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Course

MATHEMATICAL MODELS FOR CALCULATING MAGNETIC FORCES IN ELECTROTECHNICAL DEVICES IN THEORY OF ELECTROMAGNETIC FIELDS

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FOREWORD

The purpose of the work we present in this document is theoretical and summary in nature, focusing on the various existing models for evaluating forces based on electromagnetic quantities. This work is of great significance in the modeling and optimization of electrotechnical applications, such as rotary and linear electric machines, transformers, actuators, contactors, relays, etc.

Forces, referred to as magnetic or electromagnetic in origin, play a crucial role in electro-magneto-mechanical energy conversion, particularly in terms of displacement or structural deformation. The physical laws of electromagnetism provide a straightforward explanation as long as ferromagnetic materials are not considered. In the case of electrical conductors, for instance, there is no ambiguity—the local force density f follows the classical Laplace law: $\mathbf{f} = \mathbf{J} \wedge \mathbf{B}$. However, due to their unique properties, ferromagnetic materials are extensively used in electromagnetism, as the performance of electromagnetic systems is directly dependent on them. It is therefore essential to express these forces both qualitatively and quantitatively, as they act within these materials and, consequently, within the aforementioned structures.

Various methods can be used to evaluate the overall force acting on a given body. The determination of this resultant force can be achieved either with or without knowledge of the force distribution within the material or object, depending on whether the situation involves a simple displacement or a deformation—whether minor or significant—of the subject in question. However, the computational models employed each provide their own approach to the concept of force density.

In the first part of this document, we will present a systematic analysis of the various force calculation methods. We will successively develop:

- The method of magnetic coenergy and energy variation,
- The virtual work method,
- The Maxwell stress tensor method,
- The magnetic energy derivative method,
- Methods based on conventional representations of ferromagnetism, which highlight formulations derived from the principle of equivalent sources in magnetic materials. We will mention:
    - The case of equivalent currents,
    - The case of equivalent magnetic charges,
    - The case of a combination of equivalent currents and equivalent magnetic charges.

Finally, through comparison, we will identify the method that seems to be the most effective in addressing commonly encountered electrotechnical problems. More specifically, we will focus on the ease of its implementation in numerical computation tools, which are generally accessible to and required by users.

1. METHOD OF VARIATION OF MAGNETIC COENERGY AND ENERGY

The magnetic force can be calculated by differentiating the magnetic coenergy ($\overline{W}$) with respect to displacement at constant current, or by differentiating its corresponding form, the magnetic energy (W), with respect to displacement at constant flux (Figure 1.1).

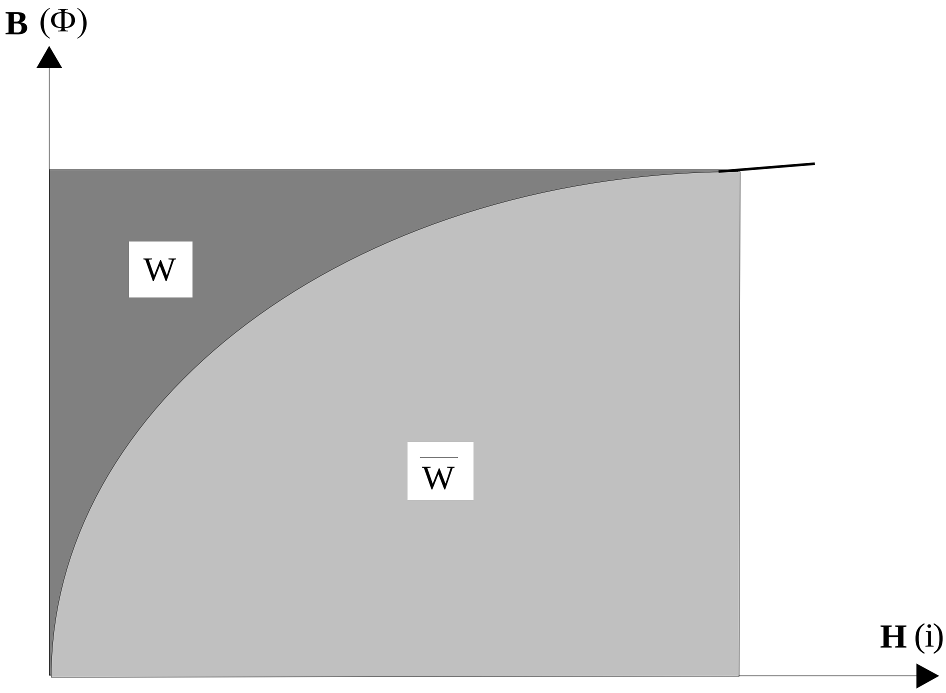

Figure 1.1. Magnetization curve showing magnetic coenergy and magnetic energy.

If F_s represents the component of the total magnetic force along a given direction **s**, it can be expressed as:

$$F_s = + \left.\frac{\partial \overline{W}}{\partial s}\right|_{i=Cte} \tag{1.1}$$

$$F_s = - \left.\frac{\partial W}{\partial s}\right|_{\Phi=Cte} \tag{1.2}$$

where W and $\overline{W}$ are, respectively, the magnetic coenergy and energy, knowing that:

$$\overline{W} = \int_{\Omega} \left(\int_{0}^{H} B \, dH \right) d\Omega \qquad (1.3)$$

$$W = \int_{\Omega} \left(\int_{0}^{B} H \, dB \right) d\Omega \qquad (1.4)$$

where Ω is the total study domain.

Numerically, the derivatives can be obtained using the following relations:

$$F_s = \left. \frac{\overline{W}_{s+\Delta s} - \overline{W}_s}{\Delta s} \right|_{i = Cte} \qquad (1.5)$$

$$F_s = - \left. \frac{W_{s+\Delta s} - W_s}{\Delta s} \right|_{\Phi = Cte} \qquad (1.6)$$

where Δs represents a displacement increment.

2. VIRTUAL WORK METHOD

2.1. *Principle of the Method*

In the numerical modeling of electromagnetic phenomena in electrical devices, it is proposed to use the finite element method (FEM) to directly evaluate the total magnetic force. Indeed, the "deformed" domain V_d is decomposed into sub-domains V_e (finite elements), on which all integrations are conducted based on the local coordinates (u, v, w).

By using the coenergy (1.3), the expression for the force is written as:

$$F_s = \frac{\partial}{\partial s}\left(\int_V \left(\int_0^H B dH\right) dV\right)$$

$$= \frac{\partial}{\partial s}\sum_e \left(\int_{V_e} \left(\int_0^H B dH\right) d\Omega_e\right) \qquad (2.1)$$

In the local coordinate system, the expression (2.1) for the force takes the following form:

$$F_s = \frac{\partial}{\partial s}\sum_e \left(\int_{V_{e_{local}}} \left(\int_0^H B dH\right) |G|\ dV_e\right)$$

$$= \frac{\partial}{\partial s}\sum_e \left(\int_{V_{e_{local}}} \left(\int_0^H B dH\right) |G|\ dudvdw\right) \qquad (2.2)$$

$|G|$ is the determinant of the Jacobean matrix of the coordinate transformation.

The differentiation with respect to displacement **s**, introduced under the volume integral, leads to the following expression:

$$F_s = \sum_{\text{é lé ments}} \int_{V_{\text{é lé local}}} \left[-B^T.G^{-1}.\frac{\partial G}{\partial s}.H + \int_0^H B.dH.|G|^{-1}\frac{\partial |G|}{\partial s}\right] dV_e \qquad (2.3)$$

Thus, to determine the force, only the Jacobean matrix and its determinant vary with displacement, so they are the only terms that need to be calculated.

Observation:

The main advantages of this force calculation method are the accuracy of the calculations and its good adaptability to finite element analysis.

3. METHOD OF THE MAXWELL STRESS TENSOR

3.1. *Preamble*

We first present the principle of the formulation using tensors, taking as an example the case of the force acting on a non-magnetic electrical conductor. Then, we will generalize this formulation to calculate the forces acting on a magnetic medium.

3.2. *Case of an Electrical Conductor (Non-Magnetic Medium*)

The example commonly used is the formulation of the force acting on a non-magnetic electrical conductor. The Laplace equation, derived from J.C. Maxwell's equations, provides the expression for the force acting on an ideal conductor placed in a magnetic induction field (Figure 3.1).

$$\mathrm{d}\mathbf{F} = \mathrm{i}\,\mathrm{d}\mathbf{l} \wedge \mathbf{B} \tag{3.1}$$

This equation can be expressed in a local form. Let $\mathbf{f}$ be the force per unit volume:

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{F}}{\mathrm{dV}} \tag{3.2}$$

$$\mathbf{f} = \frac{\mathrm{i}\,\mathrm{d}\mathbf{l} \wedge \mathbf{B}}{\mathrm{dV}} = \frac{\mathrm{i}\,\mathrm{d}\mathbf{l} \wedge \mathbf{B}}{\mathrm{S}\,\mathrm{dl}} \tag{3.3}$$

Let:

$$\mathbf{f} = \mathbf{J} \wedge \mathbf{B} \tag{3.4}$$

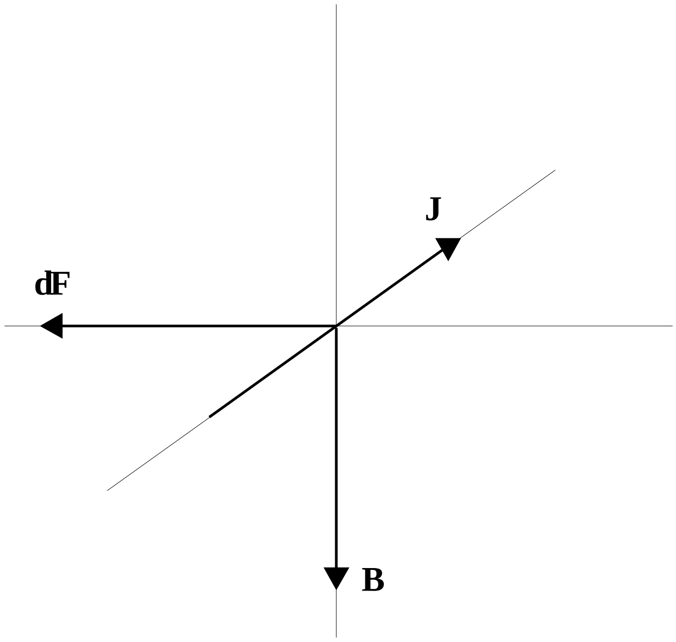

Figure 3.1. Vector representation of the Laplace equation.

The total force will be given by:

$$\mathbf{F} = \int_{v} \mathbf{f} \, dV = \int_{v} (\mathbf{J} \wedge \mathbf{B}) dV \qquad (3.5)$$

V is the volume of the conductor.

Using the equations of magnetostatics:

$$\mathbf{rot}(H) = J \qquad (3.6)$$

$$div(\mathbf{B}) = 0 \qquad (3.7)$$

$$\mathbf{B} = \mu_0 \mathbf{H} \qquad (3.8)$$

By substituting equations (3.6) and (3.8) into (3.4), we obtain:

$$\mathbf{f} = \mu_0 (\mathbf{rot}(H)) \wedge H \qquad (3.9)$$

By using the notation involving the Nabla symbol, we can write:

$$\mathbf{f} = \mu_0 \left(\nabla \wedge \mathbf{H}\right) \wedge \mathbf{H} \tag{3.10}$$

$$= \mu_0 \left(\left[(\mathbf{H}.\nabla).\mathbf{H} \right] - \left[\frac{1}{2} \nabla \left(\mathbf{H}^2 \right) \right] \right) \tag{3.11}$$

For each of the coordinates x, y and z, the relation (3.11) becomes:

$$f_x = \mu_0 \left[\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y} + H_z \frac{\partial H_x}{\partial z} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(H_x^2 + H_y^2 + H_z^2 \right) \right]$$

$$f_y = \mu_0 \left[\left(H_x \frac{\partial H_y}{\partial x} + H_y \frac{\partial H_y}{\partial y} + H_z \frac{\partial H_y}{\partial z} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left(H_x^2 + H_y^2 + H_z^2 \right) \right] \tag{3.12}$$

$$f_z = \mu_0 \left[\left(H_x \frac{\partial H_z}{\partial x} + H_y \frac{\partial H_z}{\partial y} + H_z \frac{\partial H_z}{\partial z} \right) - \frac{1}{2} \frac{\partial}{\partial z} \left(H_x^2 + H_y^2 + H_z^2 \right) \right]$$

To condense the notation, the following change is proposed:

$$x_1 = x \quad ; \qquad x_2 = y \quad ; \qquad x_3 = z \tag{3.13}$$

$$f_1 = f_x \quad ; \qquad f_2 = f_y \quad ; \qquad f_3 = f_z \tag{3.14}$$

Under these conditions, the generalized component of the force density given by equations (3.12) becomes:

$$f_m = \mu_0 \left(\sum_{n=1}^{3} \left(H_n \frac{\partial H_m}{\partial x_n} \right) - \frac{1}{2} \frac{\partial}{\partial x_m} \left(\mathbf{H}^2 \right) \right) \tag{3.15}$$

In this last expression, the quantity $\mathbf{H}^2$ is given by the relation:

$$\mathbf{H}^2 = H_1^2 + H_2^2 + H_3^2 \tag{3.16}$$

By using the Kronecker coefficient δ_{mn} such that:

$$\delta_{mn} = \begin{cases} 1 & \text{si} \quad m = n \\ 0 & \text{si} \quad m \neq n \end{cases} \tag{3.17}$$

We can then write:

$$f_m = \mu_0 \sum_{n=1}^{3} \left[\frac{\partial}{\partial x_n} \left(H_n H_m - \frac{1}{2} \delta_{mn} \left(\mathbf{H}^2 \right) \right) - H_m \frac{\partial H_n}{\partial x_n} \right] \tag{3.18}$$

The last term of this equation can be written as:

$$\mu_0 \sum_{n=1}^{3} \left[H_m \frac{\partial H_n}{\partial x_n} \right] = \sum_{n=1}^{3} \left[H_m \frac{\partial \left(\mu_0 \, H_n \right)}{\partial x_n} \right] = H_m \, \mathrm{div}\left(\mathbf{B} \right) = 0 \tag{3.19}$$

The expression for the generalized force component then simplifies as follows:

$$f_m = \mu_0 \sum_{n=1}^{3} \left[\frac{\partial}{\partial x_n} \left(H_n H_m - \frac{1}{2} \delta_{mn} \left(\mathbf{H}^2 \right) \right) \right] \tag{3.20}$$

3.3. *Definition of the Maxwell Stress Tensor*

Tensor analysis is a mathematical formalism particularly useful for studying a wide variety of physical systems. The Maxwell stress tensor will allow the study of the force acting on a material of volume (V), by knowing only the distribution of the field at different points on a closed surface (S) surrounding the volume in question.

The Maxwell stress tensor is defined through its components τ_{mn}. These components are given by the following formula:

$$\tau_{mn} = \mu_0 \left(H_n H_m - \frac{1}{2} \, \delta_{mn} \left(\mathbf{H}^2 \right) \right) \tag{3.21}$$

By substituting into the formula (3.20), we obtain:

$$f_m = \sum_{n=1}^{3} \left(\frac{\partial \tau_{mn}}{\partial x_n} \right) = \mathrm{div}\left(\mathbf{T}_m\right) \tag{3.22}$$

The vector $\mathbf{T}_m$ consists of three components of the Maxwell stress tensor:

$$\mathbf{T}_m = \tau_{m1}\,\mathbf{i} + \tau_{m2}\,\mathbf{j} + \tau_{m3}\,\mathbf{k} \tag{3.23}$$

The Maxwell stress tensor $\overline{\mathbf{T}}$ can then take the following form:

$$\overline{\mathbf{T}} = \begin{Bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{Bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} . \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} \tag{3.24}$$

3.4. *Determination of Forces Using the Maxwell Stress Tensor*

The forces acting on a volume (V) are determined by the following relations, resulting from the expression (3.22):

$$F_m = \int_v f_m \, dV = \int_v \mathrm{div}\left(\mathbf{T}_m\right) dV \tag{3.25}$$

By the divergence theorem, we obtain:

$$F_m = \oint_s \mathbf{T}_m . \, d\mathbf{S} = \oint_s \left(\mathbf{T}_m . \mathbf{n}\right) dS \tag{3.26}$$

More generally, we note:

$$F = \oint_s \left(\left[\overline{\mathbf{T}}\right] . \mathbf{n} \right) dS \tag{3.27}$$

(S) being the closed surface surrounding the object of volume (V) on which we wish to determine the force, **n** is the normal to the surface of the conductor and **F** the total force.

If we take, for example, the component of the total force along the (**Oz**), axis, it will be given by the following expression according to the generalized equation (3.20):

$$F_z = \mu_0 \oint_s \left(\left(H_x . n_x + H_y . n_y + H_z . n_z \right) H_z - \frac{1}{2} \left(\mathbf{H}^2 \right) n_z \right) dS$$

$$= \mu_0 \oint_s \left(\left(\mathbf{H} . \mathbf{n} \right) H_z - \frac{1}{2} \left(\mathbf{H}^2 \right) n_z \right) dS \qquad (3.28)$$

The total force will then be:

$$\mathbf{F} = \mu_0 \oint_s \left(\left(\mathbf{H} . \mathbf{n} \right) \mathbf{H} - \frac{1}{2} \left(\mathbf{H}^2 \right) \mathbf{n} \right) dS \qquad (3.29)$$

In terms of the magnetic flux density vector **B**:

$$\mathbf{F} = \frac{1}{\mu_0} \oint_s \left(\left(\mathbf{B} . \mathbf{n} \right) \mathbf{B} - \frac{1}{2} \left(\mathbf{B}^2 \right) \mathbf{n} \right) dS \qquad (3.30)$$

3.5. *Special Case: Calculation of the Axial Force Component in the Case of Cylindrical Axisymmetrical Systems*

In the case *of cylindrical axisymmetrical systems*, the mathematical developments will be simpler: the currents being perpendicular to the plane of study, the magnetic field and magnetic flux density will be given by:

$$\mathbf{B} = \begin{Bmatrix} B_r \\ 0 \\ B_z \end{Bmatrix} , \quad \mathbf{H} = \begin{Bmatrix} H_r \\ 0 \\ H_z \end{Bmatrix} \tag{3.31}$$

From expression (3.18), the axial component f_z of the force density is expressed as:

$$f_z = \mu_0 \sum_{n=1}^{2} \left[\frac{\partial}{\partial x_n} \left(H_n H_z - \frac{1}{2} \delta_{zn} \left(\mathbf{H}^2 \right) \right) - H_z \frac{\partial H_n}{\partial x_n} \right] \tag{3.32}$$

In this case, the quantity $\mathbf{H}^2$ is given by the following relation:

$$\mathbf{H}^2 = H_r^{\,2} + H_z^{\,2} \quad : \; \left(H_\varphi = 0 \right) \tag{3.33}$$

The last term of equation (3.32) can be written as:

$$\mu_0 \sum_{n=1}^{2} \left[H_z \frac{\partial H_n}{\partial x_n} \right] = \sum_{n=1}^{2} \left[H_z \frac{\partial \left(\mu_0 H_n \right)}{\partial x_n} \right]$$

$$= H_z \left(\frac{\partial \left(\mu_0 H_r \right)}{\partial r} + \frac{\partial \left(\mu_0 H_z \right)}{\partial z} \right)$$

$$= H_z \left(\frac{\partial \left(B_r \right)}{\partial r} + \frac{\partial \left(B_z \right)}{\partial z} \right)$$

$$= H_z \frac{\partial \left(B_r \right)}{\partial r} + H_z \frac{\partial \left(B_z \right)}{\partial z} \tag{3.34}$$

In c*ylindrical axisymmetrical configurations*, naturally and without imposing gauge conditions, we have:

$$\operatorname{div}\left(\mathbf{B} \right) = \frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} + \frac{B_r}{r} = 0 \quad \text{(Maxwell equation)} \tag{3.35}$$

Therefore:

$$\frac{B_r}{r} = -\frac{\partial B_r}{\partial r} - \frac{\partial B_z}{\partial z} = -\mu_0 \left(\frac{\partial H_r}{\partial r} + \frac{\partial H_z}{\partial z} \right) \tag{3.36}$$

and :

$$\frac{H_r}{r} = -\left(\frac{\partial H_r}{\partial r} + \frac{\partial H_z}{\partial z} \right) \tag{3.37}$$

Under these conditions, let's revisit the expression for the magnetic force density given in (3.32):

$$f_z = \mu_0 \sum_{n=1}^{2} \left[\frac{\partial}{\partial x_n} \left(H_n H_z - \frac{1}{2} \delta_{zn} \left(\mathbf{H}^2 \right) \right) - H_z \frac{\partial H_n}{\partial x_n} \right]$$

$$= \mu_0 \left[\left(\frac{\partial}{\partial r} \left(H_r H_z \right) + \frac{\partial}{\partial z} \left(H_z^2 - \mathbf{H}^2 \right) \right) - \left(H_z \frac{\partial H_r}{\partial r} + H_z \frac{\partial H_z}{\partial z} \right) \right]$$

$$= \mu_0 \left(\frac{\partial}{\partial r} \left(H_r H_z \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(H_z^2 - H_r{}^2 \right) - H_z \frac{\partial H_r}{\partial r} + H_z \frac{\partial H_z}{\partial z} \right)$$

$$= \mu_0 \left(\frac{\partial}{\partial r} \left(H_r H_z \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(H_z^2 - H_r{}^2 \right) - H_z \left(\frac{\partial H_r}{\partial r} + \frac{\partial H_z}{\partial z} \right) \right) \tag{3.38}$$

By introducing equation (3.37) into (3.38), the expression for the force density given in (3.38) takes the form and becomes:

$$f_z = \mu_0 \left(\frac{\partial}{\partial r} \left(H_r H_z \right) + \frac{1}{2} \frac{\partial}{\partial z} \left(H_z^2 - H_r{}^2 \right) + H_z \frac{H_r}{r} \right)$$

$$= \frac{\partial}{\partial r} \left(\mu_0 H_r H_z \right) + \frac{\partial}{\partial z} \left[\frac{1}{2} \mu_0 \left(H_z^2 - H_r{}^2 \right) \right] + \frac{\mu_0 H_z H_r}{r} \tag{3.39}$$

In the same coordinate system, the third component of the tensor will be given by:

$$\mathbf{T}_z = \tau_{zr}\,\mathbf{e}_r + \tau_{z\varphi}\,\mathbf{e}_\varphi + \tau_{zz}\,\mathbf{e}_z \tag{3.40}$$

with:

$$\begin{cases} \tau_{zr} = \mu_0\left(H_r H_z\right) \\ \tau_{z\varphi} = 0 \\ \tau_{zz} = \dfrac{1}{2}\mu_0\left(H_z^2 - H_r^2\right) \end{cases} \tag{3.41}$$

and the divergence of the vector is given by:

$$\operatorname{div}\left(\mathbf{T}_z\right) = \frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}$$

$$= \frac{\partial}{\partial r}\left(\mu_0 H_r H_z\right) + \frac{\partial}{\partial z}\left[\frac{1}{2}\mu_0\left(H_z^2 - H_r{}^2\right)\right] + \frac{\mu_0 H_z H_r}{r} \tag{3.42}$$

In terms of the components of the vector $\mathbf{T}_z$, the force density f_z, can be expressed as follows:

$$f_z = \frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} \tag{3.43}$$

By comparing equations (3.42) and (3.43), the expression for the axial component of the force density simplifies as follows:

$$f_z = \operatorname{div}\left(\mathbf{T}_z\right) \tag{3.44}$$

The final expression for the axial component of the global force exerted on the entire volume and expressed in the (**n**, **t**) reference frame is identical to the one we found in the case of the three-dimensional Cartesian system. Therefore, this expression is general and valid regardless of the chosen coordinate system.

3.6. *Observations:*

- The forces acting on a volume can be determined from the components of the Maxwell Tensor.
- The transition from volumetric calculation to a surface integral is of paramount importance in numerical calculations.
- The surface calculation of the global force from magnetic or electromagnetic origin remains unchanged if, instead of Laplace forces, we consider forces on ferromagnetic materials. This remains true as long as the integration is performed over a closed surface within a physically linear medium, typically an air gap (air or vacuum).
- In other words, it can be deduced and generalized that the force exerted on an object can be evaluated and determined from the calculation of the magnetic field on its surface.

3.7. *Case of a Magnetic Medium (Ferromagnetic Materials)*

In the case of magnetic media, more specifically, in the presence of ferromagnetic materials, and due to the discontinuities in the magnetic field at the interface between different materials when calculating the forces acting on a magnetic medium, the question arises as to which side of the surface (S) the field should be calculated from. To address this, we adopt the following notations:

$\left[\overline{\mathbf{T}}\right]^{+}$: the terms of the tensor are calculated outside,

$\left[\overline{\mathbf{T}}\right]^{-}$: the terms of the tensor are calculated inside,

S^{+} : the surface S on the outside,

S^{-} : the surface S on the inner side.

We will now generalize the expression of force (3.29) or (3.30) for the calculation of forces on magnetic media.

Suppose we have a heterogeneous medium composed of different magnetic materials: air, copper, iron, etc. Inside each magnetic material, we have:

$$\mathbf{rot}\ (\mathbf{H}) = \mathbf{J} \tag{3.45}$$

$$\mathrm{div}(\mathbf{B}) = 0 \tag{3.46}$$

where **J**, **H** and **B** represent the volume current density, the magnetic field, and the magnetic flux density, respectively.

At the surface of the different materials, we have:

$$\mathbf{n} \wedge \left(\mathbf{H}^{+} - \mathbf{H}^{-}\right) = \mathbf{K} \tag{3.47}$$

$$\mathbf{n}.\left(\mathbf{B}^{+} - \mathbf{B}^{-}\right) = 0 \tag{3.48}$$

where $\left(\mathbf{H}^{+},\ \mathbf{B}^{+}\right)$ are the magnetic field and magnetic flux density, respectively, near the exterior surface of the material, $\left(\mathbf{H}^{-}, \mathbf{B}^{-}\right)$ are the magnetic field and magnetic flux density, respectively, near the interior surface of the material and **K** is the surface current density.

To these two relations, we must add the relation that characterizes the material:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \tag{3.49}$$

where **M** is the magnetization of the material. It is given by the following relation:

$$\mathbf{M} = \mathbf{M}_i + \mathbf{M}_r \tag{3.50}$$

$\mathbf{M}_i$ is the induced magnetization and $\mathbf{M}_r$ is the rigid magnetization.

Taking into account this relation, equations (3.45) and (3.46) inside the material take the following form:

$$\mathbf{rot}\left(\frac{\mathbf{B}}{\mu_0}\right) = \mathbf{J} + \mathbf{rot}\left(\frac{\mathbf{M}}{\mu_0}\right) \tag{3.51}$$

$$\mathrm{div}\left(\mathbf{B}\right) = 0 \tag{3.52}$$

and at the outer surface of the material:

$$\mathbf{n} \wedge \left(\frac{\mathbf{B}^+}{\mu_0} - \frac{\mathbf{B}^-}{\mu_0}\right) = \mathbf{K} + \mathbf{n} \wedge \left(\frac{\mathbf{M}^+}{\mu_0} - \frac{\mathbf{M}^-}{\mu_0}\right) \tag{3.53}$$

$$\mathbf{n} \,.\, \left(\mathbf{B}^+ - \mathbf{B}^-\right) = 0 \tag{3.54}$$

By comparing equations (3.45), (3.46), (3.47), and (3.48), equations (3.51), (3.52), (3.53), and (3.54) indicate that a magnetic medium can be replaced by a non-magnetic medium carrying a volume current density $\mathbf{J}_e$ and a surface current density $\mathbf{K}_e$, given by:

$$\mathbf{J}_e = \mathbf{J} + \mathbf{rot}\left(\frac{\mathbf{M}}{\mu_0}\right) \tag{3.55}$$

$$\mathbf{K}_e = \mathbf{K} + \mathbf{n}\wedge\left(\frac{\mathbf{M}^+}{\mu_0} - \frac{\mathbf{M}^-}{\mu_0}\right) \tag{3.56}$$

Theoretical considerations show that the force exerted on this medium is equal to:

$$\mathbf{F} = \int_v \left(\mathbf{J}_e \wedge \mathbf{B}\right)dV \ + \ \oint_s \left(\mathbf{K}_e \wedge \mathbf{B}\right)dS \tag{3.57}$$

According to the relations (3.51) and (3.55), we have, on one hand:

$$\mathbf{J_e} = \mathbf{rot}\left(\frac{B}{\mu_\mathbf{0}}\right) \tag{3.58}$$

and on the other hand, for the volume integral:

$$\mathbf{F}_V = \int_v \left(\mathbf{rot}\left(\frac{\mathbf{B}}{\mu_0}\right)\wedge\mathbf{B}\right) dV \tag{3.59}$$

Developments of calculation similar to those used to establish the Maxwell Tensor from Laplace's law lead to the following expression:

$$\mathbf{F}_V = \oint_s \left(\left[\overline{\mathbf{T}}\right]^- . \ \mathbf{n}\right) dS \tag{3.60}$$

The negative sign (-) indicates that the tensor $\overline{T}$ is calculated inside the volume.

According to the relations (3.53), (3.54), and (3.56), we have:

$$\mathbf{K}_e = \frac{1}{\mu_0}\left[\mathbf{n}\wedge\left(\mathbf{B}^+ - \mathbf{B}^-\right)\right] \tag{3.61}$$

and the surface integral is then written as:

$$\mathbf{F}_s = \oint_s \left(\mathbf{K}_e \wedge \mathbf{B}_s\right) d\mathbf{S}$$

$$= \frac{1}{\mu_0} \oint_s \left[\left(\mathbf{n} \wedge \left(\mathbf{B}^+ - \mathbf{B}^-\right)\right) \wedge \mathbf{B}_s\right] d\mathbf{S} \tag{3.62}$$

The magnetic induction vector $\mathbf{B}_s$ at the surface S of the medium is given, according to specialized literature, as the average of the magnetic induction vectors calculated on either side of the interface, and it is expressed in terms of $\mathbf{B}^+$ and $\mathbf{B}^-$ as follows:

$$\mathbf{B}_s = \frac{1}{2}\left(\mathbf{B}^+ + \mathbf{B}^-\right) \tag{3.63}$$

The developments of the calculations then lead to:

$$\mathbf{F}_s = \left(\oint_s \left(\left[\overline{\mathbf{T}}\right]^+ . \, \mathbf{n}\right) dS\right) - \left(\oint_s \left(\left[\overline{\mathbf{T}}\right]^- . \, \mathbf{n}\right) dS\right) \tag{3.64}$$

Or :

$$\oint_{s^+} \left(\left[\overline{\mathbf{T}}\right] . \, \mathbf{n}\right) dS = \oint_s \left(\left[\overline{\mathbf{T}}\right]^+ . \, \mathbf{n}\right) dS \tag{3.65}$$

The total force is therefore equal to:

$$\mathbf{F} = \mathbf{F}_s + \mathbf{F}_v \tag{3.66}$$

$$\mathbf{F} = \oint_s \left(\left[\overline{\mathbf{T}}\right]^+ . \, \mathbf{n}\right) dS \tag{3.67}$$

The force acting on a magnetic medium can thus be formulated using the Maxwell Tensor. This force can also be expressed by the relation:

$$\mathbf{F} = \frac{1}{\mu_0} \oint_s \left((\mathbf{B}.\mathbf{n})\mathbf{B} - \frac{1}{2}\left(\mathbf{B}^2\right)\mathbf{n} \right) dS \tag{3.68}$$

Or in terms of the magnetic induction **H**:

$$\mathbf{F} = \mu_0 \oint_s \left((\mathbf{H}.\mathbf{n})\mathbf{H} - \frac{1}{2}\left(\mathbf{H}^2\right)\mathbf{n} \right) dS \tag{3.69}$$

Thus, we demonstrate that the expression of the force given by relation (3.29), or also by (3.30), can be generalized and applied to the calculation of forces on magnetic media. Therefore, the force acting on a magnetic medium can be formulated using the Maxwell Tensor and expressed in terms of the magnetic field vector **H**, or also in terms of the magnetic induction vector **B**.

It can be observed that the expression of the global force resulting from the surface calculation of the magnetic force remains unchanged if, instead of the Laplace forces, we need to consider forces on ferromagnetic materials.

4. OTHER METHODS

4.1. *Method Based on the Derivative of Magnetic Energy*

In an isotropic medium without current sources, the variation of energy δW during a displacement δs, can be expressed in terms of the variation of magnetic permeability $\delta\mu$ by the following relation:

$$\delta W = \frac{1}{2} \int_V \left(\delta\mu \, H^2 \right) dV \tag{4.1}$$

where V is the volume of the medium on which the force is calculated.

The variation in magnetic permeability $\delta\mu$ can be attributed to the physical in homogeneity and mechanical stresses exerted on the medium. If we neglect the variation due to mechanical stresses, the expression for $(\delta\mu)$during a displacement δu along the direction sis given by the following relation:

$$\delta\mu = \frac{\partial \mu}{\partial s}\,\delta s \tag{4.2}$$

This relation is valid only if the permeability varies continuously within the domain of study. For a volume contained in an isotropic, undeformable medium, without currents, and where the magnetic permeability varies continuously, the component of the total force along the chosen direction is given by the following relation:

$$F_v = -\frac{1}{2}\int_v \left(\frac{\partial \mu}{\partial s} H^2\right) dV \tag{4.3}$$

Similarly, we can obtain the other components of the total force along the other directions of the working coordinate system. Thus, the total force exerted on the entire volume will be given by:

$$\mathbf{F}_v = -\frac{1}{2}\int_v \left(\mathbf{H}^2\ \mathbf{grad}(\mu)\right) dV \tag{4.4}$$

and its density $\mathbf{f}_v$ is given by:

$$\mathbf{f}_v = -\frac{1}{2}\mathbf{H}^2\ \mathbf{grad}(\mu) \tag{4.5}$$

If the volume (V) is traversed by currents with a volumetric current density $\mathbf{J}_v$, it is necessary to account for them by adding a term representing these currents to the previous relation. The new volumetric force density will then be given by the following relation:

$$\mathbf{f}_{\mathrm{v}} = -\frac{1}{2}\mathbf{H}^2\,\mathbf{grad}(\mu) + \mathbf{J}_{\mathrm{v}} \wedge (\mu\mathbf{H}) \qquad (4.6)$$

When we want to calculate the force exerted on a volume (V) surrounded by a surface (S), and where the magnetic permeability is discontinuous, the expression for the volumetric force distribution given by relation (4.6) is insufficient. We must then add the force exerted on the discontinuity surfaces (the interface between two media with different physical properties). To overcome this difficulty, we will rely on assumptions formulated in various references. Here, the hypothesis is made that the surface (S) has a nonzero infinitesimal thickness, thus forming a fictitious volume where the magnetic permeability is continuous. To determine the force exerted on this surface, we must integrate the expression for its density over the corresponding fictitious volume, denoted as V_s (Figure 4.1).

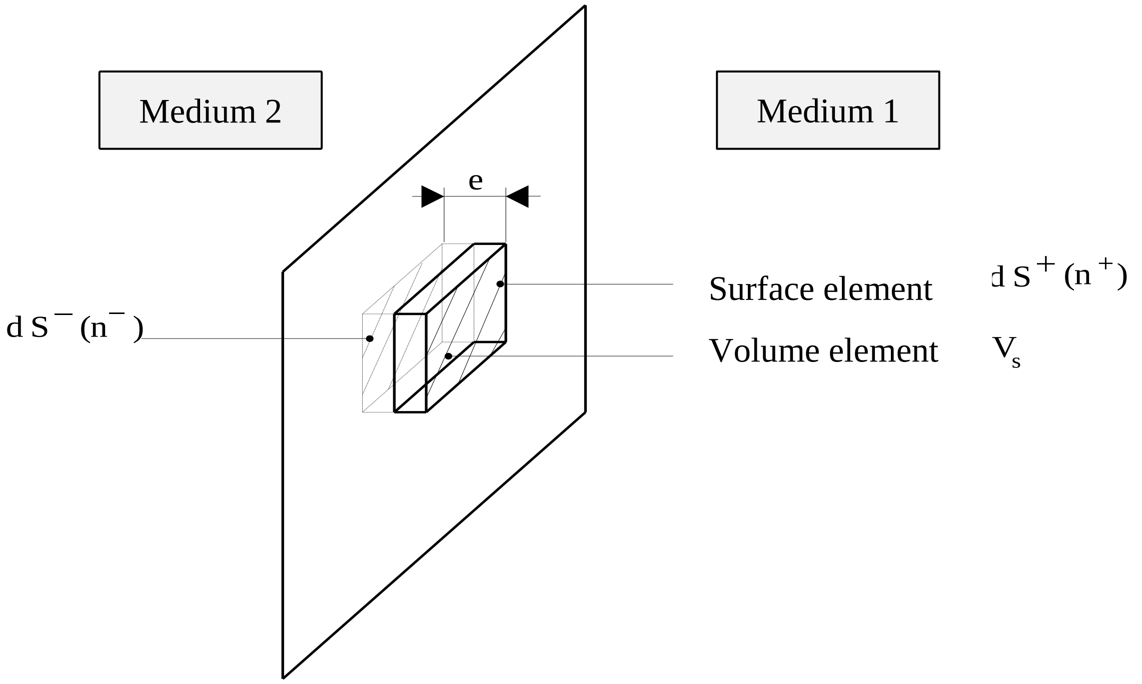

Figure 4.1. Fictitious volume (V_S) for calculating the force densities at the interface between two media with different physical properties.

The divergence theorem then allows calculating the force exerted on the volume V_s by integrating the following expression over the surface surrounding this volume:

$$\mathbf{f}_s = (\mathbf{B}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{B}\,\mathbf{H})\mathbf{n} \qquad (4.7)$$

In order to obtain the force exerted on the surface (S), the thickness e of the volume (V_s) is reduced to zero. The integral is then calculated over the two faces of the surface (S), namely (S^-) and (S^+) whose normal vectors are, respectively, $\mathbf{n}^-$ and $\mathbf{n}^+$, such that: $\mathbf{n}^- = -\mathbf{n}^+$. This leads to:

$$\begin{aligned}\mathbf{F}_s &= \left(\int_{s^+}\left[(\mathbf{B}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{B}.\mathbf{H})\mathbf{n}^+\right]dS\right) + \left(\int_{s^-}\left[(\mathbf{B}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{B}.\mathbf{H})\mathbf{n}^-\right]dS\right)\\ &= \left(\int_{s^+}\left[(\mathbf{B}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{B}.\mathbf{H})\mathbf{n}\right]dS\right) - \left(\int_{s^-}\left[(\mathbf{B}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{B}.\mathbf{H})\mathbf{n}\right]dS\right) \qquad (4.8)\end{aligned}$$

$\mathbf{n}$ being the normal to the surface (S), such that: $\mathbf{n} = \mathbf{n}^+$.

The difference between these two terms is explained by the fact that the total force results from the difference in forces exerted on either side of the separating surface (S).

<u>Development of Calculations:</u>

In this section, we have presented the demonstration of the following mathematical equivalence:

$$\mathbf{F}_s = \int_s \left(\left[\left((\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1 - \frac{1}{2}(\mathbf{B}_1.\mathbf{H}_1)\right) - \left((\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2 - \frac{1}{2}(\mathbf{B}_2.\mathbf{H}_2)\right)\right]\mathbf{n}\right) dS$$

$$= \frac{1}{2}\int_{s}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)-\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n}\,dS$$

where the indices 1 and 2 identify the two media on either side of the separating surface (S).

The expression for the total force can be written as:

$$\mathbf{F}_s = \int_{s}\left(\left[\left((\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1-\frac{1}{2}(\mathbf{B}_1.\mathbf{H}_1)\right)-\left((\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2-\frac{1}{2}(\mathbf{B}_2.\mathbf{H}_2)\right)\right]\mathbf{n}\right)dS \qquad \textbf{(D.1)}$$

And its surface density is given by:

$$\mathbf{f}_s = \left[(\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1-(\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2\right] - \frac{1}{2}\left[(\mathbf{B}_1.\mathbf{H}_1)-(\mathbf{B}_2.\mathbf{H}_2)\right]\mathbf{n} \qquad \textbf{(D.2)}$$

Let **n** and **t** be the unit normal and tangential vectors to the surface S. In a (**n**, **t**) reference frame, the magnetic field and induction vectors can be expressed as follows:

$$\mathbf{B} = \mathbf{B}_n + \mathbf{B}_t \Rightarrow \begin{cases}\mathbf{B}_1 = \mathbf{B}_{1n}+\mathbf{B}_{1t}\\ \mathbf{B}_2 = \mathbf{B}_{2n}+\mathbf{B}_{2t}\end{cases} \qquad \textbf{(D.3)}$$

$$\mathbf{H} = \mathbf{H}_n + \mathbf{H}_t \Rightarrow \begin{cases}\mathbf{H}_1 = \mathbf{H}_{1n}+\mathbf{H}_{1t}\\ \mathbf{H}_2 = \mathbf{H}_{2n}+\mathbf{H}_{2t}\end{cases} \qquad \textbf{(D.4)}$$

with the following interface conditions:

$$\begin{cases}\mathbf{B}_1.\mathbf{n} = \mathbf{B}_2.\mathbf{n} = \mathbf{B}.\mathbf{n} = B_n\\ \mathbf{H}_1.\mathbf{t} = \mathbf{H}_2.\mathbf{t} = \mathbf{H}.\mathbf{t} = H_t\end{cases} \qquad \textbf{(D.5)}$$

Let's take **the first term** of equation (D.2):

$$\left[(\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1 - (\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2\right] \quad \textbf{(D.6)}$$

From equations (D.3), (D.4), and (D.5), we can write:

1- $$(\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1 = \mathrm{B}_1(\mathbf{H}_{1n} + \mathbf{H}_{1t}) = \mathrm{B}_n\left(\frac{\mathbf{B}_{1n}}{\mu_1} - \mathbf{H}_{1t}\right) = \mathrm{B}_n\left(\frac{\mathbf{B}_n}{\mu_1} - \mathbf{H}_t\right) \quad \textbf{(D.7)}$$

2- $$(\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2 = \mathrm{B}_1(\mathbf{H}_{2n} + \mathbf{H}_{2t}) = \mathrm{B}_n\left(\frac{\mathbf{B}_{2n}}{\mu_2} - \mathbf{H}_{2t}\right) = \mathrm{B}_n\left(\frac{\mathbf{B}_n}{\mu_2} - \mathbf{H}_t\right) \quad \textbf{(D.8)}$$

Thus :

$$\left[(\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1 - (\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2\right] = \mathrm{B}_n\left(\frac{\mathbf{B}_n}{\mu_1} - \mathbf{H}_t\right) - \mathrm{B}_n\left(\frac{\mathbf{B}_n}{\mu_2} - \mathbf{H}_t\right)$$

$$\left[(\mathbf{n}.\mathbf{B}_1)\mathbf{H}_1 - (\mathbf{n}.\mathbf{B}_2)\mathbf{H}_2\right] = \mathrm{B}_n^2\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\mathbf{n} \quad \textbf{(D.9)}$$

<u>The second term:</u>

$$\frac{1}{2}\left[(\mathbf{B}_1 . \mathbf{H}_1) - (\mathbf{B}_2 . \mathbf{H}_2)\right]\mathbf{n} \quad \textbf{(D.10)}$$

From equations (D.3), (D.4), and (D.5), we can write:

1/ $$\mathbf{B}_1 . \mathbf{H}_1 = (\mathbf{B}_{1n} + \mathbf{B}_{1t}).(\mathbf{H}_{1n} + \mathbf{H}_{1t})$$

$$= (\mathbf{B}_{1n} + \mu_1 \mathbf{H}_t).\left(\frac{\mathbf{B}_{1n}}{\mu_1} + \mathbf{H}_{1t}\right)$$

$$= (\mathbf{B}_n + \mu_1 \mathbf{H}_t).\left(\frac{\mathbf{B}_n}{\mu_1} + \mathbf{H}_t\right)$$

$$= \left(\frac{\mathbf{B}_n^2}{\mu_1} + \mathbf{H}_t \mathbf{B}_n + \mu_1 \mathbf{H}_t^2 \right) \quad \textbf{(D.11)}$$

2/ $\mathbf{B}_2 . \mathbf{H}_2 = \left(\mathbf{B}_{2n} + \mathbf{B}_{2t} \right) . \left(\mathbf{H}_{2n} + \mathbf{H}_{2t} \right)$

$$= \left(\mathbf{B}_{2n} + \mu_2 \mathbf{H}_t \right) . \left(\frac{\mathbf{B}_{2n}}{\mu_2} + \mathbf{H}_{2t} \right)$$

$$= \left(\mathbf{B}_n + \mu_2 \mathbf{H}_t \right) . \left(\frac{\mathbf{B}_n}{\mu_2} + \mathbf{H}_t \right)$$

$$= \left(\frac{\mathbf{B}_n^2}{\mu_2} + 2 \mathbf{H}_t \mathbf{B}_n + \mu_2 \mathbf{H}_t^2 \right) \quad \textbf{(D.12)}$$

The two equations (D.11) and (D.12) give:

$$\left(\mathbf{B}_1 . \mathbf{H}_1 \right) - \left(\mathbf{B}_2 . \mathbf{H}_2 \right) = \left(\frac{\mathbf{B}_n^2}{\mu_1} + 2 \mathbf{H}_t \mathbf{B}_n + \mu_1 \mathbf{H}_t^2 - \frac{\mathbf{B}_n^2}{\mu_2} - 2 \mathbf{H}_t \mathbf{B}_n - \mu_2 \mathbf{H}_t^2 \right)$$

$$= \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \left(\mathrm{B}_n^2 \right) + \left(\mu_1 - \mu_2 \right) \left(\mathrm{H}_t^2 \right) \quad \textbf{(D.13)}$$

Thus:

$$\frac{1}{2} \left[\left(\mathbf{B}_1 . \mathbf{H}_1 \right) - \left(\mathbf{B}_2 . \mathbf{H}_2 \right) \right] \mathbf{n} = \frac{1}{2} \left[\left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \left(\mathrm{B}_n^2 \right) + \left(\mu_1 - \mu_2 \right) \left(\mathrm{H}_t^2 \right) \right] \mathbf{n} \quad \textbf{(D.14)}$$

From the two expressions (D.9) and (D.14), we obtain:

$$\left[\left(\mathbf{n} . \mathbf{B}_1 \right) \mathbf{H}_1 - \left(\mathbf{n} . \mathbf{B}_2 \right) \mathbf{H}_2 \right] - \frac{1}{2} \left[\left(\mathbf{B}_1 . \mathbf{H}_1 \right) - \left(\mathbf{B}_2 . \mathbf{H}_2 \right) \right] \mathbf{n}$$

$$= B_n^2\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\mathbf{n}-\frac{1}{2}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)+\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n}$$

$$= \frac{1}{2}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)-\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n} \quad \textbf{(D.15)}$$

Finally, we arrive at the following equality:

$$\left[\left(\mathbf{n}.\mathbf{B}_1\right)\mathbf{H}_1-\left(\mathbf{n}.\mathbf{B}_2\right)\mathbf{H}_2\right]-\frac{1}{2}\left[\left(\mathbf{B}_1.\mathbf{H}_1\right)-\left(\mathbf{B}_2.\mathbf{H}_2\right)\right]\mathbf{n}$$

$$= \frac{1}{2}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)-\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n} \quad \textbf{(D.16)}$$

This leads us to a simplified expression for the surface force density $\mathbf{f}_s$, expressed in terms of the continuous components of the magnetic field, the magnetic induction, and the magnetic permeabilities on both sides of the interface, given by the following relation:

$$\mathbf{f}_s = \frac{1}{2}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)-\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n} \quad (4.9)$$

The corresponding total force will be:

$$\mathbf{F}_s = \int_s \mathbf{f}_s \, dS = \frac{1}{2}\int_s\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(B_n^2\right)-\left(\mu_1-\mu_2\right)\left(H_t^2\right)\right]\mathbf{n}\, dS \quad (4.10)$$

From this easily exploitable numerical form of the total force, two essential results can be derived:

1/ The force is always normal to the surface of separation between the two media with different permeabilities.

2/ The force is directed from the medium with the highest permeability (magnetic materials) to the one with the lowest permeability (vacuum, air gap, or air).

4.2. *Methods Based on Conventional Representations of Ferromagnetism (Principle of Equivalent Sources for Magnetic Materials)*

4.2.1. *Introduction*

Recently, methods for calculating forces based on a surface force density derived from magnetization currents have been proposed in several references. Although these methods are recent, the underlying theory is classical, and it is only after the evolution of computational tools that their relevance has become apparent. These methods are based on the principle of equivalent sources for ferromagnetic materials, such as equivalent magnetic charges or equivalent currents, or even a combination of both.

These methods involve replacing a magnetic medium, on which the force calculation needs to be performed, with a non-magnetic medium to which two types of field sources are assigned:

- a volumetric distribution inside the non-magnetic medium,
- a surface distribution on the surface of the non-magnetic medium.

These field source distributions can either be current distributions or magnetic charge distributions.

4.2.2. *Case of Equivalent Currents*

The magnetic medium with permeability μ is replaced by a non-magnetic medium to which two equivalent current sources are associated: a volumetric current density $\mathbf{J}_v$ and a surface current density $\mathbf{J}_s$, whose expressions are given in terms of the magnetization vector **M** as follows:

$$\mathbf{J}_v = \mathbf{rot}(\mathbf{M}_1) \qquad (4.11)$$

$$\mathbf{J}_s = \mathbf{M}_1 \wedge \mathbf{n} \qquad (4.12)$$

n is the unit normal vector to the surface (S) limiting the medium.

The magnetization vector can be expressed in terms of the magnetic field **H** alone:

$$\mathbf{M} = (\mu_r - 1)\mathbf{H} \qquad (4.13)$$

By invoking Laplace's law, the volumetric force density $\mathbf{f}_v$ and the surface force density $\mathbf{f}_s$ are given by the following relations:

$$\mathbf{f}_v = \mathbf{J}_v \wedge \mathbf{B}_1 \qquad (4.14)$$

$$\mathbf{f}_s = \mathbf{J}_s \wedge \mathbf{B}_s \qquad (4.15)$$

$\mathbf{B}_s$ represents the magnetic flux density vector at the surface (S) of the medium.

Considering relations (4.11) and (4.13), the volumetric density of the equivalent current can be given in terms of the magnetic field as follows:

$$\mathbf{J}_v = \mathbf{rot}\left[(\mu_r - 1)\,\mathbf{H}\right]$$

$$= \left[(\mu_r - 1)\mathbf{rot}(\mathbf{H}_1)\right] + \left[\mathbf{grad}(\mu_r - 1) \wedge \mathbf{H}_1\right]$$

$$= \left[(\mu_r - 1)\mathbf{rot}(\mathbf{H}_1)\right] + \left[\mathbf{grad}(\mu_r) \wedge \mathbf{H}_1\right] \qquad (4.16)$$

In the specific case of a physically homogeneous and magnetically linear medium, the expression for the equivalent current density is simplified. It is given by the following equation:

$$\mathbf{J}_{\mathrm{v}} = (\mu_{\mathrm{r}} - 1)\,\mathbf{rot}(\mathbf{H}_1) \tag{4.17}$$

Moreover, if the medium in question does not contain any real conduction currents (i.e., its electrical conductivity σ is zero), then the curl of the magnetic field is null (according to Maxwell's equations). Consequently, the volumetric density of the equivalent current is zero, leaving only the surface density of the equivalent current.

To evaluate the force, it is sufficient to consider the force density created by the surface current density alone. If we take its expression: $\mathbf{f}_{\mathrm{s}} = \mathbf{J}_{\mathrm{s}} \wedge \mathbf{B}_{\mathrm{s}}$, the magnetic induction vector $\mathbf{B}_{\mathrm{s}}$ at the surface (S) of medium 1 is given as equal to the average of the magnetic induction vectors on both sides of the interface, and it is expressed in terms of $\mathbf{B}_1$ and $\mathbf{B}_2$ as :

$$\mathbf{B}_{\mathrm{s}} = \frac{1}{2}(\mathbf{B}_1 + \mathbf{B}_2) \tag{4.18}$$

Given the physical-mathematical conditions for the transition from one physical medium to another, which involve the continuous components of the field $(\mathbf{H}_{\mathrm{t}})$ and the magnetic induction $(\mathbf{B}_{\mathrm{n}})$, as well as the relations (4.12), (4.15), and (4.18), the surface force density will be given by:

$$\mathbf{f}_{\mathrm{s}} = \frac{1}{2\mu_0}\left(\mathrm{B}_{2\mathrm{t}} - \mathrm{B}_{1\mathrm{t}}\right)\mathbf{k} \wedge (\mathbf{B}_2 + \mathbf{B}_1) \tag{4.19}$$

k is the unit vector tangent to the surface S at the considered point.

To obtain the total force, it is sufficient to integrate the expression for the force density over the entire surface (S) surrounding the medium in question.

4.2.3. *Case of equivalent magnetic charges*

In this second case, the magnetic medium with permeability μ is replaced by a non-magnetic medium to which two sources of the field are associated. These sources are a volumetric distribution of magnetic charges with a density ρ_v, inside the medium, and a surface distribution with density ρ_s spread over its surface S.

The mathematical expressions for these distributions are given in terms of the magnetization vector **M** as follows:

$$\rho_v = -\mu_0 \, \mathrm{div}\left(\mathbf{M}_1\right) \qquad (4.20)$$

$$\rho_s = \mu_0 \, \mathbf{n}.\mathbf{M} \qquad (4.21)$$

The magnetization vector can be expressed solely in terms of the magnetic induction **B**:

$$\mathbf{M} = \left(1 - \frac{1}{\mu_r}\right)\frac{\mathbf{B}}{\mu_0} \qquad (4.22)$$

These equivalent distributions of magnetic charges give rise to distributions of magnetic forces: a volumetric distribution of magnetic forces with density $\mathbf{f}_v$ and a surface distribution with density $\mathbf{f}_s$, whose expressions are given by the following relations:

$$\mathbf{f}_v = \rho_v \, \mathbf{H}_1 \qquad (4.23)$$

$$\mathbf{f}_S = \rho_S \, \mathbf{H}_S \tag{4.24}$$

Given relations (4.20) and (4.22), the volumetric density of equivalent magnetic charges is expressed in terms of the magnetic induction as follows:

$$\rho_V = \left[-\left(1 - \frac{1}{\mu_r}\right) \mathrm{div}\left(\mathbf{B}_1\right) \right] + \left[\mathbf{grad}\left(\frac{1}{\mu_r}\right) . \mathbf{B}_1 \right] \tag{4.25}$$

However, since the divergence of the magnetic induction vector is zero (Maxwell's equation), the expression simplifies. Thus, we obtain:

$$\rho_V = \mathbf{grad}\left(\frac{1}{\mu_r}\right) . \mathbf{B}_1 \tag{4.26}$$

For the particular case of a magnetically linear and physically homogeneous medium, the expression for the volumetric density of equivalent magnetic charges is zero, and only the surface density remains.

To evaluate the magnetic forces in question, it is therefore sufficient to consider the surface density of magnetic forces created by the surface distribution of charges. Similarly to the previous case, the magnetic field vector $\mathbf{H}_S$ at the surface (S) of the medium is given as the average of the magnetic field vectors on either side of the interface, expressed in terms of $\mathbf{H}_1$ and $\mathbf{H}_2$ as follows:

$$\mathbf{H}_S = \frac{1}{2}\left(\mathbf{H}_1 + \mathbf{H}_2\right) \tag{4.27}$$

Given the transition conditions from one medium to another, and considering relations (4.21), (4.24), and (4.27), the surface density of the force is then expressed as follows:

$$\mathbf{f}_s = \frac{1}{2\mu_0}\left(H_{2n} - H_{1n}\right)\left(\mathbf{H}_2 + \mathbf{H}_1\right) \tag{4.28}$$

To obtain the total force, it is sufficient to integrate the expression for the force density over the entire surface (S) surrounding the medium in question.

4.2.4. *Other Distributions of Equivalent Sources*

Other distributions of equivalent sources can be deduced from the combination of magnetic charge distributions and equivalent current distributions. One of the most interesting is the combination of a surface distribution of magnetic charges with density ρ_s and a surface distribution of equivalent currents with density $\mathbf{J}_s$, whose expressions are given by the following relations:

$$\rho_s = \mathbf{n}.\mathbf{B}_2 \tag{4.29}$$

$$\mathbf{J}_s = \frac{1}{\mu_0}\mathbf{n} \wedge \mathbf{B}_2 \tag{4.30}$$

In this case, the equivalent sources are purely surface sources. Therefore, only a surface distribution of forces with density $\mathbf{f}_s$ will be created, and its expression is given by:

$$\mathbf{f}_s = \left[\left(\mathbf{n}.\mathbf{B}_2\right)\frac{\mathbf{B}_s}{\mu_0}\right] + \left[\left(\mathbf{n} \wedge \frac{\mathbf{B}_2}{\mu_0}\right) \wedge \mathbf{B}_s\right] \tag{4.31}$$

Similarly to the previous two cases, the magnetic induction vector $\mathbf{B}_s$ at the surface (S) of the medium is given as the average of the magnetic induction vectors on either side of the interface. However, in this particular case, the induction inside the medium is zero. Therefore, the expression for the induction at the surface is a function of the only quantity $\mathbf{B}_2$, and its expression is given by:

$$\mathbf{B}_s = \frac{1}{2}\mathbf{B}_2 \tag{4.32}$$

Substituting relation (4.32) into (4.31), and after developing the latter (4.31), we obtain the expression for the surface density of magnetic forces exerted on the surface of the medium as follows:

$$\mathbf{f}_s = \frac{1}{\mu_0}\left[\left(\mathbf{n}.\mathbf{B}_2\right)\mathbf{B}_2 - \frac{1}{2}\left(\mathbf{B}_2^2\right).\mathbf{n}\right] \tag{4.33}$$

To obtain the total force, it is sufficient to integrate the expression for the force density over the entire surface (S) surrounding the medium in question.

5. COMPARISON OF DIFFERENT METHODS AND APPROPRIATE METHODS

Based on what we have previously presented in this report, we have several methods for determining the force distribution exerted on a given magnetic medium. These methods are summarized in the following table:

Table 5.1. Methods for Calculating Magnetic Forces.

Method	**Expression of the Force**
Variation of Magnetic Coenergy	$\left.\frac{\overline{W}_{s+\Delta s} - \overline{W}_s}{\Delta s}\right\vert_{i=Cte}$
Variation of Magnetic Energy	$\left.-\frac{W_{s+\Delta s} - W_s}{\Delta s}\right\vert_{\Phi=Cte}$
Virtual Work and Finite Elements	$\frac{\partial}{\partial s}\sum_e\left(\int_{V_{e_{local}}}\left(\int_0^H BdH\right)\lvert G\rvert\, dV_e\right)$

Maxwell Tensor	$\mu_0 \oint_s \left((\mathbf{H}.\mathbf{n})\mathbf{H} - \frac{1}{2}(\mathbf{H}^2)\mathbf{n} \right) dS$
Derivative of Magnetic Energy	$\frac{1}{2}\int_s \left[\left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right)\left(B_n^2\right) - (\mu_1 - \mu_2)\left(H_t^2\right) \right] \mathbf{n}\, dS$
Equivalent Currents	$\frac{1}{2\mu_0} \int_s \left[\left(B_{2t} - B_{1t}\right)\mathbf{k} \wedge (\mathbf{B}_2 + \mathbf{B}_1) \right] dS$
Equivalent Charges	$\frac{1}{2\mu_0} \int_s \left[\left(H_{2n} - B_{1n}\right)(\mathbf{H}_2 + \mathbf{H}_1) \right] dS$
Equivalent Currents and Charges	$\frac{1}{\mu_0} \int_s \left[(\mathbf{n}.\mathbf{B}_2)\mathbf{B}_2 - \frac{1}{2}\left(\mathbf{B}_2^2\right).\mathbf{n} \right] dS$

If we take the specific case of a magnetically linear and physically homogeneous medium, the mentioned methods reveal only surface force densities, whose expressions differ from one method to another (Table 5.1). The integration of the force density given by each of these methods leads to the same global force.

For us, as electrical engineers, only the global force exerted on magnetic media is of interest, as it represents the crucial step and essential parameter in the study of phenomena in electrotechnical applications that we are generally required to address, model, or even optimize. For this reason, we recommend the three methods for evaluating the global force exerted on a magnetic medium. These are:

- The method of virtual work,
- The Maxwell stress tensor method,
- The method of the derivative of magnetic energy.

To choose among these three methods, we based our decision on the criterion of adaptability to the problems to be addressed and the ease of implementing the mathematical models on numerical tools.

In the case of the method of the derivative of magnetic energy, the global force is obtained by adding the forces acting on the sides of the finite elements placed at the interface (Air – Ferromagnetic Materials). To do this, we use the expression of the force that involves the continuous components of the field $(\mathbf{H}_t)$ and the magnetic induction $(\mathbf{B}_n)$. However, the finite element method, based on the vector magnetic potential formulation, ensures only the continuity of the normal component of the magnetic induction at the interface and does not ensure the continuity of the tangential component of the magnetic field.

To overcome this difficulty and minimize the resulting error, we will be forced to refine the mesh at the interfaces, particularly at the air gaps. Furthermore, the regions of the domains that separate the fixed and movable parts of electrotechnical structures (such as the air gap in the case of electrical machines) are generally very sensitive areas, where the variation of magnetic phenomena is significant. This leads users to apply fine, and sometimes even very fine, meshing. This approach improves the quality of the results from the method known as the derivative of magnetic energy and results in obtaining values very close to those derived from the virtual work method.

The results obtained with the Maxwell stress tensor method on an integration line considered on a layer of finite elements discretizing the space surrounding the ferromagnetic materials will be close to those provided by the virtual work method. The global force obtained by the Maxwell stress tensor method depends very little on the chosen integration surface, although each chosen surface corresponds to a different force density distribution.

As for the method of virtual work, which is considered as the reference for comparison, its advantages are numerous. Notably, its general applicability and remarkable accuracy in calculations, confirmed by experiments conducted on typical electrotechnical systems, stand out. Its main and major disadvantage lies in the complexity of evaluation and implementation when it is applied to numerical computing tools.

In conclusion, when the calculation of the global force is the sole and primary objective, it is entirely justified to use the Maxwell stress tensor method, and this is for the sole reason of its ease of implementation and integration into numerical computing codes.

GENERAL CONCLUSION

In this document, we have presented a theoretical analysis of the different methods for calculating the forces exerted on magnetic media.

Although each of these methods adopts a different approach to determining the force density, their integration ultimately leads, in practice, to the same global force. This can be explained by the fact that the various formulas are merely attempts to approximate a better representation of the same physical reality, aiming to provide the most accurate possible depiction of global macroscopic quantities.

Finally, we have compared the different methods for evaluating force in a magnetic medium. This comparison has allowed electrotechnical users to select the Maxwell stress tensor method due to its suitability for most applications and its ease of implementation in both existing and future computational codes.

Bibliography

[1] **E. DURANT**, "Magnétostatique", Masson et Cie, Editeurs, Paris, France 1968.

[2] **J. A. EDMINISTER**, "Electromagnétisme", Série Schaum, 1985.

[3] **A. ANGOT**, "Compléments de Mathématiques à l'usage des Ingénieurs de l'Electrotechnique et des Télécommunications", Sixième Edition, Masson et Cie, Editeurs, 1972.

[4] **M. JUFER**, "Electromécanique", Traité d'Electricité, d'Electronique et d'Electrotechnique", Ecole Polytechnique Fédérale de Lausanne, Dunod, Editions Georgi, 1979.

[5] **G. FOURNET**, "Electromagnétisme à Partir des Equations Locales", Masson Editeurs, Paris, France 1985.

[6] **A. NICOLET** and **R. BELMANS**, "Electric and Magnetic Fields from Numerical Models to Industrial Applications", Plenum Press, New York, USA, 1995.

[7] **K. J. BINNS**, **P. J. LAWRENSON** and **C. W. TROWBRIDGE**, "The Analytical and Numerical Solution of Electric and Magnetic Fields", John Wiley & Sons Publishers, New York, USA, 1992.

[8] **C. PUSTERLE**, "Analyse Vectorielle des Champs", Masson Editeur, Paris, France 1991.

[9] **G. DHATT** et **G. TOUZOT**, "Une Présentation de la Méthode des Eléments Finis", Maloine S. A. Editeur Paris, 1984.

[10] **O. C. ZIENKIEWICZ**, "La Méthode des Eléments Finis Appliquée à l'Art de l'Ingénieur", Edi science, Paris, 1973.

[11] **J. L. COULOMB** et **J. C. SABONNADIERE**, "C.A.O. en Electrotechnique", Hermes Publishing, 1985.

[12] **D. EUVRARD**, "Résolution Numérique des Equations aux Dérivées Partielles de la Physique, de la Mécanique et des Sciences de l'Ingénieur, Différence Finis, Eléments Finis, Problèmes en Domaine Non Borné", Masson Editeurs, Troisième Edition, Editeurs, Paris, France 1994.

[13] **C. BREZINSKI**, "Analyse Numérique Discrète", Publications du Laboratoire de Calcul, Université des Sciences et Techniques de Lille, France.

[14] **C. BREZINSKI**, "Algorithmique Numérique", Publications du Laboratoire de Calcul, Université des Sciences et Techniques de Lille, France.

[15] **J. L. COULOMB** et **G. MEUNIER**, "Calcul des Forces, Couples et Raideurs dans les Machines Electriques par la Méthode des Eléments Finis", Rapport Interne, L. E. G., I.N.P. Grenoble, France.

[16] **J. B. ALBERTINI**, "Calcul de Forces Electromagnétiques sur des Conducteurs en 3D et Maillage d'Objets Complexes", Rapport Interne, L. E. G., I.N.P. Grenoble, France.

[17] **J. H. BODMER** and **D. E. LIMBERT**, "Force Computation by the Magnetizing Current Method", Digital Equipment Corporation, Maynard-Ma, University of New Hampshire, Durham NH, USA, Rapport interne.

[18] **K. REICHERT**, **H. FREUNDL** and **W. VOGT**, "The Calculation of Forces and Torques within Numerical Magnetic Field Calculation Methods", AG Brown Boveri, Baden, Switzerland, Rapport Interne.

[19] **F. DELINCÉ**, **P. DULAR**, **A. GENON**, **W. LEGROS**, **N. NICOLET** and **M. UMÉ**, "Modelization of the Non-Linear Moving Parts of an Electromechanical Relay", Rapport Interne, University of Liège, Department of Electrical Engineering, Institute Montefort-SartTilman, Liège, Belgium.