People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Mohamed Khider University of Biskra

Faculty of Exact Sciences and Siences of Nature and Life

Department of Mathematics

Dissertation presented for the purpose of obtaining the Diploma:

Master's degree in mathematics

Option : **Probability and Statistics**

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Title :

A Numerical Method for Solving Stochastic Linear Quadratic Problem with a Finance Application

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02/06/2025

Dedication

To the one who planted the seeds of ambition in my heart and nurtured me with love and prayers, to my dear father **Abdelhamid** it is for you that I stood tall with knowledge,and for you that I wrote this achievement.

To my mother **Khamsa Kasmiouri**, the pulse of my heart and soul, your prayers lit my path and your patience guided my steps. Every word in this thesis is the fruit of your love and sacrifices.

To my beloved grandmother **Salhaoui Zineb**, may Allah have mercy on her, who always believed in me and was proud of all that I did.

To my dear uncle **Lakhder**, may Allah have mercy on him, who was always my supporter, believing in every step I took. His voice still echoes in my heart; his memory lives on, even in his absence.

To my dear brother **Haythem** and my beloved sisters **Khouloud**, **Shilia**, and **Sirta**,you were the wings that helped me soar,and your presence was my safe haven.

To **my uncles** and **maternal uncles**, you have always been a support and strength, with your prayers, encouraging words, and sincere love.

To **my dearest friends**, companions of the journey, through lectures and laughter, you were the most beautiful part of this path.

To those who supported me in moments of weakness, to those who understood me without a word, to everyone who taught me even a single letter—To all whom I mentioned

I dedicate this graduation as a token of love, gratitude, and loyalty.

Acknowledgment

First and foremost, I thank Allah, the Almighty, for granting me the strength to persevere and the courage to overcome all difficulties throughout this journey.

I would like to express my deepest gratitude to my supervisor, Pr.**CHIGHOUB FARID**, for his invaluable guidance, unwavering support, and patience during the preparation of this thesis. His insightful advice and constant encouragement have played a crucial role in the successful completion of this work.

I am sincerely grateful to *the jury members Dr.TAMER LAZHAR & Dr.NINOUH ABDELHAKIM* for taking the time to evaluate my work and for their constructive feedback, which helped improve the quality of this thesis.

Special thanks go to all *the professors and staff* of the *Department of Mathematics* at the **University of Biskra** for the knowledge they have imparted and the support they have provided throughout my academic journey.

My heartfelt appreciation goes to **my family**, especially *my parents*, for their unconditional love, endless support, and encouragement at every stage of my studies. Finally, I extend my thanks to **my friends** and **fellow classmates** for their collaboration, support, and the memorable moments we shared during these years.

Thank's

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Introduction

Stochastic control theory plays a central role in the mathematical modeling of decision-making under uncertainty, with wide-ranging applications in engineering, finance, and physics. At the heart of this field lies the linear-quadratic (LQ) control problem, which has long been regarded as a foundational model due to its analytical tractability and rich structure. Seminal contributions such as those by Bismut [2], Wonham [18], and McLane [10] have laid the theoretical groundwork for the analysis of LQ control systems, particularly in stochastic environments where noise and randomness affect both state dynamics and control processes.

The stochastic maximum principle (SMP) and dynamic programming principle (DPP) are two cornerstones in this theory. The duality-based approaches introduced in Bismut [2], and further developed in Haussmann [6], provide powerful tools for characterizing optimal controls, while the Hamilton–Jacobi–Bellman (HJB) framework, as detailed in Yong and Zhou [19], remains fundamental to solving time-consistent optimization problems. For models involving Brownian motion and stochastic calculus, the standard references Karatzas and Shreve [9] and Oksendal [11] offer rigorous treatments of the mathematical foundations.

In recent decades, mean-variance portfolio optimization has emerged as a prominent application of stochastic LQ control, extending Markowitz's original idea to continuous-time settings. The work by Zhou and Li [19] represents a significant advance, recasting the mean-variance problem within the stochastic LQ framework. Simultaneously, the notion of stochastic differential utility, as formulated in Duffie and Epstein [5], has enabled more flexible preference modeling in continuous-time finance, further motivating advanced control formulations.

Despite their elegant theoretical formulation, practical implementation of stochastic control models, especially in high-dimensional or nonlinear settings, demands efficient numerical methods. Techniques for solving large-scale Lyapunov and Riccati equations are well-documented in Benner et al. [1], while various integral equation approaches—such as the wavelet-Galerkin methods in Heydari et al. [7] and iterative schemes in Saffarzadeh et al. [14]—demonstrate promise for solving stochastic integral and Volterra-type equations. Complementary numerical tools, including function approximation and spline techniques [7,8], further enhance computational capabilities. The goal of this study is to propose a numerical approach to a class of stochastic LQR problems with control-dependent diffusion coefficients. The proposed method involves solving nonlinear stochastic Itô-Volterra integral equations using a combination of fixed-point iteration and linear spline interpolation [7], [8], [14]. This approach offers a practical and efficient alternative, especially considering the limited number of numerical examples for such problems found in the literature. To evaluate the performance of the method, relative error is used as a convergence criterion. The structure of this thesis is as follows: **Chapter 1:** provides a general introduction to stochastic calculus. **Chapter 2:** presents a numerical approach to the stochastic LQR problem. **Chapter 3 :**discusses numerical applications of the stochastic LQR problem in finance.

Chapter 1

Introduction to Stochastic Computation

Let's embark on a journey through the intriguing realm of stochastic computation. This chapter serves as our gateway, introducing foundational terms and concepts that will pave the way for deeper exploration in subsequent sections. For more details see e.g [9], [11] and [12] .

1.1 Stochastic Processes

Definition 1.1.1 (Stochastic Process) *A stochastic process, symbolized by* $X = (X(t))_{t \in \mathcal{T}}$*, emerges as a collection of random variables defined within a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with values spanning* $\mathbb{R}^n$*.*

Now, let's elucidate some key terminology,

1. The variable t typically denotes time.

2. If the index set $\mathcal{T}$ is countable, we classify X as a discrete-time stochastic process; for a continuous index set, it's a continuous-time stochastic process.

3. Common index sets include the half-line $[0, \infty)$ or a finite interval $[0, T]$, where $T > 0$.

4. Each $X(t)$ represents a random variable, mapping $\omega \longrightarrow X(t, \omega)$ for $\omega \in \Omega$.

5. Fixing $\omega \in \Omega$ transforms $X(\omega)$ into a function $t \longrightarrow X(t, \omega)$, known as a path of X.

Definition 1.1.2 (Modification of Process) *We term a stochastic process* $(X(t))_{t\in\mathcal{T}}$ *a modification of another process* $(\bar{X}(t))_{t\in\mathcal{T}}$ *if the probability* $\mathbb{P}(X(t) = \bar{X}(t))$ *equals* 1 *for all* $t \in \mathcal{T}$.

Definition 1.1.3 (Indistinguishable Processes) *When* $\mathbb{P}(X(t) = \bar{X}(t), \forall t \in \mathcal{T})$ *equals* 1*, we refer to two stochastic processes* $(X(t))_{t\in\mathcal{T}}$ *and* $(\bar{X}(t))_{t\in\mathcal{T}}$ *as indistinguishable.*

Remark 1.1.1 *Indistinguishable processes are modifications of each other, but the converse isn't always true.*

Definition 1.1.4 (Measurable stochastic process) *A stochastic process* $X = (X(t))_{t\in\mathcal{T}}$ *is measurable if the mapping* $X : [0, T] \times \Omega \to \mathbb{R}^n$ *is* $(\mathcal{B}([0, T]) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$ *measurable.*

Definition 1.1.5 (Filtration) *A filtration comprises a sequence of* σ*-algebras* $\{\mathcal{F}_t\}_{t\geq 0}$ *defined over a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, wherein* $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ *for all* $0 \leq s \leq t$*. This structure captures the evolving information accessible to an observer over time, with* $\mathcal{F}_t$ *representing distinguishable events up to time* t.

- A filtration is right continuous if $\mathcal{F}_{t^+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \geq 0$.

- A filtration is complete if $\mathcal{F}_0 \subset \mathcal{F}_t$, and it's termed to satisfy the usual conditions if it's both right continuous and complete.

Definition 1.1.6 (Adapted stochastic process) *We say that a stochastic process $X = (X(t))_{t\in\mathcal{T}}$ is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each t.*

Definition 1.1.7 (Natural filtration) *The natural filtration of a stochastic process $(X(t))_{t\in\mathcal{T}}$ comprises the collection of σ-algebras $\{\mathcal{G}(t)\}_{t\geq 0}$, where $\mathcal{G}(t) = \sigma\{X(s) : 0 \leq s \leq t\}$ for all $t \geq 0$. It's the minimal augmented filtration generated by $(X(t))_{t\in\mathcal{T}}$, characterized by being both right continuous and complete.*

Definition 1.1.8 (Stopping time) *A stopping time τ with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ is a random variable $\tau : \Omega \to [0, +\infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathcal{T}$.*

Definition 1.1.9 (σ-algebra of events prior to $\mathbb{T}$) *For a stopping time τ, the σ-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \in \mathcal{T}\}$ captures events preceding $\mathcal{T}$.*

1.1.1 Brownian Motion

Definition 1.1.10 (Standard Brownian Motion) *The Standard Brownian Motion, also known as Wiener process, manifests as a stochastic process $(B(t))_{t\geq 0}$ characterized by independent and identically distributed increments,*

1. *It starts at 0 almost surely, $B(0) = 0$.*

2. *For all $0 \leq s < t$, the increment $B(t) - B(s)$ follows a normal distribution with mean 0 and variance $t - s$ (i.e., $B(t) - B(s) \sim N(0, t - s)$).*

3. *The sample paths of $B(t)$ are almost surely continuous.*

Definition 1.1.11 (d-dimensional Brownian Motion) *A d-dimensional Brownian motion, denoted by $B = \left(B^{(1)}, B^{(2)}, \ldots, B^{(d)}\right)$, is defined by considering $B^{(i)}$ as independent standard Brownian motions for $i = 1, 2, \ldots, d$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a Brownian motion B is defined as,*

$$\mathcal{F}_t = \sigma(B(s) : s \leq t), \quad t \geq 0,$$

and it is called the natural filtration of B or Brownian filtration.

1.2 Martingales

Definition 1.2.1 (Martingale) *A continuous-time martingale (resp. submartingale, supermartingale) is a stochastic process $\{X(t), t \geq 0\}$ satisfying the following conditions,*

1. *$X(t)$ is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., $X(t)$ is measurable with respect to $\mathcal{F}_t$ for all $t \geq 0$,*
2. *$X(t)$ is integrable for all $t \geq 0$,*
3. *For all $0 \leq s \leq t$, $\mathrm{E}\left(X(t)|\mathcal{F}_s\right) =$ (resp. $\leq, \geq$)$X(s)$ almost surely.*

Remark 1.2.1 *A process X is a martingale if it is both a submartingale and a supermartingale. If X is a martingale, then $\mathrm{E}(X(t)) = \mathrm{E}(X(0))$ for all $t \in \mathcal{T}$.*

Example 1.2.1 *If B is a Brownian motion, then $B(t)$, $B^2(t) - t$, and $\exp\left(\sigma B(t) - \frac{\sigma^2 t}{2}\right)$ for $t \in \mathcal{T}$ are martingales. Conversely, if X is a continuous process such that $\{X(t)\}_{t\geq 0}$ and $\{X^2(t) - t\}_{t\geq 0}$ are martingales, then X is a Brownian motion.*

Definition 1.2.2 (Local Martingale) *A stochastic process* $\{M(t)\}_{t\in\mathbb{R}^+}$ *adapted and caglad (right-continuous with left limits) is a local martingale if there exists an increasing sequence of stopping times* (τ_n) *such that* $\tau_n \to +\infty$ *as* $n \to \infty$ *and* $M(t \wedge \tau_n)$ *is a martingale for all* n.

Remark 1.2.2 *A positive local martingale is a supermartingale. A locally uniformly integrable martingale is a martingale.*

Theorem 1 (Burkholder-Davis-Gundy) For any $1 \leq p < \infty$ there exist positive constants c_p, C_p such that, for all local martingales X with $X_0 = 0$ and stopping times τ, the following inequality holds.

$$c_p\mathbb{E}\left[[X]_\tau^{p/2}\right] \leq \mathbb{E}\left[(X_\tau^*)^p\right] \leq C_p\mathbb{E}\left[[X]_\tau^{p/2}\right].$$

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$.

Definition 1.2.3 (Semimartingale) *A semimartingale is a cadlag adapted process* X *admitting a decomposition of the form,* $X = A + M$, *where* M *is a cadlag local martingale null at* 0 *and* A *is an adapted process of finite variation and null at* 0.

A continuous semimartingale is a semimartingale X such that in the decomposition $X = A + M$, M and A are continuous. Such a decomposition where M and A are continuous is unique.

1.3 Stochastic Integration and Itô's Formula

In this section, we consider a positive real number T, and aim to define the integral

$$\mathbb{I}(\theta) = \int_0^T \theta(t) dB(t) \tag{1.1}$$

Here, $(\theta(t))_{t\geq 0}$ represents any process, and $(B(t))_{t\geq 0}$ denotes a Brownian motion. The challenge lies in giving meaning to the differential element $dB(s)$ since the function $s \rightarrow B(s)$ is not differentiable.

1.3.1 Wiener Integral

The Wiener integral is an integral of the form

$$\mathbb{I}(\theta) = \int_0^T \theta(t) dB(t) \tag{1.2}$$

with θ being a deterministic function, meaning it does not depend on the random variable ω. Define

$$L^2([0,T], \mathbb{R}) = \left\{ \theta : [0,T] \rightarrow \mathbb{R} \text{ such that } \int_0^T |\theta(s)|^2 ds < \infty \right\}.$$

Suppose θ_n is a deterministic step function defined as

$$\theta_n(t) = \sum_{i=1}^{p_n} \alpha_i 1_{[t_i^n, t_{i+1}^n]}(t),$$

where $p_n \in \mathbb{N}$, the α_i are real numbers, and $\{t_i^n\}$ is an increasing sequence in $\mathcal{T} = [0, T]$. Then, the Wiener integral is defined as

$$\mathbb{I}(\theta_n) = \int_0^T \theta_n(s) dB(s) = \sum_{i=1}^{p_n} \alpha_i \left(B(t_{i+1}) - B(t_i) \right).$$

Due to the Gaussian nature of Brownian motion and the independence of its increments, the random variable $\mathbb{I}(\theta_n)$ is a Gaussian variable with zero mean and variance

$$\begin{aligned} \operatorname{Var}(\mathbb{I}(\theta_n)) &= \sum_{i=1}^{p_n} \operatorname{Var}\left(\alpha_i \left(B(t_{i+1}) - B(t_i)\right)\right) \\ &= \sum_{i=1}^{p_n} \alpha_i^2 (t_{i+1} - t_i) \\ &= \int_0^T \theta_n^2(s) ds \end{aligned}$$

Remark 1.3.1 *We observe that $\theta \to \mathbb{I}(\theta)$ is a linear function. Moreover, if b and g are two step functions, we have*

$$\mathrm{E}(\mathbb{I}(b)\mathbb{I}(g)) = \int_0^T b(s)g(s)ds.$$

We then refer to the isometry property of the Wiener integral. Now let $\theta \in L^2([0, T], \mathbb{R})$. Therefore, there exists a sequence of step functions $\{\theta_n, n \geq 0\}$ that converges in $L^2([0, T], \mathbb{R})$ to θ. According to the previous paragraph, we can construct the Wiener integrals $\mathbb{I}(\theta_n)$, which are centered Gaussians forming a Cauchy sequence by isometry. Since the space $L^2([0, T], \mathbb{R})$ is complete, this sequence converges to a Gaussian random variable denoted by $\mathbb{I}(\theta)$. It can be shown that the limit does not depend on the choice of the sequence $\theta_n, n \geq 0$. $\mathbb{I}(\theta)$ is called the Wiener integral of θ with respect to $(B(t))_{t \in \mathbb{R}}$.

1.3.2 The Itô's integral

Our objective now is to define the integral given by equation (1.2). To achieve this, we construct $\mathbb{I}(\theta)$ using discretization, similar to the approach used for the Wiener integral. Let's start by examining step processes represented by,

$$\theta_n(t) = \sum_{i=0}^{p_n} \alpha_i 1_{\left[t_i^n, t_{i+1}^n\right]}(t), \tag{1.3}$$

where $p_n \in \mathbb{N}$, (t_i^n) forms an increasing sequence in $\mathcal{T} = [0, T]$, and $\alpha_i \in L^2\left(\Omega, \mathcal{F}_{t_i}, \mathbb{P}\right)$ for all $i = 0, \cdots, p_n$. We define $\mathbb{I}\left(\theta_n\right)$ as

$$\mathbb{I}\left(\theta_n\right) = \sum_{i=0}^{p_n} \alpha_i \left(B\left(t_{i+1}\right) - B\left(t_i\right)\right).$$

It can be confirmed that $\mathrm{E}\left(\mathbb{I}\left(\theta_n\right)\right) = 0$, and

$$\operatorname{Var}\left(\mathbb{I}\left(\theta_n\right)\right) = \mathrm{E}\left(\int_0^T \theta_n^2(s) ds\right).$$

Let H denote the space of caglad (left-continuous and right-limited), $\mathcal{F}_t$ adapted processes θ such that

$$\|\theta\|^2 = \mathrm{E}\left(\int_0^T |\theta(s)|^2 ds\right) < \infty.$$

We can define $\mathbb{I}(\theta)$ for any $\theta \in H$. We approximate θ using a sequence of step processes given by equation (1.3), and the limit exists in $L^2(\Omega, [0, T])$. The integral $\mathbb{I}(\theta)$ is then defined as $\lim_{n \to +\infty} \mathbb{I}\left(\theta_n\right)$, where $\mathrm{E}\left(\mathbb{I}(\theta)\right) = 0$, and

$$\operatorname{Var}(\mathbb{I}(\theta)) = \mathrm{E}\left(\int_0^T \theta^2(s) ds\right).$$

Definition 1.3.1 (Itô Process) *An Itô process is defined as a real-valued process $(X(t))_{t\in\mathcal{T}}$ satisfying the following conditions almost surely,*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s), \quad \text{for } 0 \leq t \leq T. \tag{1.4}$$

Alternatively, it can be expressed differentially as,

$$dX(t) = b(t)dt + \sigma(t)dB(t).$$

Here, $X(0)$ is $\mathcal{F}_0$-measurable, and b and σ are two progressively measurable processes, which satisfy almost surely,

$$\int_0^T |b(s)|ds < \infty, \quad \text{and} \quad \int_0^T |\sigma(s)|^2 ds < \infty.$$

In other words, $b \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$ and $\sigma \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$. The coefficient b represents the drift or derivative, and σ is the diffusion coefficient.

Definition 1.3.2 (Integration by Parts Formula) *If X and $\bar{X}$ are two Itô processes, where*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s),$$

and

$$\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{b}(s)ds + \int_0^t \bar{\sigma}(s)dB(s),$$

then the integration by parts formula states that

$$X(t)\bar{X}(t) = X(0)\bar{X}(0) + \int_0^t X(s)d\bar{X}(s) + \int_0^t \bar{X}(s)dX(s) + \langle X, \bar{X}\rangle_t,$$

where

$$\langle X, \bar{X} \rangle_t = \int_0^t \sigma(s)\bar{\sigma}(s)ds.$$

1.4 Itô's Formula

Definition 1.4.1 (Itô's Formula) *Let* $b \in L^1_{\mathcal{F}_t}[0,T]$, $\sigma \in L^2_{\mathcal{F}_t}[0,T]$, *and let* X *be an Itô process defined as in (1.4). Define* $\langle X(t) \rangle = \int_0^t |\sigma(s)|^2 ds$. *If* $h \in C^{1,2}(\mathcal{T} \times \mathbb{R}, \mathbb{R})$, *then*

$$\begin{aligned} dh(t, X(t)) &= \partial_t h(t, X(t))dt + \partial_x h(t, X(t))dX(t) + \frac{1}{2}\partial_{xx} h(t, X(t))d\langle X \rangle_t \\ &= (\partial_t h(t, X(t)) + \partial_x h(t, X(t))b(t) + \frac{1}{2}|\sigma(t)|^2 \partial_{xx} h(t, X(t)))dt \\ &+ \partial_x h(t, X(t))\sigma(t)dB(t). \end{aligned}$$

1.5 Stochastic Differential Equation

The proof of existence is carried out by successive approximations, similar to that for ordinary differential equations (Picard iterations). It can be found in [**?**]. Let $B = (B(t))_{t \geq 0}$ denote a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. For $T > 0$, consider two functions

$$b : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \qquad \text{and} \qquad \sigma : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}.$$

Here, $\|\sigma\|$ denotes the trace of $\sigma\sigma^\top$. We aim to solve the following stochastic differential equation,

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dB(t)$$

Definition 1.5.1 *Let* $b \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$ *and* $\sigma \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$. *A (strong) solution to this stochastic differential equation is a continuous stochastic process* $X = (X(t))_{t\geq 0}$ *that satisfies*

$$X(t) = X(0) + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dB(s),\ \mathbb{P}-a.s.$$

Existence and Uniqueness of Strong Solutions

Consider the SDE

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dB(t)$$

Theorem 1.5.1 *Under conditions*

1. ***Lipschitz continuity,*** *There exist constants* $L, K > 0$ *such that for all* $t \geq 0,\ x, y \in \mathbb{R}^n$,

$$|b(t,x) - b(t,y)|^2 + \|\sigma(t,x) - \sigma(t,y)\|^2 \leq L|x-y|^2$$

2. ***Linear growth,*** *For* $t \geq 0$,

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 \leq K(1 + |x|^2),$$

3. $X(0)$ *is independent of* B, *and* $\mathrm{E}\,|X(0)|^2 < \infty$.

The SDE has a unique solution X. *Additionally, for every* $t \geq 0$, *the solution satisfies*

$$\mathrm{E}\left(\sup_{t\in[0,T]} |X(t)|^2\right) \leq C\mathrm{E}\left(1 + |X(0)|^2\right) < +\infty.$$

where constant C depends only on K and T.

Chapter 2

Numerical Approach to the Stochastic LQR Problem

2.1 Solution to LQR Problem

The stochastic Linear Quadratic Regulator (LQR) problem is one of the most important problems in optimal control. It aims to determine control strategies for a dynamic system represented by linear differential equations with a random component. This problem involves optimizing a quadratic cost function depending on the system state and control, making it applicable in various fields such as engineering, economics, and finance, particularly in financial market control.

The main approach to solving this problem is through the use of Riccati differential equations, which allow the determination of optimal control strategies over time. In this section, we will present how to formulate and solve stochastic LQR equations using Riccati equations, emphasizing the importance of the existence of unique solutions to the equations' boundary conditions and the practical applications of these solutions.

Let $(\Omega, \mathcal{F}, \{F_t\}_{t_0 \leq t \leq t_f}, \mathbb{P})$ be given and fixed complete filtered probability space and let $\xi\left([t_0, t_f]; S\right)$ be the Banach space of S-valued continuous functions on $[t_0, t_f]$ endowed with the maximum norm $\| \cdot \|$ for a given Hilbert space S. Consider the following linear stochastic differential equation:

$$d\mathcal{X}(t) = [m_1(t)x(t)+m_2(t)v(t)+f(t)]dt+[m_3(t)v(t)]dW(t), \ \ t_0 \leq t \leq t_f \tag{2.1}$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0$. Here $W(\cdot)$ is an n-dimensional standard Brownian motion on the interval $[t_0, t_f]$ and $\mathcal{X}(\cdot) \in \mathbb{R}^n$ (n-dimensional space).Furthermore ,the entries of $n \times n$ matrix $m_1(t)$ and $n \times m$ matrices $m_3(t)$ and $m_2(t)$ belong to $\xi([t_0; t_f]; \mathbb{R})$.The cost functional associated with system 2.1 is:

$$\begin{aligned} &\mathbf{J}(t_0, \mathcal{X}_0, v(\cdot)) \\ &= \mathbb{E}\left\{ \frac{1}{2}\int_0^{t_f} \left[\mathcal{X}^T(t)m_4(t)\mathcal{X}(t) + v^T(t)m_5(t)v(t)\right] dt + \frac{1}{2}\mathcal{X}^T(t_f)m_6\mathcal{X}(t_f) \right\}, \end{aligned} \tag{2.2}$$

where $\mathbb{E}$ indicates the mathematical expectation and the symbol T denotes the transpose operation. Let the $n \times n$ matrices $m_4(t)$ and m_6 be symmetric positive semidefinite, the $m \times m$ matrix $m_5(t)$ be symmetric, and t_f be an exit time or a terminal time. In addition, the matrices $m_4(t)$ and $m_5(t)$ have continuous entries. Note that here we do not assume that $m_5(t)$ is positive definite.

Definition 2.1.1 *A feedback control law is a piecewise continuous function* $\mathbb{V}(\cdot;\cdot)$ *from* $\left[t_0; t_{t_f}\right] \times \mathbb{R}^n$ *into* V*, where* V *is a closed subset of* $\mathbb{R}^m$*. Here, the class of admissible controls is denoted by* v*. The control applied at time* t *by using the feedback control* $\mathbb{V}$ *is* $v(t) = \mathbb{V}(t; x(t))$*. The minimization of stochastic LQR* *optimal control problem is regarded as the task of finding an optimal control* $\tilde{v}(\cdot) \in \mathcal{V}$ *on the interval* $[t_0; t_f]$ *by minimizing* $\mathbf{J}$ *and the associated optimum performance index* $\tilde{\mathbf{J}}(t_0; \mathcal{X}_0)$ *is the value of* $\mathbf{J}$ *resulted by using the optimal control.*

Consider the matrix Riccati differential equation for $t_0 \leq t \leq t_f$

$$\dot{K}(t) = -K^T(t)m_1(t) - m_1^T(t)K(t) - m_4(t) \tag{2.3a}$$
$$+K^T(t)m_2(t)\left(m_5(t) + m_3^T(t)K(t)m_3(t)\right)^{-1} m_2^T(t)K(t),$$

with boundary condition $K(t_f) = m_6$. Also, consider the vector differential equation for $t_0 \leq t \leq t_f$

$$\dot{g}(t) = -m_1^T(t)g(t) - K(t)f(t) + K^T(t)m_2(t)\left(m_5(t) + m_3^T(t)K(t)m_3(t)\right)^{-1} m_2^T(t)g(t), \tag{2.4}$$

with boundary condition $g(t_f) = 0$, here $m_5(t) + m_3^T(t)K(t)m_3(t)$ is a positive definite matrix.

Assume that Eqs. 2.3a and 2.4 have unique solutions $K(t)$ and $g(t)$, respectively, such that the entries of $n \times n$ matrix $K(t)$ belong to $\xi([t_0; t_f]; \mathbb{R})$ and $g(t) \in \xi([t_0; t_f]; \mathbb{R}^n)$. Then the stochastic LQR problem can be reduced to solving these differential boundary problems.

Theorem 2.1.1 . *Assume that Eqs. 2.3a and 2.4 have unique solutions. Then an optimal control for problem 2.1–2.2 is*

$$\tilde{v}(t) = -P(t)^{-1}m_2^T(t)\left(K(t)\mathcal{X}(t) + g(t)\right), \qquad t_0 \leq t \leq t_f \tag{2.5}$$

where

$$P(t) = m_5(t) + m_3^T(t)K(t)m_3(t), \qquad t_0 \leq t \leq t_f. \tag{2.6}$$

Furthermore, the optimal cost value is

$$\tilde{\mathbf{J}}(t_0, \mathcal{X}_0) = \frac{1}{2}\int_{x_0}^{t_f} \left[2f^T(t)g(t) + g^T(t)m_2(t)P^{-1}(t)m_2^T(t)g(t)dt\right] dt \qquad (2.7)$$
$$+ \frac{1}{2}\mathcal{X}_0^T K(t_0)\mathcal{X}_0 + \mathcal{X}_0 g(t_0).$$

Proof. Consider the stochastic linear differential Eq. 2.1. Using Ito's formula, we have

$$d(\mathcal{X}^T(t)K(t)\mathcal{X}(t)) \qquad (2.8)$$
$$= \mathcal{X}^T(t)\dot{K}(t)\mathcal{X}(t)dt + 2\mathcal{X}^T(t)K(t)d\mathcal{X}(t) + \left(v^T(t)m_3^T(t)K(t)m_3^T(t)v(t)\right)$$
$$+ \left\{\mathcal{X}^T(t)\left(-m_4(t) + K(t)m_2(t)P^{-1}(t)m_2^T(t)K(t)\right)\mathcal{X}(t) + 2v^T(t)m_2^T(t)K(t)\mathcal{X}(t)\right.$$
$$\left. +2\mathcal{X}^T(t)K(t)f(t) + \left(v^T(t)m_3^T(t)K(t)m_3^T(t)v(t)\right)\right\} dt$$
$$+ \left\{2\mathcal{X}^T(t)K(t)m_3(t)v(t)\right\} dW(t).$$

and

$$d(\mathcal{X}^T(t)g(t)) = \left\{\mathcal{X}^T(t)K^T(t)m_2(t)P^{-1}(t)m_2^T(t)g(t) - \mathcal{X}^T(t)K(t)f(t)\right.$$
$$\left. +v^T(t)m_2^T(t)g(t) + f^T(t)g(t)\right\} dt$$
$$+ \left\{v^T(t)m_3^T(t)g(t)\right\} dW(t) \qquad (2.9)$$

The boundary conditions of 2.3a and 2.4 imply

$$\mathcal{X}^T(t_f)m_6(t_f) = \int_{t_0}^{t_f} d(\mathcal{X}^T(t)K(t)\mathcal{X}(t)) + \mathcal{X}_0^T K(t_0)\mathcal{X}_0,$$

and

$$\int_{t_0}^{t_f} d\left(\mathcal{X}^T(t)g(t)\right) + \mathcal{X}_0 g(t_0) = 0.$$

Thus by integrating both 2.8 and 2.9 fromt_0 to t_f, taking expectations, and adding them together, the cost functional 2.2 gets

$$\begin{aligned}
\mathbf{J}(t_0, \mathcal{X}_0, v(\cdot)) &= \frac{1}{2}\mathbb{E}\{\int_{t_0}^{t_f}[\mathcal{X}^T(t)m_4(t)\mathcal{X}(t) + v^T(t)\left(P(t) - m_3^T(t)K(t)m_3(t)\right)v(t)]dt \\
&+ \int_{t_0}^{t_f} d(\mathcal{X}^T(t)K(t)\mathcal{X}(t) + d\left(\mathcal{X}^T(t)\mathbf{g}(t)\right)\} + \mathcal{X}_0^T K(t_0)\mathcal{X}_0 + \mathcal{X}_0 g(t_0) \\
&= \frac{1}{2}\mathbb{E}\{\int_{t_0}^{t_f}[v^T(t)P(t)v(t) + 2v^T(t)m_2^T(t)(K(t)\mathcal{X}(t) + g(t)) \\
&+ \mathcal{X}^T(t)K(t)m_2(t)P^{-1}(t)m_2^T(t)K(t)\mathcal{X}(t) + 2f^T(t)g(t) \\
&+ 2\mathcal{X}^T(t)K(t)m_2(t)P^{-1}(t)m_2^T(t)g(t)]dt\} + \frac{1}{2}\mathcal{X}_0^T K(t_0)\mathcal{X}_0 + \mathcal{X}_0 g(t_0) \\
&+ 2f^T(t)g(t) - g(t)m_2(t)P^{-1}(t)m_2^T(t)g(t)]dt\} + \frac{1}{2}\mathcal{X}_0^T K(t_0)\mathcal{X}_0 + \mathcal{X}_0 g(t_0).
\end{aligned}$$

Then we get

$$\begin{aligned}
\mathbf{J}(t_0, \mathcal{X}_0, v(\cdot)) = \frac{1}{2}\mathbb{E}\{\int_{t_0}^{t_f}&[(v(t)P^{-1}(t)m_2^T(t)(K(t)\mathcal{X}(t) + g(t)))^T P(t) \\
&+ (v(t) + P^{-1}(t)m_2^T(t)(K(t)\mathcal{X}(t) + g(t)))
\end{aligned} \tag{2.10}$$

Here $P(t)$ is a positive definite matrix, and so:

$$(v(t)+P^{-1}(t)m_2^T(t)(K(t)\mathcal{X}(t)+g(t)))^T P(t)(v(t)+P^{-1}(t)m_2^T(t)(K(t)\mathcal{X}(t)+g(t))) \geq 0.$$

Thus, the minimum value of the cost functional occurs when:

$$v(t) + P^{-1}(t)m_2^T(t)(K(t)\mathcal{X}(t) + g(t)) = 0.$$

It follows immediately that the optimal feedback control is resulted using 2.5 and the optimal cost value is resulted using 2.7. The optimal control is given by 2.2 provided that the corresponding Eq. 1.1 under 2.5 has a solution. However, underl 2.5, Eq. 2.1 is reduced to:

$$\begin{cases} d\mathcal{X}(t) = \left[m_1(t)\mathcal{X}(t) - m_2(t)P^{-1}(t)m_2(t)\left(K(t)\mathcal{X}(t) + g(t)\right)\right]dt \\ -m_3(t)P^{-1}(t)m_2^T(t)\left(K(t)\mathcal{X}(t) + g(t)\right)dW(t), \quad t_0 \le t \le t_f, \\ \mathcal{X}_{t_0} = \mathcal{X}_0. \end{cases} \tag{2.11}$$

Note that $\mathbf{K} \in \xi([t_0; t_f]; \mathbb{R})$; $\mathbf{g} \in \xi([t_0; t_f]; \mathbb{R})$; $\mathbf{P} \in \xi([t_0; t_f]; \mathbb{R})$. Eq.2.11 is a nonhomogeneous linear stochastic differential equation, and consequently, it has one and only one solution. Thus, the proof is completed . ■

2.2 Numerical solution

In the preceding section, it was shown that the solution of the stochastic Linear-Quadratic Regulator (LQR) problem, defined by equations 2.1-2.2, is equivalent to solving a pair of differential equations, namely 2.3a and 2.4. Consequently, obtaining an approximate solution to the original control problem 2.1 necessitates solving these two auxiliary equations.

Given the initial condition $\mathcal{X}_{t_0} = \mathcal{X}_0$, equation 2.1 can be reformulated as the following stochastic integral equation:

$$\mathcal{X}(t) = \mathcal{X}_0 + \int_{t_0}^{t} [m_1(s)\mathcal{X}(s) + m_2(s)v(s) + f(s)]ds + \int_{t_0}^{t} m_3(s)v(s)dW(s), \qquad t_0 \leq t \leq t_f \tag{2.12}$$

where the control function $v(s)$ is defined by:

$$v(s) = -\left[m_5(s) + m_3^T(s)K(s)m_3(s)\right]^{-1} m_2^T(s)\left(K(s)\mathcal{X}(s) + g(s)\right).$$

and the functions $K(s)$ and $g(s)$ are solutions of equations 2.3a and 2.4, respectively.The second part of 2.12 is a stochastic integral. For any $t \geq 0$, the Brownian motion is almost definitely continuous but not differentiable at t. Thus, in this section, a new approximations method is presented, which is based on the linear spline interpolation.

The Successive Approximation Method (SAM), is recognized as one of the classical and widely used techniques for addressing initial value problems and integral equations. This method is frequently employed to establish the existence and uniqueness of solutions to such equations.

In the context of the SAM, the following recurrence relation is defined for $t_0 \leq t \leq t_f$,

$$\mathcal{X}_n(t) = \mathcal{X}_0 + \int_{t_0}^{t} [m_1(s)\mathcal{X}_{n-1}(s) + m_2(s)v_{n-1}(s) + f(s)]ds + \int_{t_0}^{t} m_3(s)v_{n-1}(s)dW(s), \tag{2.13}$$

where the control function v_{n-1} is given by:

$$v_{n-1} = -\left[m_5(s) + m_3^T(s)K(s)m_3(s)\right]^{-1} m_2^T(s)\left(K(s)\mathcal{X}_{n-1}(s) + g(s)\right).$$

Theorem 2.2.1 *Let $\{\mathcal{X}_n(t)\}_{n\geq 0}$ be the sequence generated by the successive approximation method defined in equation 2.13. Then $\{\mathcal{X}_n(t)\}$ converges to a unique solution $\mathcal{X}(t)$ of the stochastic differential equation 2.12 as $n \to \infty$.*

Proof. We assume that the entries of the matrices $m_1(t)$, $m_2(t)$, and $m_3(t)$ are continuous on $[t_0, t_f]$. Consequently, the integrands in equation 2.13 satisfy both Lipschitz and linear growth conditions. In particular, for any two $\mathcal{X}(t)$ and $\mathcal{Y}(t)$, we have:

$$\| (m_1(s)\mathcal{X}(s) + m_2(s)v(s, \mathcal{X}(s))) - (m_1(s)\mathcal{Y}(s)m_2(s)vs, \mathcal{Y}(s)) \|^2 \leq L \| \mathcal{X}(s) - \mathcal{Y}(s) \|^2,$$
$$\| m_3(s)v(s, \mathcal{X}(s)) - m_3(s)v(s, \mathcal{Y}(s)) \|^2 \leq L \| \mathcal{X}(s) - \mathcal{Y}(s) \|^2,$$

and

$$\| m_1(s)\mathcal{X}(s) + m_2(s)v(s, \mathcal{X}(s)) \|^2 \leq L_1 \left(1 + \| \mathcal{X}(s) \|^2\right),$$
$$\| m_3(s)v(s, x\mathcal{X}(s)) \|^2 \leq L_1 \left(1 + \| \mathcal{X}(s) \|^2\right),$$

for all $s \in [t_0, t_f]$ where L, L_1 are positive constants, and

$$v(s, \mathcal{X}(s)) = P^{-1}(s)m_2^T(s)\left(K(s)\mathcal{X}(s)(s) + g(s)\right).$$

To construct an implementable numerical method, we employ a **linear spline interpolation** approach as a modification to the classical successive approximation method (SAM).

Let $\Delta = \{t_0 < t_1 < ... < t_m = t_f\}$be a uniform or non-uniform partition of the interval $[t_0, t_f]$, where $h_i = t_i - t_{i-1}$, for $i = 1, 2, ..., m$.We start with the zeroth approximation defined by the constant function: $x_0(t) = x_0$ Using the recurrence relation 2.13, the first iteration yields for $t_0 \leq t \leq t_f$,

$$\mathcal{X}_1(t) = \mathcal{X}_0 + \int_{t_0}^{t} [m_1(s)\mathcal{X}_0 + m_2(s)v_0(s) + f(s)]ds + \int_{t_0}^{t} m_3(s)v_0(s)dW(s), \quad (2.14)$$

where

$$v_0(s) = -\left(m_5(s) + m_3^T(s)K(s)m_3(s)\right)^{-1} m_2^T(s)\left(K(s)\mathcal{X}_0 + g(s)\right)$$

We approximate $\mathcal{X}_1(t)$ using linear splines over Δ as:

$$\mathcal{X}_1(t) \approx S_\Delta^1(t) = \sum_{i=1}^{m-1} \Psi_i^1(t) X_{[t_i,t_{i+1}]}(t), \quad (2.15)$$

with

$$\Psi_i^1(t) = \frac{1}{h_{i+1}}\left[\mathcal{X}_1(t_i)(t_{i+1} - t) + \mathcal{X}_1(t_{i+1})(t - t_i)\right], \qquad i = 0, 1, 2, ..., m-1. \quad (2.16)$$

By evaluating equation 2.14 at the grid points t_i, the values $\mathcal{X}_1(t_i)$ are given by $t_0 \leq t \leq t_f$,

$$\mathcal{X}_1(t_i) = \mathcal{X}_0 + \int_{t_0}^{t} [m_1(s)\mathcal{X}_0 + m_2(s)v_0(s) + f(s)]ds + \int_{t_0}^{t} m_3(s)v_0(s)dW(s), \quad (2.17)$$

By substituting 2.15 in 2.13 for $n = 2$, we can obtain $t_0 \leq t \leq t_f$,

$$\begin{aligned}\mathcal{X}_2(t) &= \mathcal{X}_0 + \int_{t_0}^{t} [m_1(s)\mathcal{X}_1(s) + m_2(s)v_1(s) + f(s)]ds + \int_{t_0}^{t} m_3(s)v_1(s)dW(s) \\ &\approx \mathcal{X}_0 + \int_{t_0}^{t} \left[m_1(s)S_\Delta^1(s) + m_2(s)v_\Delta^1(s) + f(s)\right] ds + \int_{t_0}^{t} m_3(s)v_\Delta^1(s)dW(s),\end{aligned} \tag{2.18}$$

where

$$v_\Delta^1(s) = -\left(m_5(s) + m_3^T(s)K(s)m_3(s)\right)^{-1} m_2^T(s)\left(K(s)S_\Delta^1(s) + g(s)\right).$$

Again $\mathcal{X}_2(t)$ is approximated by a linear spline:

$$\mathcal{X}_2(t) \approx S_\Delta^2(t) = \sum_{i=1}^{m-1} \Psi_i^2(t) X_{[t_i,t_{i+1}]}(t), \tag{2.19}$$

with

$$\Psi_i^2(t) = \mathcal{X}_2(t_i)(t_{i+1} - t) + \mathcal{X}_2(t_{i+1})(t - t_i)/h_{i+1}, \quad i = 0, 1, 2, ..., m-1.$$

Similarly, we can obtain the unknown coefficients $\mathcal{X}_2(t_i)$,$K = 0, 1, ..., m$, by substituting the grid points t_i, $i = 0, 1, ..., m$, in 2.18 , as follows $t_0 \leq t \leq t_f$.

$$\mathcal{X}_2(t_i) \approx \mathcal{X}_0 + \int_{t_0}^{t_i} \left[m_1(s)S_\Delta^1(s) + m_2(s)v_\Delta^1(s) + f(s)\right] ds + \int_{t_0}^{t_i} m_3(s)v_\Delta^1(s)dW(s), \tag{2.20}$$

Generally, based on the aforementioned structure, we can approximate the function $\mathcal{X}_n(t)$ on $[t_0, t_f]$, for $n \geq 2$, as

$$\mathcal{X}_n(t) \approx S_\Delta^n(t) = \sum_{i=1}^{m-1} \Psi_i^n(t) X_{[t_i,t_{i+1}]}(t) \tag{2.21}$$

where

$$\Psi_i^n(t) = \frac{1}{h_{i+1}} \left[\mathcal{X}_n(t_i)(t_{i+1} - t) + \mathcal{X}_n(t_{i+1})(t - t_i)\right], \qquad i = 0, 1, 2, ..., m-1 \tag{2.22}$$

In addition, $t_0 \leq t \leq t_f$,

$$\mathcal{X}_n(t_i) \approx \mathcal{X}_0 + \int_{t_0}^{t_i} \left[m_1(s) S_\Delta^{n-1}(s) + m_2(s) v_\Delta^{n-1}(s) + f(s)\right] ds + \int_{t_0}^{t_i} m_3(s) v_\Delta^{n-1}(s) dW(s), \tag{2.23}$$

where

$$v_\Delta^{n-1}(s) = \left(m_5(s) + m_3^T(s) K(s) m_3(s)\right)^{-1} m_2^T(s) \left(K(s) S_\Delta^{n-1}(s) + g(s)\right).$$

■

Definition 2.2.1 *Let f be a function on $[a, b]$. The modulus of continuity of f is defined by*

$$\omega(f, \delta) = \sup_{\mathcal{X}, \mathcal{Y} \in [a,b], |\mathcal{X} - \mathcal{Y}| < \delta} |\ f(\mathcal{X}) - f(\mathcal{Y})\ | .$$

Lemma 2.2.1 *The function $f(t)$ is uniformly continuous on $[a, b]$ if and only if*

$$\lim_{\delta \to 0} \omega(f, \delta) = 0.$$

Theorem 2.2.2 *Let $f(t) \in m_3^1([a, b])$ and $S_\Delta^n(t)$ be approximations of $f(t)$ by a* *linear spline interpolation. Then*

$$\| f(t) - S_{\Delta}^{n}(t) \|_{\infty} \leq \omega(f, \delta).$$

Proposition 2.2.1 *Let $\{S_{\Delta}^{n}(t)\}_{n=1}^{\infty}$ be the solution sequence produced by the numerical successive approximations. Then this sequence converges to the solution $\mathcal{X}(t)$ to 2.12.*

Proof. The entries of matrices $m_1(t)$, $m_2(t)$, and$m_3(t)$ involve continuous functions. Therefore, the terms

in integrals satisfy Lipschitz and linear growth conditions. Thus the error function $E_{\Delta}^{n}(t)$ and residual function $R_{\Delta}^{n}(t)$ are defined as

$$E_{\Delta}^{n}(t) = \mathcal{X}_n(t) - S_{\Delta}^{n}(t), \quad n \geq 1,$$
$$R_{\Delta}^{n}(t) = \mathcal{X}_n(t) - Z_{\Delta}^{n}(t), \quad n \geq 2,$$

where $t_0 \leq t \leq t_f$,

$$Z_n(t) = \mathcal{X}_0 + \int_{t_0}^{t_i} \left[m_1(s)S_{\Delta}^{n-1}(s) + m_2(s)v_{\Delta}^{n-1}(s) + f(s)\right] ds + \int_{t_0}^{t_i} m_3(s)v_{\Delta}^{n-1}(s)dW(s),$$

First, we show that

$$\lim_{|\Delta| \to 0, n \to \infty} \| R_{\Delta}^{n}(t) \| = 0, \qquad n \geq 2.$$

Thus, for $n \geq 2$, by applying the Cauchy-Schwarz inequality and Doob's inequality, we have

$$\| R_{\Delta}^{n}(t) \|^2 = \| \mathcal{X}_n(t) - Z_{\Delta}^{n}(t) \|^2$$

$$\leq 2 \int_{t_0}^{t} \| \left[(m_1(s)\mathcal{X}_{n-1}(s) + m_2(s)v_{n-1}(s)) - \left(m_1(s)S_{\Delta}^{n-1}(s) + m_2(s)v_{\Delta}^{n-1}(s)\right)\right] \|^2 \, ds$$

$$+2 \int_{t_0}^{t} \| m_3(s)v_{n-1}(s) - v_{\Delta}^{n-1}(s) \|^2 \, ds.$$

$$\leq L_3 \int_{t_0}^{t} \| \mathcal{X}_{n-1}(s) - S_{\Delta}^{n-1} \|^2 \, ds$$

$$= L_3 \int_{t_0}^{t} \| E_{\Delta}^{n-1} \|^2 \, ds$$

$$\leq L_3 \omega^2 (\mathcal{X}_{n-1}(s), | \Delta |).$$

Furthermore,

$$\| E_{\Delta}^{n}(t) \|^2 = \| \mathcal{X}_n(t) - Z_{\Delta}^{n}(t) + Z_{\Delta}^{n}(t) - S_{\Delta}^{n}(t) \|^2$$

Hence,

$$\| E_{\Delta}^{n}(t) \|^2 \leq 2 \| Z_{\Delta}^{n}(t) - S_{\Delta}^{n}(t) \|^2 + 2 \| R_{\Delta}^{n}(t) \|^2$$

$$\leq 2\omega^2 \left(Z_{\Delta}^{n}(t), | \Delta |\right) + 2L_3\omega^2 (\mathcal{X}_{n-1}(s), | \Delta |).$$

Moreover, Z_{Δ}^{n} is uniformly continuous on $[t_0, t_f]$, and from Lemma 2.2.1 , we obtain

$$\lim_{|\Delta| \to 0, n \to \infty} \| R_{\Delta}^{n}(t) \| = 0, \quad n \geq 2.$$

Thus, by Theorem2.1.1 , we have

$$\lim_{|\Delta|\to 0, n\to\infty} \parallel \mathcal{X}_n(t) - S^n_\Delta(t) \parallel = 0, \text{ for } n \geq 2.$$

Via applying a numerical integration method (e.g., Legendre Gauss method), we can approximate the first integral part of 2.23. Regarding the stochastic integral part, we can apply the Ito approximation as follows:

$$\int_{t_0}^{t_i} m_3(t) v^1_\Delta(t) dW(t) = \sum_{j=0}^{j=i-1} m_3(t_j) v^1_\Delta(t_j) \left(W(tj_{+1}) - W(t_j)\right).$$

Therefore, we obtain the ith optimal control law as follows:

$$v_i(t) = -\left[m_5(t) + m_3^T(t)K(t)m_3(t)\right]^{-1} m_2^T(t)\left(K(t)\mathcal{X}_i(t) + g(t)\right).$$

According to Theorem 2.1.1 , the solution sequence $\mathcal{X}_i(t)$ is almost surely uniform convergence. We define $\hat{\mathcal{X}}(t)$ as the limits of sequence $\mathcal{X}_i(t)$. The control sequence $v_i(t)$ is only related to $\mathcal{X}_i(t)$; so it is also uniformly convergent. Assume $\tilde{v}(t)$ as the limit of sequence $v_i(t)$. Summarizing the above, we obtainthe following theorem.

■

Theorem 2.2.3 *Consider the problem of minimizing the cost functional 2.2 subject to system 2.3a. Then the optimal control law is obtained as follows:*

$$\tilde{v}(t) = -\left[m_5(t) + m_3^T(t)K(t)m_3(t)\right]^{-1} m_2^T(t)\left(K(t)\hat{\mathcal{X}}(t) + g(t)\right). \tag{2.25}$$

2.3 Suboptimal control design strategy

This section focuses on the practical implementation of the theoretical results discussed earlier. Since the exact optimal control law described in equation 2.25 involves a limit as $n \to \infty$, it cannot be computed directly. Therefore, **a suboptimal control law of order** $l \in \mathbb{N}$ is considered as a feasible approximation for practical purposes.

The control law of order l, denoted by $v_l(t)$, is defined as follows:

$$v_l(t) = -\left[m_5(t) + m_3^T(t)K(t)m_3(t)\right]^{-1}(t)m_2^T(t)\left(K(t)\mathcal{X}_l(t) + g(t)\right). \tag{2.26}$$

Here, the integer l is selected based on a predefined accuracy requirement. Once $l = i$ is fixed, the corresponding quadratic cost functional can be computed as:

$$\mathbf{J}_l = \mathbb{E}\left\{\frac{1}{2}\int_{t_0}^{t_f}\left[\mathcal{X}_l^T(t)m_4(t)\mathcal{X}_l(t) + v_l^T(t)m_5(t)v_l(t)\right]dt + \frac{1}{2}\mathcal{X}_l^T(t_f)m_6\mathcal{X}_l(t_f)\right\}, \tag{2.27}$$

where $\mathcal{X}_l(t)$ is the state trajectory associated with the control $v_l(t)$ from equation 2.26.

To evaluate the accuracy of the suboptimal control, the following relative error criterion is applied: $E_{r_l} = \frac{\|\mathcal{X}_l - \mathcal{X}_{l-1}\|}{\|\mathcal{X}_l\|} < \varepsilon$, where $\|\cdot\|$ denotes the Euclidean norm, and $\varepsilon > 0$ is a small tolerance value. If this condition is satisfied, the suboptimal control law is considered sufficiently close to the true optimal control $\tilde{v}(t)$, and the cost $\mathbf{J}_l$ approaches the optimal cost $\tilde{\mathbf{J}}$.

Based on this approach, the following numerical algorithm is proposed to construct a suboptimal control law with low computational complexity:

2.3.1 Algorithm 1: Successive Approximation Method (SAM) for Suboptimal Control

This algorithm describes a numerical strategy for approximating the suboptimal control law in stochastic Linear Quadratic Regulator (LQR) problems.

Input Parameters:

- Initial time $t_0 \in \mathbb{R}$.
- Final time $t_f \in \mathbb{R}$.
- Number of subdivisions $m \in \mathbb{N}$.
- Maximum number of iterations $N \in \mathbb{N}$.

Procedure:

Step 1: Solve the matrix Riccati differential equation for $K(t)$ and the associated vector differential equation for $g(t)$, as defined in equations 2.3a and 2.4.

Step 2: Initialize the state and control:

$$\mathcal{X}_0(t) = \mathcal{X}_0, \quad v_0(t) = -\left[m_5(t) + m_3^T(t)K(t)m_3(t)\right]^{-1} m_2^T(t)\left(K(t)\mathcal{X}_0 + g(t)\right).$$

Step 3: Compute the first state trajectory approximation $\mathcal{X}_1(t_i)$,for $i = 0, 1, ..., m$, using equation 2.17.

Step 4: Construct the spline basis functions $\Psi_i^1(t)$, for $i = 0, 1, ..., m-1$, using equation 2.16.

Step 5: Construct the spline approximation for the trajectory: $\mathcal{X}_1(t) \approx S^1_\Delta(t)$, based on equation 2.15.

Step 6: Determine the updated control law: $v_1(t) = v^1_\Delta(t)$.

Step 7: For $n = 2, 3, \ldots, N$, repeat the following:

7.1 Compute the state trajectory $\mathcal{X}_n(t_i)$ from equation 2.23.

7.2 Define the spline basis functions $\Psi^1_i(t)$ from equation 2.22.

7.3 Compute the spline approximation: $\mathcal{X}_n(t) \approx S^n_\Delta(t)$ using equation 2.21.

7.4 Set: $v_1(t) = v^1_\Delta(t)$.

7.5 Convergence check: If the relative error satisfies: $E_{r_n} = \frac{\|\mathcal{X}_n - \mathcal{X}_{n-1}\|}{\|\mathcal{X}_n\|} < \varepsilon$, then stop the iteration.

Step 8: Return $v_n(t)$ as the final suboptimal feedback control law.

Chapter 3

Numerical Applications of the Stochastic LQR Problem in Finance

In this chapter, we present numerical applications of the stochastic Linear Quadratic Regulator (LQR) problem using a numerical method based on the Successive Approximation Method (SAM) combined with piecewise linear spline interpolation, as described in chapter2. These examples are designed to validate the effectiveness of the proposed approach and highlight its practical relevance, particularly in financial applications. We begin with academic test problems where the exact solution is known, which allows us to assess the accuracy of the numerical method, and then proceed to more realistic financial models, where optimal investment decisions are guided by stochastic control principles and solved numerically using the proposed strategy. In what follows, we illustrate the proposed numerical scheme on a benchmark example whose analytical solution is known.

3.1 Numerical Example for a Stochastic LQR Problem

Example 3.1.1 *Consider the problem of minimizing the cost functional:*

$$\mathbf{J}(v) = E\left\{ \frac{1}{2}\int_0^1 \mathcal{X}^2(t)dt + \frac{1}{2}\mathcal{X}^2(1) \right\},$$

subject to the system of differential equation

$$d\mathcal{X} = \left(\mathcal{X}(t) + 2v(t) + e^{(t-1)}\right) dt + 2v(t)dW(t),\ \ 0 \le t \le 1.$$

with initial condition $\mathcal{X}(0) = 5$. *We have* $m_1(t) = m_4(t) = m_6 = 1$, $m_2(t) = m_3(t) = 2$, $m_5(t) = 0$, *and* $f(t) = e^{(t-1)}$. *Therefore, the Riccati differential equation becomes*

$$\dot{K}(t) = -K(t) - 1, \quad 0 \le t \le 1.$$

with boundary condition $K(1) = 1$ *, whose unique solution is* $K(t) = -1 + 2e^{1-t}$. *Also we have* $\dot{g}(t) = e^{(t-1)} - 2$, $0 \le t \le 1$, *with boundary condition* $g(1) = 0$, *whose unique solution is* $g(t) = e^{(t-1)} - 2t + 1$.

First, we take a partition Δ with nodal points on $[t_0, t_f]$as Δ: $t_0 < t_1 <\cdots< t_{m-1} < t_m = t_f$, Usin relation 2.17 for $i = 1, 2, ..., m$, we have

$$\mathcal{X}_1(t_i) = 5 + \int_0^{t_i} [5 + v_0(s) + f(s)]ds + \int_0^{t_i} v_0(s)dW(s), \qquad t_0 \le t \le t_f,$$

where

$$v_0(t) = -(-2 + 4e^{(1-t)})^{-1} \left(5(-1 + 2e^{(1-t)}) + e^{(t-1)} - 2t + 1\right)$$

Then we compute the functions $\Psi_i^1(t)$ from 2.16, and using 2.15, we obtain

$$\mathcal{X}_1(t) \approx S_\Delta^1(t) = \sum_{i=1}^{m-1} \Psi_i^1(t) X_{[t_i,t_{i+1}]}(t).$$

In the following, via applying 2.21 and 2.22, we obtain

$$\mathcal{X}_n(t) \approx S_\Delta^n(t) = \sum_{i=1}^{m-1} \Psi_i^n(t) X_{[t_i,t_{i+1}]}(t).$$

where

$$\Psi_i^n(t) = \left[\mathcal{X}_n(t_i)(t_{i+1}-t) + \mathcal{X}_n(t_{i+1})(t-t_i)\right]/h_{i+1}, \qquad i = 0, 1, 2, ..., m-1.$$

In addition

$$\mathcal{X}(t_i) \approx 5 + \int_0^{t_j} \left[S_\Delta^{n-1}(s) + v_\Delta^{n-1}(s) + e^{(t-1)}\right] ds + \int_0^{t_j} v_\Delta^{n-1}(s) dW(s), \qquad 0 \leq t \leq 1$$

Initial approximation $\mathcal{X}_0(t) = \mathcal{X}_0$ can be taken into account. Through persistent use of this process, eachiteration can result in $\mathcal{X}t)$ and $v(t)$. As a result, a suboptimal cost functional can be obtained. It should be noted that $\mathcal{X}(t)$ and $v(t)$ are considered as random variables and that the results are achieved with probability one. Thus, optimal value and suboptimal cost functionals are considered random, as well.Therefore, the simulation 100 times, optimal value, and suboptimal cost functionals are run, and the errors averages are estimated.

As 2.27 demonstrates, suboptimal cost functionals are acquired. In order to obtain an accurate enough suboptimal control law, the proposed algorithm was applied with the tolerance error bound $\varepsilon = 10^{-4}$. Table 3.1 illustrates the errors at the different iteration times. As observed in Table 3.1 , after four iterations, the convergence is achieved only; that is, $Er_4 = 7 \times 10^{-6} < 10^{-4}$.

The results demonstrate that in $l = 4$, this method converges to solution $\tilde{\mathcal{X}}(t) \cong \mathcal{X}_4(t)$ in probability. Thus, the optimal control is $\tilde{v}(t) \cong v_4(t)$. Moreover, Figures 01 and 02 show plots of $\mathcal{X}_4(t)$ and $v_4(t)$.

iteration time 1	Er_l
1	–
2	0.8707
3	0.4668
4	7×10^{-6}
⋮	⋮

Table 3.1: Errors at the different iterations for Example1

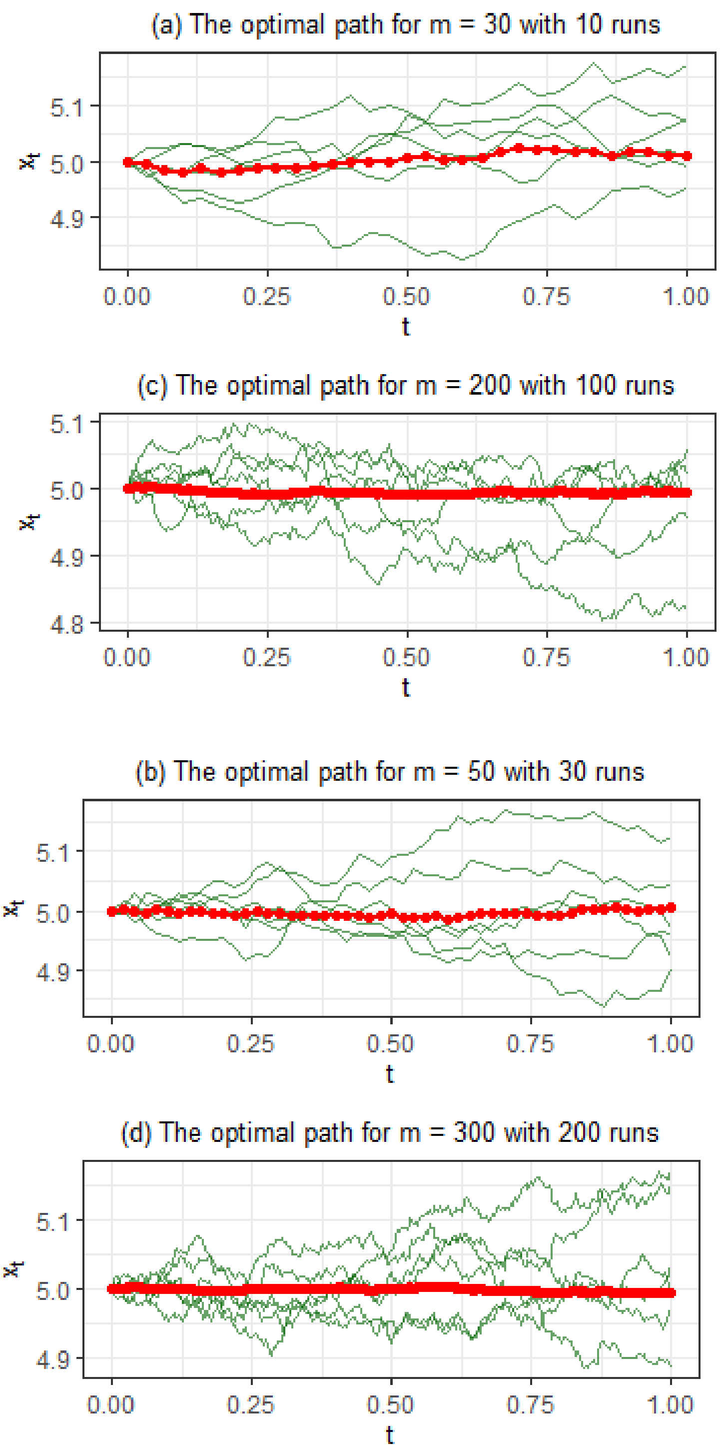
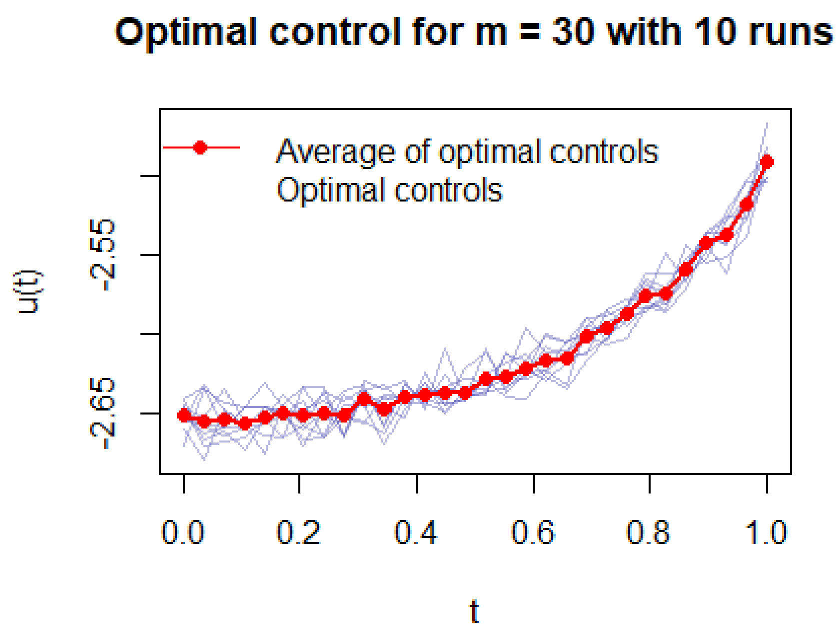

(a) The optimal path for m = 30 with 10 runs
(c) The optimal path for m = 200 with 100 runs
(b) The optimal path for m = 50 with 30 runs
(d) The optimal path for m = 300 with 200 runs
x_t
t
4.8
4.9
5.0
5.1
0.00
0.25
0.50
0.75
1.00

Fig. 01.


Optimal control for m = 30 with 10 runs
Average of optimal controls
Optimal controls
u(t)
-2.55
-2.65
0.0
0.2
0.4
0.6
0.8
1.0
t

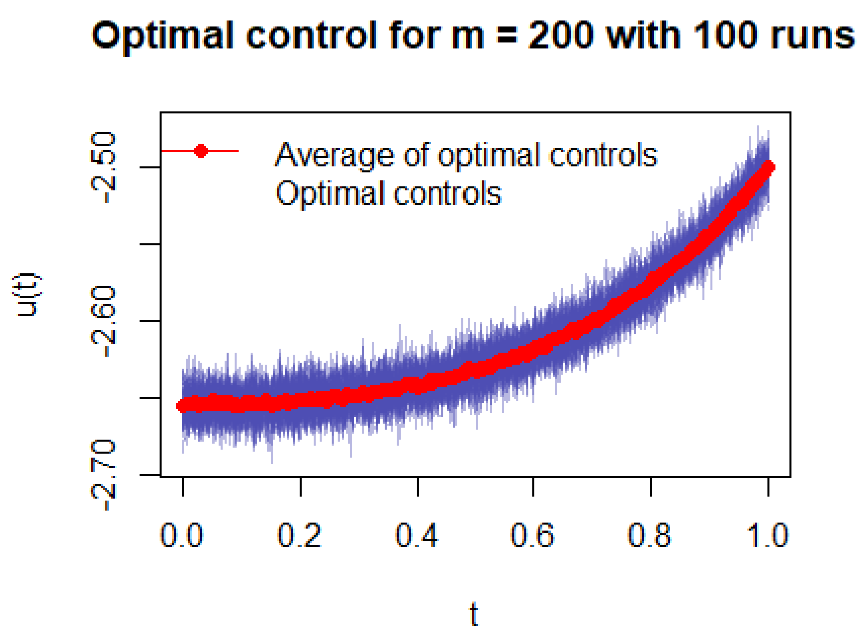

Optimal control for m = 200 with 100 runs
Average of optimal controls
Optimal controls
u(t)
-2.50
-2.60
-2.70
0.0
0.2
0.4
0.6
0.8
1.0
t

Fig.02

3.2 Application in finance

In this section, we consider a financial market with two investment opportunities:

- A **risk-free asset** (a bond or bank account) that grows at a fixed interest rate r , and
- A **risky asset** (e.g., a stock) with an expected return $\mu > r$ and time-varying volatility $\sigma(t) > 0$.

The price of the risk-free asset follows the differential equation:

$$db(t) = b(t)rdt, \quad b_0 = 1,$$

let $\pi(t)$ denote the proportion of the total wealth invested in the risky asset at time t and let $\mathcal{X}(t)$ represent the total wealth at time t. Then the wealth dynamics are governed by the stochastic differential equation (SDE)

$$d\mathcal{X}(t) = \mathcal{X}(t)\left[r + (\mu - r)\pi(t)\right]dt + \sigma(t)\pi(t)\mathcal{X}(t)dW(t), \quad t_0 \le t \le t_f, \tag{3.1}$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0 > 0$. The control variable in this model is $\pi(t)$, and the objective is to minimize a given cost functional over the interval $[t_0, t_f]$. First, we let final cost for this market be $\frac{1}{2}\mathcal{X}^2(t_f)Q$. Thus, the objective function is

$$\min_{\pi(t)} \mathbb{E}\left\{\frac{1}{2}\mathcal{X}^2(t_f)Q\right\}. \tag{3.2}$$

It is evident that the presented model has a continuous optimal control. Here, we explain how the results of previous sections were applied in order to solve this problem using the stochastic LQR problem.

Normally, we should turn this problem formula into the stochastic LQR problem 2.1–2.2. We can rewrite 3.1 as the following equation

$$d\mathcal{X}(t) = \left[r\mathcal{X}(t) + (\mu - r)\pi(t)\mathcal{X}(t)\right]dt + \sigma(t)\pi(t)\mathcal{X}(t)dW(t), \qquad t_0 \leq t \leq t_f$$

If we get $v(t) = v(t, x(t) = \pi(t)\mathcal{X}(t)$, then this problem is converted to a stochastic LQR problem, and using the obtained results, the problem can be solved. Therefore, 3.1 is rewritten as:

$$d\mathcal{X}(t) = \left[r\mathcal{X}(t) + (\mu - r)v(t)\right]dt + \sigma(t)v(t)dW(t), \quad t_0 \leq t \leq t_f, \tag{3.3}$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0$. Note that here we have $m_1(t) = r$, $m_2(t) = (\mu - r)$, $m_3(t) = \sigma(t)$, $f(t) = 0$, $m_4(t) = m_5(t) = 0$ and $m_6 = Q$.

As a result, the optimal control problem 2.1–2.2 is converted to equivalent stochastic LQR problem 3.2–3.3. Thus two differential equations 2.2 and 2.3a are obtained as below:

$$\dot{K}(t) = -2rK(t) + (\mu - r)^2(\sigma^2(t))^{-1}K(t), \; t_0 \leq t \leq t_f,$$

with boundary condition $K(t_f) = Q$ and

$$\dot{g}(t) = -rg(t) + (\mu - r)^2(\sigma^2(t))^{-1}g(t), \; t_0 \leq t \leq t_f,$$

with boundary condition $g(t_f) = 0$. It is evident that $g(t) = 0$ for all $[t_0, t_f]$. Thus, using the LQR problem, we have

$\tilde{v}(t) = -(\sigma^2(t))^{-1}(\mu - r)\mathcal{X}(t)$, where $\tilde{v}(t) = \pi^*(t)\mathcal{X}(t)$. It can be concluded $\pi^*(t) = -(\mu - r)/\sigma^2(t)$, and so the problem is solved.

Now let the cost function be total cost. Consequently, the objective function is as follows:

$$\min_{\pi(t)} \mathbb{E}\left\{ \frac{1}{2}\int_{t_0}^{t_f} \left((\pi(t)\mathcal{X}(t))^2 m_5(t)\right) dt + \frac{1}{2}\mathcal{X}^2(t_f)Q \right\}.$$

As in the previous example, we convert the problem to the stochastic LQR problem 3.3 with the cost function

$$\min_{u} \mathbb{E}\left\{ \frac{1}{2}\int_{t_0}^{t_f} \left((v(t))^2 m_5(t)\right) dt + \frac{1}{2}\mathcal{X}^2(t_f)Q \right\},$$

where $v(t) = \pi(t)\mathcal{X}(t)$. Hence the differential equation 2.2 is obtained as below:

$$\dot{K}(t) = -2rK(t) + (\mu - r)^2 K^2(t)\left(m_5(t) + \sigma^2(t)K(t)\right)^{-1},\ t_0 \le t \le t_f,$$

with boundary condition $K(t_f) = Q$. In this example, $g(t) = 0$ for all $[t_0, t_f]$. Thus, using the LQR problem, we have $\tilde{v}(t) = -\left(m_5(t) + \sigma^2(t)K(t)\right)^{-1}(\mu - r)K(t)\mathcal{X}(t)$, where $\tilde{v}(t) = \pi(t)\mathcal{X}(t)$. Consequently

$$\pi^*(t) = -\left(m_5(t) + \sigma^2(t)K(t)\right)^{-1}(\mu - r)K(t).$$

Example 3.2.1 *Consider a financial market with the following stochastic differential equation:*

$$d\mathcal{X} = \mathcal{X}(t)\left[2 + 3\pi(t)\right] dt + 2e^t\pi(t)\mathcal{X}(t)dW(t),\ 0 \le t \le 1,$$

with initial condition $\mathcal{X}(0) = 10$. The objective function is considered as

$$\min_{\pi(t)} \mathbb{E} \left\{ \frac{1}{2} \int_0^1 \left((\pi(t)\mathcal{X}(t))^2 e^{2t} \right) dt + \frac{1}{4} \mathcal{X}^2(1) \right\}.$$

Here, we have$r = 2$, $\mu = 5$, $\sigma(t) = 2e^t$, $m_5(t) = e^{2t}$, and $Q = \frac{1}{2}$. Therefore, the Riccati differential equation is

$$\dot{K}(t) = -4K(t) + 9K^2(t) \left(e^{2t} + 4e^{2t} \right)^{-1}, \ 0 \leq t \leq 1,$$

with boundary condition $K(1) = \frac{1}{2}$, whose unique solution is

$$K(t) = \frac{10e^4}{3e^{(4-2t)} - 3e^{(4t-2)} + 20e^{4t}},$$

As a result

$$\pi^*(t) = -3 \left(\frac{10e^{(2t+4)}}{3e^{(4-2t)} - 3e^{(4t-2)} + 20e^{4t}} + 4e^{-2t} \right).$$

Conclusion

In this dissertation, we addressed the stochastic Linear Quadratic Regulator (LQR) problem with a particular focus on control-dependent diffusion and financial applications. We first presented the theoretical background of stochastic calculus and optimal control theory, then derived the optimal control solution using the Riccati differential equation coupled with a backward stochastic differential equation.

A numerical method was developed based on the Successive Approximation Method (SAM) combined with linear spline interpolation to approximate the optimal solution. This method proved to be effective and convergent, as shown through a numerical simulation with a known analytical solution. Furthermore, we applied our approach to a financial portfolio problem, which demonstrated the practical relevance and flexibility of the proposed technique.

Despite the encouraging results, the method may still be improved in terms of computational cost and accuracy. In future work, one could explore adaptive time discretization, higher-order spline approximations, or machine learning techniques to enhance control approximation in high-dimensional systems.

Overall, this study contributes to the growing literature on stochastic optimal control by proposing a numerically efficient and theoretically grounded solution approach for complex financial decision-making problems.

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Abstract :

In this thesis, we deal with a stochastic optimal control problem in continuous time. We consider a stochastic Linear Quadratic Regulator (LQR) problem where the dynamics of the system are governed by a stochastic differential equation (SDE), and the cost functional is quadratic in both state and control variables.
We first present the optimal solution using the stochastic Riccati differential equation. Then, we propose a numerical method based on the Successive Approximation Method (SAM) combined with linear spline approximation to find suboptimal control laws. Finally, we provide numerical experiments to illustrate the efficiency and accuracy of the proposed approach in a financial application.

الملخص :

في هذه المذكرة، نتناول مسألة التحكم الأمثل العشوائي في الزمن المستمر. ندرس مشكلة المنظم التربيعي الخطي العشوائي (LQR) حيث توصف ديناميكية النظام بواسطة معادلة تفاضلية عشوائية(SDE) ، ويكون تابع التكلفة تربيعيًا في كل من متغيرات الحالة والتحكم.
نستعرض أولاً الحل الأمثل بالاعتماد على معادلة ريكاتي التفاضلية العشوائية. ثم نقترح طريقة عددية تعتمد على منهج التقريب المتتالي (SAM) مدمجًا مع التقريب بالسبلينات الخطية لإيجاد قوانين تحكم شبه مثلى. وأخيرًا، نقدم تطبيقات عددية لتوضيح فعالية ودقة النهج المقترح في مجال المالية.

Résumé :

Ce mémoire traite d’un problème de contrôle optimal stochastique en temps continu. Nous étudions un problème LQR stochastique dans lequel la dynamique du système est décrite par une équation différentielle stochastique (EDS), et la fonction de coût est quadratique en les variables d’état et de commande.
Nous présentons tout d’abord la solution optimale via l’équation différentielle de Riccati stochastique. Ensuite, nous proposons une méthode numérique basée sur la méthode d’approximation successive (SAM) combinée à une interpolation spline linéaire pour déterminer des lois de commande sous-optimales. Enfin, des simulations numériques illustrent l’efficacité et la précision de l’approche proposée dans une application financière.