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Course

ELECTROMAGNETIC PHENOMENA AND MATHEMATICAL MODELS IN FIELDS THEORY

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Contents

FOREWORD

In any precise and rigorous study of electrical systems, or more specifically electrotechnical systems, the formulation of mathematical equations that govern the spatial and temporal evolution of electromagnetic phenomena present in such devices constitutes the initial and essential approach. Accurately establishing these equations, which describe the underlying physical phenomena, is a fundamental preliminary step. This step is necessary both for a proper understanding of the functioning of these systems and for their modeling, as well as for enabling any required or anticipated predictive analysis. This approach is also highly relevant when aiming to optimize the physical and geometrical configuration of the systems in question.

In terms of electrotechnical applications relevant to our proposal, we may cite, by way of example, static or dynamic electrical machines operating in space, whether in rotational or linear motion, actuators, transducers, electromagnets, contactors, relays, and so on.

In line with this specific logic, and in order to achieve this well-defined objective, the present document is entirely devoted to the description and various formulations of the mathematical models used to compute electromagnetic fields in electrical systems falling within the aforementioned domain.

For this purpose, we will first present the local equations that describe electromagnetic fields, along with the magnetic vector potential formulations, in the static, harmonic dynamic, and time-transient dynamic cases. These equations will also be developed for the specific case of problems involving geometrically cylindrical and physically axisymmetrical systems, which are among the most frequently encountered applications in practice.

To generalize our study, the proposed models will be extended to voltage-fed systems. This configuration is closer to real-world conditions and is therefore commonly found in academic and research laboratories. It is essential, in the various formulations, to take into account the presence of electrical conductors that provide power to these applications.

I. ELECTROMAGNETIC EQUATIONS

I.1. Introduction

Thanks to James Clerk Maxwell, we have had, for over a century, the set of equations that govern electromagnetic phenomena. These equations must be considered in conjunction with the constitutive laws of the materials involved.

Accordingly, we focus on the development of models describing static and dynamic magnetic phenomena encountered in a wide range of electrotechnical applications. We present the local equations that define the fields, along with the magnetic vector potential formulations that enable the prediction of the electromagnetic behavior of the structures to be modeled. Special attention is given to the widely encountered case of cylindrically axisymmetrical structures, which will be addressed in detail.

I.2. Maxwell's Equations

Maxwell's equations are fundamental laws of electromagnetism and form its basic postulates.

In Maxwell’s formulation, four vector quantities characterize the electromagnetic fields (electric and magnetic). These quantities, which depend on both space and time, are the electric field **E**, the magnetic field **H**, the magnetic flux density **B**, and the electric displacement field **D**. These four quantities are governed by equations that express their spatiotemporal interdependence, as well as their relationship with the physical nature of matter more precisely, with the electromagnetic and thermal properties of the materials, also referred to as media. These equations are expressed as follows:

$$\mathrm{rot}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(I.1)}$$

$$\mathrm{rot}(\mathbf{H}) = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial \mathrm{t}} \qquad \text{(I.2)}$$

$$\mathrm{div}\left(\mathbf{B}\right) = 0 \qquad \text{(I.3)}$$

$$div\left(\mathbf{D}\right) = \rho \qquad \text{(I.4)}$$

Additional relations characterizing the different media must be added to the above equations. Thus, for isotropic media, we write:

- Constitutive relations of the medium:

$$\mathbf{D} = \varepsilon\, \mathbf{E} \qquad \text{(I.5)}$$

$$\mathbf{B} = \mu\, \mathbf{H} \qquad \text{(I.6)}$$

- Detailed expressions of current densities:

$$\mathbf{J} = \mathbf{J}_{\mathrm{s}} + \mathbf{J}_{\mathrm{ind}} \qquad \text{(I.7)}$$

$$\mathbf{J}_{\mathrm{ind}} = \sigma\mathbf{E} + \sigma\left(\mathbf{v} \wedge \mathbf{B}\right) \qquad \text{(I.8)}$$

The term $\sigma\mathbf{E}$ represents the current resulting from the electric field **E**, while the term $\sigma\left(\mathbf{v} \wedge \mathbf{B}\right)$ represents the current induced by motion.

Figure 1 shows that, at the interface between two media corresponding respectively to indices 1 and 2, the following boundary conditions apply:

$$\mathbf{n}.\mathbf{B}_1 = \mathbf{n}.\mathbf{B}_2 \qquad \text{(I.9)}$$

$$\mathbf{n}.\mathbf{J}_1 = \mathbf{n}.\mathbf{J}_2 \qquad \text{(I.10)}$$

$$\mathbf{E}_1 \wedge \mathbf{n} = \mathbf{E}_2 \wedge \mathbf{n} \qquad (I.11)$$

$$\mathbf{H}_1 \wedge \mathbf{n} - \mathbf{H}_2 \wedge \mathbf{n} = \mathbf{K} \qquad (I.12)$$

n is the unit normal vector to the interface between the two media 1 and 2, and the surface current density at the boundary between the two media.

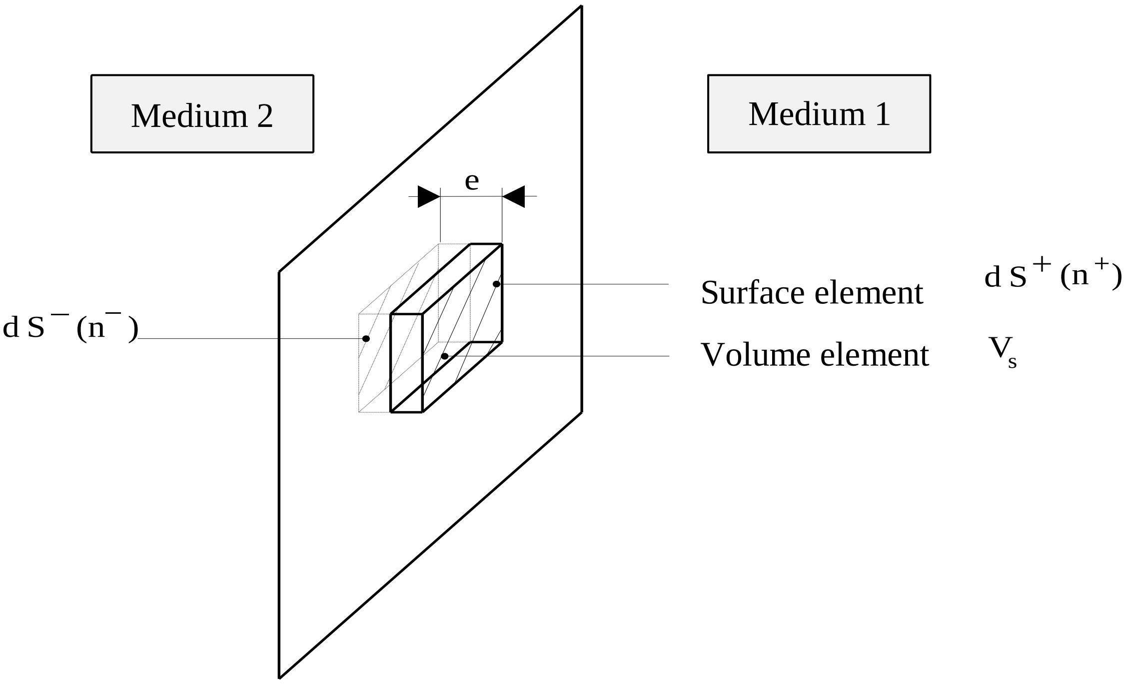

Fig. 1. Physical-geometrical interface.

Relations (I.9), (I.10), and (I.11) respectively show the continuity of the normal component of the magnetic flux density**B**, the continuity of the normal component of the current density vector**J**, and the continuity of the tangential component of the electric field vector**E**, while relation (I.12) expresses the discontinuity of the tangential components of the magnetic field**H**.

The set of equations presented so far provides a comprehensive description of all electromagnetic phenomena. However, in practice and in most cases, these equations cannot be solved directly. Moreover, depending on the specific devices under study, certain phenomena may be negligible; as a result, the equations can be decoupled, leading to simplified models.

I.3. Simplifying Assumptions

I.3.1. Displacement Currents

The electromagnetic structures under study can be characterized by dielectric properties similar to those of vacuum. Moreover, the operating frequencies are relatively low. Under these conditions, displacement currents are weak and can therefore be neglected.

I.3.2. Induced Currents and Imposed Currents

The term **J**encompasses all current density components. These currents may be induced currents (in the inductor or the load) or imposed currents (in the inductor). Formally, equation (I.2) can be rewritten as follows:

$$\mathrm{rot}(\mathbf{H}) = \mathbf{J}_{\mathrm{s}} + \mathbf{J}_{\mathrm{ind}} \qquad \text{(I.13)}$$

With the assumptions thus formulated, the Maxwell equations to be solved are:

$$\mathrm{rot}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} \qquad \text{(I.14)}$$

$$\mathrm{rot}(\mathbf{H}) = \mathbf{J} \qquad \text{(I.15)}$$

$$\mathrm{div}(\mathbf{B}) = 0 \qquad \text{(I.16)}$$

$$\mathrm{div}(\mathbf{D}) = 0 \qquad \text{(I.17)}$$

Given that:

$$\mathbf{B} = \mu\ \mathbf{H} \qquad \text{(I.18)}$$

$$\mathbf{J} = \mathbf{J}_{\mathrm{s}} + \mathbf{J}_{\mathrm{ind}} \qquad \text{(I.19)}$$

$$\mathbf{J}_{\mathrm{ind}} = \sigma\mathbf{E} + \sigma\left(\mathbf{v} \wedge \mathbf{B}\right) \text{(I.20)}$$

$\mathbf{J}_{\mathrm{s}}$ is the current density vector imposed on the inductor, and is zero elsewhere.

II. FORMULATIONS USING THE MAGNETIC VECTOR POTENTIAL

As previously stated, the system of equations (I.14, I.15, I.16, and I.17) involves many unknowns, making it poorly suited for numerical resolution. The use of the magnetic vector potential **A** and the electric scalar potential Φ helps condense and reduce the number of unknowns. These methods are now well known and widely used in the field of electromagnetic field computation. In this section, we present the magnetostatic and magnetodynamic formulations that describe our system.

II.1. Magnetostatic Equation

Figure 2 illustrates a typical model of electromagnetic devices considered in most studies in Electromagnetic Field Theory and in modeling work carried out to date. It consists of current-carrying coils (Inductor), where the current (or voltage) is imposed, of ferromagnetic regions that conduct induced currents (Load), and of the surrounding air region (Air).

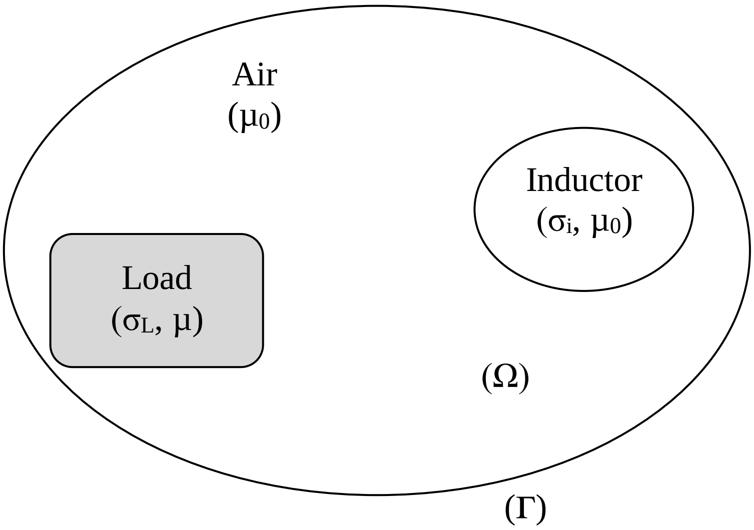

Fig. 2. Typical model of an electromagnetic problem inelectrotechnics domain.

In magnetostatics, it is assumed that the magnetic field is produced by time-independent current sources.

From equation (I.16), the magnetic vector potential A is defined, and the equations to be solved in this case are:

$$\mathbf{B} = \mathbf{rot}(\mathbf{A}) \tag{II.1}$$

$$\mathbf{rot}(\mathbf{H}) = \mathbf{J}_S \tag{II.2}$$

$$\mathbf{B} = \mu\,\mathbf{H} \tag{II.3}$$

In the most general case, there exists an infinite number of vector potentials A that can satisfy relation (II.1). Indeed, since the curl of the gradient of any scalar function f is zero, any vector of the form $\mathbf{A} + \mathbf{grad}(\mathrm{f}\,)$ also satisfies relation (II.1). Therefore, it is necessary to impose a gauge condition to ensure the uniqueness of the solution. This is not required in the particular case of 2D axisymmetrical cylindrical systems, where the Coulomb gauge condition, given by $div(\mathbf{A}) = 0$, is automatically satisfied, and the issue of solution uniqueness does not arise.

Substituting equations (II.1) and (II.3) into (II.2) yields the electromagnetic equation in the magnetostatic case, expressed by the following system of equations:

$$\begin{cases} \boldsymbol{rot}(\nu\,\boldsymbol{rot}(\boldsymbol{A})) = \boldsymbol{J}_s \\ \mathbf{div}(\boldsymbol{A}) = 0 \end{cases} \tag{II.4}$$

where $\nu = \dfrac{1}{\mu}$ denotes the magnetic reluctivity.

Depending on the geometry of the problem to be addressed, the unknowns will therefore be the non-zero components of the magnetic vector potential.

The boundary conditions must be expressed in terms of the magnetic vector potential. The tangential component of A is set to zero at infinity and along the axis of rotational symmetry. On the planes of geometric and magnetic symmetry, the condition $\frac{\partial \mathbf{A}}{\partial \mathbf{n}} = 0$ is imposed, which reflects the fact that the flux lines are orthogonal to these planes.

In certain cases, one encounters a problem in which the current sources (voltage) vary with time. In this situation, the term $\partial \mathbf{B} / \partial \mathrm{t}$ is no longer zero, and the magnetic and electric fields become coupled through the presence of induced currents (Eddy currents). In the following, we will extend the concept of the magnetic vector potential to the dynamic case, where the sources depend arbitrarily on time.

II.2. Magnetodynamic Equation

The equation we are about to establish applies to electromagnetic devices in which induced currents arise due to time-varying fields.

By replacing **B**with the term $\mathbf{rot}(\mathbf{A})$, equation (I.14) becomes:

$$\mathbf{rot}\left\{\mathbf{E} + \frac{\partial \mathbf{A}}{\partial \mathrm{t}}\right\} = 0 \qquad \text{(II.5)}$$

From this relation, an electric scalar potential Φ can be defined such that:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial \mathrm{t}} = -\mathbf{grad}(\Phi) \qquad \text{(II.6)}$$

The combination of equations (I.19), (I.20), and (II.6) gives:

$$\sigma\mathbf{E} = \mathbf{J} - \sigma\left(\mathbf{v} \wedge \mathbf{B}\right) - \mathbf{J}_s = -\sigma\left\{\frac{\partial \mathbf{A}}{\partial t} + \mathbf{grad}(\Phi)\right\} \qquad \text{(II.7)}$$

Hence:

$$\mathbf{J} = -\sigma\left\{\frac{\partial \mathbf{A}}{\partial t} - \left(\mathbf{v} \wedge \mathbf{B}\right) + \mathbf{grad}(\Phi)\right\} + \mathbf{J}_s \qquad \text{(II.8)}$$

By adopting the notation: $\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \wedge \mathbf{B}$, Equation (II.8) will therefore take the following form:

$$\mathbf{J} = -\sigma\left\{\frac{D\mathbf{A}}{Dt} + \mathbf{grad}(\Phi)\right\} + \mathbf{J}_s \qquad \text{(II.9)}$$

By substituting **J** with its expression from equation (II.9) into equation (I.15), one easily obtains the condensed vector form of the system of equations (I.14, I.15, I.16, and I.17), thus describing and representing the magnetodynamic equation in terms of the magnetic vector potential, given by the following system of equations:

$$\begin{cases} \mathbf{rot}\left(\nu\ \mathbf{rot}(\mathbf{A})\right) = -\sigma\left\{\frac{D\mathbf{A}}{Dt} + \mathbf{grad}(\Phi)\right\} + \mathbf{J}_s \\ \\ \mathrm{div}(\mathbf{A}) = 0 \end{cases} \qquad \text{(II.10)}$$

II.3. Magnetodynamic Equation in the Case of Systems with Cylindrical Geometry and Physical Axial Symmetry

In an axisymmetrical cylindrical configuration (cylindrical coordinates $(r,\ \varphi,\ z)$), the formulation based on the magnetic vector potential A offers the advantage that the system of partial differential equations involves only one unknown namely, the orthoradial or the azimuthally component $\left(A_{\varphi}\right)$ of the vector **A**, when the current itself is oriented along this azimuthally direction $(\mathbf{O}\varphi)$.

In such a configuration, the currents are perpendicular to the study plane. The various vector quantities have the following components:

$$\mathbf{J}=\begin{Bmatrix}0\\ J_{\varphi}\\ 0\end{Bmatrix},\quad \mathbf{E}=\begin{Bmatrix}0\\ E_{\varphi}\\ 0\end{Bmatrix},\quad \mathbf{A}=\begin{Bmatrix}0\\ A_{\varphi}\\ 0\end{Bmatrix},\quad \mathbf{B}=\begin{Bmatrix}B_r\\ 0\\ B_z\end{Bmatrix},\quad \mathbf{H}=\begin{Bmatrix}H_r\\ 0\\ H_z\end{Bmatrix} \qquad \text{(II.11)}$$

Since the vector A coincides with its orthoradial component, its divergence is naturally zero. Under these conditions, the system of equations (II.10) then reduces to:

$$\mathbf{rot}\left(\nu\ \mathbf{rot}(A)\right)=-\sigma\left\{\frac{D\mathbf{A}}{Dt}+\mathbf{grad}(\Phi)\right\}+\mathbf{J}_s \qquad \text{(II.12)}$$

Now, in an axisymmetrical cylindrical configuration, the term $\mathbf{grad}(\Phi)$ is zero. From this, equation (II.12) takes the following form:

$$\mathbf{rot}\left(\nu\ \mathbf{rot}(A)\right)+\sigma\,\frac{D\mathbf{A}}{Dt}=\mathbf{J}_s \qquad \text{(II.13)}$$

Knowing that in axisymmetrical cylindrical coordinates, we have:

$$\mathbf{rot}(\mathbf{A}) \ : \ \begin{cases} -\dfrac{\partial A_\varphi}{\partial z} & (\mathbf{e}_r) \\ 0 & (\mathbf{e}_\varphi) \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\,A_\varphi\right) & (\mathbf{e}_z) \end{cases} \qquad \text{(II.14)}$$

This reduces the vector form (II.13) to the following relation:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\nu r\,\frac{\partial A_\varphi}{\partial r}\right)-\frac{\partial}{\partial z}\left(\nu\frac{\partial A_\varphi}{\partial z}\right)+\frac{A_\varphi}{r^2}+\sigma\frac{DA_\varphi}{Dt}=J_{s\varphi} \qquad \text{(II.15)}$$

If we consider, as the unknown of the problem, the modified magnetic vector potential $\boldsymbol{A}$ such that:

$$\boldsymbol{A} = r.A_\varphi \qquad \text{(II.16)}$$

And if we adopt, for convenience, the notation $J_{s\varphi} = J_s$, equation (II.15) takes the following form:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\nu\,\frac{\partial A}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\,\nu\,\frac{\partial A}{\partial z}\right)-\frac{\sigma}{r}\,\frac{DA}{Dt}=-J_s \qquad \text{(II.17)}$$

III. CONSIDERATION OF CONDUCTORS: THE INDUCTORS

III.1. Introduction

Significant efforts have been made to couple Maxwell’s equations with those governing electrical conductors, as such coupling provides a more accurate physical representation of the behavior of structures subjected to electromagnetic fields. This, in turn, allows for the determination of currents and impedances in voltage-driven systems. It is worth noting that quantities such as resistance, impedance, voltage, and total current are generally more useful and even essential for engineers than surface current density or resistivity, which are typically used in local formulations. The need for a better understanding and representation of physical reality has led researchers to address these problems initially in relatively simple cases one-dimensional **(1D)** and two-dimensional **(2D)**and subsequently extend them to more complex situations, such as the three-dimensional **(3D)** case.

III.2. Types of Conductors

In the most general case, and according to applications commonly encountered in electrical engineering, two types of conductors can be distinguished. This distinction is based on their dimensions in the direction perpendicular to the current flow. The first type includes solid conductors, which have dimensions large enough relative to the frequencies involved for the skin effect to occur. The second type comprises thin conductors, or multi-filament conductors, whose dimensions are small enough for the current to be considered uniformly distributed across their cross-section. Consequently, the formulation of electromagnetic phenomena will depend on the significance of the skin effect within the conductors.

III.3. Formulation of Equations

The field and current equations in non-magnetic conducting regions are given respectively by the following relations :

$$\frac{\partial}{\partial r}\left(\frac{1}{\mu_0}\,\frac{1}{r}\,\frac{\partial A}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\,\frac{1}{r}\,\frac{\partial A}{\partial z}\right)=-J \qquad \text{(III.1)}$$

$$J=-\sigma\,\frac{DA}{Dt}+J_s \qquad \text{(III.2)}$$

$$J_s=-\sigma\,\mathrm{grad}(V) \qquad \text{(III.3)}$$

By substituting (III.2) and (III.3) into (III.1), the magnetic field equation in terms of the magnetic vector potential in conducting regions becomes:

$$\frac{\partial}{\partial r}\left(\frac{1}{\mu_0}\,\frac{1}{r}\,\frac{\partial A}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\,\frac{1}{r}\,\frac{\partial A}{\partial z}\right)-\frac{\sigma}{r}\,\frac{DA}{Dt}-\sigma\,\mathrm{grad}(V)=0 \qquad \text{(III.4)}$$

with : $\boldsymbol{A} = r.A_\varphi$

The term $\mathrm{grad}(V)$ represents the gradient of the electric potential within the conductors.

In axisymmetrical systems, the currents flowing through the conductors have only one component, which is the orthoradial component. Under these conditions, the magnetic field is always parallel to the (ORZ) plane and therefore has two components, and the term $grad(V)$ is constant across the cross-section of each conductor.

If we denote by L_k the length of a conductor k in the (φ) direction, the expression $L_k\,\left[\mathbf{grad}(V)\right]_k$ will represent the voltage (ΔV_k) applied across its terminals:

$$\Delta V_k = L_k\,\left[\mathrm{grad}(V)\right]_k \qquad \text{(III.5)}$$

Let :

$$J_s = -\sigma \frac{\Delta V_k}{L_k} \qquad \text{(III.6)}$$

$L_k = 2\pi r_k$ in the case of a cylindrical geometry, r_k is the radius of loop k and J_s is the source current density.

By substituting equation (III.6) into (III.2), the expression for the current density over the cross-section S_k of the conductor becomes:

$$J = -\sigma_k \frac{DA}{Dt} - \frac{\sigma_k}{L_k} \Delta V_k \qquad \text{(III.7)}$$

By integrating the current density over the cross-section of the conductor, we obtain the total current I_k flowing through the conductor, whose expression is given by the following relation:

$$I_k = \int_{s_k} J\, dS = -\int_{s_k} \sigma_k \frac{DA}{Dt}\, dS - \int_{s_k} \frac{\sigma_k}{L_k} \Delta V_k\, dS \qquad \text{(III.8)}$$

In the case of a significant skin effect, the cross-section S_k to be considered is the effective cross-section of the conductor. This is defined as a function of the conductor’s cross-sectional radius r_{sk} and the skin depth ξ :

$$S_k = \pi.\xi \left(2.r_{sk} - \xi\right) \qquad \text{(III.9)}$$

Finally, the equations that describe electromagnetic phenomena in the presence of electrical conductors are written as follows:

$$\begin{cases} \dfrac{\partial}{\partial r}\left(\nu\,\dfrac{1}{r}\,\dfrac{\partial A}{\partial r}\right)+\dfrac{\partial}{\partial z}\left(\nu\,\dfrac{1}{r}\,\dfrac{\partial A}{\partial z}\right)-\dfrac{\sigma}{r}\,\dfrac{DA}{Dt}-\sigma\left[\mathrm{grad}(V)\right]_k=0 \\ \\ \left[\mathrm{grad}(V)\right]_k=\dfrac{\Delta V_k}{L_k} \end{cases} \qquad \text{(III.10)}$$

IV. MAGNETIC FLUX THROUGH A WINDING

Magnetic fluxes in windings can be determined from energy considerations, without making any assumptions about the magnetic behavior of the materials.

Indeed, the expression for the electrical power P_k transmitted through electrical circuit number k is written as:

$$P_k = \Delta V_k . I_k = R_k . I_k^2 + \frac{\partial \phi_k}{\partial t} . I_k \qquad \text{(IV.1)}$$

ΔV_k : potential difference across the circuit terminals,

I_k : current flowing through the circuit,

R_k : circuit resistance,

ϕ_k : magnetic flux through the circuit.

The overall term $\left(\Delta V_k \, I_k\right)$ can be expressed in terms of the electric potential V_k and the current density **J**as a surface integral over the conductor's cross-section S_k given by:

$$\Delta V_k . I_k = -\int_{S_k} V_k \, \mathbf{J}.\mathbf{dS} = -\int_{S_k} V_k \, \mathbf{J}.\mathbf{n} dS \qquad \text{(IV.2)}$$

n: the normal to the surface of the conductor.

Assuming that the potential V_k is constant on the input and output sections of the current, and taking $J_n = 0$ (normal component of the electric current density vector J), equation (IV.2) can be expressed using the divergence theorem:

$$\Delta V_k . I_k = -\int_{\mathrm{V}_k} div(V_k \mathbf{J}) d\mathrm{V}$$

$$= -\left[\int_{\mathrm{V}_k} \mathbf{grad}(V).\mathbf{J}.d\mathrm{V} + \int_{\mathrm{V}_k} V.div(\mathbf{J}).d\mathrm{V} \right] \quad \text{(IV.3)}$$

V_k : the volume of the conductor.

Or :

$$\mathrm{div}(\mathbf{J}) = 0 \quad \text{(IV.4)}$$

Therefore:

$$\Delta V_k . I_k = -\int_{\mathrm{V}_k} \mathbf{grad}(V).\mathbf{J}.d\mathrm{V} \quad \text{(IV.5)}$$

Faraday's law gives the electric field vector **E** as a function of the variation of the magnetic induction vector **B**, as follows:

$$\mathbf{rot}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} = -\mathbf{rot}\left(\frac{\partial \mathbf{A}}{\partial \mathrm{t}}\right) \quad \text{(IV.6)}$$

Then :

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial \mathrm{t}} - \mathbf{grad}(\mathrm{V}) \quad \text{(IV.7)}$$

The current density J depends on the electric field vector E according to Ohm's law, given by:

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{(IV.8)}$$

σ : the electrical conductivity of the conductor.

By combining equations (IV.5), (IV.6), (IV.7), and (IV.8), we obtain:

$$\Delta V_k.I_k = \int_{\mathrm{V}_k} \frac{\mathbf{J}^2}{\sigma} d\mathrm{V} + \int_{\mathrm{V}_k} (\frac{\partial \mathbf{A}}{\partial t}.\mathbf{J}) d\mathrm{V} \qquad \text{(IV.9)}$$

The comparison of equations (IV.1) and (IV.9) yields:

$$R_k.I_k^2 + \frac{\partial \phi_k}{\partial t}.I_k = \int_{\mathrm{V}_k} \frac{\mathbf{J}^2}{\sigma} d\mathrm{V} + \int_{\mathrm{V}_k} (\frac{\partial \mathbf{A}}{\partial t}.\mathbf{J}) d\mathrm{V} \qquad \text{(IV.10)}$$

This equation allows us to derive the following two equalities:

$$R_k.I_k^2 = \int_{\mathrm{V}_k} \frac{\mathbf{J}^2}{\sigma} d\mathrm{V} \qquad \text{(IV.11)}$$

and

$$\frac{\partial \phi_k}{\partial t} I_k = \int_{\mathrm{V}_k} \frac{\partial \mathbf{A}}{\partial t}.\mathbf{J}.d\mathrm{V} \qquad \text{(IV.12)}$$

The term $\mathrm{R_k.I_k^2}$, given by equation (IV.11), represents the Joule losses in the circuit.

In the absence of skin effect and deformation of the electrical conductor, the current density in the conductor is directly related to the current flowing through it:

$$\frac{\mathrm{J}}{\mathrm{I_k}} = \frac{1}{\mathrm{S_k}} \qquad \text{(IV.13)}$$

Equation (IV.12) then takes the following form:

$$\frac{\partial \phi_k}{\partial t} = \int_{V_k} \frac{\partial}{\partial t} . \left[\mathbf{A} . \frac{\mathbf{J}}{I_k} \right] . dV \qquad \text{(IV.14)}$$

With stationary conductors (the volume V_k is time-independent), we arrive at the following expression:

$$\frac{\partial \phi_k}{\partial t} = \frac{\partial}{\partial t} \left[\int_{V_k} \left(\mathbf{A} . \frac{\mathbf{J}}{I_k} \right) dV \right] \qquad \text{(IV.15)}$$

Therefore, the final expression of the magnetic flux will be given by the following relation:

$$\phi_k = \frac{1}{I_k} \int_{V_k} (\mathbf{A}.\mathbf{J}) dV \qquad \text{(IV.16)}$$

In the absence of skin effect and deformation of the electrical conductor, the current density within the conductor is directly related to the current flowing through it:

$$\frac{J}{I_k} = \frac{1}{S_k} \qquad \text{(IV.17)}$$

From which:

$$\phi_k = \frac{1}{S_k} \int_{V_k} A_\varphi \, dV = \frac{2\pi}{S_k} \int_{s_k} A_\varphi \, r dr dz = \frac{2\pi}{S_k} \int_{s_k} A dr dz \qquad \text{(IV.18)}$$

CONCLUSION

In this document, we focused on the description and development of various mathematical formulations used in the computation of electromagnetic fields in electrical systems related to electrotechnical applications.

Local equations were presented, describing the fields and the formulations in terms of the magnetic vector potential in both static and general dynamic regimes.

The transient dynamic case was addressed separately, as it is a particular scenario frequently encountered in laboratory experiments and also in industrial contexts.

To better approximate or even accurately reflect physical reality, the mathematical developments proposed in this work take into account the presence of electrical conductors (inductors) through voltage sources.

The case of applications with cylindrical geometry and physical axial symmetry, often encountered in practice, was treated comprehensively.

Finally, a method based on energy considerations without simplifying assumptions regarding the magnetic behavior of materialswas proposed for the calculation of one of the most fundamental quantities in electromagnetism: the magnetic flux through windings.

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