x \to x_0} f_1(x) = \ell$, then $\lim_{x \to x_0} g_1(x) = \ell$.

Classical Equivalences Near 0

$$e^x - 1 \sim x, \qquad \sin x \sim x, \qquad \tan x \sim x,$$
$$1 - \cos x \sim \frac{x^2}{2}, \quad \ln(1+x) \sim x, \quad (1+x)^\alpha - 1 \sim \alpha x.$$

3.6 Exercises and Solutions

Exercise 3.1.

Define the following definition set of functions:

$$1. f(x) = \frac{\ln(x+1)}{\ln x}$$

$$2. g(x) = \sqrt{\ln(x+2)}$$

$$3. h(x) = \frac{\sqrt{x-1}}{|x-3|}$$

$$4. k(x) = \frac{\sqrt{x^2 - 3x + 2}}{x^2 - 4}$$

Exercise 3.2.

Calculate the limits if they exist:

$$\lim_{x \to 0} \frac{x^2 + 2|x|}{x}$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 3x + 2}$$

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{\sqrt[3]{1+x} - 1}$$

$$\lim_{x \to +\infty} \frac{x^2 - 3}{x^4 - 2}$$

$$\lim_{x \to 0} \frac{\sqrt{x+1} - \sqrt{1 - x^2}}{x}$$

$$\lim_{x \to 0} \ln x \ln(1-x)$$

Exercise 3.3.

Study the continuity of the following functions:

$$f(x) = \begin{cases} \ln(e^2 + x), & x > 0 \\ x + 2, & x \leq 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 + x, & x \leq 0 \\ \sin x, & 0 < x \leq \pi \\ 1 + \cos x, & x > \pi \end{cases}$$

$$h(x) = \begin{cases} \frac{\sqrt{x-1}-1}{x-5}, & x < 5 \\ x + 2, & x \geq 5 \end{cases}$$

Exercise 3.4.

We define the following function: $f(x) = \frac{x^3+1}{x^2+3x+2}$

1. Determine the definition set of the function f .
2. Calculate the limit of f at $x_0 = -1$. Is f continuous there?
3. Does f accept extension by continuity at $x_0 = -2$?

Exercise 3.5.

Using the derived number definition, calculate the following limits:

1. $\lim_{x\to 0} \frac{e^{3x+2}-e^2}{x}$
2. $\lim_{x\to 0} \frac{\cos(x)-1}{x}$
3. $\lim_{x\to 1} \frac{\ln(2-x)}{x-1}$
4. $\lim_{x\to \frac{\pi}{2}} \frac{\exp(\cos x)-1}{x-\frac{\pi}{2}}$

Exercise 3.6.

Do the following functions accept differentiation?

$$f(x) = \begin{cases} \frac{e^x-1}{e^x+1}, & x > 0 \\ 1 - e^{-x}, & x \leq 0 \end{cases}$$

$$g(x) = \begin{cases} e^{1/x}, & x < 0 \\ 0, & x = 0 \\ x \ln x - x, & x > 0 \end{cases}$$

$$h(x) = \begin{cases} \frac{\sqrt{x^2+2}-\sqrt{2}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Exercise 3.7.

Let $a, b \in \mathbb{R}$ and $f(x) = \begin{cases} ax + b & , x \leq 0 \\ \frac{1}{1+x} & , x > 0 \end{cases}$

1. Give a condition for b such that f is continuous on $\mathbb{R}$.
2. Find a and b such that f is differentiable on $\mathbb{R}$. Find $f'(0)$

Solution — Exercise 3.1 — Domains of Definition.

Function 1: $f(x) = \dfrac{\ln(x+1)}{\ln x}$

Step 1 — Numerator: $\ln(x+1)$ requires $x+1>0 \Rightarrow x>-1$.

Step 2 — Denominator: $\ln x$ requires $x>0$ and $\ln x \neq 0 \Rightarrow x \neq 1$.

Step 3 — Intersection: $x>0$ and $x \neq 1$:

$$\boxed{D_f = (0,1) \cup (1,+\infty)}$$

Function 2: $g(x) = \sqrt{\ln(x+2)}$

Step 4 — Inner logarithm requires $x+2>0 \Rightarrow x>-2$.

Step 5 — Square root requires $\ln(x+2) \geq 0 \Rightarrow x+2 \geq 1 \Rightarrow x \geq -1$.

$$\boxed{D_g = [-1,+\infty)}$$

Function 3: $h(x) = \dfrac{\sqrt{x-1}}{|x-3|}$

Step 6 — Numerator: $x-1 \geq 0 \Rightarrow x \geq 1$.

Step 7 — Denominator: $|x-3| \neq 0 \Rightarrow x \neq 3$.

$$\boxed{D_h = [1,3) \cup (3,+\infty)}$$

Function 4: $k(x) = \dfrac{\sqrt{x^2-3x+2}}{x^2-4}$

Step 8 — Numerator: $x^2-3x+2 = (x-1)(x-2) \geq 0 \Rightarrow x \leq 1$ or $x \geq 2$.

Step 9 — Denominator: $x^2-4 \neq 0 \Rightarrow x \neq \pm 2$.

Step 10 — At $x=-2$: numerator $= 4+6+2 = 12 > 0$, denominator $= 0$ — exclude. At $x=2$: both numerator and denominator equal 0 — undefined.

$$\boxed{D_k = (-\infty,-2) \cup (-2,1] \cup (2,+\infty)}$$

Solution — Exercise 3.2 — Limits.

Limit (1): $\displaystyle\lim_{x\to 0} \frac{x^2+2|x|}{x}$

Step 1 — Right limit. For $x > 0$: $\dfrac{x^2+2x}{x} = x+2 \xrightarrow{x\to 0^+} 2.$

Step 2 — Left limit. For $x < 0$: $\dfrac{x^2-2x}{x} = x-2 \xrightarrow{x\to 0^-} -2.$

Conclusion. Left and right limits differ. **The limit does not exist.**

Limit (2): $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-3x+2}$

Step 3 — Factor.

$$\frac{x^2-4}{x^2-3x+2} = \frac{(x-2)(x+2)}{(x-2)(x-1)} = \frac{x+2}{x-1} \xrightarrow{x\to 2} \frac{4}{1} = \boxed{4}.$$

Limit (3): $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{\sqrt[3]{1+x}-1}$

Step 4 — Use standard equivalences near $x = 0$.
$\sqrt{x+1}-1 = (1+x)^{1/2}-1 \sim \dfrac{x}{2}$ and $\sqrt[3]{1+x}-1 = (1+x)^{1/3}-1 \sim \dfrac{x}{3}$.

Step 5 — Thus the limit equals $\dfrac{x/2}{x/3} = \boxed{\dfrac{3}{2}}$.

Limit (4): $\lim\limits_{x\to +\infty} \dfrac{x^2-3}{x^4-2}$

Step 6 — Divide by x^4.

$$\frac{x^2-3}{x^4-2} = \frac{1/x^2-3/x^4}{1-2/x^4} \xrightarrow{x\to +\infty} \frac{0}{1} = \boxed{0}.$$

Limit (5): $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-\sqrt{1-x^2}}{x}$

Step 7 — Rationalize.

$$\sqrt{x+1}-\sqrt{1-x^2} = \frac{(x+1)-(1-x^2)}{\sqrt{x+1}+\sqrt{1-x^2}} = \frac{x+x^2}{\sqrt{x+1}+\sqrt{1-x^2}}.$$

Step 8 — Dividing by x:

$$\frac{x(1+x)}{x(\sqrt{x+1}+\sqrt{1-x^2})} \xrightarrow{x\to 0} \frac{1}{1+1} = \boxed{\frac{1}{2}}.$$

Limit (6): $\lim\limits_{x\to 0^+} \ln x \cdot \ln(1-x)$

Step 9 — As $x \to 0^+$: $\ln(1-x) \sim -x$, so:

$$\ln x \cdot \ln(1-x) \sim -x \ln x.$$

Step 10 — Standard limit: $\lim_{x\to 0^+} x\ln x = 0$. Therefore the limit equals $\boxed{0}$.

Solution — Exercise 3.3 — Continuity of Piecewise Functions.

Function f.

Step 1 — At $x = 0$: $f(0) = 0 + 2 = 2$ and $\lim_{x\to 0^+} \ln(e^2 + x) = \ln(e^2) = 2$. The limits match $f(0)$. Both pieces are continuous on their respective domains. So f is continuous on $\mathbb{R}$.

Function g.

Step 2 — At $x = 0$: $g(0) = 0$ and $\lim_{x\to 0^+} \sin x = 0$. Continuous at 0.

Step 3 — At $x = \pi$: $\lim_{x\to\pi^-} \sin x = 0$, $g(\pi) = \sin\pi = 0$, and $\lim_{x\to\pi^+}(1+\cos x) = 1+(-1) = 0$. All values agree. So g is continuous on $\mathbb{R}$.

Function h.

Step 4 — At $x = 5$: $h(5) = 7$. Compute the left limit:

$$\lim_{x\to 5^-} \frac{\sqrt{x-1}-1}{x-5} = \frac{\sqrt{4}-1}{0^-} = \frac{1}{0^-} = -\infty \neq 7.$$

So h is **discontinuous** at $x = 5$.

Solution — Exercise 3.4 — Analysis of $f(x) = \dfrac{x^3+1}{x^2+3x+2}$.

Part 1 — Domain.

Step 1 — Denominator: $(x+1)(x+2) = 0 \Rightarrow x = -1$ or $x = -2$.

$$\boxed{D_f = \mathbb{R} \setminus \{-1, -2\}}$$

Part 2 — Limit at $x_0 = -1$.

Step 2 — **Factor.** $x^3 + 1 = (x+1)(x^2 - x + 1)$, so for $x \neq -1$:

$$f(x) = \frac{(x+1)(x^2-x+1)}{(x+1)(x+2)} = \frac{x^2-x+1}{x+2}.$$

Step 3 — Compute the limit at $x_0 = -1$.

$$\lim_{x \to -1} f(x) = \frac{1+1+1}{-1+2} = 3.$$

Since $f(-1)$ is undefined, f has a **removable discontinuity** at -1; it can be extended by setting $f(-1) = 3$.

Part 3 — Behaviour at $x_0 = -2$.

Step 4 — Numerator at -2: $(-8+1) = -7 \neq 0$; denominator $= 0$.

Step 5 — One-sided limits.

- $x \to -2^+$: denominator $\to 0^-$, numerator $\to -7 < 0$, so $f(x) \to +\infty$.
- $x \to -2^-$: denominator $\to 0^+$, numerator $\to -7 < 0$, so $f(x) \to -\infty$.

The limit is $\pm\infty$: **no continuous extension** exists at $x = -2$.

Solution — Exercise 3.5 — Limits via the Derivative Definition.

Recall: $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.

Limit 1: $\lim_{x \to 0} \dfrac{e^{3x+2} - e^2}{x}$

Step 1 — Write as $e^2 \cdot \dfrac{e^{3x} - 1}{x}$. Let $f(x) = e^{3x}$, $f(0) = 1$, $f'(x) = 3e^{3x}$, $f'(0) = 3$.

$$\text{Limit} = e^2 \cdot 3 = \boxed{3e^2}.$$

Limit 2: $\lim_{x \to 0} \dfrac{\cos x - 1}{x}$

Step 2 — This equals $\dfrac{\cos x - \cos 0}{x - 0}$. Let $f(x) = \cos x$, $f'(0) = -\sin 0 = 0$.

$$\text{Limit} = \boxed{0}.$$

Limit 3: $\lim_{x \to 1} \dfrac{\ln(2-x)}{x-1}$

Step 3 — Write as $\dfrac{\ln(2-x) - \ln 1}{x-1}$. Let $f(x) = \ln(2-x)$, $f(1) = 0$, $f'(x) = \dfrac{-1}{2-x}$, $f'(1) = -1$.

$$\text{Limit} = \boxed{-1}.$$

Limit 4: $\lim_{x \to \pi/2} \dfrac{e^{\cos x} - 1}{x - \pi/2}$

Step 4 — Write as $\dfrac{e^{\cos x}-e^0}{x-\pi/2}$. Let $f(x)=e^{\cos x}$, $f(\pi/2)=1$, $f'(x)=-\sin x\cdot e^{\cos x}$, $f'(\pi/2)=-1$.

$$\text{Limit} = \boxed{-1}.$$

Solution — Exercise 3.6 — Differentiability of Piecewise Functions.

Function 1: $f(x)=\begin{cases}\dfrac{e^x-1}{e^x+1}, & x>0\\ 1-e^{-x}, & x\le 0\end{cases}$

Step 1 — Continuity at 0**.**
$f(0)=1-1=0$ and $\lim_{x\to 0^+}\dfrac{e^x-1}{e^x+1}=\dfrac{0}{2}=0$. Continuous. ✓

Step 2 — Left derivative. $(1-e^{-x})'=e^{-x}$, so $f'_-(0)=1$.

Step 3 — Right derivative. $\left(\dfrac{e^x-1}{e^x+1}\right)'=\dfrac{2e^x}{(e^x+1)^2}$, so $f'_+(0)=\dfrac{2}{4}=\dfrac{1}{2}$.

Conclusion. $f'_-(0)=1\neq\dfrac{1}{2}=f'_+(0)$: f is **not differentiable** at 0.

Function 2: $g(x)=\begin{cases}e^{1/x}, & x<0\\ 0, & x=0\\ x\ln x-x, & x>0\end{cases}$

Step 4 — Continuity at 0**.**
$\lim_{x\to 0^-}e^{1/x}=0$ (since $1/x\to-\infty$); $\lim_{x\to 0^+}(x\ln x-x)=0$ (standard); $g(0)=0$. Continuous. ✓

Step 5 — Right derivative.

$$\lim_{h\to 0^+}\frac{g(h)}{h}=\lim_{h\to 0^+}\frac{h\ln h-h}{h}=\lim_{h\to 0^+}(\ln h-1)=-\infty.$$

Conclusion. g is **not differentiable** at 0.

Function 3: $h(x)=\begin{cases}\dfrac{\sqrt{x^2+2}-\sqrt{2}}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Step 6 — Continuity at 0**.**

$$\lim_{x\to 0}\frac{\sqrt{x^2+2}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{x^2}{x(\sqrt{x^2+2}+\sqrt{2})}=\lim_{x\to 0}\frac{x}{\sqrt{x^2+2}+\sqrt{2}}=0=h(0). \quad ✓$$

Step 7 — Derivative at 0**.**

$$\frac{h(x)-h(0)}{x}=\frac{\sqrt{x^2+2}-\sqrt{2}}{x^2}=\frac{1}{\sqrt{x^2+2}+\sqrt{2}}\xrightarrow{x\to 0}\frac{1}{2\sqrt{2}}.$$

Conclusion. h is differentiable at 0 with $h'(0) = \dfrac{1}{2\sqrt{2}}$.

Solution — Exercise 3.7 — Finding a and b for Differentiability.

Let $f(x) = \begin{cases} ax+b, & x \le 0 \\ \dfrac{1}{1+x}, & x > 0 \end{cases}$.

Part 1 — Condition on b for continuity.

Step 1 — At $x = 0$: $f(0) = b$ and $\lim_{x\to 0^+} \dfrac{1}{1+x} = 1$. For continuity:

$$\boxed{b = 1}.$$

Part 2 — Condition on a for differentiability.

Step 2 — Left derivative (with $b = 1$):

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{(ah+1) - 1}{h} = a.$$

Step 3 — Right derivative:

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0^+} \frac{-h}{h(1+h)} = -1.$$

Step 4 — For differentiability, left and right derivatives must match: $a = -1$.

$$\boxed{a = -1, \quad b = 1, \quad f'(0) = -1.}$$

CHAPTER 4

Applications to Elementary Functions

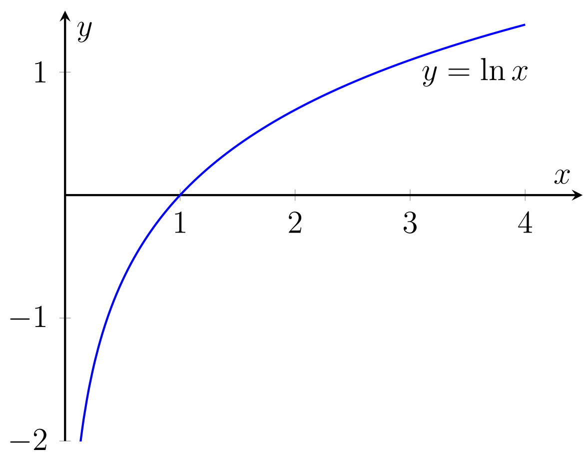
4.1 Logarithm, Exponential, and Power Functions

Logarithm Function

Definition 4.1 (Natural Logarithm).

The **natural logarithm** is the unique primitive that vanishes at 1 of the function $x \mapsto \frac{1}{x}$ defined on $\mathbb{R}_+^*$:

$$\ln : \mathbb{R}_+^* \to \mathbb{R}, \quad x \mapsto \ln x = \int_1^x \frac{dt}{t}.$$



Remark.

The function $x \mapsto \ln x$ is continuous, strictly increasing, and defines a bijection from $\mathbb{R}_+^*$ to $\mathbb{R}$.

Theorem 4.1 (Properties of Logarithms).

For $a, b > 0$ and $\alpha \in \mathbb{R}$:

- ***Product:*** $\ln(ab) = \ln a + \ln b$
- ***Inverse:*** $\ln\left(\frac{1}{a}\right) = -\ln a$
- ***Quotient:*** $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- ***Power:*** $\ln(a^\alpha) = \alpha \ln a$
- ***Square root:*** $\ln(\sqrt{a}) = \frac{1}{2}\ln a$
- $\ln a = \ln b \iff a = b$
- $\ln a \geq \ln b \iff a \geq b$
- $\ln a < \ln b \iff a < b$
- $\ln a \leq 0 \iff 0 < a \leq 1$ *and* $\ln a > 0 \iff a > 1$

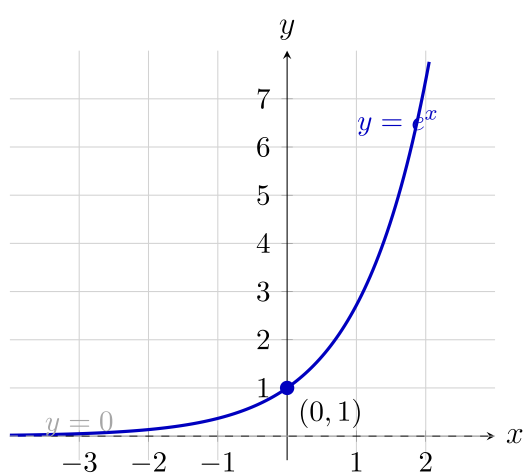
Theorem 4.2 (Special Limits).

$$\lim_{x\to+\infty} \ln x = +\infty, \qquad \lim_{x\to 0^+} \ln x = -\infty, \quad \lim_{x\to+\infty} \frac{\ln x}{x} = 0,$$
$$\lim_{x\to 0^+} x\ln x = 0, \qquad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Exponential Function

Definition 4.2 (Exponential Function).

The inverse function of $\ln : \mathbb{R}_+^* \to \mathbb{R}$ is called the **exponential function**, denoted $\exp : \mathbb{R} \to \mathbb{R}_+^*$ or $e^x : \mathbb{R} \to \mathbb{R}_+^*$.



Remark.

The function $\exp : \mathbb{R} \to \mathbb{R}_+^*$ is continuous, strictly increasing, differentiable on $\mathbb{R}$, and

$$(\exp x)' = \exp x, \quad \text{for all } x \in \mathbb{R}.$$

Theorem 4.3 (Properties of Exponentials).

For $a, b \in \mathbb{R}$ and $n \in \mathbb{R}$:

- ***Product:*** $e^a \times e^b = e^{a+b}$
- ***Inverse:*** $\frac{1}{e^a} = e^{-a}$
- ***Quotient:*** $\frac{e^a}{e^b} = e^{a-b}$
- ***Power:*** $(e^a)^n = e^{na}$
- $\ln(e^a) = a$
- $e^{\ln a} = a > 0$
- $e^a = b \iff a = \ln b$
- $a^b = e^{b \ln a}$

Theorem 4.4 (Equations and Inequalities with Exponentials).

- $e^a = e^b \iff a = b$
- $e^a \geq e^b \iff a \geq b$
- $e^a < e^b \iff a < b$
- $e^a \geq b > 0 \iff a \geq \ln b$
- $e^a < b \iff a < \ln b$, *with* $b > 0$

Theorem 4.5 (Special Limits).

$$\lim_{x \to +\infty} e^x = +\infty, \qquad \lim_{x \to -\infty} e^x = 0, \quad \lim_{x \to +\infty} \frac{e^x}{x} = +\infty,$$
$$\lim_{x \to -\infty} xe^x = 0, \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

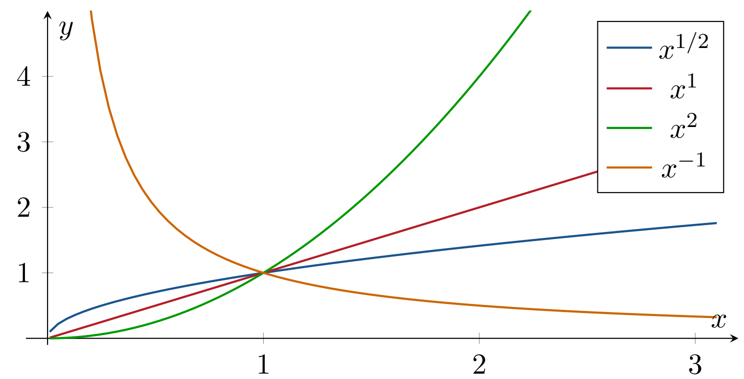
Power Function

Definition 4.3 (Power Function).

The **power function** x^m with real exponent m is defined for $x > 0$ by

$$y = x^m = e^{m \ln x}.$$

Example 4.1. Graphs of power functions for several values of m:



Theorem 4.6 (Variation of x^m).

If $m > 0$, x^m is strictly increasing; if $m < 0$, it is strictly decreasing.

4.2 Trigonometric Functions and Their Inverses

Basic Trigonometric Functions

Definition 4.4 (Sine and Cosine).

For an angle θ (in radians) measured counterclockwise from the positive x-axis to the terminal side, if (x, y) is the point on the unit circle corresponding to θ, then:

$$\cos\theta = x, \quad \sin\theta = y$$

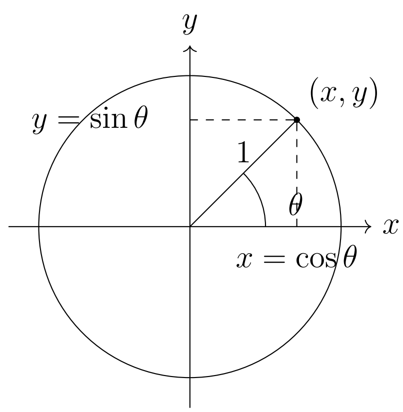
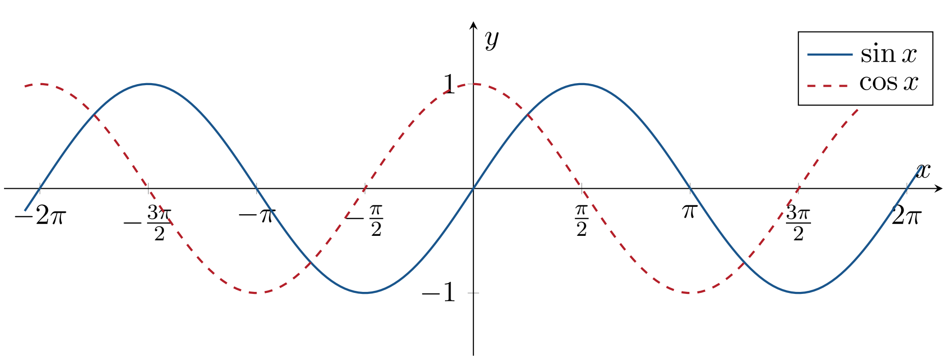


Figure 4.1: Definition of sine and cosine on the unit circle

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1

Table 4.1: Values of sine and cosine



Theorem 4.7 (Fundamental Properties).

The sine and cosine functions satisfy the following properties for all $x \in \mathbb{R}$*:*

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2 \cos x \sin x$$

Theorem 4.8 (Addition Formulas).

For all $x, y \in \mathbb{R}$*:*

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Theorem 4.9 (Product-to-Sum Formulas).

$$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$$

Theorem 4.10 (Sum-to-Product Formulas).

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$
$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$
$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$
$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

Definition 4.5 (Tangent and Cotangent).

The **tangent** function tan (or tg) is defined by:

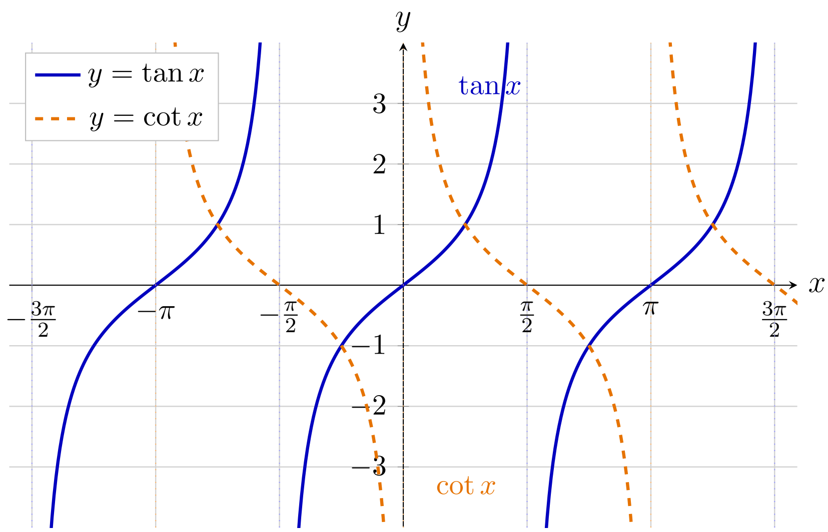
$$\tan x = \frac{\sin x}{\cos x}, \quad \text{for all } x \in \mathbb{R} \setminus A,$$

where $A = \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$.

The **cotangent** function cot is defined by:

$$\cot x = \frac{\cos x}{\sin x}, \quad \text{for all } x \in \mathbb{R} \setminus B,$$

where $B = \{k\pi : k \in \mathbb{Z}\}$.



Theorem 4.11 (Properties of Tangent and Cotangent).

- *For all* $x \in \mathbb{R} \setminus (A \cup B)$, $\cot x \cdot \tan x = 1$.
- *Both functions are periodic with period* π. *We can restrict their study to intervals of length* π, *e.g.,* $]-\frac{\pi}{2}, \frac{\pi}{2}[$ *for tangent and* $]0, \pi[$ *for cotangent.*
- *Tangent and cotangent are continuous and differentiable on their domains, with:*

$$\tan'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x, \quad \cot'(x) = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x).$$

Inverse Circular Functions

Arcsin Function

The sine function has a strictly positive derivative on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, hence it is a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$. The inverse bijection is called the **arcsine** function and is denoted arcsin:

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad x \mapsto \arcsin x.$$

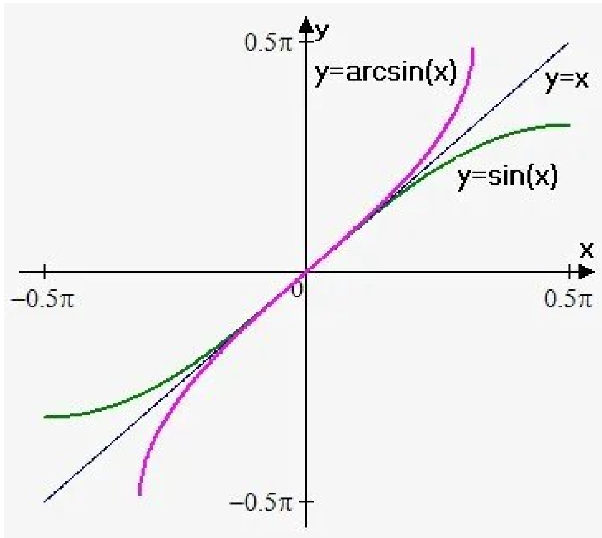


Figure 4.2: Graph of $y = \arcsin(x)$ and its relation to $y = \sin(x)$ (reflected about $y = x$).

Theorem 4.12 (Properties of Arcsin).

1. $\forall x \in [-1, 1]$, $\sin(\arcsin x) = x$.
2. $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arcsin(\sin x) = x$.
3. $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin x = y \iff x = \arcsin y$.
4. $\forall x \in [-1, 1]$, $\cos(\arcsin x) = \sqrt{1 - x^2}$.
5. $\arcsin$ *is differentiable on* $]-1, 1[$, *and*

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

Arccos Function

The cosine function has a strictly negative derivative on $]0, \pi[$, hence it is a bijection from $[0, \pi]$ to $[-1, 1]$. The inverse bijection is called the **arccosine** function and is denoted arccos:

$$\arccos : [-1, 1] \to [0, \pi], \quad x \mapsto \arccos x.$$

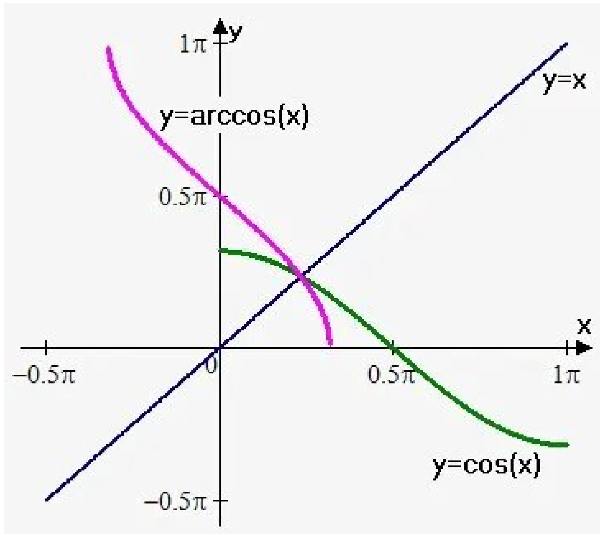


Figure 4.3: Graph of $y = \arccos(x)$ and its relation to $y = \cos(x)$ (reflected about $y = x$).

Theorem 4.13 (Properties of Arccos).

1. $\forall x \in [-1, 1]$, $\cos(\arccos x) = x$.
2. $\forall x \in [0, \pi]$, $\arccos(\cos x) = x$.
3. $\forall x \in [0, \pi]$, $\cos x = y \iff x = \arccos y$.
4. $\forall x \in [-1, 1]$, $\sin(\arccos x) = \sqrt{1 - x^2}$.
5. arccos *is differentiable on* $]-1, 1[$, *and*

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

Arctan Function

The tangent function has a strictly positive derivative on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, hence it is a bijection from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ to $\mathbb{R}$. The inverse bijection is called the **arctangent** function and is denoted arctan:

$$\arctan : \mathbb{R} \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \quad x \mapsto \arctan x.$$

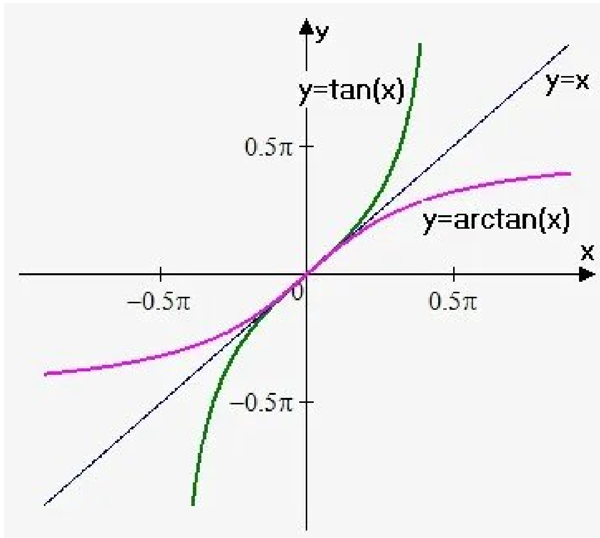


Figure 4.4: Graph of $y = \arctan(x)$ and its relation to $y = \tan(x)$ (reflected about $y = x$).

Theorem 4.14 (Properties of Arctan).

1. $\forall x \in \mathbb{R}$, $\tan(\arctan x) = x$.
2. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\arctan(\tan x) = x$.
3. $\arctan$ *is differentiable on* $\mathbb{R}$, *and*

$$(\arctan x)' = \frac{1}{1+x^2}.$$

Arccot Function

The cotangent function has a strictly negative derivative on $]0, \pi[$, hence it is a bijection from $]0, \pi[$ to $\mathbb{R}$. The inverse bijection is called the **arccotangent** function and is denoted arccot:

$$\text{arccot} : \mathbb{R} \to]0, \pi[\,, \quad x \mapsto \text{arccot}\, x.$$

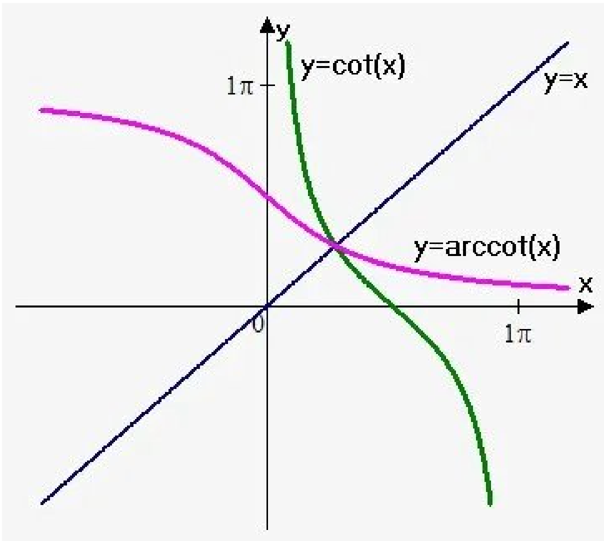


Figure 4.5: Graph of $y = \text{arccot}(x)$ and its relation to $y = \cot(x)$ (reflected about $y = x$).

Theorem 4.15 (Properties of Arccot).

1. $\forall x \in \mathbb{R}$, $\cot(\textit{arccot}\, x) = x$.
2. $\forall x \in]0, \pi[$, $\textit{arccot}(\cot x) = x$.
3. *arccot is differentiable on* $\mathbb{R}$, *and*

$$(\textit{arccot}\, x)' = -\frac{1}{1+x^2}.$$

4. *For all* $x \in \mathbb{R}$: $\arctan x + \textit{arccot}\, x = \frac{\pi}{2}$.

4.3 Hyperbolic Functions and Their Inverses

Hyperbolic Functions

Definition 4.6 (Hyperbolic Functions).

For $x \in \mathbb{R}$, we define:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1},$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad (x \neq 0).$$

These are called, respectively, hyperbolic cosine, hyperbolic sine, hyperbolic tangent, and hyperbolic cotangent.

Theorem 4.16 (Properties of Hyperbolic Functions).

1. cosh *is even;* sinh, tanh, *and* coth *are odd.*

2. *For all* $x \in \mathbb{R}$*:*

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$$
$$\cosh^2 x - \sinh^2 x = 1, \qquad 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \coth^2 x - 1 = \frac{1}{\sinh^2 x}.$$

3. *For all* $x, y \in \mathbb{R}$*:*

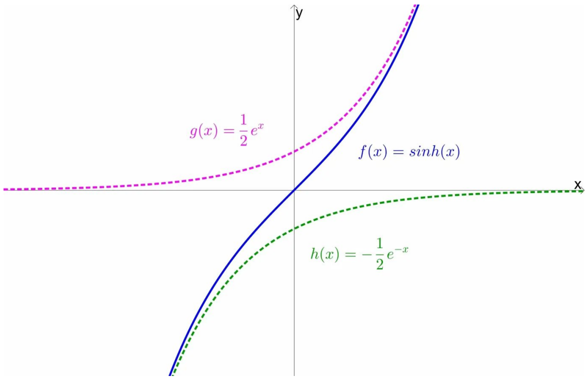
$$\begin{aligned} \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y, \\ \tanh(x+y) &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}. \end{aligned}$$

4. cosh, sinh, tanh *are infinitely differentiable on* $\mathbb{R}$*:*

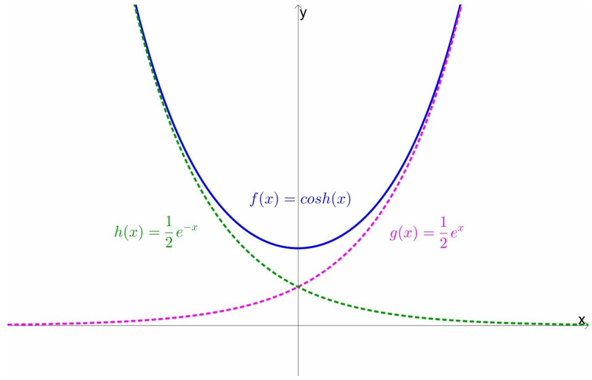
$$(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x, \quad (\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x.$$

5. coth *is infinitely differentiable on* $\mathbb{R}^*$*:*

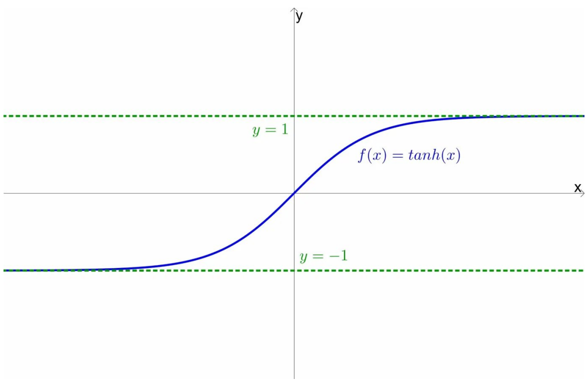
$$(\coth x)' = -\frac{1}{\sinh^2 x}.$$



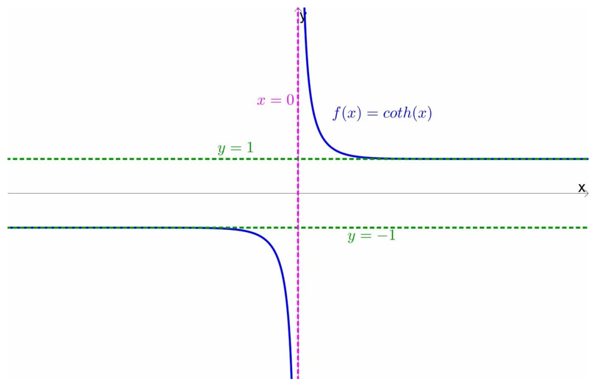
(a) $y = \sinh(x) = \dfrac{e^x - e^{-x}}{2}$, shown via graphical addition of $\frac{1}{2}e^x$ and $-\frac{1}{2}e^{-x}$.



(b) $y = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$, shown via graphical addition of $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.



(c) $y = \tanh(x)$, with horizontal asymptotes $y = \pm 1$.



(d) $y = \coth(x)$, with vertical asymptote $x = 0$ and horizontal asymptotes $y = \pm 1$.

Figure 4.6: Graphs of the four basic hyperbolic functions.

Argsh Function

The hyperbolic sine function has a strictly positive derivative on $\mathbb{R}$, hence it is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. The inverse function is called the **argument of hyperbolic sine** and is denoted Argsh:

$$\operatorname{Argsh} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \operatorname{Argsh} x.$$

Theorem 4.17 (Properties of Argsh).

1. $\forall x \in \mathbb{R}$, $\sinh(\operatorname{Argsh} x) = x$ *and* $\operatorname{Argsh}(\sinh x) = x$.
2. $\forall x \in \mathbb{R}$, $\operatorname{Argsh} x = \ln\left(\sqrt{x^2+1} + x\right)$.
3. Argsh *is continuous and differentiable on* $\mathbb{R}$, *with*

$$(\operatorname{Argsh} x)' = \frac{1}{\sqrt{x^2+1}}.$$

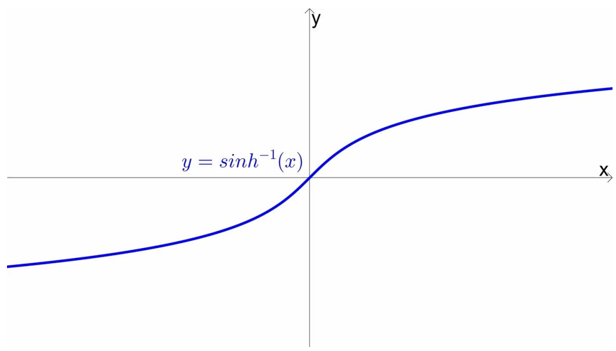


Figure 4.7: Graph of $y = \sinh^{-1}(x) = \operatorname{Argsh}(x)$, defined for all $x \in \mathbb{R}$.

Argch Function

The hyperbolic cosine function has a strictly positive derivative on $\mathbb{R}_+^*$, hence it is a bijection from $\mathbb{R}^+$ onto $[1, +\infty[$. The inverse function is called the **argument of hyperbolic cosine** and is denoted Argch:

$$\operatorname{Argch} : [1, +\infty[\to [0, +\infty[, \quad x \mapsto \operatorname{Argch} x.$$

Theorem 4.18 (Properties of Argch).

1. $\forall x \in [1, +\infty[$, $\cosh(\operatorname{Argch} x) = x$.
2. $\forall x \in [0, +\infty[$, $\operatorname{Argch}(\cosh x) = x$.
3. $\forall x \in [1, +\infty[$, $\operatorname{Argch} x = \ln\left(\sqrt{x^2 - 1} + x\right)$.
4. Argch *is continuous on* $[1, +\infty[$ *and differentiable on* $]1, +\infty[$, *with*

$$(\operatorname{Argch} x)' = \frac{1}{\sqrt{x^2 - 1}}.$$

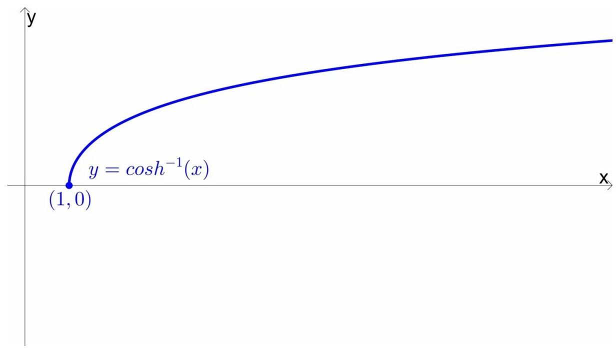


Figure 4.8: Graph of $y = \cosh^{-1}(x) = \operatorname{Argch}(x)$, defined for $x \geq 1$, with starting point $(1, 0)$.

Argth Function

The hyperbolic tangent function has a strictly positive derivative on $\mathbb{R}$, hence it is a bijection from $\mathbb{R}$ onto $]-1, 1[$. The inverse function is called the **argument of hyperbolic tangent** and is denoted Argth:

$$\operatorname{Argth} :]-1, 1[\to \mathbb{R}, \quad x \mapsto \operatorname{Argth} x.$$

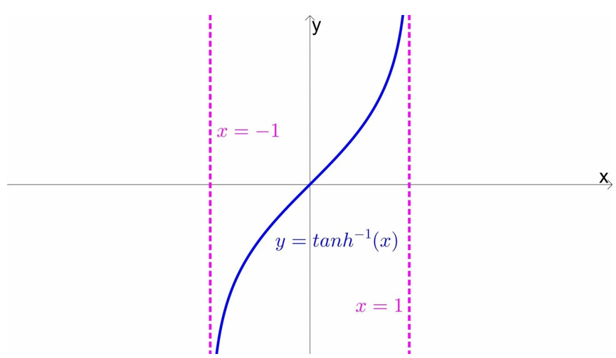


Figure 4.9: Graph of $y = \tanh^{-1}(x) = \operatorname{Argth}(x)$, defined on $]-1, 1[$, with vertical asymptotes $x = \pm 1$.

Theorem 4.19 (Properties of Argth).

1. $\forall x \in]-1,1[$, $\tanh(\operatorname{Argth} x) = x$.
2. $\forall x \in \mathbb{R}$, $\operatorname{Argth}(\tanh x) = x$.
3. $\forall x \in]-1,1[$, $\operatorname{Argth} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.
4. Argth *is continuous and differentiable on* $]-1,1[$, *with*

$$(\operatorname{Argth} x)' = \frac{1}{1-x^2}.$$

Argcoth Function

The hyperbolic cotangent function has a strictly positive derivative on $\mathbb{R}^*$, hence it is a bijection from $\mathbb{R}^*$ onto $]-\infty,-1[\cup]1,+\infty[$. The inverse function is called the **argument of hyperbolic cotangent** and is denoted Argcoth:

$$\operatorname{Argcoth} :]-\infty,-1[\cup]1,+\infty[\to \mathbb{R}^*, \quad x \mapsto \operatorname{Argcoth} x.$$

Theorem 4.20 (Properties of Argcoth).

1. $\forall x \in \mathbb{R}^*$, $\coth(\operatorname{Argcoth} x) = x$.
2. $\forall x \in]-\infty,-1[\cup]1,+\infty[$, $\operatorname{Argcoth}(\coth x) = x$.
3. $\forall x \in]-\infty,-1[\cup]1,+\infty[$, $\operatorname{Argcoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$.
4. Argcoth *is continuous and differentiable on its domain, with*

$$(\operatorname{Argcoth} x)' = \frac{1}{x^2-1}.$$

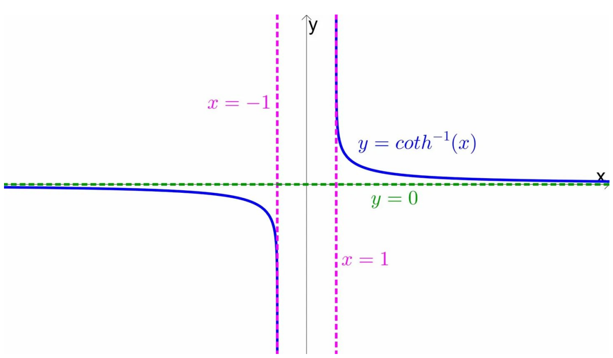


Figure 4.10: Graph of $y = \coth^{-1}(x) = \operatorname{Argcoth}(x)$, defined on $]-\infty,-1[\cup]1,+\infty[$, with vertical asymptotes $x = \pm 1$ and horizontal asymptote $y = 0$.

4.4 Exercises and Solutions

Exercise 4.1.

I. Let f and g be two functions defined by:

$$f(x) = 2\arcsin(1-x) - 5\arccos\left(\frac{1}{\sqrt{x}}\right),$$

$$g(x) = \arctan\left(\frac{2x}{1-x^2}\right) + x\arcsin\left(2-x^2\right).$$

1. Find the domain of definition of f and g .
2. find the derivatives of the given functions f and g .

II. Solve the following trigonometric equation:

$$\pi + 3\arccos(x+1) = 0.$$

Exercise 4.2.

1. Calculate $\cosh^2 x - \sinh^2 x$.
2. Find the derivatives of the given functions:

$$f(x) = 3\,\text{Argsh}\left(\sqrt{x}\right) + \text{Argth}\,(1+5x)\,,$$

$$g(x) = \text{Argch}\left(\sqrt{x}\right) - \text{Argth}\left(x^2\right).$$

Solution — Exercise 4.1 — Inverse Trigonometric Functions.

Part I — Domains and Derivatives.

Function $f(x) = 2\arcsin(1-x) - 5\arccos\left(\frac{1}{\sqrt{x}}\right)$**:**

Step 1 — Domain of $\arcsin(1-x)$**.**
Requires $-1 \le 1-x \le 1 \Rightarrow 0 \le x \le 2$.

Step 2 — Domain of $\arccos\left(\frac{1}{\sqrt{x}}\right)$**.**
Requires $x > 0$ and $0 < \frac{1}{\sqrt{x}} \le 1 \Rightarrow \sqrt{x} \ge 1 \Rightarrow x \ge 1$.

Step 3 — Intersection.

$$\boxed{D_f = [1,2]}$$

Step 4 — Derivative of f**.**
Using $(\arcsin u)' = \dfrac{u'}{\sqrt{1-u^2}}$ and $(\arccos u)' = \dfrac{-u'}{\sqrt{1-u^2}}$:

$$f'(x) = \frac{-2}{\sqrt{1-(1-x)^2}} + \frac{5}{2x^{3/2}\sqrt{1-\frac{1}{x}}} = \frac{-2}{\sqrt{2x-x^2}} + \frac{5}{2\sqrt{x(x-1)}}, \quad x \in (1,2).$$

Function $g(x) = \arctan\left(\frac{2x}{1-x^2}\right) + x\arcsin(2-x^2)$**:**

Step 5 — Domain of $\arctan\left(\frac{2x}{1-x^2}\right)$**.**
Requires $x^2 \neq 1$, i.e., $x \neq \pm 1$.

Step 6 — Domain of $\arcsin(2-x^2)$**.**
Requires $-1 \leq 2-x^2 \leq 1 \Rightarrow 1 \leq x^2 \leq 3 \Rightarrow x \in [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$.

Step 7 — Intersection (excluding $x = \pm 1$**).**

$$\boxed{D_g = [-\sqrt{3}, -1) \cup (1, \sqrt{3}]}$$

Step 8 — Derivative of g**.**
Using the identity $\arctan\left(\frac{2x}{1-x^2}\right) = 2\arctan x$ (valid for $|x| < 1$; must be applied with care at the boundary):

$$\frac{d}{dx}\arctan\left(\frac{2x}{1-x^2}\right) = \frac{2}{1+x^2}.$$

Hence:

$$g'(x) = \frac{2}{1+x^2} + \arcsin(2-x^2) - \frac{2x^2}{\sqrt{1-(2-x^2)^2}}.$$

Part II — Solving $\pi + 3\arccos(x+1) = 0$**.**

Step 9 — Solve for x**.**

$$\arccos(x+1) = -\frac{\pi}{3}.$$

The range of arccos is $[0, \pi]$, but $-\dfrac{\pi}{3} < 0$ is outside this range. **No real solution.**

Solution — Exercise 4.2 — Hyperbolic Identity and Derivatives.

Part (1) — Verify $\cosh^2 x - \sinh^2 x = 1$**.**

Step 1 — Expand using definitions.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}.$$

Step 2 — Use difference of squares.

$$(e^x+e^{-x})^2-(e^x-e^{-x})^2 = [(e^x+e^{-x})+(e^x-e^{-x})][(e^x+e^{-x})-(e^x-e^{-x})] = 2e^x \cdot 2e^{-x} = 4.$$

Step 3 — Therefore $\cosh^2 x - \sinh^2 x = \dfrac{4}{4} = 1$. ✓

Part (2) — Derivatives.

Recall: $(\operatorname{Argsh} u)' = \dfrac{u'}{\sqrt{1+u^2}}$, $(\operatorname{Argch} u)' = \dfrac{u'}{\sqrt{u^2-1}}$, $(\operatorname{Argth} u)' = \dfrac{u'}{1-u^2}$.

Function $f(x) = 3\operatorname{Argsh}(\sqrt{x}) + \operatorname{Argth}(1+5x)$**:**

Step 4 — Domain of Argth$(1+5x)$**.**
Requires $|1+5x|<1 \Rightarrow -\frac{2}{5}<x<0$. Combined with Argsh$(\sqrt{x})$ requiring $x \geq 0$: domain reduces to $\{0\}$, but practically $x \in (-\frac{2}{5}, 0)$ for the Argth part.

Step 5 — Compute $f'(x)$**.**

$$f'(x) = \frac{3 \cdot \frac{1}{2\sqrt{x}}}{\sqrt{1+x}} + \frac{5}{1-(1+5x)^2} = \frac{3}{2\sqrt{x}\sqrt{1+x}} + \frac{5}{1-(1+5x)^2}.$$

Function $g(x) = \text{Argch}(\sqrt{x}) - \text{Argth}(x^2)$**:**

Step 6 — Using $(\text{Argch } u)' = \dfrac{u'}{\sqrt{u^2-1}}$ with $u = \sqrt{x}$, $u' = \dfrac{1}{2\sqrt{x}}$:

$$g'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x-1}} - \frac{2x}{1-x^4} = \frac{1}{2\sqrt{x(x-1)}} - \frac{2x}{1-x^4}.$$

CHAPTER 5

Taylor Expansions

5.1 Taylor's Formula

Taylor-Young Formula

Theorem 5.1 (Taylor-Young Formula).

Let $f :]a,b[\to \mathbb{R}$, $x_0 \in]a,b[$. *Suppose* f *is of class* C^{n-1} *and* $f^{(n)}(x_0)$ *exists (and is finite). Then for all* $x \in [a,b]$,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (x-x_0)^n\varepsilon(x),$$

where ε *is a function defined on* $]a,b[$ *such that* $\lim_{x\to x_0}\varepsilon(x) = 0$.

Remark.

The term $(x-x_0)^n\varepsilon(x)$ with $\varepsilon(x) \to 0$ as $x \to x_0$ is often abbreviated as $o((x-x_0)^n)$.

Maclaurin-Young Formula

When $x_0 = 0$ in the previous formula, we obtain the Maclaurin formula of order n with Young's remainder:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x),$$

with $\varepsilon(x) \to 0$ as $x \to 0$.

5.2 Taylor Expansions Near a Point

Definition and Existence

Definition 5.1 (Taylor Expansion).

Let I be an open interval. For $a \in I$ and $n \in \mathbb{N}$, we say that f has a **Taylor expansion (LE)** at a of order n if there exist real numbers $c_0, c_1, \ldots, c_n$ and a function $\varepsilon : I \to \mathbb{R}$ such that $\lim_{x \to a} \varepsilon(x) = 0$ and for all $x \in I$,

$$f(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + (x-a)^n \varepsilon(x).$$

- The equality above is called a Taylor expansion of f near a of order n.
- The term $P_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$ is called the polynomial part of the Taylor expansion.
- The term $(x-a)^n \varepsilon(x)$ is called the remainder of the Taylor expansion.

Example 5.1. The function $f(x) = e^x$ is defined on $\mathbb{R}$ and $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$. Hence

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + x^n \varepsilon(x).$$

Example 5.2. The function $\varphi(x) = \sin x$ is defined on $\mathbb{R}$. For all $n \in \mathbb{N}$,

$$\varphi^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), \quad \varphi^{(n)}(0) = \sin\left(\frac{n\pi}{2}\right).$$

Thus

$$\varphi^{(n)}(0) = \begin{cases} 0, & \text{if } n = 2p, \\ (-1)^p, & \text{if } n = 2p+1. \end{cases}$$

Therefore

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^p \frac{x^{2p+1}}{(2p+1)!} + x^{2p+2} \varepsilon(x).$$

Standard Taylor Expansions at the Origin

The following Taylor expansions at 0 come from the Maclaurin-Young formula:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + x^n\varepsilon(x)$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x)$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x)$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x)$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + x^n\varepsilon(x)$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + x^n\varepsilon(x)$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + x^n\varepsilon(x)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n)}x^n + x^n\varepsilon(x)$$

Taylor Expansion at an Arbitrary Point

Proposition 5.2 (Change of Variable).

f has a Taylor expansion of order n at a if and only if the function $g(x) = f(x+a)$ has a Taylor expansion of order n at 0.

Example 5.3. Find the Taylor expansion of $f(x) = e^x$ at 1. Let $h = x - 1$. As x approaches 1, h approaches 0. Then

$$\begin{aligned} f(x) &= e^x = e^{1+h} = e\cdot e^h \\ &= e\left(1 + h + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!} + h^n\varepsilon(h)\right) \\ &= e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \cdots + \frac{e}{n!}(x-1)^n + (x-1)^n\varepsilon(x-1), \end{aligned}$$

where $\lim_{x\to 1}\varepsilon(x-1) = 0$.

Example 5.4. Find the Taylor expansion of $f(x) = \cos x$ at $\frac{\pi}{2}$. We know $\cos x = -\sin(x-\frac{\pi}{2})$. Let $h = x - \frac{\pi}{2} \to 0$. Then

$$\begin{aligned} \cos x = -\sin h &= -h + \frac{h^3}{3!} + \cdots + (-1)^{n+1}\frac{h^{2n+1}}{(2n+1)!} + h^{2n+2}\varepsilon(h) \\ &= -\left(x - \frac{\pi}{2}\right) + \cdots + \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1} + \left(x - \frac{\pi}{2}\right)^{2n+2}\varepsilon\left(x - \frac{\pi}{2}\right). \end{aligned}$$

5.3 Operations on Taylor Expansions

Suppose f and g have Taylor expansions at 0 of order n:

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon_1(x) = P_n(x) + x^n \varepsilon_1(x),$$
$$g(x) = b_0 + b_1 x + \cdots + b_n x^n + x^n \varepsilon_2(x) = Q_n(x) + x^n \varepsilon_2(x),$$

where $\lim_{x\to 0} \varepsilon_1(x) = \lim_{x\to 0} \varepsilon_2(x) = 0$.

Sum and Product

- $f + g$ has a Taylor expansion of order n at 0 given by:

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + x^n \varepsilon(x).$$

- $f \cdot g$ has a Taylor expansion of order n at 0 given by:

$$(f \times g)(x) = P_n(x) Q_n(x) + x^n \varepsilon(x),$$

  where we keep only terms of degree $\leq n$.

Example 5.5. Let $f(x) = \frac{1}{1-x} - e^x$ defined on $]-\infty, 1[$. Find its expansion at order 3 near 0.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^3 \varepsilon_1(x), \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3 \varepsilon_2(x).$$

Hence

$$f(x) = \frac{x^2}{2} + \frac{5}{6} x^3 + x^3 \varepsilon(x).$$

Example 5.6. Find the expansion at order 5 of $\varphi(x) = \cos x \sin x$ near 0.

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = x - \frac{2x^3}{3} + \frac{2x^5}{15} + x^5 \varepsilon(x).$$

Quotient

To compute the quotient of two Taylor expansions, we can use polynomial division in increasing powers.

Example 5.7. Find the expansion of $\tan x = \frac{\sin x}{\cos x}$ at order 3.

$$\sin x = x - \frac{x^3}{6} + x^4 \varepsilon(x), \quad \cos x = 1 - \frac{x^2}{2} + x^3 \varepsilon(x).$$

Performing division:

$$\tan x = x + \frac{x^3}{3} + x^3 \varepsilon(x).$$

Integration

Proposition 5.3 (Integration of Taylor Expansions).

If f is continuous on an interval containing 0 and has a Taylor expansion $f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$, then any primitive F of f with $F(0) = 0$ has the Taylor expansion:

$$F(x) = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} + o(x^{n+1}).$$

Example 5.8. From $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + o(x^n)$, integrating from 0 to x gives:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n).$$

Composition

If $g(0) = 0$ (i.e., $b_0 = 0$), then $f \circ g$ has a Taylor expansion of order n at 0 whose polynomial part is obtained by composing the polynomials.

Example 5.9. Find the expansion of $f(x) = e^{\sin x}$ at 0 of order 4.

$$\sin x = x - \frac{x^3}{6} + x^4 \varepsilon(x).$$

Let $u = x - \frac{x^3}{6} + x^4 \varepsilon(x)$. Then

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + u^4 \varepsilon(u).$$

Substituting and keeping terms up to degree 4:

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + x^4 \varepsilon(x).$$

Taylor Expansion at Infinity

Definition 5.2 (Taylor Expansion at Infinity).

A function f defined on an interval $I =]x_0, +\infty[$ has a Taylor expansion at $+\infty$ of order n if there exist real numbers $c_0, c_1, \ldots, c_n$ such that

$$f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{1}{x^n} \varepsilon\left(\frac{1}{x}\right),$$

where $\lim_{x \to +\infty} \varepsilon\left(\frac{1}{x}\right) = 0$.

Example 5.10. For $f(x) = e^{1/x}$ defined on $]0, +\infty[$, let $u = 1/x$. As $x \to +\infty$, $u \to 0$. Then

$$e^u = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + u^n \varepsilon(u),$$

so

$$e^{1/x} = 1 + \frac{1}{x} + \frac{1}{2!x^2} + \cdots + \frac{1}{n!x^n} + \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right).$$

5.4 Applications of Taylor Expansions

Computing Limits

Taylor expansions are particularly useful for resolving indeterminate forms.

Example 5.11. Compute $\lim_{x\to 0} \frac{\sin x - x}{x^3}$. Using $\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x)$, we have

$$\frac{\sin x - x}{x^3} = -\frac{1}{6} + \varepsilon(x) \to -\frac{1}{6}.$$

Position of the Curve Relative to its Tangent

Suppose f has a Taylor expansion at x_0 of order $n \geq 2$:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x).$$

This implies that f (or its extension if not defined at x_0) is continuous and differentiable at x_0, with $f(x_0) = a_0$ and $f'(x_0) = a_1$.

The equation of the tangent is $y = a_0 + a_1(x - x_0)$. The sign of $f(x) - (a_0 + a_1(x - x_0))$ near x_0 is determined by the first non-zero coefficient among $a_2, \ldots, a_n$.

Let m be the smallest integer such that $a_m \neq 0$. Then:

- If m is even, the sign of $f(x) - (a_0 + a_1(x - x_0))$ is locally the same as a_m:
  - If $a_m > 0$, the curve is locally above its tangent.
  - If $a_m < 0$, the curve is locally below its tangent.
- If m is odd, the curve crosses its tangent at $(x_0, f(x_0))$; this is an inflection point.

Example 5.12. For $f(x) = \sin x$, $f(x) = x - \frac{x^3}{6} + x^3\varepsilon(x)$. The tangent at 0 is $y = x$, and since $m = 3$ is odd, the curve crosses the tangent.

Example 5.13. For $f(x) = \frac{x}{1-x}$, $f(x) = x + x^2 + x^3\varepsilon(x)$. The tangent at 0 is $y = x$, and since $m = 2$ is even with $a_2 = 1 > 0$, the curve is above its tangent.

Position of the Curve Relative to an Asymptote

Suppose f has an asymptote $y = a_0 + a_1 x$. To find a_0 and a_1, we compute:

$$a_1 = \lim_{x\to+\infty} \frac{f(x)}{x}, \quad a_0 = \lim_{x\to+\infty} (f(x) - a_1 x).$$

Using Taylor expansions, we can find these coefficients by expanding $f(1/y)$ near 0.
If $yf(1/y) = a_0 + a_1 y + \cdots + a_n y^n + y^n \varepsilon(y)$ near 0, then

$$f(x) = a_0 + a_1 x + \frac{a_2}{x} + \cdots + \frac{a_n}{x^{n-1}} + \frac{1}{x^{n-1}} \varepsilon\left(\frac{1}{x}\right) \text{ near } +\infty.$$

Let m be the smallest integer ≥ 2 such that $a_m \neq 0$. Then:

- If $a_m > 0$, $f(x) - (a_0 + a_1 x) > 0$ near $+\infty$, so the curve is above the asymptote.
- If $a_m < 0$, the curve is below the asymptote.

Example 5.14. Consider $f(x) = xe^{1/x}$ for $x > 0$. Let $y = 1/x$. Then

$$f(1/y) = \frac{1}{y} e^y = \frac{1}{y}\left(1 + y + \frac{y^2}{2!} + \cdots\right) = \frac{1}{y} + 1 + \frac{y}{2!} + \cdots$$

Thus $yf(1/y) = 1 + y + \frac{y^2}{2!} + \cdots$, so $a_0 = 1$, $a_1 = 1$, $a_2 = 1/2$, etc. Therefore $f(x) = x + 1 + \frac{1}{2x} + \cdots$. The line $y = x + 1$ is an asymptote, and since $a_2 > 0$, the curve is above the asymptote.

Remark.

In this example, the expansion $e^y = 1 + y + \frac{y^2}{2!} + \cdots$ is a **Taylor series** (an infinite sum). The purpose here is to identify the asymptote and the position of the curve relative to it, which only requires reading off the coefficients a_0, a_1, and the sign of a_2. The rest of the course deals with **finite Taylor expansions** (truncated sums with a remainder), not infinite series.

5.5 Exercises and Solutions

Exercise 5.1.

Find the Taylor expansion of the given functions $\mathrm{LD}_3(0)$:

$$e^{\cos x}, \quad \frac{e^{\sin x} - e^x}{1 + \ln(1 + x)}, \quad \frac{\ln(1 + 2x)}{\sin(2x)}, \quad \frac{e^{2x} - 1}{\arctan(4x)}.$$

Exercise 5.2.

Using the Taylor expansion to calculate the limits:

$$\frac{e^{x^2} - \cos x}{x^2}, \quad \frac{1 - \cos x}{\sin^2 x}, \quad \frac{e^{\sin x} - e^x}{\sin x^3}.$$

Hint:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o\left(x^3\right), \quad \sin x = x - \frac{x^3}{6} + o\left(x^3\right),$$

$$\cos x = 1 - \frac{x^2}{2} + o\left(x^3\right), \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{6} + o\left(x^3\right), \quad \arctan x = x - \frac{x^3}{3} + o\left(x^3\right).$$

Solution — Exercise 5.1 — Taylor Expansions $\mathrm{LD}_3(0)$**.**

Standard expansions at 0 (order 3):

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(u^3), \quad \sin x = x - \frac{x^3}{6} + o(x^3), \quad \cos x = 1 - \frac{x^2}{2} + o(x^3),$$

$$\ln(1+v) = v - \frac{v^2}{2} + \frac{v^3}{3} + o(v^3), \quad \arctan v = v - \frac{v^3}{3} + o(v^3).$$

Expression 1: $e^{\cos x}$

Step 1 — Write $\cos x = 1 + u$ where $u = -\frac{x^2}{2} + o(x^3)$, so $e^{\cos x} = e \cdot e^u$.

Step 2 — Since $u^2 = O(x^4)$, at order 3:

$$e^u = 1 + u + O(x^4) = 1 - \frac{x^2}{2} + o(x^3).$$

$$\boxed{e^{\cos x} = e - \frac{e}{2}x^2 + o(x^3).}$$

Expression 2: $\dfrac{e^{\sin x} - e^x}{1 + \ln(1+x)}$

Step 3 — Numerator.

$\sin x = x - \frac{x^3}{6} + o(x^3)$, so $e^{\sin x} = e^x \cdot e^{\sin x - x}$. Now $\sin x - x = -\frac{x^3}{6} + o(x^3)$, giving:

$$e^{\sin x} = e^x\left(1 - \frac{x^3}{6} + o(x^3)\right) \Rightarrow e^{\sin x} - e^x = -\frac{x^3}{6} + o(x^3).$$

Step 4 — Denominator.

$$1 + \ln(1+x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \xrightarrow{x\to 0} 1.$$

Step 5 — Divide.

$$\boxed{\frac{e^{\sin x} - e^{x}}{1 + \ln(1+x)} = -\frac{x^3}{6} + o(x^3).}$$

Expression 3: $\dfrac{\ln(1+2x)}{\sin(2x)}$

Step 6 — Expand.

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + o(x^3) = 2x - 2x^2 + \frac{8x^3}{3} + o(x^3).$$

$$\sin(2x) = 2x - \frac{(2x)^3}{6} + o(x^3) = 2x - \frac{4x^3}{3} + o(x^3).$$

Step 7 — Divide. Factor $2x$ from both:

$$\frac{2x\left(1 - x + \frac{4x^2}{3}\right)}{2x\left(1 - \frac{2x^2}{3}\right)} = \left(1 - x + \frac{4x^2}{3}\right)\left(1 + \frac{2x^2}{3} + o(x^2)\right) = 1 - x + 2x^2 + o(x^3).$$

$$\boxed{\frac{\ln(1+2x)}{\sin(2x)} = 1 - x + 2x^2 + o(x^3).}$$

Expression 4: $\dfrac{e^{2x} - 1}{\arctan(4x)}$

Step 8 — Expand.

$$e^{2x} - 1 = 2x + 2x^2 + \frac{4x^3}{3} + o(x^3),$$

$$\arctan(4x) = 4x - \frac{(4x)^3}{3} + o(x^3) = 4x - \frac{64x^3}{3} + o(x^3).$$

Step 9 — Divide. Factor $2x$ and $4x$:

$$\frac{2x\left(1 + x + \frac{2x^2}{3}\right)}{4x\left(1 - \frac{16x^2}{3}\right)} = \frac{1}{2}\left(1 + x + \frac{2x^2}{3}\right)\left(1 + \frac{16x^2}{3} + o(x^2)\right) = \frac{1}{2} + \frac{x}{2} + \frac{x^2 \cdot 17}{6} + o(x^3).$$

$$\boxed{\frac{e^{2x} - 1}{\arctan(4x)} = \frac{1}{2} + \frac{x}{2} + 3x^2 + o(x^3).}$$

Solution — Exercise 5.2 — Limits Using Taylor Expansions.

Limit 1: $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$

Step 1 — Expand numerator to order 4.

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4), \qquad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4).$$

Step 2 — Subtract the expansions.

$$e^{x^2} - \cos x = \frac{3x^2}{2} + \frac{11x^4}{24} + o(x^4).$$

Step 3 — Divide by x^2.

$$\frac{e^{x^2} - \cos x}{x^2} = \frac{3}{2} + \frac{11x^2}{24} + o(x^2) \xrightarrow{x\to 0} \boxed{\frac{3}{2}.}$$

Limit 2: $\lim_{x\to 0} \dfrac{1-\cos x}{\sin^2 x}$

Step 4 — Expand to order 4.

$$1 - \cos x = \frac{x^2}{2} - \frac{x^4}{24} + o(x^4), \qquad \sin^2 x = \left(x - \frac{x^3}{6}\right)^2 + o(x^4) = x^2 - \frac{x^4}{3} + o(x^4).$$

Step 5 — Divide and take the limit.

$$\frac{1-\cos x}{\sin^2 x} = \frac{\frac{1}{2} - \frac{x^2}{24}}{1 - \frac{x^2}{3}} \cdot \frac{x^2}{x^2} = \left(\frac{1}{2} - \frac{x^2}{24}\right)\left(1 + \frac{x^2}{3} + o(x^2)\right) \xrightarrow{x\to 0} \boxed{\frac{1}{2}.}$$

Limit 3: $\lim_{x\to 0} \dfrac{e^{\sin x} - e^x}{\sin(x^3)}$

Step 6 — Numerator (from Exercise 5.1, Expression 2).

$$e^{\sin x} - e^x = -\frac{x^3}{6} + o(x^3).$$

Step 7 — Denominator.

$$\sin(x^3) = x^3 + o(x^3).$$

Step 8 — Divide.

$$\frac{e^{\sin x} - e^x}{\sin(x^3)} = \frac{-\frac{x^3}{6} + o(x^3)}{x^3 + o(x^3)} \xrightarrow{x\to 0} \boxed{-\frac{1}{6}.}$$

CHAPTER 6

Linear Algebra

6.1 Internal Composition Laws

Definition 6.1 (Internal Composition Law).

Let G be a set. An **internal composition law** on G is a map from $G \times G$ to G. If we denote it by $(a, b) \mapsto a * b$, we speak of the law $*$ and say that $a * b$ is the composite of a and b under the law $*$.

Example 6.1.

- On $G = \mathbb{Z}$, addition $+$ and multiplication $\times$ are internal composition laws.
- On $G = \mathbb{R}^2$, addition $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $((x_1, y_1), (x_2, y_2)) \mapsto (x_1 + x_2, y_1 + y_2)$ is an internal law.

Example 6.2. On $\mathbb{R}^*$, define the law δ by:

$$x\delta y = x + y + \ln|xy|.$$

Then δ is an internal law on $\mathbb{R}^*$. Indeed, let $x, y \in \mathbb{R}^*$. We show that $x\delta y \in \mathbb{R}^*$:

$$\begin{aligned}(x\delta y = 0) &\Leftrightarrow (x + y + \ln|xy| = 0)\\ &\Leftrightarrow (\ln|xy| = -(x+y))\\ &\Leftrightarrow (|xy| = e^{-(x+y)})\\ &\Rightarrow (x \neq 0 \text{ and } y \neq 0)\end{aligned}$$

Thus $x\delta y \in \mathbb{R}^*$, so δ is an internal law.

Definition 6.2 (Properties of Internal Laws).

Let $*$ be an internal law on a set G. We say that:

1. $*$ is **commutative** if $\forall x, y \in G$, $x * y = y * x$.
2. $*$ is **associative** if $\forall x, y, z \in G$, $(x * y) * z = x * (y * z)$.
3. $*$ admits an **identity element** on G, denoted e, if $\exists e \in G$, $\forall x \in G$, $x * e = e * x = x$.

4. If, moreover, $*$ is commutative, it suffices to check that $\forall x \in G$, $x * e = x$.

Example 6.3. On $\mathbb{R} \setminus \{\frac{1}{2}\}$, define the internal law $*$ by:

$$x * y = x + y - 2xy.$$

- **Internal:** For $x, y \in \mathbb{R} \setminus \{\frac{1}{2}\}$, we have

$$x * y = \frac{1}{2} \iff (1-2y)(x - \frac{1}{2}) = 0 \iff y = \frac{1}{2} \text{ or } x = \frac{1}{2}.$$

  Since $x, y \neq \frac{1}{2}$, we have $x * y \neq \frac{1}{2}$, so $*$ is internal.
- **Commutative:** $x * y = x + y - 2xy = y + x - 2yx = y * x$.
- **Associative:**

$$\begin{aligned}(x * y) * z &= (x + y - 2xy) * z = (x + y - 2xy) + z - 2(x + y - 2xy)z \\ &= x + y + z - 2xy - 2xz - 2yz + 4xyz \\ &= x + (y + z - 2yz) - 2x(y + z - 2yz) \\ &= x * (y * z).\end{aligned}$$

- **Identity:** Let e be such that $x * e = x$ for all x. Then $x + e - 2xe = x \Rightarrow e(1-2x) = 0 \Rightarrow e = 0$. Thus $e = 0$ is the identity element.

Definition 6.3 (Symmetric Element).

Let $*$ be an internal law on a set G, possessing an identity element e. Let $x \in G$. We say that x admits a **symmetric element** x' under the law $*$ if

$$x * x' = x' * x = e.$$

Example 6.4. On $\mathbb{R} \setminus \{\frac{1}{2}\}$ with the law $x * y = x + y - 2xy$, the identity is $e = 0$. For $x \in \mathbb{R} \setminus \{\frac{1}{2}\}$, we look for x' such that $x * x' = 0$:

$$x + x' - 2xx' = 0 \Rightarrow x'(1-2x) = -x \Rightarrow x' = \frac{x}{2x-1}.$$

We need to check that $x' \neq \frac{1}{2}$. Indeed,

$$x' = \frac{1}{2} \iff \frac{x}{2x-1} = \frac{1}{2} \iff 2x = 2x - 1 \iff 0 = -1,$$

which is impossible. Thus every element has a symmetric element $x' = \frac{x}{2x-1}$.

Definition 6.4 (Distributivity).

Let G be a set equipped with two internal composition laws, denoted $\triangle$ and $*$. We say that $*$ is **distributive** over $\triangle$ if

$$\forall x, y, z \in G, \quad x * (y \triangle z) = (x * y) \triangle (x * z).$$

6.2 Group Structure

Definition 6.5 (Group).

Let G be a set equipped with an internal composition law $*$. We say that $(G, *)$ is a **group** if the law $*$ satisfies the following three conditions:

1. $*$ is associative.
2. $*$ admits an identity element.
3. Every element of G admits a symmetric element under $*$.

If, in addition, the law is commutative, we say that the group is **commutative** or **abelian**.

Example 6.5.

- $(\mathbb{Z}, +)$ is a commutative group.
- $(\mathbb{R}, \times)$ is not a group because 0 has no symmetric element.
- $(\mathbb{R}^*, \times)$ is a commutative group.

Definition 6.6 (Subgroup).

Let $(G, *)$ be a group. A non-empty subset $H \subset G$ is a **subgroup** of G if the restriction of the operation $*$ to H endows H with a group structure.

Proposition 6.1 (Characterization of Subgroups).

Let H be a non-empty subset of a group G. Then H is a subgroup of G if and only if:

1. *for all $x, y \in H$, $x * y \in H$,*
2. *for all $x \in H$, the symmetric element x' of x belongs to H.*

Example 6.6.

- $(\mathbb{R}^+, \times)$ is a subgroup of $(\mathbb{R}^*, \times)$. Indeed:
  - If $x, y \in \mathbb{R}^+$, then $x \times y \in \mathbb{R}^+$.
  - If $x \in \mathbb{R}^+$, then $x^{-1} = \frac{1}{x} \in \mathbb{R}^+$.
- $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$. Indeed:
  - If $x = 2x_1$, $y = 2y_1 \in 2\mathbb{Z}$, then $x + y = 2(x_1 + y_1) \in 2\mathbb{Z}$.
  - If $x = 2x_1 \in 2\mathbb{Z}$, then $-x = -2x_1 = 2(-x_1) \in 2\mathbb{Z}$.

6.3 Ring Structure

Definition 6.7 (Ring).

Let A be a set equipped with two internal composition laws, which we will denote by $\triangle$ and $*$. We say that $(A, \triangle, *)$ is a **ring** if the following conditions are satisfied:

1. $(A, \triangle)$ is a commutative group.
2. The law $*$ is associative.
3. The law $*$ is distributive over the law $\triangle$.

If, in addition, the law $*$ is commutative, we say that the ring $(A, \triangle, *)$ is **commutative**. If the law $*$ admits an identity element, we say that the ring is **unitary**.

Example 6.7. $(\mathbb{Z}, +, \cdot)$ is a commutative unitary ring.

Definition 6.8 (Subring).

If $(A, \triangle, *)$ is a ring and B is a subset of A, we say that B is a **subring** of A if, equipped with the laws induced by A, it is itself a ring, i.e., $(B, \triangle, *)$ is a ring.

Proposition 6.2 (Characterization of Subrings).

A subset B of a ring A is a subring of A if and only if:

1. *for all $a, b \in B$, $a - b \in B$,*
2. *for all $a, b \in B$, $a \times b \in B$.*

Example 6.8. $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$ is a subring of $(\mathbb{Z}, +, \cdot)$. Indeed, for $x = 2n$, $y = 2m \in 2\mathbb{Z}$, we have $x - y = 2(n - m) \in 2\mathbb{Z}$ and $xy = 2(2nm) \in 2\mathbb{Z}$.

6.4 Field Structure

Definition 6.9 (Field).

Let $\mathbb{K}$ be a set equipped with two internal composition laws, still denoted $\triangle$ and $*$. We say that $(\mathbb{K}, \triangle, *)$ is a **field** if the following conditions are satisfied:

1. $(\mathbb{K}, \triangle, *)$ is a ring.
2. $(\mathbb{K} \setminus \{e\}, *)$ is a group, where e is the identity element of $\triangle$.

If, in addition, $*$ is commutative, we say that $(\mathbb{K}, \triangle, *)$ is a **commutative field**.

Example 6.9. $(\mathbb{R}, +, \cdot)$ is a commutative field.

Definition 6.10 (Subfield).

If $\mathbb{K}$ is a field and H is a non-empty subset of $\mathbb{K}$, then H is said to be a **subfield** of $\mathbb{K}$ if the restrictions of the two operations of $\mathbb{K}$ endow H with a field structure.

Proposition 6.3 (Characterization of Subfields).

If H is a non-empty subset of a field $\mathbb{K}$, then H is a subfield of $\mathbb{K}$ if and only if:

1. $a \in H$ *and* $b \in H \Rightarrow a - b \in H$,
2. $a \in H$ *and* $b \in H \setminus \{0\} \Rightarrow a \cdot b^{-1} \in H$.

Example 6.10.

- $\mathbb{R}$ is a subfield of $(\mathbb{C}, +, \times)$.
- $\mathbb{Q}$ is a subfield of $(\mathbb{R}, +, \times)$ and hence of $(\mathbb{C}, +, \times)$.

6.5 Vector Spaces

Definitions and Elementary Properties

Let $\mathbb{K}$ be a commutative field (generally $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty set equipped with an internal operation denoted $+$:

$$(+) : E \times E \to E, \quad (x, y) \mapsto x + y,$$

and an external operation denoted $\cdot$:

$$(\cdot) : \mathbb{K} \times E \to E, \quad (\lambda, x) \mapsto \lambda x.$$

Definition 6.11 (Vector Space).

A **vector space** over the field $\mathbb{K}$, or a $\mathbb{K}$-vector space, is a triple $(E, +, \cdot)$ such that:

1. $(E, +)$ is a commutative group (its identity element is denoted 0_E).
2. $\forall \lambda \in \mathbb{K}$, $\forall x, y \in E$, $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$.
3. $\forall \lambda, \mu \in \mathbb{K}$, $\forall x \in E$, $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$.
4. $\forall \lambda, \mu \in \mathbb{K}$, $\forall x \in E$, $(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x)$.
5. $\forall x \in E$, $1_{\mathbb{K}} \cdot x = x$.

The elements of a vector space are called **vectors**, and the elements of $\mathbb{K}$ are called **scalars**.

Example 6.11.

- $(\mathbb{R}, +, \cdot)$ is an $\mathbb{R}$-vector space.
- $(\mathbb{C}, +, \cdot)$ is a $\mathbb{C}$-vector space.
- $(\mathbb{C}, +, \cdot)$ is also an $\mathbb{R}$-vector space.
- $\mathbb{R}^n$, equipped with component-wise addition and scalar multiplication, is an $\mathbb{R}$-vector space.

Proposition 6.4 (Properties of Vector Spaces).

If E is a $\mathbb{K}$-vector space, then we have the following properties:

$$\begin{aligned}
&(1) \quad \forall x \in E,\ 0_{\mathbb{K}} \cdot x = 0_E,\\
&(2) \quad \forall x \in E,\ (-1_{\mathbb{K}}) \cdot x = -x,\\
&(3) \quad \forall \lambda \in \mathbb{K},\ \lambda \cdot 0_E = 0_E,\\
&(4) \quad \forall \lambda \in \mathbb{K},\ \forall x, y \in E,\ \lambda(x-y) = \lambda x - \lambda y,\\
&(5) \quad \forall \lambda \in \mathbb{K},\ \forall x \in E,\ \lambda x = 0_E \Leftrightarrow \lambda = 0_{\mathbb{K}} \text{ or } x = 0_E.
\end{aligned}$$

Subspaces

Definition 6.12 (Subspace).

Let $(E, +, \cdot)$ be a $\mathbb{K}$-vector space and let F be a non-empty subset of E. We say that F is a **subspace** of E if $(F, +, \cdot)$ is also a $\mathbb{K}$-vector space.

Remark.

1. If F is a subspace of E, then $0_E \in F$.
2. If $0_E \notin F$, then F cannot be a subspace of E.

Theorem 6.5 (Characterization of Subspaces).

Let $(E, +, \cdot)$ be a $\mathbb{K}$-vector space and $F \subset E$, $F \neq \emptyset$. The following are equivalent:

1. *F is a subspace of E.*
2. *F is closed under addition and scalar multiplication, i.e.:*

$$\forall \lambda \in \mathbb{K},\ \forall x, y \in F,\ \lambda x \in F \text{ and } x + y \in F.$$

3. *$\forall \lambda, \mu \in \mathbb{K},\ \forall x, y \in F,\ \lambda x + \mu y \in F$.*

In other words,

$$F \text{ is a subspace } \Leftrightarrow \begin{cases} F \neq \emptyset, \\ \forall \lambda, \mu \in \mathbb{K},\ \forall x, y \in F,\ \lambda x + \mu y \in F. \end{cases}$$

Example 6.12. Let $F = \{(x, y) \in \mathbb{R}^2 : x - y = 0\} \subset \mathbb{R}^2$. Then F is a subspace.

- $0_{\mathbb{R}^2} = (0, 0) \in F$, because $0 - 0 = 0$.
- $\forall \lambda, \mu \in \mathbb{R}$, $\forall (x, y), (x', y') \in F$, we have $x - y = 0$ and $x' - y' = 0$. Then

$$\lambda(x - y) + \mu(x' - y') = (\lambda x + \mu x') - (\lambda y + \mu y') = 0,$$

  i.e., $\lambda(x, y) + \mu(x', y') \in F$. Hence F is a subspace of $\mathbb{R}^2$.

Proposition 6.6.

The intersection of any non-empty family of subspaces is a subspace.

Remark.

The union of two subspaces is not necessarily a subspace. For example, let $F_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $F_2 = \{(0, y) : y \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^2$. Then $F_1 \cup F_2$ is not a subspace because $(1, 0) \in F_1$, $(0, 1) \in F_2$, but $(1, 0) + (0, 1) = (1, 1) \notin F_1 \cup F_2$.

Sum of Two Subspaces

Definition 6.13 (Sum of Subspaces).

Let E_1 and E_2 be two subspaces of a $\mathbb{K}$-vector space E. The **sum** of E_1 and E_2, denoted $E_1 + E_2$, is the set

$$E_1 + E_2 = \{x \in E : \exists x_1 \in E_1, \exists x_2 \in E_2 \text{ such that } x = x_1 + x_2\}.$$

Example 6.13. Let $E_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $E_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ be subspaces of $\mathbb{R}^2$. For any $(x, y) \in \mathbb{R}^2$,

$$(x, y) = \underbrace{(0, y)}_{\in E_1} + \underbrace{(x, 0)}_{\in E_2},$$

so $(x, y) \in E_1 + E_2$. Thus $E_1 + E_2 = \mathbb{R}^2$.

Proposition 6.7.

The sum of two subspaces E_1 and E_2 (of the same $\mathbb{K}$-vector space E) is a subspace of E containing $E_1 \cup E_2$, i.e.,

$$E_1 \cup E_2 \subset E_1 + E_2.$$

Direct Sum of Two Subspaces

Definition 6.14 (Direct Sum).

Let E_1 and E_2 be two subspaces of the same $\mathbb{K}$-vector space E. The sum $E_1 + E_2$ is said to be **direct** if $E_1 \cap E_2 = \{0_E\}$. We write $E_1 \oplus E_2$.

Proposition 6.8.

Let E_1 and E_2 be two subspaces of the same $\mathbb{K}$-vector space E. The sum $E_1 + E_2$ is direct if and only if every vector $x \in E_1 + E_2$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in E_1$, $x_2 \in E_2$.

Example 6.14. Let $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ and $F_2 = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ be subspaces of $\mathbb{R}^3$.

- For any $(x, y, z) \in \mathbb{R}^3$,

  $$(x, y, z) = \underbrace{(0, y, z)}_{\in F_1} + \underbrace{(x, 0, 0)}_{\in F_2},$$

  so $F_1 + F_2 = \mathbb{R}^3$.

- Let $(x, y, z) \in F_1 \cap F_2$. Then $(x, y, z) \in F_1 \Rightarrow x = 0$, and $(x, y, z) \in F_2 \Rightarrow y = z = 0$. Thus $(x, y, z) = (0, 0, 0) = 0_{\mathbb{R}^3}$, so $F_1 \cap F_2 = \{0\}$.

Therefore, $\mathbb{R}^3 = F_1 \oplus F_2$.

Generating Families, Free Families, and Bases

Definition 6.15 (Linear Independence, Generating Families, Bases).

Let E be a vector space and $e_1, e_2, \dots, e_n$ elements of E.

1. $\{e_1, e_2, \dots, e_n\}$ are said to be **linearly independent** (or **free**) if for all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$:

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0_E \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_{\mathbb{K}}.$$

   Otherwise, they are said to be **linearly dependent**.
2. $\{e_1, e_2, \dots, e_n\}$ is called a **generating family** of E, or that E is **generated** by $\{e_1, e_2, \dots, e_n\}$, if

$$\forall x \in E, \ \exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{K} \text{ such that } x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n.$$

3. If $\{e_1, e_2, \dots, e_n\}$ is a free and generating family of E, then $\{e_1, e_2, \dots, e_n\}$ is called a **basis** of E.

Example 6.15. On $\mathbb{R}^2$, let $u_1 = (1, 0)$, $u_2 = (1, -1)$. Then $\{u_1, u_2\}$ is a basis of $\mathbb{R}^2$.

- **Linear independence:** For $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\alpha_1 u_1 + \alpha_2 u_2 = 0 \Rightarrow (\alpha_1 + \alpha_2, -\alpha_2) = (0, 0) \Rightarrow \alpha_2 = 0, \ \alpha_1 = 0.$$

- **Generating:** For any $(x, y) \in \mathbb{R}^2$,

$$(x, y) = \alpha_1 u_1 + \alpha_2 u_2 = (\alpha_1 + \alpha_2, -\alpha_2) \Rightarrow \alpha_2 = -y, \ \alpha_1 = x + y.$$

Remark.

In a vector space E, any non-zero vector by itself is free.

Example 6.16. In the set of polynomials of degree at most 2 with real coefficients:

$$\mathbb{R}_2[x] = \{P(x) = a + bx + cx^2 : a, b, c \in \mathbb{R}\},$$

$\{p_1(x) = 1, \ p_2(x) = x, \ p_3(x) = x^2\}$ is a basis.

- **Linear independence:** Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\alpha + \beta x + \gamma x^2 = 0$. This implies $\alpha = \beta = \gamma = 0$.
- **Generating:** Any polynomial $a + bx + cx^2$ can be written as $a \cdot 1 + b \cdot x + c \cdot x^2$.

6.6 Linear Maps

Definition

Definition 6.16 (Linear Map).

Let E and F be two $\mathbb{K}$-vector spaces. A map $f : E \to F$ is called **linear** (or a **homomorphism**) if it satisfies:

$$\forall x, y \in E,\ \forall \lambda \in \mathbb{K},\ f(x+y) = f(x) + f(y) \quad \text{and} \quad f(\lambda x) = \lambda f(x).$$

Equivalently,

$$\forall x, y \in E,\ \forall \lambda, \mu \in \mathbb{K},\ f(\lambda x + \mu y) = \lambda f(x) + \mu f(y).$$

Remark.

The set of linear maps from E to F is denoted $\mathcal{L}(E, F)$.

Example 6.17. The map f defined by

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto f(x, y, z) = (2x + y, y - z) \end{aligned}$$

is linear. Indeed, for $(x, y, z), (x', y', z') \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned} f((x, y, z) + (x', y', z')) &= f(x + x', y + y', z + z') \\ &= (2(x + x') + (y + y'), (y + y') - (z + z')) \\ &= (2x + 2x' + y + y', y + y' - z - z') \\ &= (2x + y, y - z) + (2x' + y', y' - z') \\ &= f(x, y, z) + f(x', y', z') \end{aligned}$$

and

$$f(\lambda(x, y, z)) = f(\lambda x, \lambda y, \lambda z) = (2\lambda x + \lambda y, \lambda y - \lambda z) = \lambda(2x + y, y - z) = \lambda f(x, y, z).$$

Remark.

Not all maps are linear. For example, $f(x) = x^2$ is not linear.

Definition 6.17 (Isomorphism, Endomorphism, Automorphism).

Let E and F be two $\mathbb{K}$-vector spaces, and let $f \in \mathcal{L}(E, F)$.

1. f is an **isomorphism** from E to F if f is bijective.
2. f is an **endomorphism** if $E = F$ (i.e., $f : E \to E$).
3. f is an **automorphism** if f is both an isomorphism and an endomorphism.

Example 6.18. The map f defined by

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = -2x \end{aligned}$$

is an automorphism. It is linear, and its inverse is $f^{-1}(x) = -\frac{1}{2}x$.

The zero map $0_{\mathcal{L}(E,F)}$ is given by $f(x) = 0_F$ for all $x \in E$. The identity map id_E is given by $id_E(x) = x$ for all $x \in E$.

Proposition 6.9 (Properties of Linear Maps).

Let f be a linear map from E to F. Then:

$$1)\ f(0_E) = 0_F, \qquad 2)\ \forall x \in E : f(-x) = -f(x).$$

Proof. For $x \in E$,

$$1)\ f(0_E) = f(0_{\mathbb{K}} \cdot 0_E) = 0_{\mathbb{K}} \cdot f(0_E) = 0_F,$$
$$2)\ f(-x) = f((-1) \cdot x) = (-1) \cdot f(x) = -f(x).$$

□

Kernel, Image, and Rank of a Linear Map

Definition 6.18 (Kernel and Image).

Let $f : E \to F$ be a linear map.

1. The set $f(E)$ is called the **image** of f and is denoted $\operatorname{Im} f$:

$$\operatorname{Im} f = \{f(x) : x \in E\} \subset F.$$

2. The set $f^{-1}(\{0_F\})$ is called the **kernel** of f and is denoted $\operatorname{Ker} f$:

$$\operatorname{Ker} f = \{x \in E : f(x) = 0_F\} \subset E.$$

Example 6.19. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x - y$.

- $\operatorname{Ker} f = \{(x, y) \in \mathbb{R}^2 : x - y = 0\} = \{(x, x) : x \in \mathbb{R}\} = \operatorname{Span}\{(1, 1)\}$, so $\dim \operatorname{Ker} f = 1$.
- $\operatorname{Im} f = \{x - y : (x, y) \in \mathbb{R}^2\} = \mathbb{R}$, so $\dim \operatorname{Im} f = 1$.

Proposition 6.10.

Let $f : E \to F$ be a linear map. Then:

1. $\operatorname{Im} f$ is a subspace of F.
2. $\operatorname{Ker} f$ is a subspace of E.

Definition 6.19 (Rank).

Let $f : E \to F$ be a linear map. If $\dim \operatorname{Im} f = n < +\infty$, then n is called the **rank** of f and is denoted $\operatorname{rg}(f)$.

Proposition 6.11 (Injectivity and Surjectivity).

Let $f : E \to F$ be a linear map. Then:

- *f is **surjective** if and only if $\operatorname{Im} f = F$.*
- *f is **injective** if and only if $\operatorname{Ker} f = \{0_E\}$.*

Example 6.20. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (y, x)$.

$$\operatorname{Im} f = \{(y, x) : (x, y) \in \mathbb{R}^2\} = \mathbb{R}^2, \quad \operatorname{Ker} f = \{(0, 0)\}.$$

Thus f is bijective.

Linear Maps on Finite-Dimensional Spaces

Proposition 6.12 (Linear Maps are Determined by Images of Basis).

Let E and F be two $\mathbb{K}$-vector spaces, and let f and g be two linear maps from E to F. If E has finite dimension n and $\{e_1, e_2, \ldots, e_n\}$ is a basis of E, then

$$\forall k \in \{1, 2, \ldots, n\} : f(e_k) = g(e_k) \Leftrightarrow \forall x \in E : f(x) = g(x).$$

Proof. The implication ($\Leftarrow$) is obvious. For ($\Rightarrow$), since E is generated by $\{e_1, \ldots, e_n\}$, any $x \in E$ can be written as $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$. Then

$$f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n) = \lambda_1 g(e_1) + \cdots + \lambda_n g(e_n) = g(x).$$

□

Example 6.21. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(1, 0) = -1$ and $f(0, 1) = 4$. Then for any $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = xf(1, 0) + yf(0, 1) = -x + 4y.$$

Theorem 6.13 (Rank-Nullity Theorem).

Let $f : E \to F$ be a linear map, with E finite-dimensional. Then

$$\dim E = \dim \operatorname{Ker} f + \dim \operatorname{Im} f.$$

Example 6.22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = -x + 5y$.

$$\operatorname{Ker} f = \{(x, y) \in \mathbb{R}^2 : -x + 5y = 0\} = \{5y, y) : y \in \mathbb{R}\} = \operatorname{Span}\{(5, 1)\},$$

so $\dim \operatorname{Ker} f = 1$. Since $\dim \mathbb{R}^2 = 2$, we have $\dim \operatorname{Im} f = 2 - 1 = 1$.

Theorem 6.14 (Characterization of Isomorphisms).

Let $f : E \to F$ be a linear map with $\dim E = \dim F = n$. Then the following are *equivalent:*

$$\begin{aligned} f \textit{ is an isomorphism} &\Leftrightarrow f \textit{ is surjective} \Leftrightarrow \dim \operatorname{Im} f = \dim F \\ &\Leftrightarrow f \textit{ is injective} \Leftrightarrow \operatorname{Im} f = F \\ &\Leftrightarrow \dim \operatorname{Ker} f = 0 \Leftrightarrow \operatorname{Ker} f = \{0_E\}. \end{aligned}$$

Remark.

From this theorem, if f is an isomorphism from E to F and E is finite-dimensional, then necessarily $\dim E = \dim F$. In other words, if $\dim E \neq \dim F$, then f cannot be an isomorphism.

Example 6.23. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (2x - y, x)$.

$$\operatorname{Ker} f = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0 \text{ and } x = 0\} = \{(0, 0)\}.$$

Since $\dim \mathbb{R}^2 = 2$ and $\operatorname{Ker} f = \{0\}$, f is an isomorphism.

6.7 Exercises and Solutions

Exercise 6.1.

On $G =]-1,1[$, define the internal law $*$ as:

$$\forall (x,y) \in G \times G : x * y = \frac{x+y}{1+xy}.$$

Show that $(G, *)$ is a commutative group.

Exercise 6.2.

Consider $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that $(\mathbb{Z}[\sqrt{2}], +, \times)$ is a ring.

Exercise 6.3.

Let $\mathbb{D}$ be the set of decimal numbers:

$$\mathbb{D} = \left\{ \frac{n}{10^k} : n \in \mathbb{Z}, k \in \mathbb{N} \right\}.$$

1. Show that $(\mathbb{D}, +, \times)$ is a ring.
2. What are its invertible elements?

Exercise 6.4.

Let $d \in \mathbb{N}$ such that $\sqrt{d} \notin \mathbb{Q}$. Define

$$\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}.$$

Show that $(\mathbb{Q}[\sqrt{d}], +, \times)$ is a field.

Exercise 6.5.

Define two laws $\oplus$ and $\otimes$ on $\mathbb{R}^2$ as follows:

$$\forall (x,y),(x',y') \in \mathbb{R}^2 : (x,y) \oplus (x',y') = (x+x', y+y'),$$

$$\forall (x,y),(x',y') \in \mathbb{R}^2 : (x,y) \otimes (x',y') = (xx', yy').$$

Is $(\mathbb{R}^2, \oplus, \otimes)$ a commutative field?

Exercise 6.6.

Consider the subset E of $\mathbb{R}^3$ defined by:

$$E = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

1. Show that E is a subspace of $\mathbb{R}^3$
2. Give a basis of E

Exercise 6.7.

Let $E = \{(x, y, z) \in \mathbb{R}^3 : x + y - 2z = 2x - y - z = 0\}$ be a subset of $\mathbb{R}^3$

1. Show that E is a subspace of $\mathbb{R}^3$
2. Find a generating family of E and extract a basis.
3. Let $F = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$ be a subspace of $\mathbb{R}^3$
   a) Find a generating family of F
   b) Do we have $E \oplus F = \mathbb{R}^3$

Exercise 6.8.

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$f(x, y, z) = (x + y + z, 2x + y - z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

1. Show that f is linear.
2. Find Ker f

Exercise 6.9.

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$f(x, y) = (x - y, -3x + 3y).$$

1. Show that f is a linear map.
2. Give a basis of its kernel and a basis of its image.
3. Find $f \circ f$

Exercise 6.10.

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$f(x, y, z) = (-2x + y + z, x - 2y + z, x + y - 2z).$$

1. Show that f is a linear map.
2. Give a basis of Ker f , and deduce $\dim \operatorname{Im} f$.
3. Give a basis of Im f .

Exercise 6.11.

Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be defined for all $(x, y, z, t) \in \mathbb{R}^4$ by:

$$f(x, y, z, t) = (x - 2y, x - 2y, 0, x - y - z - t).$$

1. Show that f is a linear map.
2. Find the kernel and the image of f .
3. Do we have $\operatorname{Ker} f \oplus \operatorname{Im} f = \mathbb{R}^4$?

Solution — Exercise 6.1 — Commutative Group on $]-1,1[$.

We verify the five group axioms for $(G,*)$ where $x * y = \dfrac{x+y}{1+xy}$.

Step 1 — Internal law (closure).
Let $x, y \in]-1,1[$. We must show $|x*y| < 1$, i.e., $\left|\dfrac{x+y}{1+xy}\right| < 1$.

Since $-1 < x < 1$ and $-1 < y < 1$, we have $(1-x)(1-y) > 0$ and $(1+x)(1+y) > 0$, so

$$(1-x^2)(1-y^2) > 0 \implies 1 - x^2 - y^2 + x^2y^2 > 0 \implies (x+y)^2 < (1+xy)^2.$$

Taking square roots: $|x+y| < |1+xy|$, which gives $|x*y| < 1$. Hence $x*y \in]-1,1[$.

Step 2 — Associativity.
A direct computation shows:

$$(x*y)*z = \frac{\frac{x+y}{1+xy} + z}{1 + \frac{x+y}{1+xy} \cdot z} = \frac{x+y+z+xyz}{1+xy+yz+xz} = x*(y*z).$$

So $*$ is associative.

Step 3 — Identity element.
For any $x \in G$:

$$x * 0 = \frac{x+0}{1+x\cdot 0} = x \quad \text{and} \quad 0 * x = \frac{0+x}{1+0\cdot x} = x.$$

Therefore $0 \in]-1,1[$ is the identity element.

Step 4 — Inverse element.
For $x \in G$, set $x^{-1} = -x$. Since $|-x| = |x| < 1$, we have $-x \in G$, and:

$$x * (-x) = \frac{x+(-x)}{1+x(-x)} = \frac{0}{1-x^2} = 0.$$

So every element has an inverse $-x \in G$.

Step 5 — Commutativity.

$$x * y = \frac{x+y}{1+xy} = \frac{y+x}{1+yx} = y * x.$$

Conclusion. $(G,*)$ is a commutative (abelian) group. $\square$

Solution — Exercise 6.2 — $\mathbb{Z}[\sqrt{2}]$ is a Ring.

We show $\mathbb{Z}[\sqrt{2}]$ is a subring of $(\mathbb{R},+,\times)$ using the subring criterion. Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ with $a,b,c,d \in \mathbb{Z}$.

Step 1 — Non-empty.
$0 = 0 + 0 \cdot \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. ✓

Step 2 — Stable under subtraction.

$$x - y = (a - c) + (b - d)\sqrt{2} \in \mathbb{Z}[\sqrt{2}],$$

since $a - c$, $b - d \in \mathbb{Z}$.

Step 3 — Stable under multiplication.

$$x \times y = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Z}[\sqrt{2}],$$

since $ac + 2bd$, $ad + bc \in \mathbb{Z}$.

Step 4 — Multiplicative identity.

$$1 = 1 + 0 \cdot \sqrt{2} \in \mathbb{Z}[\sqrt{2}].$$

Conclusion. By the subring criterion, $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$, hence a ring. □

Solution — Exercise 6.3 — The Ring $\mathbb{D}$ and its Invertible Elements.

Part 1 — $(\mathbb{D}, +, \times)$ is a ring.

Let $x = \dfrac{n}{10^k}$ and $y = \dfrac{m}{10^\ell}$ with $n, m \in \mathbb{Z}$, $k, \ell \in \mathbb{N}$.

Step 1 — Stable under subtraction.

$$x - y = \frac{n}{10^k} - \frac{m}{10^\ell} = \frac{n \cdot 10^\ell - m \cdot 10^k}{10^{k+\ell}} \in \mathbb{D}.$$

Step 2 — Stable under multiplication.

$$x \times y = \frac{n}{10^k} \cdot \frac{m}{10^\ell} = \frac{nm}{10^{k+\ell}} \in \mathbb{D}.$$

Step 3 — Multiplicative identity.

$$1 = \frac{10^k}{10^k} \in \mathbb{D}.$$

Hence $\mathbb{D}$ is a subring of $(\mathbb{Q}, +, \times)$, so it is a ring.

Part 2 — Invertible elements of $\mathbb{D}$.

Step 1 — Necessary condition.
Let $x = \dfrac{n}{10^k} \in \mathbb{D}$ be invertible with inverse $y = \dfrac{m}{10^\ell} \in \mathbb{D}$. Then:

$$xy = 1 \implies nm = 10^{k+\ell} = 2^{k+\ell} \cdot 5^{k+\ell}.$$

So n must divide a power of 10, meaning $n = \pm 2^p \cdot 5^q$ for some $p, q \in \mathbb{N}$.

Step 2 — Sufficient condition.
Conversely, if $x = \dfrac{\pm 2^p \cdot 5^q}{10^k}$, set $y = \dfrac{\pm 10^k}{2^p \cdot 5^q} = \dfrac{\pm 10^k}{2^p \cdot 5^q}$. Since $10^k = 2^k \cdot 5^k$, we get $y = \pm 2^{k-p} \cdot 5^{k-q}$ if $k \geq p, k \geq q$, which belongs to $\mathbb{D}$, and $xy = 1$.

Conclusion. The invertible elements of $\mathbb{D}$ are exactly the numbers of the form $\pm \dfrac{2^p \cdot 5^q}{10^k}$ with $p, q, k \in \mathbb{N}$.

Solution — Exercise 6.4 — $\mathbb{Q}[\sqrt{d}]$ is a Field.

We show $\mathbb{Q}[\sqrt{d}]$ is a subfield of $(\mathbb{R}, +, \times)$. Let $x = a_1 + b_1\sqrt{d}$, $y = a_2 + b_2\sqrt{d}$ with $a_i, b_i \in \mathbb{Q}$.

Step 1 — Non-empty. $0 = 0 + 0 \cdot \sqrt{d} \in \mathbb{Q}[\sqrt{d}]$. ✓

Step 2 — Stable under subtraction.

$$x - y = (a_1 - a_2) + (b_1 - b_2)\sqrt{d} \in \mathbb{Q}[\sqrt{d}],$$

since $a_1 - a_2$, $b_1 - b_2 \in \mathbb{Q}$.

Step 3 — Inverse for multiplication.
Let $y = a_2 + b_2\sqrt{d} \neq 0$. Note that $a_2^2 - b_2^2 d \neq 0$ because $\sqrt{d} \notin \mathbb{Q}$ (if $a_2^2 = b_2^2 d$ and $b_2 \neq 0$ then $\sqrt{d} = |a_2/b_2| \in \mathbb{Q}$, contradiction). So:

$$y^{-1} = \frac{1}{a_2 + b_2\sqrt{d}} = \frac{a_2 - b_2\sqrt{d}}{a_2^2 - b_2^2 d} = \frac{a_2}{a_2^2 - b_2^2 d} + \frac{-b_2}{a_2^2 - b_2^2 d}\sqrt{d} \in \mathbb{Q}[\sqrt{d}].$$

Step 4 — Stable under multiplication.
Since $x \times y^{-1} \in \mathbb{Q}[\sqrt{d}]$ by a similar computation (product of two elements of $\mathbb{Q}[\sqrt{d}]$).

Conclusion. $\mathbb{Q}[\sqrt{d}]$ is a subfield of $\mathbb{R}$, hence a field. □

Solution — Exercise 6.5 — $(\mathbb{R}^2, \oplus, \otimes)$ is Not a Field.

Step 1 — $(\mathbb{R}^2, \oplus, \otimes)$ is a commutative ring.

- $(\mathbb{R}^2, \oplus)$ is an abelian group: identity $(0, 0)$, inverse $-(x, y) = (-x, -y)$.
- $\otimes$ is associative and commutative: $(x, y) \otimes (x', y') = (xx', yy') = (x'x, y'y)$.
- $\otimes$ distributes over $\oplus$: $(x, y) \otimes ((x', y') \oplus (x'', y'')) = (xx' + xx'', yy' + yy'')$.
- The identity for $\otimes$ is $(1, 1)$.

Step 2 — It is NOT a field.
A field requires every *non-zero* element to have a multiplicative inverse. Consider $(2,0) \neq (0,0)$. Its inverse (x', y') would satisfy $(2x', 0 \cdot y') = (1,1)$, i.e., $2x' = 1$ **and** $0 = 1$. The second equation is impossible.

More generally, any element of the form $(x, 0)$ or $(0, y)$ with $x, y \neq 0$ has no inverse under $\otimes$.

Conclusion. $(\mathbb{R}^2, \oplus, \otimes)$ is a commutative ring but **not** a field.

Solution — Exercise 6.6 — Subspace and Basis of $E = \{x + y + z = 0\}$.

Part 1 — E is a subspace of $\mathbb{R}^3$.

Step 1 — Zero vector. $\mathbf{0} = (0,0,0)$ satisfies $0 + 0 + 0 = 0$, so $\mathbf{0} \in E$.

Step 2 — Stability under linear combinations.
Let $(x,y,z), (x',y',z') \in E$ and $\lambda, \mu \in \mathbb{R}$. Then:

$$\lambda(x,y,z) + \mu(x',y',z') = (\lambda x + \mu x',\ \lambda y + \mu y',\ \lambda z + \mu z').$$

The sum of coordinates:

$$(\lambda x + \mu x') + (\lambda y + \mu y') + (\lambda z + \mu z') = \lambda \underbrace{(x + y + z)}_{=0} + \mu \underbrace{(x' + y' + z')}_{=0} = 0.$$

So $\lambda(x,y,z) + \mu(x',y',z') \in E$. Hence E is a subspace.

Part 2 — Basis of E.

Step 3 — Parametric form.
From $x + y + z = 0$ we get $z = -x - y$, so:

$$E = \{(x, y, -x - y) \mid x, y \in \mathbb{R}\} = \{x(1,0,-1) + y(0,1,-1) \mid x, y \in \mathbb{R}\}.$$

Step 4 — Linear independence.
Set $\alpha(1,0,-1) + \beta(0,1,-1) = (0,0,0)$: this gives $\alpha = 0, \beta = 0, -\alpha - \beta = 0$. Independent. ✓

Conclusion. $\mathcal{B} = \{(1,0,-1),\ (0,1,-1)\}$ is a basis of E, so $\dim E = 2$.

Solution — Exercise 6.7 — Subspaces E, F and Direct Sum.

Let $E = \{(x,y,z) \in \mathbb{R}^3 : x + y - 2z = 0 \text{ and } 2x - y - z = 0\}$.

Part 1 — E is a subspace.

E is the intersection of two hyperplanes (kernels of linear maps), so it is a subspace. Explicitly:

- $\mathbf{0} \in E$: $0 + 0 - 0 = 0$ and $0 - 0 - 0 = 0$. ✓
- If $u, v \in E$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda u + \mu v$ satisfies both equations by linearity.

Part 2 — Basis of E.

Step 1 — Solve the system.

$$\begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases}$$

Adding: $3x - 3z = 0 \Rightarrow x = z$. Substituting into the first: $z + y - 2z = 0 \Rightarrow y = z$.

Step 2 — Parametric form.

$$E = \{(z, z, z) \mid z \in \mathbb{R}\} = \text{Span}\{(1, 1, 1)\}.$$

Conclusion. $\mathcal{B}_E = \{u_1 = (1, 1, 1)\}$ is a basis of E, and $\dim E = 1$.

Part 3 — Basis of $F = \{x + y - z = 0\}$.

Step 3 — From $z = x + y$:

$$F = \{(x, y, x + y) \mid x, y \in \mathbb{R}\} = \{x(1, 0, 1) + y(0, 1, 1)\}.$$

So $\mathcal{B}_F = \{u_2 = (1, 0, 1),\ u_3 = (0, 1, 1)\}$ is a basis of F, and $\dim F = 2$.

Part 4 — Direct sum $E \oplus F = \mathbb{R}^3$?

Step 4 — Check $E \cap F = \{0\}$.
A vector in $E \cap F$ must have the form (z, z, z) and satisfy $z + z - z = 0$, i.e., $z = 0$. So $E \cap F = \{(0, 0, 0)\}$.

Step 5 — Dimension check. $\dim E + \dim F = 1 + 2 = 3 = \dim \mathbb{R}^3$.

Step 6 — Linear independence of $\{u_1, u_2, u_3\}$.

$$\lambda_1(1, 1, 1) + \lambda_2(1, 0, 1) + \lambda_3(0, 1, 1) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{cases} \quad \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Conclusion. $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, hence $E \oplus F = \mathbb{R}^3$.

Solution — Exercise 6.8 — Linear Map $f : \mathbb{R}^3 \to \mathbb{R}^2$, Kernel.

Let $f(x, y, z) = (x + y + z,\ 2x + y - z)$.

Part 1 — f is linear.

Step 1 — Let $u = (x, y, z)$, $v = (x', y', z')$, $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} f(\alpha u + \beta v) &= \big(\alpha x + \beta x' + \alpha y + \beta y' + \alpha z + \beta z', \\ &\qquad 2(\alpha x + \beta x') + (\alpha y + \beta y') - (\alpha z + \beta z')\big) \\ &= \alpha(x + y + z,\, 2x + y - z) + \beta(x' + y' + z',\, 2x' + y' - z') \\ &= \alpha f(u) + \beta f(v). \end{aligned}$$

Hence f is linear.

Part 2 — Kernel of f.

Step 2 — Set up system.

$$\operatorname{Ker} f = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{cases} x + y + z = 0 \\ 2x + y - z = 0 \end{cases} \right\}.$$

Step 3 — Solve.
Subtracting the first from the second: $x - 2z = 0 \Rightarrow x = 2z$.
Substituting into the first: $2z + y + z = 0 \Rightarrow y = -3z$.

So every solution has the form $(x, y, z) = (2z, -3z, z) = z(2, -3, 1)$.

Step 4 — Conclusion.

$$\operatorname{Ker} f = \operatorname{Span}\{(2, -3, 1)\}, \quad \dim \operatorname{Ker} f = 1.$$

By the rank-nullity theorem: $\dim \operatorname{Im} f = 3 - 1 = 2$, so f is surjective.

Solution — Exercise 6.9 — Linear Map $f : \mathbb{R}^2 \to \mathbb{R}^2$, Kernel, Image, Composition.

Let $f(x, y) = (x - y,\, -3x + 3y)$.

Part 1 — f is linear.

Step 1 — $f(\alpha(x, y) + \beta(x', y')) = f(\alpha x + \beta x', \alpha y + \beta y') = ((\alpha x + \beta x') - (\alpha y + \beta y'), -3(\alpha x + \beta x') + 3(\alpha y + \beta y')) = \alpha(x - y, -3x + 3y) + \beta(x' - y', -3x' + 3y') = \alpha f(x, y) + \beta f(x', y')$. ✓

Part 2 — Kernel and Image.

Step 2 — Kernel.

$$\operatorname{Ker} f = \{(x, y) : x - y = 0 \text{ and } -3x + 3y = 0\} = \{(x, y) : x = y\} = \operatorname{Span}\{(1, 1)\}.$$

Basis of $\operatorname{Ker} f$: $\{(1, 1)\}$.

Step 3 — Image.

$$\operatorname{Im} f = \{(x-y,\ -3(x-y)) \mid x,y \in \mathbb{R}\} = \{(t,-3t) \mid t \in \mathbb{R}\} = \operatorname{Span}\{(1,-3)\}.$$

Basis of $\operatorname{Im} f$: $\{(1,-3)\}$.

Part 3 — Compute $f \circ f$.

Step 4 — Verify $f \circ f = 4f$.

$$\begin{aligned}(f \circ f)(x,y) &= f(x-y,\ -3x+3y)\\ &= \big((x-y)-(-3x+3y),\ -3(x-y)+3(-3x+3y)\big)\\ &= (4x-4y,\ -12x+12y)\\ &= 4(x-y,\ -3x+3y) = 4\,f(x,y).\end{aligned}$$

Conclusion. $f \circ f = 4f$. In particular, f is a *dilatation* in the sense that applying f twice equals scaling f by 4.

Solution — Exercise 6.10 — Linear Map $f : \mathbb{R}^3 \to \mathbb{R}^3$, Kernel, Image.

Let $f(x,y,z) = (-2x+y+z,\ x-2y+z,\ x+y-2z)$.

Part 1 — f is linear.

Step 1 — Each component of f is a linear combination of x,y,z, so f is linear (linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are exactly those defined by matrix multiplication). ✓

Part 2 — Kernel of f.

Step 2 — Set up and solve.

$$\begin{cases}-2x+y+z=0\\ x-2y+z=0\\ x+y-2z=0\end{cases}$$

Adding all three equations: $0=0$ (dependent system). From equations (1) and (2): adding gives $-x-y+2z=0$, i.e., $x+y=2z$. Equation (3) says $x+y=2z$. These are identical, so the system reduces to one independent equation. Set $z=t$: then $x+y=2t$. From (1): $-2x+y+t=0 \Rightarrow y=2x-t$. Substituting: $x+(2x-t)=2t \Rightarrow 3x=3t \Rightarrow x=t$, so $y=t$.

Step 3 — Describe $\operatorname{Ker} f$ and find its basis.

$$\operatorname{Ker} f = \{(t,t,t) \mid t \in \mathbb{R}\} = \operatorname{Span}\{(1,1,1)\}.$$

Basis of $\operatorname{Ker} f$: $\{(1,1,1)\}$, and $\dim \operatorname{Ker} f = 1$.

Part 3 — Dimension and Basis of $\operatorname{Im} f$.

Step 4 — Rank-Nullity.

$$\dim \operatorname{Im} f = \dim \mathbb{R}^3 - \dim \operatorname{Ker} f = 3 - 1 = 2.$$

Step 5 — Generating family.

$$\operatorname{Im} f = \operatorname{Span}\{f(e_1), f(e_2), f(e_3)\} = \operatorname{Span}\{(-2,1,1),\ (1,-2,1),\ (1,1,-2)\}.$$

Note: $(-2,1,1) + (1,-2,1) + (1,1,-2) = (0,0,0)$, so the three are linearly dependent. But any two are independent.

Step 6 — Extract a basis.
Check $(-2,1,1)$ and $(1,-2,1)$ are independent: $\lambda(-2,1,1) + \mu(1,-2,1) = (0,0,0)$ gives $\lambda = \mu = 0$. ✓

Conclusion. A basis of $\operatorname{Im} f$ is $\{(-2,1,1),\ (1,-2,1)\}$.

Solution — Exercise 6.11 — Linear Map $f : \mathbb{R}^4 \to \mathbb{R}^4$, Kernel, Image, Direct Sum.

Let $f(x,y,z,t) = (x - 2y,\ x - 2y,\ 0,\ x - y - z - t)$.

Part 1 — f is linear.

Step 1 — Each component is a linear combination of x, y, z, t, so f is linear. ✓

Part 2 — Kernel and Image.

Step 2 — Kernel.

$$\operatorname{Ker} f = \left\{ (x,y,z,t) : \begin{cases} x - 2y = 0 \\ x - y - z - t = 0 \end{cases} \right\}.$$

From equation (1): $x = 2y$. Substituting into (2): $2y - y - z - t = 0 \Rightarrow t = y - z$. So the free variables are y and z:

$$(x,y,z,t) = (2y, y, z, y - z) = y(2,1,0,1) + z(0,0,1,-1).$$

Basis of $\operatorname{Ker} f$: $\mathcal{B}_{\text{ker}} = \{v_1 = (2,1,0,1),\ v_2 = (0,0,1,-1)\}$, and $\dim \operatorname{Ker} f = 2$.

Step 3 — Image.

$$\operatorname{Im} f = \{(x - 2y,\ x - 2y,\ 0,\ x - y - z - t) \mid (x,y,z,t) \in \mathbb{R}^4\}.$$

Setting $\lambda = x - 2y$ and $\mu = x - y - z - t$ (which are independent):

$$\operatorname{Im} f = \{\lambda(1,1,0,0) + \mu(0,0,0,1) \mid \lambda, \mu \in \mathbb{R}\}.$$

Basis of $\operatorname{Im} f$: $\mathcal{B}_{\text{im}} = \{w_1 = (1,1,0,0),\ w_2 = (0,0,0,1)\}$, and $\dim \operatorname{Im} f = 2$.

Verification: $\dim \operatorname{Ker} f + \dim \operatorname{Im} f = 2 + 2 = 4 = \dim \mathbb{R}^4$. ✓

Part 3 — Direct sum $\operatorname{Ker} f \oplus \operatorname{Im} f = \mathbb{R}^4$?

Step 4 — Check intersection.
A vector in $\operatorname{Ker} f \cap \operatorname{Im} f$ has the form $(2a, a, b, a-b)$ (from $\operatorname{Ker} f$) and also $(\lambda, \lambda, 0, \mu)$ (from $\operatorname{Im} f$). Equating components:

$$2a = \lambda, \quad a = \lambda, \quad b = 0, \quad a - b = \mu.$$

From the first two: $2a = a \Rightarrow a = 0$, then $b = 0, \lambda = 0, \mu = 0$. So $\operatorname{Ker} f \cap \operatorname{Im} f = \{0\}$. ✓

Step 5 — Verify $\{v_1, v_2, w_1, w_2\}$ is a basis of $\mathbb{R}^4$.

$$\lambda_1(2,1,0,1) + \lambda_2(0,0,1,-1) + \lambda_3(1,1,0,0) + \lambda_4(0,0,0,1) = (0,0,0,0)$$

System:

$$\begin{cases} 2\lambda_1 + \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_2 = 0 \\ \lambda_1 - \lambda_2 + \lambda_4 = 0 \end{cases} \Rightarrow \lambda_3 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_4 = 0.$$

The four vectors are linearly independent, hence form a basis of $\mathbb{R}^4$.

Conclusion. $\operatorname{Ker} f \oplus \operatorname{Im} f = \mathbb{R}^4$. □

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