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Ministry of Higher Education and Scientific Research
University Mohamed Khider of Biskra

Faculty of Exact Sciences and Science of Nature and Life
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**Ref:**……..

Thesis Presented to obtain the degree of

**Doctorate in sciences**

Option: Partial differential equations

Entitled:

# Asymptotic Behavior of Solutions of Some Viscoelastic Problems

Presented by:

**Soumia Hamdi**

Publicly defended on: 30/06/ 2024

**In front of the Jury composed of:**

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| **Mr. Menacer Tidjani** | Professor | University of Biskra | President |
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Département de Mathématiques

**Ref:**……..

Mémoire présenté pour obtenir le diplôme de
**Doctorat en sciences**

Spécialité: Équations aux dérivées partielles

Intitulé:

# Comportement asymptotique de la solution de quelques problèmes viscoélastiques

Présenté par:

**Soumia Hamdi**

Soutenue publiquement le: 30 /06/ 2024

**Devant le jury composé de:**

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| **Mlle. Laadjal Baya** | MCA | Université de Biskra | Examinatrice |

# Dedication

*I dedicate this Work :*

*To my Parents*

*To my Husband.*

*To my Children Nardjes and Kossai.*

*To all my Sisters and Their kids,  Brothers and Friends.*

*Thanks to my professors and colleagues.*

*To all Persons Those who have Encouraged me Throughout this Work.*

# Acknowledgement

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I would like to express my deep sincere gratitude and respect to my thesis supervisor, Dr Amirouche Berkani, professor at the university of Bouira, for providing invaluable guidance throughout this thesis. His dynamism, vision, advice, motivation, support, and encouragement have deeply inspired me to carry out this work and present it as clearly as possible. His helpful suggestions and guidance have greatly contributed in the development of this work. I am extremely grateful for what he has offered me.

I am extending my thanks to the members of committee discussion: Professor Tidjani Menacer and Dr. Baya Laadjal from university of Mohamed Khider of Biskra, Dr. Athmane Abdallaoui from Ecole Normale Supérieure of Bou Saâda.

I would like to express my heartfelt gratitude to my parents for their efforts and support since my infancy. I am extremely grateful for their care, sacrifice, prayers, and patience in my university education. They never gave up encouraging me and supporting me with all the material and moral help when I needed throughout my life and during the writing of this monograph. They were eager to motivate me to keep working so as to achieve this goal. Their presence and their support are very valuable to me.

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My special thanks also go to my husband Salim and my children for their strong support, empathy, tolerance, and understanding throughout the recent years.

Finally, my thanks go to all the people who supported me in completing this work.

# Abstract

The main purpose of this thesis is to investigate the existence and uniqueness of solutions, as well as the asymptotic behavior of some viscoelastic problems in one-dimensional space, precisely, this work addresses the problem of undesirable vibrations and the control of attitude stabilization of a flexible satellite during the maneuvers. In view of this, viscoelastic materials are suggested to attenuate or suppress the unwanted vibrations of a flexible satellite. The flexible satellite system consists of a central rigid hub and two large symmetric flexible appendages. Mathematically, the problem can be modeled by a set of partial differential equations (PDEs) taking into account the dynamic boundary condition. Our research utilizes Lyapunov's direct method to study some viscoelastic systems. The results obtained in this thesis aim to enhance much of the previous scientific research.

**Key Words:** PDEs; dynamic boundary condition; Euler-Bernoulli beam; existence and uniqueness of solutions; Galerkin approximation method; arbitrary decay; viscoelasticity; relaxation function.

# Résumé

Le but principal de cette thèse est d'étudier l'existence et l'unicité des solutions, ainsi que le comportement asymptotique de certains problèmes viscoélastiques dans un espace unidimensionnel. Plus précisément, ce travail aborde le problème des vibrations indésirables et le contrôle de la stabilisation de l'attitude d'un satellite flexible pendant les manœuvres. En vue de cela, des matériaux viscoélastiques sont proposés pour atténuer ou supprimer les vibrations indésirables d'un satellite flexible. Le système de satellite flexible se compose d'un moyeu central rigide et de deux grands appendices flexibles symétriques. Mathématiquement, le problème peut être modélisé par un ensemble d'équations aux dérivées partielles (EDP) tenant en compte les conditions limites dynamiques. Notre recherche utilise la méthode directe de Lyapunov pour étudier certains systèmes viscoélastiques. Les résultats obtenus dans cette thèse visent à améliorer la plupart des recherches scientifiques précédentes.

**Mots Clés:** EDP ; conditions dynamiques aux limites ; poutre d'Euler-Bernoulli ; existence et unicité des solutions ; méthode d'approximation de Galerkin ; décroissance arbitraire ; viscoélasticité ; fonction de relaxation.

# ملخص


الغرض الرئيسي من هذه الأطروحة هو دراسة وجود الحلول ووحدانيتها، وكذلك السلوك التقاربي لبعض المشاكل اللزجة المرنة في الفضاء أحادي البعد، وعلى وجه التحديد، يتضمن هذا العمل مشكلة الاهتزازات غير المرغوب فيها والتحكم في استقرار موقف قمر صناعي مرن أثناء المناورات. وفي ضوء ذلك، يُقترح استخدام مواد لزجة مرنة لتخفيف أو كبح الاهتزازات غير المرغوب فيها في قمر صناعي مرن. يحتوي نظام القمر الصناعي المرن على محور مركزي صلب وملحقين كبيرين مرنين متماثلين. رياضيًا، يمكن تمثيل المشكلة بنمذجته حيث يظهر بمجموعة من المعادلات التفاضلية الجزئية, مع الاخذ بعين الاعتبار الشرط الحدودي الديناميكي.

يعتمد بحثنا على طريقة ليابونوف المباشرة لدراسة بعض الأنظمة اللزجة المرنة, النتائج التي تم الحصول في هذه الاطروحة تعزز الكثير من الأبحاث العلمية السابقة.



**الكلمات المفتاحية**: المعادلات التفاضلية الجزئية ؛ شرط الحدود الديناميكي ؛اولر بارنولي ؛ الوجود وتفرد الحلول؛ طريقة تقريب جالوركا؛ الاضمحلال التعسفي؛ اللزوجة المرنة؛ دالة الاسترخاء,

# Contents

# Introduction

The attitude control problem for a flexible satellite has received increasing attention from numerous researchers in recent years due to its important applications, in addition, the satellite plays a vital role in our daily activities, offering various services including telecommunications, weather forecasting, earth observation, navigation, and scientific research. The effective functioning of these sophisticated technologies relies on their capability to endure various environmental obstacles such as temperature fluctuations, radiation, and vibration. Vibrations during launch or in orbit can cause significant harm to the fragile equipment, therefore, it is crucial to minimize and suppress these undesirable vibrations to guarantee the longevity and reliability of the satellite or spacecraft, given the potential consequences of uncontrolled vibrations, the need for accurate and reliable monitoring in this field is imperative. Consequently, several articles addressed the issue of vibration control in satellite, highlighting the need for effective monitoring and mitigation measures. In this regards, we refer some previous studies, early in the 1980s, Breakwell [18] and Ben-Asher *et al.* [8] showed the most effective control strategy for flexible spacecraft issues with vibration suppression, afterwards, several robust control methods have been researched for the spacecraft attitude control system to enhance robust performance in the presence of unmodeled dynamics, disturbances, model uncertainty, and structural vibrations of flexible appendages. In [67] and [53], the researchers studied the $H_\infty$ control scheme systematically for the attitude control system to solve the disturbance and the vibration problems, whereas it cannot address the large model uncertainties of satellite. Also in [54], a sliding mode controller has been proposed combining with input shaping which is used to reduce residual vibration while the controller is used to deal with disturbance, although the control strategy mentioned above has achieved good control performance. The bounded of disturbance and uncertainty need to be known in advance, based on quintic polynomial transition a novel trajectory planning scheme has been proposed in [99].

In [48], the paper has presented a new method for reducing the vibration of flexible spacecraft during attitude maneuver by utilizing the theory of variable structural control (VSC) to design switching logic for thruster firing and lead zirconate titanate (PZT) as sensor and actuator for active vibration suppression. Later, in [52], the authors have addressed an important question involving the suppression of vibrations in flexible satellites, they have studied a flexible satellite system with a rigid hub and long flexible solar panels that are subject to some undesirable vibrations and frictional damping that acts on the transverse displacements of the left and right panels

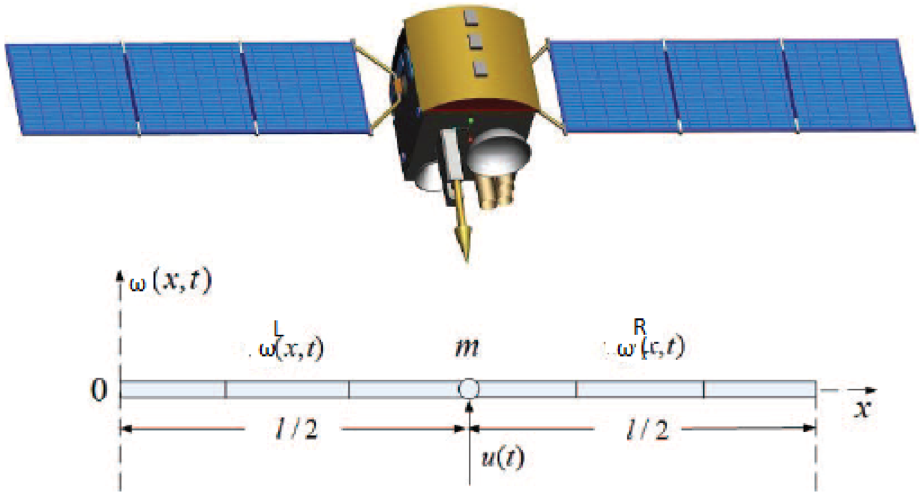


Figure 1: The flexible satellite system.

In figure 1, the functions $\omega^L(x,t)$ and $\omega^R(x,t)$ represent the transverse displacements of the left and right panels at the position $x$ for the time $t$, and $\omega\left(l/2,t\right)$ is the transverse displacements of a lumped mass. Specifically, they have derived and analyzed the problem by utilizing the Hamilton's, they have been obtained the following system,

$$\begin{cases} \rho A\omega_{tt}^{L}\left(x,t\right)+EI\omega_{xxxx}^{L}\left(x,t\right)+\gamma_1\omega_t^L\left(x,t\right)=0, & (x,t)\in[0,l/2]\times[0,\infty), \\ \rho A\omega_{tt}^{R}\left(x,t\right)+EI\omega_{xxxx}^{R}\left(x,t\right)+\gamma_2\omega_t^R\left(x,t\right)=0, & (x,t)\in[l/2,l]\times[0,\infty), \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \omega_x^L\left(l/2,t\right)=\omega_x^R\left(l/2,t\right)=0,\ \omega_{xx}^L\left(0,t\right)=\omega_{xx}^R\left(l,t\right)=0,\ \omega_{xxx}^L\left(0,t\right)=\omega_{xxx}^R\left(l,t\right)=0, \\ \omega^L\left(l/2,t\right)=\omega^R\left(l/2,t\right)=\omega\left(l/2,t\right), \\ m\omega_{tt}\left(l/2,t\right)=\ u\left(t\right)+EI\omega_{xxx}^L\left(l/2,t\right)-EI\omega_{xxx}^R\left(l/2,t\right). \end{cases} \tag{2}$$

The positive constants $\rho$, $A$, $EI$, $m$ and $\gamma_1$, $\gamma_2$ are: the density of the beam material, the cross-sectional area, bending stiffness, the mass of the center body and the coefficients of viscous damping, respectively. $u(t)$ is a single-point control force applied on the center body of the satellite, which defined as follows:

$$u(t) = -k\frac{\alpha}{\beta}\omega(l/2,t) - k\omega_t(l/2,t) - \frac{\alpha}{\beta}m\omega_t(l/2,t) - k_p\omega(l/2,t), \quad t \geq 0, \tag{3}$$

where $\alpha$ and $\beta$ represent positive weighting constants, $k$ and $k_p$ are the control gains. Then, they showed a result of exponential decay for the closed-loop system.

Subsequently, in [56], the researchers have improved the previous results by taking into account the unknown distributed disturbance in the system. They then have proved the stability of the following model for all $t \geq 0$

$$\begin{cases} \rho A\omega_{tt}^L(x,t) + EI\omega_{xxxx}^L(x,t) + \gamma_1\omega_t^L(x,t) = f^L(x,t) + U^L(x,t), & (x,t) \in [0,l/2] \times \mathbb{R}^+ \\ \rho A\omega_{tt}^R(x,t) + EI\omega_{xxxx}^R(x,t) + \gamma_2\omega_t^R(x,t) = f^R(x,t) + U^R(x,t), & (x,t) \in [l/2,l] \times \mathbb{R}^+ \end{cases} \tag{4}$$

with the boundary conditions

$$\begin{cases} \omega_x^L(l/2,t) = \omega_x^R(l/2,t) = 0, \ \omega_{xx}^L(0,t) = \omega_{xx}^R(l,t) = 0, \ \omega_{xxx}^L(0,t) = \omega_{xxx}^R(l,t) = 0, \\ \omega^L(l/2,t) = \omega^R(l/2,t) = \omega(l/2,t), \\ m\omega_{tt}(l/2,t) = u(t) + EI\omega_{xxx}^L(l/2,t) - EI\omega_{xxx}^R(l/2,t) + d(l/2,t), \end{cases} \tag{5}$$

where $f^L$, $f^R$ and $d$ represent the distributed disturbance in the left and the right panels and the disturbance imported on the centrebody, such that $U^L$ and $U^R$ represent the distributed control inputs to the system. Under some adaptive control laws, the authors have been established the stability of the system in the presence of a partial actuator fault.

Later in [57], the authors investigated techniques for controlling vibrations in satellites with adaptive actuator fault-tolerant and input quantization. There is a large set of other works in this regards, we mention among them some interested readers [66, 4, 70, 107, 93, 81, 26].

In the present work, we handle with the undesirable vibration problem in a viscoelastic satellite system, namely, we investigate the influence of a viscoelastic damping on the performance of satellite and drive it to the equilibrium position.

Viscoelastic damping is characterized by the ability to absorb and dissipate vibration energy, as viscoelastic materials work by converting mechanical energy (vibration) into heat energy through the process of internal friction, this process helps to reduce the amplitude of the vibrations and prevent them from propagating through the structure. Due to their viscoelastic component, making them effective in reducing the amplitude of vibrations, absorbing shock, resisting deformation, or recovering its shape after being subjected to a load. Due to their excellent damping properties, viscoelastic materials become an important topic of research in various engineering fields such as aerospace, automotive and civil engineering.

Viscoelastic materials are ideal for use in satellites as they are lightweight and have a long lifespan. They also provide a cost-effective solution for vibration suppression as they do not require any complex mechanical components or electrical systems.

Several scientists, such as Boltzmann, Maxwell, Kelvin and Voigt, have contributed to model the viscoelastic phenomena. In 1874, Boltzmann provided the first formulation of an isotropic viscoelasticity three-dimensional theory. Dafermos [29, 30] studied a one-dimensional viscoelastic model in the early 1970s, and he demonstrated numerous existence and asymptotic stability results for smooth, monotonically decreasing relaxation functions. Nevertheless, no rate of decay has been given, after that, many researchers have focused on viscoelastic problems and various findings concerning long-term behavior and existence have been obtained. To the best of our knowledge, Dassios and Zafirapoulos [31] offered the first work that investigate the uniform decay of solutions, in their research, they presented a viscoelastic problem in $\mathbb{R}^3$ and demonstrated a polynomial decay for exponentially decaying kernels. In 1994, Muñoz Rivera [82] considered equations for linear isotropic homogeneous viscoelastic solids, in a bounded domain, with exponentially decaying memory kernels and proved that, in the absence of body forces, the solutions decay exponentially in the bounded-domain case, whereas, in $\mathbb{R}^n$ the whole space case, the decay is of a polynomial rate. In [21] Cabanillas and Muñoz Rivera studied problems, where the kernels are of algebraic (but not exponential) decay rates and proved that the decay of solutions is algebraic at a rate that may be determined by the rate of the decay of the relaxation function and by the regularity of solutions. Later, Muñoz Rivera, et al. [7] improved this result by addressing equations related to linear viscoelastic plates. For viscoelastic problems with localized frictional dampings, Cavalcanti et al. [22] studied the following problem:

$$\begin{cases} \omega_{tt} - \Delta\omega + \int_0^t \zeta\left(t-\tau\right)\Delta\omega\left(\tau\right)d\tau + a(x)\Delta\omega_t + f\left(x,t,\omega\right) = 0, \quad \text{in } \Omega \times \left(0,\infty\right), \\ \omega = 0 \quad \text{on } \partial\Omega \times \left(0,\infty\right), \\ \omega\left(x,0\right) = \omega_0\left(x\right), \quad \omega_t\left(x,0\right) = \omega_1\left(x\right), \quad \text{in } \Omega, \end{cases} \tag{6}$$

where $\Omega$ is a bounded domain and $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ with a regular boundary $\partial\Omega$, $\zeta$ is a nonegative nonincreasing function satisfying, for $m_1$, $m_2$ two positive constants, the conditions:

$$-m_1\zeta\left(t\right) \leqslant \zeta'\left(t\right) \leqslant -m_2\zeta\left(t\right), \; t \geqslant 0.$$

and $a(x) \geqslant a_0 > 0$, with $meas(\upsilon) > 0$ and satisfying some geometry restrictions, they showed an exponential rate of decay. Also, Cavalcanti et al. [23] have studied a quasilinear equation, in a bounded domain, of the form:

$$\left|\omega_t\right|^{\rho}\omega_{tt} - \Delta\omega - \Delta\omega_{tt} + \int_0^t \zeta\left(t-\tau\right)\Delta\omega\left(\tau\right)d\tau - m_3\Delta\omega_t = 0,$$

with $\rho > 0$, and a global existence result for $m_3 \geqslant 0$, as well as an exponential decay for $m_3 > 0$ have been proved. Messaoudi and Tatar [76, 77] addressed the case when $m_3 = 0$ and offered polynomial and exponential decay results in the presence, also in the absence, with a nonlinear source term. For more general decaying kernels, Messaoudi [78, 79] investigated

$$\begin{cases} \omega_{tt} - \Delta\omega + \int_0^t \zeta\left(t-\tau\right)\Delta\omega\left(\tau\right)d\tau = b\left|\omega\right|^{m-2}\omega, \quad \text{in } \Omega \times \left(0,\infty\right), \\ \omega = 0 \quad \text{on } \partial\Omega \times \left(0,\infty\right), \\ \omega\left(x,0\right) = \omega_0\left(x\right), \quad \omega_t\left(x,0\right) = \omega_1\left(x\right), \quad \text{in } \Omega, \end{cases} \tag{7}$$

with $b = 0$, $b = 1$ and for the relaxation functions satisfying

$$\zeta\prime(t) \leqslant h(t)\zeta(t), \tag{8}$$

where $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing positive differentiable function, he observed that the energy decay rate is precisely equal to the decay rate of $\zeta$, which is not always an exponential or polynomial decay type, after that, Messaoudi [80] proved the general stability of the problem as follows:

$$\begin{cases} \omega_{tt} - \Delta\omega + \displaystyle\int_0^t \zeta(t-\tau)\Delta\omega(\tau)\,d\tau + a\left|\omega_t\right|^{m-2}\omega_t = 0, \quad \text{in } \Omega \times (0,\infty), \\ \omega = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \text{in } \Omega, \end{cases} \tag{9}$$

where $a > 0$ is a constant, $\Omega$ is a bounded regular domain of $\mathbb{R}^n$, and $\zeta$ is a non negative and non increasing function, he obtained a general decay estimate, which depended on the behavior of $\zeta$ and $m$, which usual rates of exponential and polynomial decay are only particular cases.

In addition to the above, let us now mention other known results that appeared in the literature treating the well-posedness and asymptotic behavior of solutions for wave equations with finite memory, Rivera and Salvatierra [80] considered

$$\begin{cases} \omega_{tt} - \Delta\omega + \displaystyle\int_0^t \zeta(t-\tau)\,\mathrm{div}[a(x)\nabla\omega(\tau)]d\tau = 0, \quad \text{in } \Omega \times (0,\infty), \\ \omega = 0 \quad \text{on } \Gamma \times (0,\infty), \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \text{in } \Omega, \end{cases} \tag{10}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n\,(n \in \mathbb{N}^*)$, with smooth boundary $\Gamma, \omega_0(x)$ and $\omega_1(x)$ are the initial data, $a(x)$ is a nonnegative $C^2$ function defined over $\Omega$ such that $a(x) = 0$ on $\upsilon\backslash\upsilon_\varepsilon$ and $a(x) = 1$ on $\upsilon_{\varepsilon/2}$ for $\varepsilon > 0$, where $\upsilon_\varepsilon$ is a subset of $\Omega$ and checked some assumptions. Under some hypotheses on the kernels $\zeta$, they obtained the existence and the exponential decay rate of the solution for the system (10). Many results concerning the stability of this problem have been obtained, in this regard; the reader is referred to previous studies Mustafa [84] and Tatar [102]. All these results have been improved by many authors, in [83], Mustafa and Messaoudi studied the problem (10) with $a(x) = 1$

on $\Omega$, for a wider class of the relaxation functions $\zeta$ satisfying

$$\zeta'(t) \leq -H(\zeta(t)), \quad t \geq 0,$$

where $H$ is a nonnegative function, satisfied $H(0) = H'(0) = 0$ and $H$ is strictly increasing and convex on $(0, r]$ for some $0 < r < 1$. By some arguments of convexity which introduced and developed by Cavalcanti *et al.* [25], Tatar [102] and Lasiecka *et al.* [63], the authors showed a the general decay rate for the system in which the exponential and polynomial decay are only special cases.

Regarding the viscoelastic damper effect in the Euler-Bernoulli beam, Let's introduce a few papers that are related to the boundary stabilization and boundary controllability of this problem. In [86], the investigators studied the viscoelastic Euler-Bernoulli beam, with one end fixed and the other end subjected to a nonlinear control force $f$, they considered

$$\omega_{tt}(x,t)+\omega_{xxxx}-\int_0^t \zeta(t-s)\,\omega_{xxxx}(x,s)\,ds+g(\omega_t(x,t))=0, \quad (x,t)\in[0,L]\times\mathbb{R}^+, \tag{11}$$

with the boundary conditions

$$\begin{cases} \omega(0,t)=\omega_x(0,t)=\omega_{xx}(0,t)=\omega_{xx}(L,t)=\omega_{xxx}(0,t)=0, & t>0, \\ \omega_{xxx}(L,t)-\displaystyle\int_0^t \zeta(t-s)\,\omega_{xxx}(L,s)\,ds=f(\omega(L,t)), & t>0, \end{cases} \tag{12}$$

They established the well-posedness of the system by using the Faedo-Galerkin method and showed the uniform decay rate using the multiplier technique under specific conditions on the kernels $\zeta$ and the functions $g$ and $f$. It is worth mentioning that in reference [87] , a similar problem studied in (11) with alternative boundary conditions

$$\begin{cases} \omega(0,t)=\omega_x(0,t)=\omega_{xx}(L,t)=0, \quad t\geq 0, \\ \omega_{xxx}(L,t)-\displaystyle\int_0^t \zeta(t-\tau)\omega_{xxx}(L,t)d\tau=u(t)-\tilde{\theta}\sin(t), \ t\geq 0, \\ \omega_{out}=\omega_t(L,t) \quad t\geq 0. \end{cases} \tag{13}$$

where $\tilde{\theta}$ is a positive constant, and they proved the exponential stability of the problem.

For exponentially decaying kernels, they showed the existence of solutions by using the Faedo-Galerkin method, and they established the exponential stability under the following adaptive output feedback controller

$$\begin{cases} u(t) = h(t)\omega_t(L,t) + \theta(t)\sin t, \quad t \geq 0, \\ h_t(t) = r\omega_t^2(L,t), \quad h(0) = h_0 > 0, \quad t \geq 0, \quad r > 0, \\ \theta_t(t) = \omega_t(L,t)\sin t, \quad \theta(0) = \theta_0. \end{cases} \tag{14}$$

## 0.1 The main objective of this work

The main goal of this work is to enhance the performance of a satellite, for this, we focus on studying the stability of systems by using viscoelastic materials, precisely, we consider the panels as two symmetrical viscoelastic Euler-Bernoulli beams subject to undesirable vibrations.

Firstly, under a suitable control in the central body of a flexible satellite system and assuming there are no unknown distributed disturbances during attitude maneuvering, we prove the arbitrary stabilization of the problem. Secondly, by applying a control force at the center body of the spacecraft, we establish the well posedness and arbitrary results of the system under unknown distributed disturbances during attitude maneuvering. Finally, we study a viscoelastic flexible satellite problem with unknown distributed disturbances, taking into account the tension of the system, and establish the uniform stability of the system.

## 0.2 Outline of the thesis

The dissertation is divided into five chapters:

**Chapter 1:** In this chapter, we recall properties of viscoelastic materials and their role in vibrations damping in mechanical systems, as well as functional analysis reminders that will be utilized throughout this thesis.

**Chapter 2:** In this chapter, we investigate the viscoelastic flexible satellite system under unwanted vibrations yielding during the movement, we prove the well-posedness of the problem as well as the arbitrary decay of the system. The content of this chapter has been accepted in International Journal of Control, namely

**Berkani A., Hamdi S. and Ait Abbas H. *Vibration control of a flexible satellite system with viscoelastic damping.* Int. J. Control. 2023. DOI: 10.1080/00207179.2023.2165163**

**Chapter 3:** We study the stabilization of a flexible satellite system with viscoelastic panels and under unknown distributed disturbances , to this, we use the multiplier method to show the uniform stability of our system. The content of this chapter has been published in the Journal of Mathematical Methods in the Applied Sciences, namely:

**Hamdi S. and Berkani A. *A new stability result for a flexible satellite system with viscoelastic damping.* Math. Meth. Appl. Sci. 2022;45(16):10070-10098. DOI: 10.1002/mma.8356**

**Chapter 4:** In this chapter, we discuss the stabilization of a flexible satellite system with viscoelastic panels subject to an unknown distributed disturbances and taking into account the tension of system, in this regard , we prove the arbitrary decay of the system.

**Appendix:** This chapter is devoted to deriving the constitutive equations of motion for a flexible satellite with two symmetric flexible panels by using Hamilton's principle.

# Chapter 1

# Preliminary results

This chapter introduces fundamental concepts in viscoelasticity and functional analysis, both of which play a crucial role in this work. We begin by presenting some characters and definitions of viscoelastic materials to understand the nature of these materials and their properties, after that we present preliminary materials from functional analysis that shall be utilized in subsequent chapters.

## 1.1 Basic notions in viscoelasticity linear

### 1.1.1 Viscoelastic materials

The study of viscoelastic behavior is of interest in widely contexts. First, materials used for structural applications of practical interest may exhibit viscoelastic behavior which has a profound influence on the performance of that material. Materials used in engineering applications may exhibit viscoelastic behavior as an unintentional side effect. In applications, one may deliberately make use of the viscoelasticity of certain materials in the design process.

#### Stress

Stress is the amount of force per unit area that is applied to a material. Mathematically, stress is defined as the force applied divided by the cross-sectional area of the material. Stress is usually measured in units of pressure, such as Pascals (Pa) or pounds per square inch (psi).

$$\text{Stress}\,(\sigma) = \frac{\text{Force}}{\text{Cross-sectional area}}.$$

**Strain**

Strain is the amount of deformation or elongation that occurs in a material as a result of the applied stress. Mathematically, strain is defined as the change in length or deformation of the material per unit of its original length. Strain is a dimensionless quantity and is usually expressed as a percentage or a decimal.

$$\text{Strain}\,(\varepsilon) = \frac{\text{Change in length}}{\text{Original length}}.$$

**Definition of viscoelasticity**

A viscoelastic materials are exhibited by containing both viscous and elastic behavior in varying degrees. Also, viscoelastic materials are those for which the relationship between stress and strain depends on time, where, the viscous word leads to energy dissipation. The elastic word to energy storage, some of the characteristics of viscoelastic materials are their capacity to creep, recover, undergo, stress relaxation and absorb energy. For a viscoelastic material, internal stresses are a function not only of the immediate deformation but also depend on the whole past history of deformation.

Viscoelastic phenomena are addressed for many different types of materials including polymers, metals, high damping alloys, piezoelectric materials, cosmetics, rocks, dense composite materials, cellular solids, and biological materials. Viscoelastic materials show properties of both solids and liquids in response to any force. The application of viscoelasticity are earplugs, automobile bumpers, computer disks, gaskets, medical diagnosis, satellite stability, injury prevention, vibration abatement, tire performance, spacecraft explosions, sports, and music, etc. The main mechanical models of viscoelasticity, the Maxwell and Kelvin models. For a specific material, the material will seem to be viscous, if the experiment is done slowly, whilst if the experiment is done rapidly it will show to be elastic.

### 1.1.2 Hooke's law

An English mathematician named Robert Hooke carried out various experiments and came that stress is proportional to deformation up to elastic limit. This is known as Hooke's law. Thus, Hooke's law can be defined as follows:

$$\sigma = E\varepsilon$$

where $\sigma$ is stress, $\varepsilon$ is strain and $E$ is the modulus of elasticity or Young's modulus.

### 1.1.3 Linear viscoelasticity

Linear viscoelastic materials are those for which there is a linear relationship between stress and strain (at any given time). Linear viscoelasticity is a theory representing the behavior of such perfect materials, in addittion, in the linear theory of viscoelasticity, the differential equations are linear.

If a material is exposed to deformations or stresses tiny enough and its rheological functions don't depend on the value of the deformation or stress, the material response is considered in the linear viscoelasticity domain. Linear viscoelasticity represents the simplest response of a viscoelastic material. Linear viscoelasticity is a reasonable approximation to the time-dependent behavior of polymers, and metals and ceramics at relatively low temperatures and under relatively low stress.

**Remark 1.1** *the solution of any one-dimensional linear viscoelastic problem is obtained from the knowledge of the creep function or the relaxation function.*

### 1.1.4 Fundamental uniaxial tests in one-dimensional viscoelastic behavior

#### Creep test

In material sciences and mechanics, creep refers to the slow and continuous deformation of a solid material under a constant load or stress over a period of time. It occurs when a material is subjected to a constant load or stress that is below its yield strength or

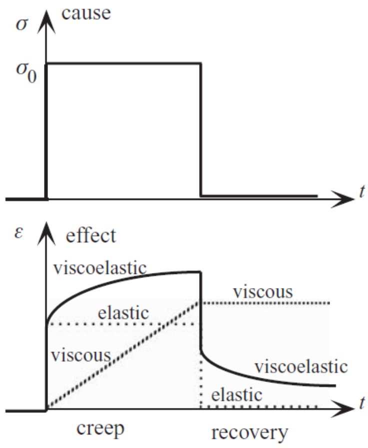


Figure 1.1: Creep and recovery, stress and strain versus time t.

breaking point, but is high enough to make the material deforms over time. Creep tests are an experimental methods commonly used in materials science to determine the creep properties of various materials, including metals, ceramics, and polymers, the test is by applying a constant load or stress to a specimen of the material and then measuring the resulting deformation or strain as a function of time. The results of creep tests can provide valuable informations about a material's mechanical behavior and help in the design and analysis of structures and components that are subjected to sustained loads over long periods of time.

In one-dimension, we assume that the history of stress, as it depends on time t, is a function. Under the effect of a stress $\sigma_0$ applied instantaneously to the material at time $t_0$ and then maintained constant during the test, creep results in an instantaneous strain $\varepsilon_0$ and then an increase in strain $\varepsilon_t$ over time (Figure 1.1) (see [73]).

Thus we write

$$\sigma(t) = \sigma_0 H_{t_0}$$

where $H_{t_0}$ is the Heaviside function, $H_{t_0} = H\left(t - t_0\right),\, H_{t_0} = \begin{cases} 0 \text{ if } t < t_0 \\ 1 \text{ if } t > t_0 \end{cases}$

The viscoelastic response, in terms of time-dependent strain is then expressed by the following formula:

$$\varepsilon(t) = J(t_0, t)\sigma_0$$

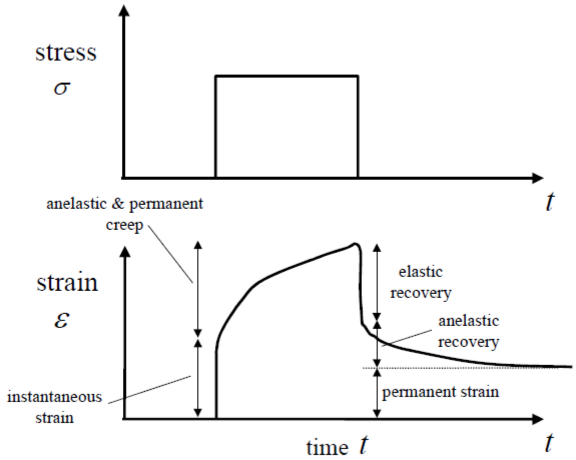


Figure 1.2: Creep-recovery test

where $J(t_0, t)$ represents the creep function.

The creep response is shown in Fig 1.1 starting at the same time as the stress history.

### Relaxation test

A relaxation test, also known as a stress relaxation test, represents a type of material testing technique utilized to measure the relaxation behavior of a material. During a relaxation test, a constant strain is applied to the material, and the resulting stress is measured over time. The strain is then held constant, and the decrease in stress is observed. The test is designed to measure how the stress in the material decreases over time while being subjected to a constant strain. This measurement can provide valuable information about the material's viscoelastic properties, including its stiffness, strength, and ability to resist deformation over time.

This relaxation behavior occurs because the properties of material viscoelastic , which causes it to respond to applied loads in both an elastic (instantaneous) and a viscous (time-dependent) manner. The elastic response causes an immediate deformation of the material, while the viscous response causes the relax of the material over time.

Also, we can say that stress relaxation is the progressive decrease of stress when the material is held at constant strain $\varepsilon$. In a relaxation test, a strain of amplitude $\varepsilon_0$ is instantaneously imposed at time $t_0$ and held constant during the test. The stress in a viscoelastic material will be decreased, as shown in (Figure 1.4) (see [73]), Thus we write

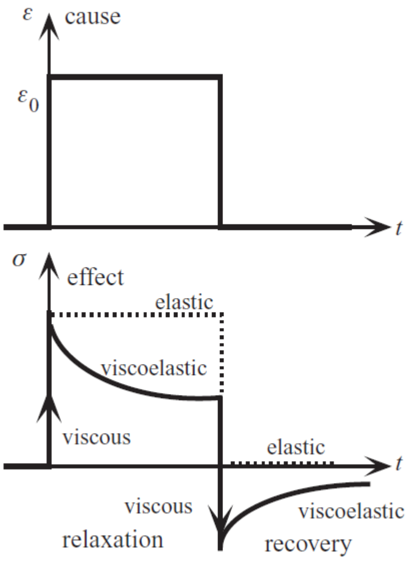


Figure 1.3: Relaxation and recovery.

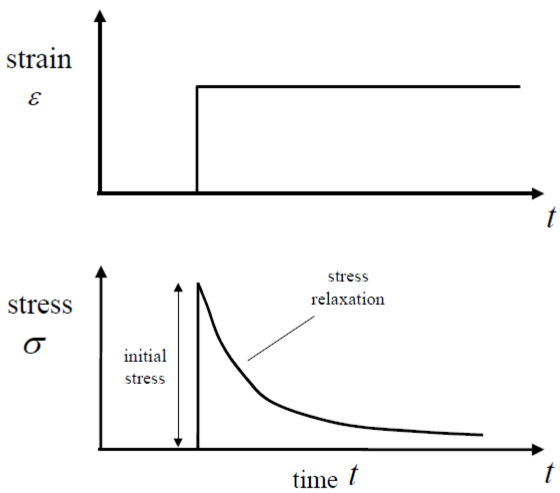


Figure 1.4: relaxation test.

$$\varepsilon(t) = \varepsilon_0 H_{t_0}$$

The corresponding response, in terms of stress, is then

$$\sigma(t) = h(t_0, t)\varepsilon_0$$

where $h(t_0, t)$ represents the relaxation function.

In Figure 1.3 the relaxation response is shown starting at the same time as the strain history.

### 1.1.5 Solids and liquids

Elastic solids constitute a particular case where the creep compliance is, $J(t) = J_0 H(t)$, $J_0$ is a constant, which is, the elastic compliance. Elastic materials respond instantly to zero strain after the load is released.

Viscoelastic materials that make a full recovery after enough time following creep or relaxation are named inelastic materials. Viscous fluids involve another specific case where the creep compliance is, $J(t) = (1/\eta)tH(t)$, here $\eta$ is the viscosity. In viscous materials, creep deformation is unbounded. A viscoelastic solid is a material at which $h(t)$ tends to a finite, nonzero limit when the time t increases to infinity; in the case of a viscoelastic liquid, the function $h(t)$ tends to zero, those properties in the modulus formulation but in the compliance formulation, a viscoelastic solid is the material which $J(t)$ function tends to a finite limit when time t increases to infinity; in the case of a viscoelastic fluid, $J(t)$ function increases without bound when t increases.

### 1.1.6 Behavior law of linear viscoelastic material

The basic assumption made for linear viscoelastic materials is that the stress at the current time is a linear function of the entire deformation history.

Two static tests are most often used to define, for long times, the coefficients of the behavior laws: the creep test and the relaxation test. The creep test consists in imposing a constant stress on a specimen and following its deformations as a function of time. In the relaxation test, an instantaneous deformation is imposed, it is maintained constant and the variations of the stress are measured as a function of the time. Based on the Boltzmann superposition principle (see [27]), the behavior law of any linear viscoelastic material can be written in integral form as

$$\sigma(t) = \int_{-\infty}^{t} h\,(t-s)\,\varepsilon'\,(s)\,ds.$$

where $\sigma$ represents the stress, $\varepsilon$ the strain, $h$ the relaxation function and $s$ is the time integration variable. If the material is at rest initially ($\varepsilon(t) = 0$ for $t < 0$), the previous equation is written as follows

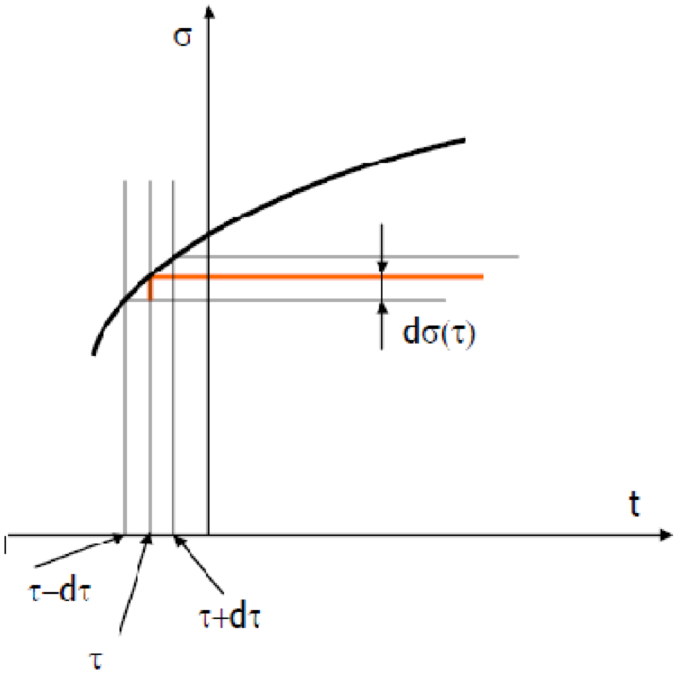


Figure 1.5: Boltzmann Superposition

$$\sigma(t) = h\,(t)\,\varepsilon\,(0) + \int_0^t h\,(t-s)\,\frac{d\varepsilon}{dt}\,(s)\,ds.$$

### 1.1.7 Boltzmann superposition principle

Using the linearity property and knowing the creep function, it is possible to determine the strain response to any (uniaxial) stress history. The stress history can be discretized as a succession of steps of amplitude $d\sigma\,(\tau)$ applied at time $t$ .

The response to such a step is

$$d\varepsilon\,(\tau) = d\sigma\,(\tau)\,J\,(\tau, t)$$

hence

$$\begin{aligned}\varepsilon\,(t) &= \int_{t_0}^t J\,(\tau, t)\,d\sigma\,(\tau) \\ &= \int_{t_0}^t J\,(\tau, t)\,\frac{d\sigma\,(\tau)}{d\tau}d\tau \\ &= \sigma\,(t_0)\,J\,(t_0, t) + \int_{t_0}^t J\,(\tau, t)\,\frac{d\sigma\,(\tau)}{d\tau}d\tau\end{aligned}$$

Integrating by parts, we obtain

$$\varepsilon(t) = \sigma(t_0) J(t_0, t) - \int_{t_0}^{t} \sigma(\tau) \frac{dJ(\tau, t)}{d\tau} d\tau$$

This constitutes the Boltzmann formula.

If we permute $\sigma$ and $\varepsilon$, we obtain the stress response to any strain history :

$$\sigma(t) = \varepsilon(t_0) h(t_0, t) - \int_{t_0}^{t} \varepsilon(\tau) \frac{dh(\tau, t)}{d\tau} d\tau$$

### 1.1.8 Viscoelastic models

Viscoelastic models are widely used in materials science, mechanical engineering, and other fields to describe the behavior of polymers, gels, and other complex materials. These models are also used in rheology, the study of the flow and deformation of materials under stress, to characterize the viscoelastic properties of fluids and soft solids. There are several types of viscoelastic models that have been developed to describe different types of materials, each with their own set of parameters and assumptions. Some commonly used viscoelastic models include: Maxwell model, Kelvin-Voigt model.

#### Basic elements

A classical approach to the description of the linear viscoelastic behavior of real materials which exhibit combined viscous and elastic properties is based upon an analogy with the response of combinations of certain mechanical elements (a spring for elasticity and a dashpot for viscosity) (see Figure 1.6 and Figure 1.7). Such models are, of course, idealized and purely hypothetical, and are useful for representing the behavior of real materials only to the extent that the observed response of the real material can be approximated by that of the model (see [95]).

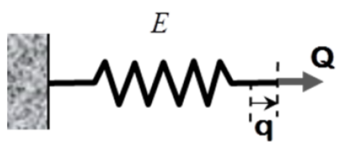


Figure 1.6: Linear elastic element

The force variable acting on a model will be denoted by Q and its associated geometrical variable by q, $Q(t) = Eq(t)$, spring of characteristic $E$, the stress is proportional to the strain $\varepsilon$ and they have the same sign, it simulates the ideal linear elasticity. Hooke elastic solid model given by $\sigma = E\varepsilon$, with $E \geqslant 0$.

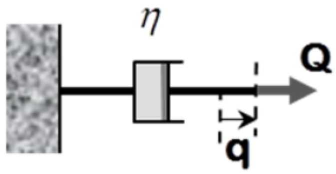


Figure 1.7: Linear viscous element

Notice that $\sigma = \eta \frac{d\varepsilon}{dt}$ is damper of characteristic $\eta$, the stress is proportional to the strain rate $\varepsilon$ and they have same sign, It can only absorb energy and simulates the so-called Newtonian normal viscosity. Newtonian fluid model is defined as follows $\sigma = \eta \frac{d\varepsilon}{dt}$, with $\eta \geqslant 0$, $(\sigma = Q)$.

### Classical models

The rheological models that are commonly referred to the linear viscoelastic are built up from linear elastic or viscous elements that are connected in series or in parallel. It must be underscored that the corresponding graphical representations are just analogical and symbolical, which imply, for instance, that they can be subject to any anamorphosis and must only be interpreted as one-dimensional.

**The Maxwell model (Viscoelastic fluid)** The Maxwell model is just the combination of a linear elastic element and a linear viscous element connected in series (Figure 1.8). In any experiment performed on this model, the same force $Q$ that is exerted on the model is exerted on each constituent element, while the stretch $q$ of the model is the result of the sum of the element stretches (see [95]).

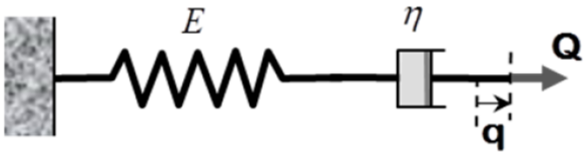


Figure 1.8: Maxwell model

**Parallel assembly rule**:

$$\sigma = \sigma_1 = \sigma_2, \ \varepsilon = \varepsilon_1 + \varepsilon_2$$

We have :

$$\sigma_1 = E\varepsilon_1, \ \sigma_2 = \eta \frac{d\varepsilon_2}{dt}$$

hence, combining all the relations above, yields to:

$$\frac{1}{E}\frac{d\sigma}{dt} + \frac{1}{\eta}\sigma = \frac{d\varepsilon}{dt}$$

which constitutes the rheological equation of Maxwell fluid, it allows to calculate the response to any type of solicitation (creep, relaxation, oscillation, . .). It follows that the creep function for a Maxwell model is obtained through the general basic rule, independent of any linearity or non-aging assumption, stating that: the creep function for a model that is made up of elements connected in series is the result of the sum of the creep functions of the constituent elements.

**The Kelvin -Voigt model (Viscoelastic solid)** The Kelvin model is the combination of a linear elastic element and a linear viscous element connected in parallel (Figure 1.9)(see [95]).

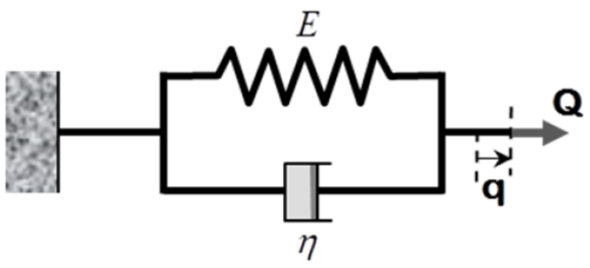


Figure 1.9: Kelvin model

In any experiment performed on this model, the stretch q of the model is the stretch of both constituent elements, while the force Q exerted on the model is the result of the sum of the forces that are exerted in the linear elastic element and the linear viscous element respectively. Then it follows that the relaxation function is obtained from the general rule, independent of any linearity or non-aging assumption, which states that the relaxation function for a model that is made up of elements connected in parallel is the result of the sum of the relaxation functions of the constituent elements.
**Parallel assembly rule**:

$$\sigma = \sigma_1 + \sigma_2,\ \varepsilon = \varepsilon_1 = \varepsilon_2$$

we have :

$$\sigma_1 = E\varepsilon_1,\ \sigma_2 = \eta\frac{d\varepsilon_2}{dt}$$

hence, combining all the relations above, leads to:

$$\eta\frac{d\varepsilon}{dt} + E\varepsilon = \sigma$$

which constitutes the rheological equation of **Kelvin Voigt solid** , it allows to calculate the response to any type of solicitation (creep, relaxation, oscillation, . .)

## 1.2 Reminders in functional analysis

In this section, we have introduced basic tools from functional analysis which will be used in the next chapters. Most of the results are represented without proofs, because considered standard and can be found in numerous references like [19]. We start with a review of concepts and results on weak topologie and weak-star topology, including definitions and properties of $C^k(\Omega)$ spaces, reflexive spaces, separable spaces, $L^p(\Omega)$ spaces. We then recall some properties of Vector-valued Functions Spaces. Finally, we present some well-known inequalities.

### 1.2.1 Weak topology and weak-star topology

#### The weak topology

**Definition 1.1** *(See [60]) Let $V$ be a Banach space. The **weak topology** on $V$ is the coarsest (i.e. smallest) topology such that every element of $V^*$ is continuous, where $V^*$ is a dual topology. Open (respectively, closed) sets in the weak topology will be called weakly open (respectively, weakly closed) sets.*

A basic neighbourhood system for the weak topology is the collection of sets of the form

$$U = \{x \in V, \;\; |f_i(x - x_o)| < \varepsilon : \text{for all } i \in I\}$$

where $x_0 \in V$, $\varepsilon > 0$, $I$ is a finite indexing set and $f_i \in V^*$ for all $i \in I$.

The set $U$ described above forms a weakly open neighbourhood of the point $x_0 \in V$.

Notation: Given a sequence $\{x_n\}$ in $V$, we write $x_n \longrightarrow x$ if the sequence converges to $x \in V$ in the norm topology, i.e. if $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. If the sequence converges to x in the weak topology, we write $x_n \rightharpoonup x$ .

**Proposition 1.1** *Let $V$ be a Banach space and let $\{x_n\}$ be a sequence in $V$.*

**1** *$x_n \rightharpoonup x$ in $V$ if and only if $f(x_n) \to f(x)$ for all $f \in V^*$.*

**2** *If $x_n \to x$ in $V$, then $x_n \rightharpoonup x$.*

**3** *If $x_n \rightharpoonup x$ in $V$, then $\{\|x_n\|\}$ is bounded and*

$$\|x\| \leqslant lim\ inf_{n\to\infty} \|x_n\| .$$

**4** *If $x_n \rightharpoonup x$ in $V$ and $f_n \to f$ in $V^*$, then $f_n(x_n) \to f(x)$.*

***Proof.*** *See ([60]).* ■

**Proposition 1.2** *If $V$ is a finite dimensional space, then the norm and weak topologies coincide.*

**Proposition 1.3 (Schur's Lemma)** *(See [60])In the space $\ell_1$ a sequence is convergent in the weak topology if and only if it converges in the norm topology.*
*$\ell_1$ is a sequence spaces $f : \mathbb{N} \to \mathbb{R}$ such that $\|f\| = \sum_{j \in \mathbb{N}} |f(j)| < \infty$ .*
***Proof.*** *See [60].* ■

### The weak* topology

Let $V$ be a Banach space. Then its dual space, $V^*$, has its natural norm topology. It also is endowed with its weak topology, viz. the coarsest topology such that all the elements of $V^{**}$ are continuous. In the following, we define an even coarser topology on $V^*$.

**Definition 1.2** *(See [60]) The **weak*topology** on $V^*$ is the coarsest topology such that the functionals $\{J_x,\ x \in V\}$ are all continuous, where $J : x \longrightarrow J_x$ is the canonical imbedding of $V$ into $V^{**}$ .*

Clearly, the weak* topology is coarser than the weak topology on $V^*$. Thus, if $S, W$ and $W^*$ denote the norm, weak and weak* topologies, respectively, on $V^*$, we have

$$W^* \subset W \subset S.$$

**Remark 1.2** *It is clear that if $V$ is a reflexive Banach space, then the weak and weak* topologies on $V^*$ are coincide.*

**Proposition 1.4** *(See [60]) Let $V$ be a Banach space and let $\{f_n\}$ be a sequence in $V^*$.*

**1** $f_n \rightharpoonup^* f$ *in* $V^*$ *if and only if* $f_n(x_n) \to f(x)$ *for all* $x \in V$.

**2** *If* $f_n \to f \Rightarrow f_n \rightharpoonup f \Longrightarrow f_n \rightharpoonup^* f$ .

**3** *If* $f_n \rightharpoonup^* f$ *in* $V^*$ *and* $x_n \to x$ *in* $V$, *then* $f_n(x_n) \to f(x)$.

### 1.2.2 $C^k(\Omega)$ Spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \longrightarrow \mathbb{R}$ and $k = 1, 2, .....$

We denote :

$C^k(\Omega)$ : the space of real-valued functions on $\Omega$ that are continuously differentiable up to order k, $0 \leqslant k \leqslant \infty$, with finite norm

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{C(\Omega)} \,.$$

and

$$C(\Omega) = C^0(\Omega) = \{f : f \text{ continuous in } \Omega\} \,.$$

where

$$\|u\|_{C(\Omega)} = \sup_{x \in \Omega} |u(x)| \,.$$

$C_0^k(\Omega)$ : the subspace of functions in $C^k(\Omega)$ with compact support in $\Omega$, such that

$$supp\ f = \overline{\{x \in \Omega : \ f(x) \neq 0\}} = \text{the support of } f.$$

and

$$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega),$$
$$C^\infty(\Omega) = \cap_{k=1}^{\infty} C^k(\Omega),$$

$C_0^\infty(\Omega)$ : the space $D(\Omega)$ ($C_0^\infty(\Omega)$ is called compactly supported smooth functions or test functions)

$$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega),$$

$C(\overline{\Omega})$ : the space of continuous functions on $\overline{\Omega}$.

$C_b(\Omega)$ : the Banach space of bounded and continuous functions on $\Omega$.

### 1.2.3 Reflexive spaces

**Definition 1.3** *Let $E$ be a Banach space and let $J : E \to E^{**}$ be the canonical injection from $E$ into $E^{**}$. The space $E$ is said to be reflexive if $J$ is surjective, i.e., $J(E) = E^{**}$.*

When $E$ is reflexive, $E$ is usually identified with $E^{**}$.

**Theorem 1.1 (Kakutani)** *Let $E$ be a Banach space. Then $E$ is reflexive* if

$$B_E = \{x \in E; x \leqslant 1\}$$

*is compact in the weak topology $\sigma(E, E^*)$.*

**Proof.** See [98] ∎

### 1.2.4 Separable spaces

**Definition 1.4** *We say that a metric space $E$ is separable if there exists a subset $D \subset E$ that is countable and dense.*

**Corollary 1.1** *Let $E$ be a separable Banach space and let $(f_n)$be a bounded sequence in $E$ . Then there exists a subsequence $(f_{n_k})$ that converges in the weak topology $\sigma(E, E^*)$.*

### 1.2.5 Definitions and elementary properties of $L^p(\Omega)$ spaces

**Definition 1.5** *Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \left\{ f :\ \Omega \to \mathbb{R}; f \quad \textit{is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[ \int_\Omega |f(x)|^p \, d\mu \right]^{1/p}$$

**Definition 1.6** *(See [19] )We set*

$$L^\infty(\Omega) = \left\{ f :\ \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} f \quad \text{is measurable and there is a constant } C\ |f|^p \in L^1(\Omega) \\ \text{such that } |f(x)| \leq C \text{ a. e.on } \Omega \end{array} \right\} \text{with}$$

$$\|f\|_{L^\infty} = \|f\|_\infty = inf\ \{C;\ |f(x)| \leq C \quad \text{a.e. on } \Omega\}.$$

**Theorem 1.2 (Fischer–Riesz)** *See [19] $L^p$ is a Banach space for every $p$, $1 \leqslant p \leqslant \infty$.*

The following table summarizes the main properties of the space $L^p(\Omega)$ when $\Omega$ is a measurable subset of $\mathbb{R}^N$:

| | Reflexive | Separable | Dual space |
|---|---|---|---|
| $L^p$ with $1 < p < \infty$ | YES | YES | $L^q$ |
| $L^1$ | NO | YES | $L^\infty$ |
| $L^\infty$ | NO | NO | Strictly greater than $L^1$ |

$q$ represent the conjugate exponent, where $\frac{1}{p} + \frac{1}{q} = 1$.

### 1.2.6 Definition and elementary properties of Sobolev spaces

Let us start with a motivation for definition of weak derivative.

#### Weak derivatives

Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Integration by parts given as follows

$$\int_\Omega u \frac{\partial \varphi}{\partial x_j} dx = -\int_\Omega \frac{\partial u}{\partial x_j} \varphi dx.$$

There is no boundary term, since $\varphi$ has a compact support in $\Omega$ and thus vanishes near $\partial\Omega$.

Then let $u \in C^k(\Omega)$, $k = 1, 2, ...$, and let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ (we use the convention that $0 \in N$) be a multi-index such that the order of multi-index $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ is at most $k$. We denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_j \partial x_j \partial x_j}$$

**Remark 1.3** (*see [61]*)*A coordinate of a multi-index indicates how many times a function is differentiated with respect to the corresponding variable. The order of a multi-index* *indicates the total number of differentiations. Successive integration by parts gives*

$$\int_{\Omega} uD^{\alpha}\varphi dx = (-1)^{|\alpha|}\int_{\Omega} D^{\alpha}u\varphi dx.$$

Notice that the left-hand side makes sense even under the assumption $u \in L^1_{loc}(\Omega)$.

**Definition 1.7** *Assume that $u \in L^1_{loc}(\Omega)$ and let $\alpha \in \mathbb{N}^n$ be a multi-index. Then $v \in L^1_{loc}(\Omega)$ is the $\alpha$th weak partial derivative of $u$, written $D^{\alpha}u = v$, if*

$$\int_{\Omega} uD^{\alpha}\varphi dx = (-1)^{|\alpha|}\int_{\Omega} v\varphi dx.$$

for every test function $\varphi \in C_0^{\infty}(\Omega)$. We denote $D^0u = D^{(0,\ldots,0)} = u$.

**Remark 1.4** *If $u \in C^k(\Omega)$, then the classical partial derivatives up to order $k$ are also the corresponding weak derivatives of $u$. In this sense, weak derivatives generalize classical derivatives.*

**Remark 1.5** *(see [20])If $u \in C^1(a,b)$ and $\varphi \in C_0^1(a,b)$, then*

$$\int_a^b u\varphi' dx = -\int_a^b u'\varphi dx.$$

*This formula for the one-dimensional case.*

**Sobolev spaces $W^{k,p}(\Omega)$**

**Definition 1.8** *(see [61])Assume that $\Omega$ is an open subset of $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index $\alpha$ with $|\alpha| \leqslant k$, the weak derivative $D^{\alpha}u$ exists and $D^{\alpha}u \in L^p(\Omega)$ Thus*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega),\ \ |\alpha| \leqslant k\ \}.$$

**Definition 1.9** *If $u \in W^{k,p}(\Omega)$, we define its norm*

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leqslant k}\int_{\Omega} |D^{\alpha}u|^p\, dx\right)^{\frac{1}{p}},\ 1 \leqslant p < \infty,$$

*and*

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leqslant k} ess_{\Omega} \sup |D^{\alpha}u| .$$

**Remark 1.6** *(see [61])* $\|u\|_{W^{k,\infty}(\Omega)}$ *is also equivalent to*

$$\max_{|\alpha|\leqslant k} \|D^{\alpha}u\|_{L^{\infty}(\Omega)} .$$

*2) For* $k = 1$ *we use the norm*

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &= \left(\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \\ &= \left(\int_{\Omega} |u|^p\, dx + \int_{\Omega} |Du|^p\, dx\right)^{\frac{1}{p}},\ 1 \leqslant P < \infty, \end{aligned}$$

*and*

$$\|u\|_{W^{1,p}(\Omega)} = ess_{\Omega} \sup |u| + ess_{\Omega} \sup |Du| .$$

**Theorem 1.3 (Completeness)** *The Sobolev space* $W^{k,p}(\Omega)$*,* $1 \leqslant P < \infty$*,* $k = 1, 2, ...,$*is a Banach space.*

**Remark 1.7** *The space* $W^{k,2}(\Omega)$ *or* $H^2(\Omega)$*is a Hilbert space with the inner product*

$$\langle u, v\rangle_{W^{k,2}(\Omega)} = \sum_{|\alpha|\leqslant k} \langle D^{\alpha}u, D^{\alpha}v\rangle_{L^2(\Omega)} .$$

*where*

$$\langle D^{\alpha}u, D^{\alpha}v\rangle_{L^2(\Omega)} = \int_{\Omega} D^{\alpha}uD^{\alpha}vdx.$$

*note that*

$$\|u\|_{W^{k,2}(\Omega)} = \sqrt{\langle u, u\rangle}_{W^{k,2}(\Omega)}.$$

**Sobolev space** $W^{1,p}(\Omega)$

Let $\Omega = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \leqslant p \leqslant \infty$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}$$

where

$$W^{1,p} = \{u \in L^p(\Omega) : \ u' \in L^p(\Omega)\}$$

or sometimes, if $1 < p < \infty$, with the equivalent norm $(\|u\|_{L^p}^p + \|u'\|_{L^p}^p)^{1/p}$. The space $H^1$ is equipped with the scalar product

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle u', v'\rangle_{L^2} = \int_a^b (uv + u'v')$$

and with the associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\right)^{1/2}.$$

**Proposition 1.5** *The space $W^{1,p}$ is a Banach space for $1 \leqslant p \leqslant \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \leqslant p < \infty$. The space $H^1$ is a separable Hilbert space.*

**Proof.** (See [19] ). ■

**Remark 1.8** *1) For $p \geqslant 2$, $1 \leqslant p \leqslant \infty$,we have*

$$W^{m,p}(\Omega) = \left\{u \in L^p(\Omega) : u^{(j)} \in L^p(\Omega),\ j \leqslant m\right\}.$$

*assotiated with the norm*

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{j\leqslant m} \left\|u^{(j)}\right\|_{L^p(\Omega)}.$$

*2) If $p = 2, W^{m,p}(\Omega) = H^m(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v\rangle_{H^m(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \sum_{j\leqslant m} \left\langle u^{(j)}, v^{(j)}\right\rangle_{L^2(\Omega)},\ u, v \in H^m(\Omega).$$

### 1.2.7 Spaces of vector-valued functions

It is required to introduce the spaces of vector-valued functions to study the time-dependent variational problems. In the following, if it is not specified otherwise,we denote by $(X,\cdot\|.\|_X)$ and $[0,T]$ a real Banach space and time interval of interest for $T>0$, respectively. We define $C([0,T];X)$ to be the space of functions $v:[0,T]\to X$ that are continuous on the closed interval $[0,T]$. With the norm

$$\|v\|_{C([0,T];X)} = max_{t\in[0,T]}\|v(t)\|_X\,,$$

the space $C([0,T];X)$ is a Banach space.

**Definition 1.10** (*see [98]*) *A function $v:[0,T]\to X$ is said to be (strongly) differentiable at $t_0\in[0,T]$ if there exists an element in $X$, denoted as $v\prime(t_0)$ and called the (strong) derivative of $v$ at $t_0$, such that*

$$\lim_{h\to 0}\left\|\frac{1}{h}(v(t_0+h)-v(t_0))-v\prime(t_0)\right\|_X=0$$

where the limit is taken with respect to $h$ with $t_0+h\in[0,T]$. The derivative at $t_0=0$ is defined as a right-side limit and that at $t_0=T$ as a left-sided limit. The function $v$ is said to be differentiable on $[0,T]$ if it is differentiable at every $t_0\in[0,T]$. It is differentiable a.e. if it is differentiable a.e. on $[0,T]$.

In this case, the function $v\prime$ is called the (strong) derivative of $v$. Higher derivatives $v^{(j)}, j\geqslant 2$, are defined recursively by $v(j)=(v^{(j-1)})\prime$. Usually we use the notation $v^\bullet=v\prime$ and we understand $v^{(0)}$ to be $v$.

For an integer $m\geqslant 0$, we define the space

$$C^m([0,T];X)=\{v\in C([0,T];X): v^{(j)}\in C([0,T];X),\ j=1,...,m\}.$$

This is a Banach space with the norm

$$\|v\|_{C^m([0,T];X)}=\sum_{j=0}^{m} max_{t\in[0,T]}\left\|v^{(j)}(t)\right\|_X.$$

In particular, $C^1([0,T];X)$ denotes the space of continuously differentiable functions on $[0,T]$ with values in $X$. This is a Banach space with the norm

$$\|v\|_{C^1([0,T];X)} = max_{t\in[0,T]} \|v(t)\|_X + max_{t\in[0,T]} \|v^\bullet(t)\|_X .$$

We also set

$$\begin{aligned} C^\infty([0,T];X) &= \cap_{m=0}^{\infty} C^m([0,T];X) \\ &= \{v \in C([0,T];X) : v \in C^m([0,T];X) \forall m \in Z_+\}, \end{aligned}$$

the space of infinitely differentiable functions defined on $[0,T]$ with values in $X$.

The spaces $L^p([0,T]\,;X)$. For $p \in [1,\infty)$, we define $L^p([0,T]\,;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $\int_0^T \|v(t)\|_X^p \, dt < \infty$. With the norm

$$\|v\|_{L^p([0,T];X)} = \left( \int_0^T \|v(t)\|_X^p \, dt \right)^{1/p}$$

the space $L^p([0,T]\,;X)$ becomes a Banach space. We define $L^\infty([0,T]\,;X)$ to be the space of all measurable functions

$v : [0,T] \to X$ such that $t \to \|v(t)\|_X$ is essentially bounded on $[0,T]$. The space $L^\infty([0,T]\,;X)$ is a Banach space with the norm

$$\|v\|_{L^\infty([0,T];X)} = ess \sup_{t\in[0,T]} \|v(t)\|_X .$$

When $(X, \langle\cdot,\cdot\rangle_X)$ is a Hilbert space, $L^2([0,T]\,;X)$ is also a Hilbert space with the inner product given by

$$\langle u, v\rangle_{L^2([0,T];X)} = \int_0^T \langle u\,(t)\,, v\,(t)\rangle_X \, dt.$$

In what follows, the space $L^p(0,T)$ is denoted by $L^p([0,T]\,;\mathbb{R})$.

### 1.2.8 Duality and weak convergence

We have the following important result, on separable Banach spaces.

**Theorem 1.4** (*see [98]*)*If $X$ is a separable Banach space, then each bounded sequence in $X^*$has a weakly * convergent subsequence.*

It is deduced from the previous theorem that if $X$ is a separable Banach space and the sequence $\{u\prime_n\} \subset X^*$ verified $sup_n \|u\prime_n\|_{X*} < \infty$, then we can find a subsequence $\{u\prime_{n_k}\} \subset \{u\prime_n\}$ and an element $u\prime \in X*$ such that $u\prime_{n_k} \rightharpoonup^* u\prime$ in $X^*$.
If $Y$ is a subspace of a normed space $(X, \|.\|_X)$, then $(Y,\cdot\|.\|_X)$ is a normed space, too. If it is not explicitly stated otherwise, the norm over a subspace is taken to be the norm of the original normed space. Moreover, we have the following theorem.

**Theorem 1.5** (*See [98]*) *Let* $(X,\cdot\|.\|_X)$ *be a Banach space and let* $Y \subset X$ *be a closed subspace of* $X$*. The following results hold :*
*(1)* $Y$ *is a Banach space with the norm* $\|.\|_X$*.*
*(2) If* $X$ *is separable, then* $Y$ *is separable.*
*(3) If* $X$ *is reflexive, then* $Y$ *is reflexive.*

### 1.2.9 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$ .

$$\begin{cases} u''(t) + A(t)\,u(t) = f(t) & t \ in \ [0,T], \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where $u$ and $f$ are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space $H$. $A(t) \quad (0 \le t \le T)$ are linear bounded operators in $H$ acting in the energy space $V \subset H$.
Assume that $\langle A(t)\,u(t), v(t)\rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t;.,.)$ is a bilinear continuos in $V$. The problem (1.1) can be formulated as: Found the solution $u(t)$ such that

$$\begin{cases} u \in C([0,T]; V), \ u' \in C([0,T]; H) \\ \langle u''(t), v\rangle + a(t; u(t), v) = \langle f, v\rangle & t \ in \ D'([0,T]), \\ u_0 \in V, \quad u_1 \in H, \end{cases} \tag{1.2}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.
Let $V_m$ a sub-space of $V$ with the finite dimension $d_m$, and let$\{w_{jm}\}$ one basis of $V_m$ such that

1. $V_m \subset V$ $(\dim V_m < \infty)$, $\forall m \in \mathbb{N}$
2. $V_m \longrightarrow V$ such that, there exist a dense subspace $\mathcal{V}$ in $V$ and for all $v \in \mathcal{V}$ we can get sequence $\{u_m\}_{m\in\mathbb{N}} \in V_m$ and $u_m \longrightarrow u$ in $V$.
3. $V_m \subset V_{m+1}$ and $\overline{U_{m\in\mathbb{N}} V_m} = V$.

We define the solution $u_m$ of the approximate problem

$$\begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm}, \\ u_m \in C([0,T]; V_m), u'_m \in C([0,T]; V_m), \; u_m \in L^2([0,T]; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}, \; u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm}, \end{cases} \tag{1.3}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in } V \text{ as } m \longrightarrow \infty$$
$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in } V \text{ as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations, the system (1.3) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

**Lemma 1.1 (Zorn's Lemma)** *[28] Let $S$ be a partially ordered set. If every totally ordered subset of $S$ has an upper bound in $S$, then $S$ contains a maximal element.*

### 1.2.10 Aubin-Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis.

**Lemma 1.2** *[65] Let $X_0, X$ and $X_1$ be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that $X_0$ is compactly embedded in $X$ and that $X$ is continuously embedded in $X_1$; assume also that $X_0$ and $X_1$ are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) / \dot{u} \in L^q([0,T]; X_1)\}$$

*Then the embedding of $W$ into $L^p([0,T]; X)$ is also compact.*

### 1.2.11 Basic inequalities

**Hölder's inequality**

**Theorem 1.6** *Assume that $f \in L^p$ and $g \in L^q$ with $1 \leqslant p \leqslant \infty$. Then $fg \in L^1$ and*

$$\int |fg| \leqslant \|f\|_p \|g\|_q .$$

$q$ represent the conjugate exponent, where $\frac{1}{p} + \frac{1}{q} = 1$.

**Remark 1.9** *It is useful to keep in mind the following extension of Hölder's inequality:*

Assume that $f_1, f_2, ..., f_k$ are functions such that

$$f_i \in L^{p_i}, 1 \leqslant i \leqslant k \text{ with } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + .... + \frac{1}{p_k} \leqslant 1.$$

Then the product $f = f_1 f_2 ... f_k$ belongs to $L^p$, and

$$\|f\|_p \leqslant \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k} .$$

In particular, if $f \in L^p \cap L^q$ with $1 \leqslant p \leqslant q \leqslant \infty$, then $f \in L^r$ for all $r$, $p \leqslant r \leqslant q$, and the following "interpolation inequality" holds:

$$\|f\|_r \leqslant \|f\|_p^{\alpha} \|f\|_q^{1-\alpha} , \text{ where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \ 0 \leqslant \alpha \leqslant 1.$$

### Cauchy and Young inequalities

Let $a, b$ are any real numbers and $p$, $q$ are real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$. Then

**Lemma 1.3 (Cauchy inequality)**

$$ab \leqslant \frac{1}{2}\left(a^2 + b^2\right).$$

**Lemma 1.4 (Cauchy inequality with epsilon)**

$$ab \leqslant \epsilon a^2 + \frac{b^2}{4\epsilon},\ \forall \epsilon > 0.$$

**Lemma 1.5 (Young inequality)**

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

### Poincaré inequalities

**Lemma 1.6** *(see [52]) For any $\phi(x,t)$ continuously differentiable on $[L_1, L_2]$,we have*

$$\int_{L_1}^{L_2} [\phi(x,t)]^2 dx \leqslant 2(L_2 - L_1)\phi^2(L_2,t) + 4(L_2 - L_1)^2 \int_{L_1}^{L_2} [\phi'(x,t)]^2 dx. \tag{1.4}$$

*and*

$$\int_{L_1}^{L_2} [\phi(x,t)]^2 dx \leqslant 2(L_2 - L_1)\phi^2(L_1,t) + 4(L_2 - L_1)^2 \int_{L_1}^{L_2} [\phi'(x,t)]^2 dx. \tag{1.5}$$

Inequality is obtained in the same manner.

**Remark 1.10** *From Poincaré inequalities* (1.4) *and* (1.5), *we further have*

$$\int_{L_1}^{L_2} [\phi(x,t)]^2 dx \leqslant 2(L_2 - L_1)\phi^2(L_2,t) + 8(L_2 - L_1)^3 \phi'^2(L_2,t) + 16(L_2 - L_1)^4 \int_{L_1}^{L_2} [\phi''(x,t)]^2 dx \tag{1.6}$$

*and*

$$\int_{L_1}^{L_2} [\phi(x,t)]^2 dx \leqslant 2(L_2 - L_1)\phi^2(L_1,t) + 8(L_2 - L_1)^3 \phi'^2(L_1,t) + 16(L_2 - L_1)^4 \int_{L_1}^{L_2} [\phi''(x,t)]^2 dx. \tag{1.7}$$

**Gronwall inequality**

**Theorem 1.7** *Let* $x$, $\Psi$ *and* $\chi$ *be real continuous functions defined in* $[a,b]$, $\chi(t) \geqslant 0$ *for* $t \in [a,b]$. *We suppose that on* $[a,b]$, *we have the inequality*

$$x(t) \leqslant \Psi(t) + \int_a^t \chi(s)\, x(s) ds.$$

*Then*

$$x(t) \leqslant \Psi(t) + \int_a^t \chi(s)\,\Psi(s) \exp\left[\int_a^t x(u) du\right] ds.$$

**Proof.** For demonstration (see [34]) ■

**Corollary 1.2** *If* $\Psi$ *is differentiable, then from the last theorem it follows that, for all* $t \in [a,b]$

$$x(t) \leqslant \Psi(a)\left(\int_a^t \chi(u) du\right) + \int_a^t \exp\left(\int_a^t \chi(u) du\right) \Psi'(s)\, ds.$$

**Corollary 1.3** *If* $\Psi$ *is constant, then from*

$$x(t) \leqslant \Psi + \int_a^t \chi(s)\, x(s) ds.$$

*it follows that*

$$x(t) \leqslant \Psi \exp\left(\int_a^t \chi(u)\, du\right).$$

# Chapter 2

# Arbitrary decay of solutions for a viscoelastic flexible satellite system

In this chapter, we investigate the viscoelastic flexible satellite system under unwanted vibrations yielding during the movement, we prove the well-posedness of the problem as well as the arbitrary decay of the system.

## 2.1 Introduction

In this chapter, we investigate the stability analysis of a flexible spacecraft problem with viscoelastic damping, namely when the two symmetric flexible appendages are made with viscoelastic materials. According to Boltzmann principle and utilizing the constitutive relationship between the stress and the strain

$$\sigma(x,t) = EI\varepsilon(x,t) - EI\int_0^t \zeta(t-s)\varepsilon(x,s)ds,$$

where $\sigma(x,t), \varepsilon(x,t)$ represent the stress, the strain respectively and the function $\zeta$ is called the relaxation function, then if $f^L \equiv f^R \equiv 0$ in the problem (4.58) to (4.59), We are interested in the following problem:

$$\begin{cases} \rho A\omega_{tt}^{L}(x,t)+EI\omega_{xxxx}^{L}(x,t)-EI\int_{0}^{t}\zeta(t-s)\,\omega_{xxxx}^{L}(x,s)\,ds=0, & x\in[0,l/2]\,, \\ \rho A\omega_{tt}^{R}(x,t)+EI\omega_{xxxx}^{R}(x,t)-EI\int_{0}^{t}\zeta(t-s)\,\omega_{xxxx}^{R}(x,s)\,ds=0, & x\in[l/2,l]\,, \end{cases} \tag{2.1}$$

with the boundary conditions

$$\begin{cases} \omega_{x}^{L}(l/2,t)=\omega_{x}^{R}(l/2,t)=0,\ \omega_{xx}^{L}(0,t)=\omega_{xx}^{R}(l,t)=0,\ \omega_{xxx}^{L}(0,t)=\omega_{xxx}^{R}(l,t)=0, \\ \omega^{L}(l/2,t)=\omega^{R}(l/2,t)=\omega(l/2,t)\,, \quad t\geq 0 \\ m\omega_{tt}(l/2,t)= \ u(t)+EI\omega_{xxx}^{L}(l/2,t)-EI\int_{0}^{t}\zeta(t-s)\,\omega_{xxx}^{L}(l/2,s)\,ds \\ \qquad -EI\omega_{xxx}^{R}(l/2,t)+EI\int_{0}^{t}\zeta(t-s)\,\omega_{xxx}^{R}(l/2,s)\,ds, \quad t\geq 0 \end{cases} \tag{2.2}$$

and the initial data

$$\begin{cases} \omega^{L}(x,0)=\omega_{0}^{L}(x),\ \ \omega_{t}^{L}(x,0)=\omega_{1}^{L}(x), & x\in[0,l/2] \\ \omega^{R}(x,0)=\omega_{0}^{R}(x),\ \ \omega_{t}^{R}(x,0)=\omega_{1}^{R}(x), & x\in[l/2,l]\,, \end{cases} \tag{2.3}$$

The integral term in (2.1) represents the viscoelastic damping term. The stability of the problem (2.1)-(2.3), for all $t\in[0,\infty)$ is discussed in this chapter, where the main contributions are summarized as follows

(i) Without any dampings implemented at the transverse displacements of the left and right panels, the viscoelastic material is shown to be able to guarantee the arbitrary stabilization of the flexible satellite system under free vibration condition. Namely, when the left and the right panels are made by a viscoelastic material.

(ii) Under a suitable control force acting on the centerbody of the satellite, the arbitrary stabilization of the system (2.1)–(2.3) is established for a large class of kernels. Namely, we suppose that the kernel $\zeta(t)$ satisfies (see [104]):

$(H_1)$ For all $t \geqslant 0$, $\zeta(t) \geqslant 0$, $\zeta$ is a continuously differentiable function satisfies

$$0 < \kappa := \int_0^{+\infty} \zeta(s)\, ds < 1.$$

$(H_2)$ $\zeta(t)$ is absolutely continuous function and of bounded variation on $(0, \infty)$ and $\zeta'(t) \leq h(t)$ for some non-negative summable function $h(t)$ $(= \max\{0, \zeta'(t)\})$ and almost all $t > 0$.

$(H_3)$ There exists a non-decreasing function $\gamma(t) > 0$ satisfying $\lim\limits_{t \longrightarrow \infty} \gamma(t) = +\infty$, the ratio $\gamma'(t)/\gamma(t) =: \mu(t)$ is decreasing and

$$\int_0^{+\infty} \zeta(s)\gamma(s) ds < +\infty.$$

It is important to consider such class as it allows the use of several types of materials that could be more convenient. For more examples and for more other types of kernels, one can consult [58] and [96], [97].

**Remark 2.1** *Notice that the assumption (H3) is satisfied by a large class of functions namely functions of polynomial type and functions of exponential type. Indeed, the functions* $\gamma(t) = (1+t)^\alpha$, $\alpha > 0$ *and* $\gamma(t) = e^{\beta t}$, $\beta > 0$ *satisfy the assumption with* $\mu(t) = \alpha(1+t)^{-1}$ *and* $\mu(t) = \beta$, *respectively.*

## 2.2 Preliminary results

In this section, we establish some Lemmas and notations which will be needed later.
We use the same notations in [104], for every measurable set $\mathcal{B} \subset \mathbb{R}^+$, we define the probability measure $\widehat{\zeta}$ by

$$\widehat{\zeta}(\mathcal{B}) = \frac{1}{\kappa} \sup_{t>0} \int_{\mathcal{B}_t} \zeta(s)\, ds. \tag{2.4}$$

where $\mathcal{B}_t = \mathcal{B} \cap [0, t]$. The non-decreasingness set is defined by

$$\chi_\zeta = \left\{ t \in \mathbb{R}^+ : \ \zeta'(s) \geq 0 \right\} \tag{2.5}$$

. Let $t_\star > 0$ be a real number such that

$$\int_0^{t_\star} \zeta(s)\,ds =: \zeta_\star > 0.$$

**Lemma 2.1** *The energy functional associated to the problem* (2.1)–(2.3) *is defined by*

$$\begin{aligned} E(t) = & \frac{\rho A}{2}\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \frac{\rho A}{2}\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t(l/2,t)\right]^2 \\ & + \frac{EI}{2}\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \frac{EI}{2}\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx, \quad t \geq 0, \end{aligned} \tag{2.6}$$

*satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) = & \int_0^{l/2}\omega_{txx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,ds dx \\ & + \int_{l/2}^{l}\omega_{txx}^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,ds dx + u(t)\,\omega_t(l/2,t) \end{aligned} \tag{2.7}$$

*for all* $t \geq 0$.

**Proof.** By multiplying the first equation of the system (2.1) by $\omega_t^L(x,t)$ and integrating over $[0, l/2]$, and similarly, multiplying the second equation of (2.2) by $\omega_t^R(x,t)$ and integrating over $[l/2, l]$, then multiplying the last Eq. of (2.2) by $\omega_t(l/2,t)$, Using the conditions of the borders and summarizing the findings, we get (2.7). ■

**Remark 2.2** *For* $t \geq 0$, *we have*

$$\begin{aligned} 2\int_0^{l/2}\omega_{txx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,ds dx = & \int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx - \zeta(t)\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx \\ & - \frac{d}{dt}\left[\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx - \left(\int_0^t \zeta(s)\,ds\right)\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx\right] \end{aligned} \tag{2.8}$$

*and*

$$\begin{aligned} 2\int_{l/2}^{l}\omega_{txx}^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,ds dx = & \int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)(t)\,dx - \zeta(t)\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx \\ & - \frac{d}{dt}\left[\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx - \left(\int_0^t \zeta(s)\,ds\right)\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right]. \end{aligned} \tag{2.9}$$

Therefore, using the relations (2.8) and (2.9), the modified energy functional associated to the problem (2.1)–(2.3) is defined by:

$$\begin{aligned}\mathcal{E}(t) =& \frac{\rho A}{2}\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t(l/2,t)\right]^2 + \frac{\rho A}{2}\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx \\ &+ \frac{EI}{2}\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx + \frac{EI}{2}\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx \\ &+ \frac{EI}{2}\left[1-\left(\int_0^t \zeta(s)\,ds\right)\right]\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right)\end{aligned} \tag{2.10}$$

and satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) =& \frac{EI}{2}\left(\int_0^{l/2}(\zeta'\Box\omega_{xx}^L)(t)dx + \int_{l/2}^{l}(\zeta'\Box\omega_{xx}^R)(t)dx\right) + u(t)\,\omega_t(l/2,t) \\ &- \frac{EI}{2}\zeta(t)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right), \quad t \geqslant 0.\end{aligned} \tag{2.11}$$

Now, we state without proof the well-posedness of the problem, first we define

$$\begin{aligned}\mathbb{V} \;=\; & \Big\{(y^L,\, y^R) \in H^2(0,l/2) \times H^2(l/2,l)\,, \quad y_x^L(l/2) = y_x^R(l/2) = 0, \quad y^L(l/2) = \\ & y^R(l/2) = y(l/2)\Big\},\end{aligned}$$

and

$$\mathbb{W} = \Big\{(y^L,\, y^R) \in H^4(0,l/2)\times H^4(l/2,l)\,, \quad y_{xx}^L(0) = y_{xx}^R(l) = 0, \quad y_{xxx}^L(0) = y_{xxx}^R(l) = 0\Big\}$$

where $H^2(0,l/2)$, $H^2(l/2,l)$, $H^4(0,l/2)$ and $H^4(l/2,l)$ represent the usual Sobolev spaces.

**Theorem 2.1** *Let* $\left(\omega_0^L,\omega_0^R\right) \in \mathbb{V}$, $\left(\omega_1^L,\omega_1^R\right) \in \mathbb{W}$ *are given. Suppose that* $\zeta'(t) \leq 0$ *holds. Then, under the external force* $u(t)$ *defined in* (2.71) *there exists a unique global weak solution* $(\omega^L,\, \omega^R)$ *of the problem* $(2.1) - (2.3)$ *which satisfies*

$$(\omega^L,\, \omega^R) \in L^\infty\big([0,T);\mathbb{W}\big),\;\; (\omega_t^L,\, \omega_t^R) \in L^\infty\big([0,T);\mathbb{V}\big),$$

$$(\omega_{tt}^L,\, \omega_{tt}^R) \in L^\infty\big([0,T);L^2(0,l/2)\times L^2(l/2,l)\big)$$

*where* $T > 0$.

**Proof.** See the next chapter. ■

## 2.3 Technical lemmas

In this section, we establish some lemmas that will be used to have the main result (i.e. Theorem 2.2). Let

$$K(t) = \mathcal{E}(t) + \sum_{i=1}^{7} \lambda_i \Xi_i(t), \ t \geqslant 0 \tag{2.12}$$

Where $\lambda_i$, $i = 1, ..., 5$ are positive constants which will be selected later such that $\lambda_3 = \lambda_4 = 1$, and for all $t \geqslant 0$

$$\Xi_1(t) := \rho A \int_0^{l/2} \omega^L(x,t)\, \omega_t^L(x,t)\, dx + \rho A \int_{l/2}^{l} \omega^R(x,t)\, \omega_t^R(x,t)\, dx + m\omega_t(l/2,t)\, \omega(l/2,t), \tag{2.13}$$

$$\begin{aligned} \Xi_2(t) := & -\rho A \int_0^{l/2} \omega_t^L(x,t) \int_0^t \zeta(t-s) \left(\omega^L(x,t) - \omega^L(x,s)\right) ds dx \\ & -\rho A \int_{l/2}^{l} \omega_t^R(x,t) \int_0^t \zeta(t-s) \left(\omega^R(x,t) - \omega^R(x,s)\right) ds dx \\ & -m\omega_t(l/2,t) \int_0^t \zeta(t-s) \left(\omega(l/2,t) - \omega(l/2,s)\right) ds, \end{aligned} \tag{2.14}$$

$$\Xi_3(t) := \frac{1}{2}(\zeta \Diamond \omega)(t) + \frac{1}{2}\left[\omega(l/2,t)\right]^2, \tag{2.15}$$

$$\Xi_4(t) := \int_0^t \varphi_\alpha(t-s) \left(\omega(l/2,t) - \omega(l/2,s)\right)^2 ds, \quad , \tag{2.16}$$

$$\Xi_5(t) := EI \int_0^t \varphi_\gamma(t-s) \int_0^{l/2} \left[\omega_{xx}^L(x,s)\right]^2 dx ds + EI \int_0^t \varphi_\gamma(t-s) \int_0^{l/2} \left[\omega_{xx}^R(x,s)\right]^2 dx ds, \tag{2.17}$$

$$\Xi_6(t) := EI \int_0^t \phi_\gamma(t-s) \int_0^{l/2} \left[\omega_{xx}^L(x,s)\right]^2 dx ds + EI \int_0^t \phi_\gamma(t-s) \int_0^{l/2} \left[\omega_{xx}^R(x,s)\right]^2 dx ds, \tag{2.18}$$

and

$$\Xi_7(t) := \int_0^t \phi_\gamma(t-s)\, \omega^2(l/2,s) ds, \tag{2.19}$$

where

$$\varphi_\alpha(t) := e^{-\alpha t} \int_t^\infty \zeta(s)\, e^{\alpha s} ds, \quad \varphi_\gamma(t) := \gamma(t)^{-1} \int_t^\infty \zeta(s)\, \gamma(s)\, ds \tag{2.20}$$

$$\text{and} \quad \phi_\gamma(t) := \gamma(t)^{-1} \int_t^\infty h(s)\, \gamma(s)\, ds, \tag{2.21}$$

such that $\alpha$ is a positive constant to be selected later for all $t \geq 0$,.

**Remark 2.3** *The functionals $\Xi_i$, $i = 1, ..., 7$ should be chosen in such a way that their derivatives will provide us with similar terms to the ones in the energy $\mathcal{E}(t)$, but with negative coefficients. So these terms can be control and have the energy with a negative sign for the right-hand side of the estimate of the rate of change of the energy.*

Now, we compare $K(t)$ and $\mathcal{E}(t) + \sum\limits_{i=3}^{7} \Xi_i(t)$.

**Proposition 2.1** *There exist two positive constants $\delta_1$ and $\delta_2$, such that*

$$\delta_1 \Big( \mathcal{E}(t) + \sum_{i=3}^{7} \Xi_i(t) \Big) \leqslant K(t) \leqslant \delta_2 \Big( \mathcal{E}(t) + \sum_{i=3}^{7} \Xi_i(t) \Big), \tag{2.22}$$

*for all $t \geqslant 0$.*

**Proof.** By using Cauchy-Schwarz, Young's and Poincarés inequalities, we obtain

$$\begin{aligned} \Xi_1(t) \leqslant & \frac{\rho A}{2} \int_0^{l/2} \left[\omega^L(x,t)\right]^2 dx + \frac{\rho A}{2} \int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx + \frac{\rho A}{2} \int_{l/2}^{l} \left[\omega^R(x,t)\right]^2 dx \\ & + \frac{\rho A}{2} \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx + \frac{m}{2} \left[\omega_t(l/2,t)\right]^2 + \frac{m}{2} \left[\omega(l/2,t)\right]^2, \end{aligned} \tag{2.23}$$

and from Lemma 1.6 and Remark 1.10, we get

$$\int_0^{l/2} \left[\omega^L(x,t)\right]^2 dx \leqslant l \left[\omega(l/2,t)\right]^2 + l^4 \int_0^{l/2} \left[\omega_{xx}^L(x,t)\right]^2 dx, \tag{2.24}$$

and

$$\int_{l/2}^{l} \left[\omega^R(x,t)\right]^2 dx \leqslant l \left[\omega(l/2,t)\right]^2 + l^4 \int_{l/2}^{l} \left[\omega_{xx}^R(x,t)\right]^2 dx. \tag{2.25}$$

then, we have for all $t \geqslant 0$

$$\begin{aligned} \Xi_1(t) \leqslant & \frac{\rho A l^4}{2} \left( \int_0^{l/2} \left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[\omega_{xx}^R(x,t)\right]^2 dx \right) + \frac{m}{2} \left[\omega_t(l/2,t)\right]^2 \\ & + \frac{\rho A}{2} \left( \int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx \right) + \left( \rho A l + \frac{m}{2} \right) \left[\omega(l/2,t)\right]^2, \end{aligned} \tag{2.26}$$

for the functional $\Xi_2(t)$, we have

$$
\begin{aligned}
\Xi_2(t) \leqslant & \frac{\rho A}{2}\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right)\\
&+\frac{\rho A}{2}\left(\int_0^t \zeta(s)\,ds\right)\left(\int_0^{l/2}\left(\zeta\Box\omega^L\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega^R\right)(t)\,dx\right)\\
&+\frac{m}{2}\left[\omega_t(l/2,t)\right]^2+\frac{m}{2}\left(\int_0^t \zeta(s)\,ds\right)(\zeta\Diamond\omega)(t)\,,\ t\geqslant 0 \qquad (2.27)
\end{aligned}
$$

and use the Remark 1.10 again, we get

$$
\int_0^{l/2}\left(\zeta\Box\omega^L\right)(t)\,dx \leqslant l\,(\zeta\Diamond\omega)(t)+l^4\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx,\ t\geqslant 0 \qquad (2.28)
$$

and

$$
\int_{l/2}^{l}\left(\zeta\Box\omega^R\right)(t)\,dx \leqslant l\,(\zeta\Diamond\omega)(t)+l^4\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx,\ t\geqslant 0, \qquad (2.29)
$$

Therefore

$$
\begin{aligned}
\Xi_2(t) \leqslant & \frac{\rho A}{2}\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right)+\kappa\left(\frac{m}{2}+\rho Al\right)(\zeta\Diamond\omega)(t)\\
&+\frac{\rho Al^4\kappa}{2}\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx\right)+\frac{m}{2}\left[\omega_t(l/2,t)\right]^2 .
\end{aligned} \qquad (2.30)
$$

Now, by using (2.10), (2.26) and (2.30), we conclude that

$$
\begin{aligned}
K(t) \leqslant & \frac{1}{2}\rho A\left(1+\lambda_1+\lambda_2\right)\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right)\\
&+\left(\frac{EI}{2}+\lambda_2\frac{\rho Al^4\kappa}{2}\right)\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(x,t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(x,t)\,dx\right)+\sum_{i=4}^{7}\lambda_i\Xi_i(t)\\
&+\left[\frac{EI}{2}\left(1-\int_0^t \zeta(s)\,ds\right)+\lambda_1\frac{\rho Al^4}{2}\right]\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right)\\
&+\frac{1}{2}\left[\lambda_1\left(2\rho Al+m\right)+1\right]\left[\omega(l/2,t)\right]^2+\frac{1}{2}\left[\lambda_2\kappa\left(m+2\rho Al\right)+1\right](\zeta\Diamond\omega)(t)\\
&+\frac{1}{2}m\left(1+\lambda_1+\lambda_2\right)\left[\omega_t(l/2,t)\right]^2 .
\end{aligned} \qquad (2.31)
$$

Similarly, we have

$$\begin{aligned} 2K(t) \geqslant{}& \rho A(1-\lambda_1-\lambda_2)\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right) \\ &+\left(EI-\lambda_2\rho A l^4\kappa\right)\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(x,t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(x,t)\,dx\right) \\ &+\left[EI(1-\kappa)-\lambda_1\rho A l^4\right]\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right) \\ &+\left[1-\lambda_1(2\rho A l+m)\right]\left[\omega(l/2,t)\right]^2+\left[1-\lambda_2\kappa(m+2\rho A l)\right](\zeta\Diamond\omega)(t) \\ &+m(1-\lambda_1-\lambda_2)\left[\omega_t(l/2,t)\right]^2+2\sum_{i=4}^{7}\lambda_i\Xi_i(t). \end{aligned} \tag{2.32}$$

thus, for

$$\lambda_1<\min\left[1,\frac{1}{2\rho A l+m},\frac{EI(1-\kappa)}{\rho A l^4}\right] \quad \text{and} \quad \lambda_2<\min\left[1-\lambda_1,\frac{1}{(2\rho A l+m)\kappa},\frac{EI}{\rho A l^4\kappa}\right] \tag{2.33}$$

there exists $\delta_i>0$, $i=1,2$ where $\delta_1\Big(\mathcal{E}(t)+\sum_{i=3}^{7}\Xi_i(t)\Big)\leqslant K(t)\leqslant\delta_2\Big(\mathcal{E}(t)+\sum_{i=3}^{7}\Xi_i(t)\Big)$ for all $t\geq 0$. ■

**Lemma 2.2** (*See [104]*) *We have, for $\zeta\in C(0,\infty)$ and $\omega\in C((0,\infty);[0,L])$*

$$\begin{aligned} \int_0^L\omega\int_0^t\zeta(t-s)\,\omega(s)\,ds dx ={}& \frac{1}{2}\left(\int_0^t\zeta(s)\,ds\right)\|\omega\|_2^2+\frac{1}{2}\int_0^t\zeta(t-s)\,\|\omega(s)\|_2^2\,ds \\ &-\frac{1}{2}\int_0^L(\zeta\Box\omega)\,dx,\ t\geqslant 0. \end{aligned} \tag{2.34}$$

**Lemma 2.3** *Let $(\omega^L,\omega^R)$ be the solution of (2.1)–(2.3). Then, the functional $\Xi_1(t)$ satisfies for any positive $\eta_1$ and $\varepsilon_1$, the estimate*

$$\begin{aligned} \frac{d}{dt}\Xi_1(t) \leqslant{}& \rho A\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right)+\left(m+\frac{1}{4\eta_1}\right)\left[\omega_t(l/2,t)\right]^2 \\ &-\frac{EI}{2}\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(x,t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(x,t)\,dx\right)+\left(\eta_1 k_p^2-k_r\right)\left[\omega(l/2,t)\right]^2 \\ &+\frac{EI}{2}\left(\int_0^t\zeta(t-s)\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2 dxds+\int_0^t\zeta(t-s)\int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2 dxds\right) \\ &-EI\left(1-\frac{\kappa}{2}\right)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right) \end{aligned} \tag{2.35}$$

*for* $t \geq 0$.

**Proof.** Direct computations, using (2.71), we get

$$\begin{aligned}\frac{d}{dt}\Xi_1(t) = {} & I_1(t) + I_2(t) + \rho A \int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx + \rho A \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx + m\left[\omega_t(l/2,t)\right]^2 \\ & + EI\omega(l/2,t)\,\omega_{xxx}^L(l/2,t) - EI\omega(l/2,t)\int_0^t \zeta(t-s)\,\omega_{xxx}^L(l/2,s)\,ds \\ & + EI\omega(l/2,t)\int_0^t \zeta(t-s)\,\omega_{xxx}^R(l/2,s)\,ds - EI\omega(l/2,t)\,\omega_{xxx}^R(l/2,t) \\ & - k_p\omega(l/2,t)\,\omega_t(l/2,t) - k_r\left[\omega(l/2,t)\right]^2\end{aligned} \tag{2.36}$$

where

$$I_1(t) = -EI\int_0^{l/2} \omega^L(x,t)\,\omega_{xxxx}^L(x,t)\,dx - EI\int_{l/2}^{l} \omega^R(x,t)\,\omega_{xxxx}^R(x,t)\,dx,$$

and

$$\begin{aligned}I_2(t) = {} & EI\int_0^{l/2} \omega^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xxxx}^L(x,s)\,ds dx \\ & + EI\int_{l/2}^{l} \omega^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xxxx}^R(x,s)\,ds dx.\end{aligned}$$

Integrating by parts $I_i$, $i = 1, 2$, and in view of the boundary (2.2), we have

$$\begin{aligned}I_1(t) = {} & - EI\omega(l/2,t)\,\omega_{xxx}^L(l/2,t) - EI\int_0^{l/2} \left[\omega_{xx}^L(x,t)\right]^2 dx + EI\omega(l/2,t)\,\omega_{xxx}^R(l/2,t) \\ & - EI\int_{l/2}^{l} \left[\omega_{xx}^R(x,t)\right]^2 dx,\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}I_2(t) = {} & EI\omega(l/2,t)\left(\int_0^t \zeta(t-s)\,\omega_{xxx}^L(l/2,s)\,ds - \int_0^t \zeta(t-s)\,\omega_{xxx}^R(l/2,s)\,ds\right) \\ & + EI\left(\int_0^{l/2} \omega_{xx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,ds dx \right. \\ & \left. + \int_{l/2}^{l} \omega_{xx}^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,ds dx\right).\end{aligned} \tag{2.38}$$

Substituting the estimates (2.37) and (2.38) in (2.36) , we get

$$
\begin{aligned}
\frac{d}{dt}\Xi_1(t) &= \rho A\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right) - k_r\left[\omega(l/2,t)\right]^2 \\
&- EI\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right) + m\left[\omega_t(l/2,t)\right]^2 \\
&+ EI\left(\int_0^{l/2}\omega_{xx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,dsdx\right. \\
&+ \left.\int_{l/2}^{l}\omega_{xx}^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,dsdx\right) - k_p\omega(l/2,t)\,\omega_t(l/2,t).
\end{aligned} \tag{2.39}
$$

Now, we will estimate some terms in (2.39). For $\eta_1 > 0$ and $\varepsilon_1 > 0$, we have

$$
k_p\omega(l/2,t)\,\omega_t(l/2,t) \leqslant \eta_1 k_p^2\left[\omega(l/2,t)\right]^2 + \frac{1}{4\eta_1}\left[\omega_t(l/2,t)\right]^2, \tag{2.40}
$$

for all $t \geq 0$, and by using the Lemma 2.2, we obtain (2.35). ■

**Lemma 2.4** *Let* $(\omega^L, \omega^R)$ *be the solution of* (2.1)–(2.3). *Then, for some positive constants* $\eta_i$, $i = 2, 3, 4$ *the functional* $\Xi_2(t)$ *satisfies, the estimate*

$$
\begin{aligned}
&\frac{d}{dt}\Xi_2(t) \leqslant (\eta_4 - \zeta_\star)\,\rho A\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx\right) \\
&- \frac{\rho A l^4}{4\eta_4} BV[\zeta, \mathcal{B}]\left(\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_t}(t)\,dx + \int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)_{\tilde{\mathcal{B}}_t}(t)\,dx\right) \\
&+ \frac{\rho A l^4}{4\eta_4}\left(\int_{\chi_t} h(s)ds\right)\left(\int_0^{l/2}\left(h\Box\omega_{xx}^L\right)_{\tilde{\chi}_t}(t)\,dx + \int_{l/2}^{l}\left(h\Box\omega_{xx}^R\right)_{\tilde{\chi}_t}(t)\,dx\right) \\
&+ EI(1-\zeta_\star)\left(\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}(\chi)\right)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right) \\
&+ k_r\eta_2\left[\omega(l/2,t)\right]^2 + \left[\frac{EI(1-\zeta_\star)}{4\eta_3} + 2EI\right]\kappa\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)_{\mathcal{B}_t}(t)\,dx + \int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)_{\mathcal{B}_t}(t)\,dx\right) \\
&+ \left(k_p + \frac{k_r}{2\eta_2}\right)\kappa\widehat{\zeta}(\chi)\left(\zeta\Diamond\omega\right)_{\chi_t}(t) + \frac{1}{4\eta_4}(2\rho A l + m)\left(\int_{\chi_t} h(s)ds\right)\left(h\Diamond\omega\right)_{\tilde{\chi}_t}(t) \\
&+ \frac{EI}{2}(1-\zeta_\star)\left(\int_{\chi_t}\zeta(t-s)\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2 dxds + \int_{\chi_t}\zeta(t-s)\int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2 dxds\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2EI\kappa\widehat{\zeta}\left(\chi\right)\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^{L}\right)_{\chi_t}(t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^{R}\right)_{\chi_t}(t)\,dx\right)\\
&+\left(k_p+\frac{k_r}{2\eta_2}\right)\kappa\left(\zeta\Diamond\omega\right)_{\mathcal{B}_t}(t)-\left[\frac{\rho Al}{2\eta_4}+\frac{m}{4\eta_4}\right]BV[\zeta,\,\mathcal{B}]\left(\zeta'\Diamond\omega\right)_{\tilde{\mathcal{B}}_t}(t) \qquad (2.41)\\
&+\left[\frac{k_p}{2}+m\left(\eta_4-\zeta_\star\right)\right]\left[\omega_t\left(l/2,t\right)\right]^2
\end{aligned}
$$

*for all* $t\geq t_\star>0$.

**Proof.** The differentiation of the functional $\Xi_2\left(t\right)$, yields

$$
\begin{aligned}
\frac{d}{dt}\Xi_2\left(t\right)&=-\rho A\int_0^{l/2}\omega_{tt}^{L}\left(x,t\right)\int_0^t\zeta\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right)dsdx\\
&-\rho A\left(\int_0^t\zeta\left(s\right)ds\right)\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx-\rho A\left(\int_0^t\zeta\left(s\right)ds\right)\int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2dx\\
&-\rho A\int_0^{l/2}\omega_t^L\left(x,t\right)\int_0^t\zeta'\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right)dsdx\\
&-\rho A\int_{l/2}^{l}\omega_{tt}^{R}\left(x,t\right)\int_0^t\zeta\left(t-s\right)\left(\omega^R\left(x,t\right)-\omega^R\left(x,s\right)\right)dsdx\\
&-m\omega_{tt}\left(l/2,t\right)\int_0^t\zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)ds\\
&-\rho A\int_{l/2}^{l}\omega_t^R\left(x,t\right)\int_0^t\zeta'\left(t-s\right)\left(\omega^R\left(x,t\right)-\omega^R\left(x,s\right)\right)dsdx\\
&-m\omega_t\left(l/2,t\right)\int_0^t\zeta'\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)ds-m\left(\int_0^t\zeta\left(s\right)ds\right)\left[\omega_t\left(l/2,t\right)\right]^2.
\end{aligned} \qquad (2.42)
$$

for all $t\geqslant 0$. By using the Eqs. of (2.1), boundary condition $(2.2)_3$ and $(2.71)$ , we find

$$
\begin{aligned}
\frac{d}{dt}\Xi_2\left(t\right)=&I_3(t)+I_4(t)+I_5(t)+I_6(t)-\rho A\int_0^t\zeta\left(s\right)ds\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx\\
&-\rho A\int_0^{l/2}\omega_t^L\left(x,t\right)\int_0^t\zeta'\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right)dsdx\\
&-\rho A\int_{l/2}^{l}\omega_t^R\left(x,t\right)\int_0^t\zeta'\left(t-s\right)\left(\omega^R\left(x,t\right)-\omega^R\left(x,s\right)\right)dsdx\\
&-\rho A\int_0^t\zeta\left(s\right)ds\int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2dx\\
&-EI\omega_{xxx}^{L}\left(l/2,t\right)\int_0^t\zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)ds\\
&+EI\left(\int_0^t\zeta\left(t-s\right)\omega_{xxx}^{L}\left(l/2,s\right)ds\right)\left(\int_0^t\zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)ds\right)
\end{aligned} \qquad (2.43)
$$

$$
\begin{aligned}
&- EI\left(\int_0^t \zeta(t-s)\,\omega^R_{xxx}(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\right)\\
&+ \left(EI\omega^R_{xxx}(l/2,t)+k_p\omega_t(l/2,t)+k_r\omega(l/2,t)\right)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\
&- m\omega_t(l/2,t)\int_0^t \zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds - m\left(\int_0^t \zeta(s)\,ds\right)\left[\omega_t(l/2,t)\right]^2
\end{aligned}
\tag{2.44}
$$

where

$$
I_3(t) = EI\int_0^{l/2}\omega^L_{xxxx}(x,t)\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx,
$$

$$
I_4(t) = EI\int_{l/2}^{l}\omega^R_{xxxx}(x,t)\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx,
$$

$$
I_5(t) = -EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,\omega^L_{xxxx}(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)ds\right)dx,
$$

and

$$
I_6(t) = -EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\,\omega^R_{xxxx}(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)ds\right)dx.
$$

Integrating by parts twice and using the boundary conditions (2.2), we get

$$
\begin{aligned}
I_3(t) &= EI\omega^L_{xxx}(l/2,t)\int_0^t \zeta(t-s)\left(\omega^L(l/2,t)-\omega^L(l/2,s)\right)\\
&\quad +EI\int_0^{l/2}\omega^L_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)dsdx,
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
I_4(t) &= -EI\omega^R_{xxx}(l/2,t)\int_0^t \zeta(t-s)\left(\omega^R(l/2,t)-\omega^R(l/2,s)\right)ds\\
&\quad +EI\int_{l/2}^{l}\omega^R_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^R_{xx}(x,t)-\omega^R_{xx}(x,s)\right)dsdx,
\end{aligned}
\tag{2.46}
$$

$$
\begin{aligned}
I_5(t) &= -EI\left(\int_0^t \zeta(t-s)\,\omega^L_{xxx}(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^L(l/2,t)-\omega^L(l/2,s)\right)ds\right)\\
&\quad -EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,\omega^L_{xx}(x,s)\,ds\right)\times\\
&\quad \left(\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)ds\right)dx,
\end{aligned}
\tag{2.47}
$$

and

$$\begin{aligned} I_6(t) &= EI\left(\int_0^t \zeta(t-s)\,\omega_{xxx}^R(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^R(l/2,t)-\omega^R(l/2,s)\right)ds\right) \\ &\quad -EI\int_{l/2}^l\left(\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,ds\right)\times \\ &\quad \left(\int_0^t \zeta(t-s)\left(\omega_{xx}^R(x,t)-\omega_{xx}^R(x,s)\right)ds\right)dx. \end{aligned} \tag{2.48}$$

for $t \geq 0$.

Injecting the estimates $I_3(t)$, $I_4(t)$, $I_5(t)$ and $I_6(t)$ into (2.43), we obtain

$$\begin{aligned} \frac{d}{dt}\Xi_2(t) &= \left(k_p\omega_t(l/2,t)+k_r\omega(l/2,t)\right)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds \\ &+ EI\left(1-\int_0^t\zeta(s)\,ds\right)\int_0^{l/2}\omega_{xx}^L(x,t)\int_0^t\zeta(t-s)\left(\omega_{xx}^L(x,t)-\omega_{xx}^L(x,s)\right)dsdx \\ &+ EI\left(1-\int_0^t\zeta(s)\,ds\right)\int_{l/2}^{l}\omega_{xx}^R(x,t)\int_0^t\zeta(t-s)\left(\omega_{xx}^R(x,t)-\omega_{xx}^R(x,s)\right)dsdx \\ &+ EI\int_{l/2}^l\left(\int_0^t\zeta(t-s)\left(\omega_{xx}^R(x,t)-\omega_{xx}^R(x,s)\right)ds\right)^2dx \\ &+ EI\int_0^{l/2}\left(\int_0^t\zeta(t-s)\left(\omega_{xx}^L(x,t)-\omega_{xx}^L(x,s)\right)ds\right)^2dx \\ &- \left(\int_0^t\zeta(s)\,ds\right)\left(\rho A\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2dx+\rho A\int_{l/2}^l\left[\omega_t^R(x,t)\right]^2dx+m\left[\omega_t(l/2,t)\right]^2\right) \\ &- \rho A\int_0^{l/2}\omega_t^L(x,t)\int_0^t\zeta'(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx \\ &- \rho A\int_{l/2}^{l}\omega_t^R(x,t)\int_0^t\zeta'(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx \\ &- m\omega_t(l/2,t)\int_0^t\zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds, \quad t\geq 0. \end{aligned} \tag{2.49}$$

Now, using all of Lemma 1.4, Remark 1.10 and Young's and Cauchy-Schwarz inequalities we estimate the terms on the right-hand side of expression (2.49). First, we start with the 1st and 2nd terms, for all measurable sets $\mathcal{B}$ and $\chi$ where $\mathcal{B}=\mathbb{R}^+\setminus\chi$, we have

$$\begin{aligned} \omega_t(l/2,t)\int_0^t\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds &\leqslant \frac{1}{2}\left[\omega_t(l/2,t)\right]^2+\kappa\left(\zeta\Diamond\omega\right)_{\mathcal{B}_t}(t) \\ &\quad +\kappa\widehat{\zeta}(\chi)\left(\zeta\Diamond\omega\right)_{\chi_t}(t), \end{aligned} \tag{2.50}$$

and

$$\omega\left(l/2,t\right)\int_0^t \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)ds \leqslant \eta_2\left[\omega\left(l/2,t\right)\right]^2+\frac{\kappa}{2\eta_2}\Big(\zeta\Diamond\omega\Big)_{\mathcal{B}_t}(t) + \frac{1}{2\eta_2}\kappa\widehat{\zeta}\left(\chi\right)\Big(\zeta\Diamond\omega\Big)_{\chi_t}(t) \tag{2.51}$$

for $\eta_2>0$, and using the notation $Z_t = Z\cap[0,t]$.

Similar to Tatar [104], the 3rd, 4th, 5th and 6th terms can be handled in the following manner

$$\int_0^{l/2}\omega_{xx}^L(x,t)\int_0^t\zeta\left(t-s\right)\left(\omega_{xx}^L\left(x,t\right)-\omega_{xx}^L\left(x,s\right)\right)dsdx \leqslant \left(\eta_3+\frac{3}{2}\kappa\widehat{\zeta}\left(\chi\right)\right)\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2dx + \frac{\kappa}{4\eta_3}\int_0^{l/2}\Big(\zeta\Box\omega_{xx}^L\Big)_{\mathcal{B}_t}(t)\,dx+\frac{1}{2}\int_{\chi_t}\zeta\left(t-s\right)\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2dxds, \tag{2.52}$$

$$\int_{l/2}^l\omega_{xx}^R(x,t)\int_0^t\zeta\left(t-s\right)\left(\omega_{xx}^R\left(x,t\right)-\omega_{xx}^R\left(x,s\right)\right)dsdx \leqslant \left(\eta_3+\frac{3}{2}\kappa\widehat{\zeta}\left(\chi\right)\right)\int_{l/2}^l\left[\omega_{xx}^R(x,t)\right]^2dx + \frac{\kappa}{4\eta_3}\int_{l/2}^l\Big(\zeta\Box\omega_{xx}^R\Big)_{\mathcal{B}_t}(t)\,dx+\frac{1}{2}\int_{\chi_t}\zeta\left(t-s\right)\int_{l/2}^l\left[\omega_{xx}^R\left(x,s\right)\right]^2dxds,\quad \eta_2>0 \tag{2.53}$$

$$\int_0^{l/2}\left|\int_0^t\zeta\left(t-s\right)\left(\omega_{xx}^L\left(x,t\right)-\omega_{xx}^L\left(x,s\right)\right)ds\right|^2dx \leqslant 2\kappa\int_0^{l/2}\Big(\zeta\Box\omega_{xx}^L\Big)_{\mathcal{B}_t}(t)\,dx + 2\kappa\widehat{\zeta}\left(\chi\right)\int_0^{l/2}\Big(\zeta\Box\omega_{xx}^L\Big)_{\chi_t}(t)\,dx \tag{2.54}$$

and

$$\int_{l/2}^l\left|\int_0^t\zeta\left(t-s\right)\left(\omega_{xx}^R\left(x,t\right)-\omega_{xx}^R\left(x,s\right)\right)ds\right|^2dx \leqslant 2\kappa\int_{l/2}^l\Big(\zeta\Box\omega_{xx}^R\Big)_{\mathcal{B}_t}(t)\,dx + 2\kappa\widehat{\zeta}\left(\chi\right)\int_{l/2}^l\Big(\zeta\Box\omega_{xx}^R\Big)_{\chi_t}(t)\,dx. \tag{2.55}$$

Finally, using Lemma 1.4, hypothesis $(H_2)$ and Remark 1.10, the last 3 terms can be estimated as follows, for all $\quad t\geqslant 0,\ \ \eta_4>0$

$$\begin{aligned}&\int_0^{l/2}\omega_t^L(x,t)\int_0^t\zeta'\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right)dsdx\\ \leqslant\ &\eta_4\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2dx-\frac{l}{4\eta_4}BV[\zeta,\,A]\Big(\zeta'\Diamond\omega\Big)_{\tilde{\mathcal{B}}_t}(t)-\frac{l^4}{4\eta_4}BV[\zeta,\,\mathcal{B}]\int_0^{l/2}\Big(\zeta'\Box\omega_{xx}^L\Big)_{\tilde{\mathcal{B}}_t}(t)\,dx,\\ &+\frac{l}{4\eta_4}\left(\int_{\chi_t}h(s)ds\right)\Big(h\Diamond\omega\Big)_{\tilde{\chi}_t}(t)+\frac{l^4}{4\eta_4}\left(\int_{\chi_t}h(s)ds\right)\int_0^{l/2}\Big(h\Box\omega_{xx}^L\Big)_{\tilde{\chi}_t}(t)\,dx\end{aligned} \tag{2.56}$$

$$\int_{l/2}^{l} \omega_t^L(x,t) \int_0^t \zeta'(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right) ds dx$$
$$\leqslant \eta_4 \int_{l/2}^{l} \left[\omega_t^L(x,t)\right]^2 dx - \frac{l}{4\eta_4} BV[\zeta, \mathcal{B}]\Big(\zeta' \Diamond \omega\Big)_{\tilde{\mathcal{B}}_t}(t) - \frac{l^4}{4\eta_4} BV[\zeta, \mathcal{B}] \int_{l/2}^{l} \Big(\zeta' \Box \omega_{xx}^R\Big)_{\tilde{\mathcal{B}}_t}(t)\, dx,$$
$$+\frac{l}{4\eta_4}\left(\int_{\chi_t} h(s)ds\right)\Big(h\Diamond\omega\Big)_{\tilde{\chi}_t}(t) + \frac{l^4}{4\eta_4}\left(\int_{\chi_t} h(s)ds\right)\int_{l/2}^{l}\Big(h\Box\omega_{xx}^R\Big)_{\tilde{\chi}_t}(t)\, dx \tag{2.57}$$

and

$$\omega_t(l/2,t) \int_0^t \zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right) ds$$
$$\leqslant \eta_4 \left[\omega_t(l/2,t)\right]^2 - \frac{1}{4\eta_4} BV[\zeta, \mathcal{B}]\Big(\zeta'\Diamond\omega\Big)_{\tilde{\mathcal{B}}_t}(t) + \frac{1}{4\eta_4}\left(\int_{\chi_t} h(s)ds\right)\Big(h\Diamond\omega\Big)_{\tilde{\chi}_t}(t) \tag{2.58}$$

Plug the previous estimates (2.50)-(2.58) into (2.49), Lemma 2.4 is established. ■

**Lemma 2.5** *The derivative of the functional* $\Xi_3(t)$ *is estimated as follows*

$$\begin{aligned}\frac{d}{dt}\Xi_3(t) \leqslant{}& \frac{1}{2}(\zeta'\Diamond\omega)(t) + \eta_6\left[\omega(l/2,t)\right]^2 + 2\kappa\eta_5\Big(\zeta\Diamond\omega\Big)_{\mathcal{B}_t}(t)\\ &+\left(\frac{1}{4\eta_5}+\frac{1}{4\eta_6}\right)\left[\omega_t(l/2,t)\right]^2 + 2\eta_5\kappa\widehat{\zeta}(\chi)\Big(\zeta\Diamond\omega\Big)_{\chi_t}(t),\end{aligned} \tag{2.59}$$

*for all* $t \geqslant 0$ *and some constants* $\eta_5, \eta_6 > 0$.

**Proof.** Clearly, we have, for all $t \geqslant 0$

$$\frac{d}{dt}\Xi_3(t) = \frac{1}{2}(\zeta'\Diamond\omega)(t) + \omega_t(l/2,t)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds + \omega_t(l/2,t)\,\omega(l/2,t), \tag{2.60}$$

then, using Lemma 1.4, we can write

$$\begin{aligned}\omega_t(l/2,t)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds \leqslant{}& \frac{1}{4\eta_5}\left[\omega_t(l/2,t)\right]^2 + 2\kappa\eta_5\Big(\zeta\Diamond\omega\Big)_{\mathcal{B}_t}(t)\\ &+2\eta_5\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\Big(\zeta\Diamond\omega\Big)_{\chi_t}(t),\quad \eta_5>0,\ t\geqslant 0,\end{aligned} \tag{2.61}$$

and

$$\omega_t(l/2,t)\,\omega(l/2,t) \leqslant \frac{1}{4\eta_6}\left[\omega_t(l/2,t)\right]^2 + \eta_6\left[\omega(l/2,t)\right]^2,\ \eta_6>0,\ t\geqslant 0. \tag{2.62}$$

This completes the proof. ■

**Lemma 2.6** *For all $t \geq 0$, we have*

$$\frac{d}{dt}\Xi_4\left(t\right) \leqslant \left(2\eta_7\overline{\varphi}_\alpha - \alpha\right)\Xi_4\left(t\right) - \left(\zeta\Diamond\omega\right)\left(t\right) + \frac{1}{2\eta_7}\left[\omega_t\left(l/2,t\right)\right]^2, \tag{2.63}$$

*where $\overline{\varphi}_\alpha = \int_0^\infty \varphi_\alpha\left(s\right)ds$ and $\eta_7 > 0$ is a constant.*

**Proof.** The differentiation of $\Xi_4(t)$, yields

$$\begin{aligned}\frac{d}{dt}\Xi_4\left(t\right) = -\alpha\Xi_4\left(t\right) - \int_0^t \zeta\left(t-s\right)\left(\omega\left(l/2,t\right) - \omega\left(l/2,s\right)\right)^2 ds \\ + 2\omega_t\left(l/2,t\right)\int_0^t \varphi_\alpha\left(t-s\right)\left(\omega\left(l/2,t\right) - \omega\left(l/2,s\right)\right)ds,\end{aligned} \tag{2.64}$$

then, the last term in the right-hand side of (2.64) will be estimated when, $t \geqslant 0$

$$\omega_t\left(l/2,t\right)\int_0^t \varphi_\alpha\left(t-s\right)\left(\omega\left(l/2,t\right) - \omega\left(l/2,s\right)\right)ds \leqslant \frac{1}{4\eta_7}\left[\omega_t\left(l/2,t\right)\right]^2 + \eta_7\overline{\varphi}_\alpha\Xi_4\left(t\right), \quad \eta_7 > 0. \tag{2.65}$$

This completes the proof. ■

**Lemma 2.7** *For the functional $\Xi_5(t)$, we have*

$$\begin{aligned}\frac{d}{dt}\Xi_5\left(t\right) \leqslant &\varphi_\gamma\left(0\right)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,t\right)\right]^2 dx + \int_{l/2}^l\left[\omega_{xx}^R\left(x,t\right)\right]^2 dx\right) - \mu\left(t\right)\Xi_5\left(t\right) \\ &- \int_0^t \zeta\left(t-s\right)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2 dx + \int_{l/2}^l\left[\omega_{xx}^R\left(x,s\right)\right]^2 dx\right)ds,\end{aligned} \tag{2.66}$$

*for all $t \geq 0$.*

**Proof.** A direct differentiation of $\Xi_5(t)$ and taking into account **(H3)** gives

$$\begin{aligned}\frac{d}{dt}\Xi_5(t) = \varphi_\gamma\left(0\right)&\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,t\right)\right]^2 dx + \int_{l/2}^l\left[\omega_{xx}^R\left(x,t\right)\right]^2 dx\right) \\ &+ \int_0^t \varphi_\gamma'(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2 dx + \int_{l/2}^l\left[\omega_{xx}^R\left(x,s\right)\right]^2 dx\right)ds\end{aligned} \tag{2.67}$$

or

$$\begin{aligned}
\frac{d}{dt}\Xi_5(t) = {} & \varphi_\gamma(0)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2dx\right) \\
& - \int_0^t \zeta(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2dx\right)ds \\
& - \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}\varphi_\gamma'(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2dx\right)ds \\
\le {} & \varphi_\gamma(0)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2dx\right) \\
& - \int_0^t \zeta(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2dx\right)ds \\
& - \mu(t)\,\Xi_5(t), \quad t \ge 0
\end{aligned} \tag{2.68}$$

where we have used the assumption that $\gamma'(t)/\gamma(t) = \mu(t)$ is a decreasing function. ■

**Lemma 2.8** *For the functional $\Xi_6(t)$ and $\Xi_7(t)$, we have*

$$\begin{aligned}
\frac{d}{dt}\Xi_6(t) \leqslant & \phi_\gamma(0)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2dx\right) - \mu(t)\,\Xi_6(t) \\
& - \int_0^t h(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2dx\right)ds,
\end{aligned} \tag{2.69}$$

*and*

$$\frac{d}{dt}\Xi_7(t) \leqslant \phi_\gamma(0)[\omega(l/2,t)]^2 - \mu(t)\,\Xi_7(t) - \int_0^t h(t-s)\,[\omega(l/2,s)]^2ds, \tag{2.70}$$

*for all $t \ge 0$.*

**Proof.** Similar to the previous proof. ■

## 2.4 Asymptotic behavior

In this section, we state and prove the uniform stability of the system (2.1)–(2.3) under a suitable control force $u(t)$ applied on the center body of the spacecraft. Firstly, in order to stabilize our problem, we propose the following control force

$$u(t) = -k_p \omega_t (l/2, t) - k_r \omega (l/2, t), \; t \geqslant 0, \tag{2.71}$$

where $k_p$ and $k_r$ are a positive "control gains".

**Remark 2.4** *How to construct the control law from the Lyapunov function is the main issue of this section. We can then design the control law $u(t)$ for vibration suppression and substitute it in* (2.11)*. In turn, substitute Lemma 2.3 to Lemma 2.6 and boundary conditions in $\frac{d}{dt}K(t)$ then examine what term should be added in the Lyapunov function $K(t)$ and control law $u(t)$ in order to satisfy $\frac{d}{dt}K(t) \leq -c_0\gamma(t)K(t)$. After continuous revision and calculation of the Lyapunov function and control law, we can obtain the appropriate $K(t)$ and $u(t)$ to achieve the control objective.*

Now we are ready to state our main result.

**Theorem 2.2** *Under the assumptions (H1)–(H3) and the control force $u(t)$ defined in* (2.71)*, if $\hat{\zeta}(\chi)$ is sufficiently small, then, there exist positive constants $\Lambda$ and $\nu$ such that*

$$\mathcal{E}(t) \leq \Lambda \gamma(t)^{-\nu}, \quad t \geq 0$$

*if* $\lim_{t \longrightarrow \infty} \mu(t) = 0$ *and*

$$\mathcal{E}(t) \leq \Lambda e^{-\nu t}, \quad t \geq 0$$

*if* $\lim_{t \longrightarrow \infty} \mu(t) \neq 0$.

**Remark 2.5** *We illustrate the energy decay rate given by Theorem 2.2, if $\gamma(t) = (1+t)^{\alpha}$, $\alpha > 0$, then $\mu(t) = \alpha(1+t)^{-1}$ satisfies the condition (H3) and we have*

$$\mathcal{E}(t) \leq c/(1+t)^{\nu}$$

*for some $c > 0$ constant. If $\gamma(t) = e^{\beta t}$, $\beta > 0$ then $\mu(t) = \beta$ satisfies the assumption (H3) and we have*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 \beta t}$$

*for some $c_1, c_2 > 0$ constants.*

**Proof.** Differentiating $K(t)$ with respect to $t$, gathering the estimates in the lemma 2.3 to lemma 2.6, making use of (2.71) in the expression (2.11) we get, for any $t \geqslant t_\star > 0$,

$$\begin{aligned}
\frac{d}{dt}K(t) &\leqslant B_1\left[\omega_t\left(l/2,t\right)\right]^2 + B_2\left[\omega\left(l/2,t\right)\right]^2 + B_3\left(\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2 dx + \int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2 dx\right)\\
&+ \frac{1}{2}\left(\zeta'\lozenge\omega\right)(t) + \left(B_4-1\right)\left(\zeta\lozenge\omega\right)(t) + B_5\int_0^{l/2}\left[\omega_{xx}^L\left(x,t\right)\right]^2 dx + \left(2\eta_7\overline{\varphi}_\alpha - \alpha\right)\Xi_4\left(t\right)\\
&+ B_5\int_{l/2}^{l}\left[\omega_{xx}^R\left(x,t\right)\right]^2 dx + \frac{EI}{2}\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx\\
&- \frac{\rho A l^4}{4\eta_4}BV[\zeta,\mathcal{B}]\lambda_2\left(\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_t}(t)\,dx + \int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)_{\tilde{\mathcal{B}}_t}(t)\,dx\right)\\
&- \lambda_2\left(\frac{\rho A l}{2\eta_4} + \frac{m}{4\eta_4}\right)BV[\zeta,\mathcal{B}]\left(\zeta'\lozenge\omega\right)_{\tilde{\mathcal{B}}_t}(t) + \left[\lambda_2\left(k_p + \frac{k_r}{2\eta_2}\right) + 2\eta_5\right]\kappa\left(\zeta\lozenge\omega\right)_{\mathcal{B}_t}(t)\\
&+ \left[\frac{\lambda_1 EI}{2} + \frac{\lambda_2}{2}\left(1-\zeta_\star\right)EI - \lambda_5\right]\int_0^t \zeta\left(t-s\right)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2 dx\right.\\
&\left.+ \int_{l/2}^{l}\left[\omega_{xx}^R\left(x,s\right)\right]^2 dx\right)ds - \lambda_5\mu(t)\Xi_5\left(t\right) - \lambda_6\mu(t)\Xi_6\left(t\right) - \lambda_7\mu(t)\Xi_7\left(t\right)\\
&+ \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3} + 2\right)EI\kappa\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)_{\mathcal{B}_t}(t)\,dx + \int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)_{\mathcal{B}_t}(t)\,dx\right)\\
&+ \left[\lambda_2\frac{\rho A l^4}{2\eta_4}\left(\int_{\chi_t} h(s)ds\right) - \lambda_6\right]\int_0^t h\left(t-s\right)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2 dx\right.\\
&\left.+ \int_{l/2}^{l}\left[\omega_{xx}^R\left(x,s\right)\right]^2 dx\right)ds + \frac{EI}{2}\int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)(t)\,dx\\
&+ \left[\frac{\lambda_2}{2\eta_4}(2\rho A l + m)\left(\int_{\chi_t} h(s)d\right) - \lambda_7\right]\int_0^t h\left(t-s\right)[\omega(l/2,s)]^2 ds\\
&+ \left(2\kappa\widehat{\zeta}\left(\chi\right) - \frac{\lambda_1}{2}\right)EI\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx + \int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx\right)
\end{aligned} \tag{2.72}$$

where we have used the following estimates

$$k_r\omega\left(l/2,t\right)\omega_t\left(l/2,t\right) \leqslant \frac{k_r^2}{4\eta_8}\left[\omega_t\left(l/2,t\right)\right]^2 + \eta_8\left[\omega\left(l/2,t\right)\right]^2, \quad \eta_8 > 0,\ t \geqslant 0$$

$$\int_0^{l/2}\left(h\Box\omega_{xx}^L\right)_{\tilde{\chi}_t}(t)\,dx \leq 2\left(\int_{\chi_t} h(s)ds\right)\int_0^{l/2}[\omega_{xx}^L(x,t)]^2 dx + 2\int_0^t h(t-s)\int_0^{l/2}[\omega_{xx}^L(x,s)]^2 dx ds$$

$$\int_{l/2}^{l} \left(h\Box\omega_{xx}^{R}\right)_{\tilde{\chi}_t}(t)\,dx \le 2\left(\int_{\chi_t} h(s)ds\right)\int_{l/2}^{l}[\omega_{xx}^{R}(x,t)]^2dx + 2\int_0^t h(t-s)\int_{l/2}^{l}[\omega_{xx}^{R}(x,s)]^2dxds$$

and

$$\left(h\Diamond\omega\right)_{\chi_t}(t) \le 2\left(\int_{\tilde{\chi}_t} h(s)ds\right)[\omega(l/2,t)]^2 + 2\int_0^t h(t-s)[\omega(l/2,s)]^2ds$$

where

$$B_1 := -k_p + \frac{k_r^2}{4\eta_8} + \left(m + \frac{1}{4\eta_1}\right)\lambda_1 + \left[\frac{k_p}{2} + m\left(\eta_4 - \zeta_\star\right)\right]\lambda_2 + \frac{1}{4\eta_5} + \frac{1}{4\eta_6} + \frac{1}{2\eta_7}, \quad (2.73)$$

$$B_2 := \eta_8 + \left(\eta_1 k_p^2 - k_r\right)\lambda_1 + k_r\eta_2\lambda_2 + \eta_6 + \frac{\lambda_2}{2\eta_4}(2\rho Al + m)\left(\int_{\chi_t} h(s)ds\right)^2 + \lambda_7\phi_\gamma(0), \quad (2.74)$$

$$B_3 := \lambda_1\rho A + \lambda_2\left(\eta_4 - \zeta_\star\right)\rho A,\ \ B_4 := \left[\lambda_2\left(k_p + \frac{k_r}{2\eta_2}\right) + 2\eta_5\right]\kappa\widehat{\zeta}\left(\chi\right) \quad (2.75)$$

and

$$\begin{aligned} B_5 :=& \lambda_2 EI\left(1 - \zeta_\star\right)\left(\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}\left(\chi\right)\right) + \lambda_2\frac{\rho Al^4}{2\eta_4}\left(\int_{\chi_t} h(s)ds\right)^2 + \lambda_5\varphi_\gamma(0) + \lambda_6\phi_\gamma(0) \\ &- \lambda_1\left(1 - \frac{\kappa}{2}\right)EI. \end{aligned} \quad (2.76)$$

Now, as in [104], we introduce the sets $\mathcal{B}_n := \left\{s \in \mathbb{R}^+ :\ n\zeta'(s) + \zeta(s) \leqslant 0\right\},\ n \in \mathbb{N}$. We note that $\bigcup\limits_n \mathcal{B}_n = \mathbb{R}^+ \setminus \{\chi_\zeta \cup N_\zeta\}$ where $N_\zeta$ is the null set where $\zeta'$ is not defined. furthermore, if we denote $\chi_n := \mathbb{R}^+ \setminus \mathcal{B}_n$, then $\lim\limits_{n\to+\infty}\widehat{\zeta}\left(\chi_n\right) = \widehat{\zeta}\left(\chi_\zeta\right)$ because $\chi_{n+1} \subset \chi_n$ for all $n$ and $\bigcap\limits_n \chi_n = \chi_\zeta \bigcup N_\zeta$. We take $\mathcal{B} := \mathcal{B}_n$ and $\chi := \chi_n$ in (2.72) and notice that on $\mathcal{B}_n$ we have $\zeta'(s) \le -\zeta(s)/n$ . Therefore,

$$\begin{aligned} \int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx \le& -\frac{1}{2n}\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \frac{1}{2}\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx \\ &+ \int_0^{l/2}\left(h\Box\omega_{xx}^L\right)_{\tilde{\chi}_{nt}}(t)\,dx \\ \le& -\frac{1}{2n}\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \frac{1}{2}\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx \\ &+ 2\int_0^t h(t-s)\int_0^{l/2}[\omega_{xx}^L(x,s)]^2dxds + 2\left(\int_{\chi_{nt}} h(s)ds\right)\int_0^{l/2}[\omega_{xx}^L(x,t)]^2dx, \end{aligned}$$

$$\int_{l/2}^{l}\left(\zeta'\square\omega_{xx}^{R}\right)(t)\,dx \leq -\frac{1}{2n}\int_{l/2}^{l}\Big(\zeta\square\omega_{xx}^{R}\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \frac{1}{2}\int_{l/2}^{l}\Big(\zeta'\square\omega_{xx}^{R}\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx$$
$$+\int_{l/2}^{l}\Big(h\square\omega_{xx}^{L}\Big)_{\tilde{\chi}_{nt}}(t)\,dx$$
$$\leq -\frac{1}{2n}\int_{l/2}^{l}\Big(\zeta\square\omega_{xx}^{R}\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \frac{1}{2}\int_{l/2}^{l}\Big(\zeta'\square\omega_{xx}^{R}\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx$$
$$+2\int_{0}^{t}h(t-s)\int_{l/2}^{l}[\omega_{xx}^{R}(x,s)]^2dxds + 2\left(\int_{\chi_{nt}}h(s)ds\right)\int_{l/2}^{l}[\omega_{xx}^{R}(x,t)]^2dx$$

and

$$(\zeta'\Diamond\omega)(t) \leqslant -\frac{1}{2n}\Big(\zeta\Diamond\omega\Big)_{\tilde{\mathcal{B}}_{nt}}(t) + \frac{1}{2}\Big(\zeta'\Diamond\omega\Big)_{\tilde{\mathcal{B}}_{nt}}(t) + \Big(h\Diamond\omega\Big)_{\tilde{\chi}_{nt}}(t)$$
$$\leq -\frac{1}{2n}\Big(\zeta\Diamond\omega\Big)_{\tilde{\mathcal{B}}_{nt}}(t) + \frac{1}{2}\Big(\zeta'\Diamond\omega\Big)_{\tilde{\mathcal{B}}_{nt}}(t) + 2\left(\int_{\chi_{nt}}h(s)ds\right)[\omega(l/2,t)]^2$$
$$+2\int_{0}^{t}h(t-s)[\omega(l/2,s)]^2ds$$

For small $\varepsilon < \zeta_\star$, we select $\lambda_1 = (\zeta_\star - \varepsilon)\lambda_2$, $\lambda_5 = EI(1-\varepsilon)\lambda_2/2$, $\eta_1 = k_r/2k_p^2$, $\eta_4 = \varepsilon/2$, $\eta_2 = (\zeta_\star - \varepsilon)/4$, $\eta_6 = k_r(\zeta_\star - \varepsilon)\lambda_2/8$, $\eta_8 = k_r(\zeta_\star - \varepsilon)\lambda_2/16$ and $\eta_7 = \alpha/4\overline{\varphi}_\alpha$ for some positive constant $\alpha$. Therefore, for $t \geqslant t_\star > 0$, (2.72) becomes

$$\frac{d}{dt}K(t) \leqslant -\frac{\varepsilon}{2}\lambda_2\left(\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2dx\right) - \frac{\alpha}{2}\Xi_4(t)$$
$$+\tau_1\left[\omega_t(l/2,t)\right]^2 + \tau_2\left[\omega(l/2,t)\right]^2 + \tau_3\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2dx\right)$$
$$+\tau_5\left(\int_0^{l/2}\Big(\zeta'\square\omega_{xx}^L\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \int_{l/2}^{l}\Big(\zeta'\square\omega_{xx}^R\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx\right) + \tau_6\Big(\zeta'\Diamond\omega\Big)_{\tilde{A}_{nt}}(t) + \tau_4\left(\zeta\Diamond\omega\right)(t)$$
$$+\tau_8\left(\int_0^{l/2}\Big(\zeta\square\omega_{xx}^L\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx + \int_{l/2}^{l}\Big(\zeta\square\omega_{xx}^R\Big)_{\tilde{\mathcal{B}}_{nt}}(t)\,dx\right) + \tau_9\left(\int_0^{l/2}\left(\zeta\square\omega_{xx}^L\right)(t)\,dx\right.$$
$$\left.+\int_{l/2}^{l}\left(\zeta\square\omega_{xx}^R\right)(t)\,dx\right) + \tau_{10}\int_0^t h(t-s)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2dx\right)ds$$
$$+\tau_{11}\int_0^t h(t-s)\left[\omega(l/2,s)\right]^2ds - \mu(t)\Big(\lambda_5\Xi_5(t) + \lambda_6\Xi_6(t) + \lambda_7\Xi_7(t)\Big) + \tau_7\Big(\zeta\Diamond\omega\Big)_{\tilde{\mathcal{B}}_{nt}}(t) \tag{2.77}$$

where

$$\begin{aligned}
\tau_1 &:= -k_p + \frac{4k_r}{(\zeta_\star-\varepsilon)\lambda_2} + \left(m+\frac{1}{4\eta_1}\right)(\zeta_\star-\varepsilon)\lambda_2 + \left[\frac{k_p}{2} + m\left(\frac{\varepsilon}{2}-\zeta_\star\right)\right]\lambda_2 + \frac{1}{4\eta_5} \\
&\quad + \frac{2}{k_r(\zeta_\star-\varepsilon)\lambda_2} + \frac{2\overline{\varphi}_\alpha}{\alpha}, \\
\tau_2 &:= \left[1+\frac{\lambda_2}{\varepsilon}(2\rho Al+m)\Big(\int\limits_{\chi_{nt}} h(s)ds\Big)\right]\Big(\int\limits_{\chi_{nt}} h(s)ds\Big) + \lambda_7\phi_\gamma(0) - \frac{-K_r}{16}(\zeta_\star-\varepsilon)\lambda_2, \\
\tau_3 &:= \lambda_2 EI\,(1-\zeta_\star)\left(\eta_3+\frac{3}{2}\kappa\widehat{\zeta}\,(\chi_n)\right) + \left[EI+\lambda_2\frac{\rho Al^4}{2\eta_4}\Big(\int\limits_{\chi_{nt}} h(s)ds\Big)\right]\Big(\int\limits_{\chi_{nt}} h(s)ds\Big) \\
&\quad + \lambda_5\varphi_\gamma(0) + \lambda_6\phi_\gamma(0) - [(1-\sigma)+\sigma]\,(\zeta_\star-\varepsilon)\lambda_2\left(1-\frac{\kappa}{2}\right)EI,
\end{aligned} \tag{2.78}$$

$$\begin{aligned}
\tau_4 &:= \left[\lambda_2\left(k_p+\frac{k_r}{2\eta_2}\right)+2\eta_5\right]\kappa\widehat{\zeta}\,(\chi_n) - 1, \\
\tau_5 &:= \frac{EI}{4} - \frac{\rho Al^4}{2\varepsilon}BV[\zeta,\mathcal{B}_n]\lambda_2,\ \ \tau_6 := \frac{1}{4} - \lambda_2\left(\frac{\rho Al}{\varepsilon}+\frac{m}{2\varepsilon}\right)BV[\zeta,\mathcal{B}_n], \\
\tau_7 &:= \left[\lambda_2\left(k_p+\frac{k_r}{2\eta_2}\right)+2\eta_5\right]\kappa - \frac{1}{4n}, \quad \tau_8 = \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3}+2\right)EI\kappa - \frac{EI}{4n},, \\
\tau_9 &:= \left(2\kappa\widehat{\zeta}\,(\chi_n) - \frac{(\zeta_\star-\varepsilon)\lambda_2}{2}\right)EI, \quad \tau_{10} := \lambda_2\frac{\rho Al^4}{\varepsilon}\Big(\int\limits_{\chi_{nt}} h(s)ds\Big) - \lambda_6
\end{aligned} \tag{2.79}$$

and

$$\tau_{11} := \frac{\lambda_2}{\varepsilon}(2\rho Al+m)\Big(\int\limits_{\chi_{nt}} h(s)ds\Big) - \lambda_7. \tag{2.80}$$

Now to achieve our goal, we start selecting the different parameters in $\tau_i$, $i=1,...,11$ so that all the coefficients in the right-hand side (2.77) be negative. For small $\varepsilon$ and large values of $n$ and $t_*$, if $\widehat{\zeta}(\chi_n)$ is sufficiently small, we have

$$\kappa\widehat{\zeta}(\chi_n) - \frac{\zeta_*-\varepsilon}{2} \leqslant 0 \tag{2.81}$$

and

$$\frac{3}{2}\,(1-\zeta_\star)\,\kappa\widehat{\zeta}(\chi_n) < \sigma\,(\zeta_*-\varepsilon)\left(1-\frac{\kappa}{2}\right) \tag{2.82}$$

with

$$\sigma = \frac{3\kappa\,(1-\zeta_\star)}{4\zeta_\star\,(2-\kappa)}. \tag{2.83}$$

Observe that, for $t_*$ large enough, we have $0<\sigma<1$. For the remaining $1-\sigma$, we require that $\varphi_\gamma(0)$ satisfy

$$\frac{(1-\varepsilon)}{2}\varphi_\gamma(0) < (1-\sigma)(\zeta_* - \varepsilon)\left(1-\frac{\kappa}{2}\right), \tag{2.84}$$

then (2.84) is satisfied if $\varphi_\gamma(0) < \frac{1}{4}\left[\zeta_*(8-\kappa) - 3\kappa\right]$ and $\zeta_* > 3\kappa/(8-\kappa)$. Consequently, combining (2.82) and ( 2.84), selecting $\eta_3$ and $\varepsilon_1$ small enough so that

$$(1-\zeta_\star)\left[\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}(\chi_n)\right] + \frac{(1-\varepsilon)}{2}\varphi_\gamma(0) - [(1-\sigma)+\sigma](\zeta_\star - \varepsilon)\left(1-\frac{\kappa}{2}\right) < 0.$$

Once $\eta_3$, $n$, $t_\star$, and $\varepsilon$ are fixed, we pick $\eta_5$, and $\lambda_2$ small enough so that

$$\begin{cases} \left[\lambda_2\left(k_p + \frac{k_r}{2\eta_2}\right) + 2\eta_5\right]\kappa\widehat{\zeta}(\chi_n) - 1 < 0, \\ \frac{1}{4} - \lambda_2\left(\frac{\rho A l}{\varepsilon} + \frac{m}{2\varepsilon}\right) BV[\zeta, \mathcal{B}_n] > 0, \\ \frac{EI}{4} - \frac{\rho A l^4}{2\varepsilon} BV[\zeta, \mathcal{B}_n]\lambda_2 > 0, \\ \left[\lambda_2\left(k_p + \frac{k_r}{2\eta_2}\right) + 2\eta_5\right]\kappa - \frac{1}{4n} < 0, \\ \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3} + 2\right) EI\kappa - \frac{EI}{4n} < 0 \end{cases} \tag{2.85}$$

and for $\int_0^\infty h(s)ds$ sufficiently small, we select $\lambda_6$ and $\lambda_7$ large enough so that

$$\lambda_2\frac{\rho A l^4}{\varepsilon}\Big(\int_{\chi_{nt}} h(s)ds\Big) - \lambda_6 < 0 \quad \text{and} \quad \frac{\lambda_2}{\varepsilon}(2\rho A l + m)\Big(\int_{\chi_{nt}} h(s)ds\Big) - \lambda_7 < 0 \tag{2.86}$$

Finally, if $\phi_\gamma(0)$ is sufficiently small, we also select $k_p$ and $k_r$ large enough so that

$$\left[1 + \frac{\lambda_2}{\varepsilon}(2\rho A l + m)\Big(\int_{\chi_{nt}} h(s)ds\Big)\right]\Big(\int_{\chi_{nt}} h(s)ds\Big) + \lambda_7\phi_\gamma(0) - \frac{-K_r}{16}(\zeta_\star - \varepsilon)\lambda_2 < 0 \tag{2.87}$$

and

$$\begin{aligned} &- k_p + \frac{\varepsilon\rho A\lambda_2}{4} + \frac{2k_r}{(\zeta_\star - \varepsilon)\lambda_2} + \left(m + \frac{1}{4\eta_1}\right)(\zeta_\star - \varepsilon)\lambda_2 + \left[\frac{k_p}{2} + m\left(\frac{\varepsilon}{2} - \zeta_\star\right)\right]\lambda_2 + 4\kappa n \\ &\quad + \frac{2}{k_r(\zeta_\star - \varepsilon)\lambda_2} + \frac{2\overline{\varphi}_\alpha}{\alpha} < 0. \end{aligned} \tag{2.88}$$

Together with (2.77), those choices lead to

$$\frac{d}{dt}K(t) \leqslant -C_0\Big[\mathcal{E}(t)+\Xi_3(t)+\Xi_4(t)\Big] - \lambda_5\mu(t)\Xi_5(t) - \lambda_6\mu(t)\Xi_6(t) - \lambda_7\mu(t)\Xi_7(t), \quad t \geq t_* \tag{2.89}$$

for some positive constant $C_0$ and if $\lim_{t\longrightarrow\infty} \mu(t) = 0$ then, for this constant $C_0$, there exists a $\bar{t}(C_0) \geq t_*$ such that $\mu(t) \leq C_0$ for $t \geq \bar{t}(C_0)$. Hence, in virtue of the right-hand side of Proposition 2.1, we find

$$\frac{d}{dt}K(t) \leq -C_1\mu(t)K(t), \quad t \geq \bar{t}(C_0) \tag{2.90}$$

for some positive constant $C_1$. Integrating (2.90), gives

$$K(t) \leq K(\hat{t})e^{-C_1\int_{\hat{t}}^{t}\mu(s)ds}, \quad t \geq \bar{t}(C_0). \tag{2.91}$$

Thus, with the help of the left-hand side in Proposition 2.1, we get

$$\delta_1\Big[\mathcal{E}(t)+\sum_{i=3}^{7}\Xi_i(t)\Big] \leq K(\hat{t})e^{-C_1\int_{\hat{t}}^{t}\mu(s)ds}, \quad t \geq \bar{t}(C_0). \tag{2.92}$$

Now, we recall $\gamma'(t)/\gamma(t) = \mu(t)$ to conclude that

$$\mathcal{E}(t) \leq \varLambda\gamma^{-\nu}(t), \quad t \geq \hat{t} \tag{2.93}$$

for some positive constants $\varLambda$ and $\nu$.

If $\lim_{t\longrightarrow\infty} \mu(t) \neq 0$ then, there exist a $\hat{t} \geq t_*$ and $C_2$ such that $\mu(t) \geq C_2$ for $\hat{t} \geq t_*$. Therefore for $C_3$,

$$\frac{d}{dt}K(t) \leq -C_3K(t), \quad t \geq \hat{t}, \tag{2.94}$$

which leads to

$$\mathcal{E}(t) \leq \varLambda e^{-\nu t}, \quad t \geq \hat{t},$$

for some positive constants $\varLambda$ and $\nu$. The continuity of $\mathcal{E}(t)$ and the boundedness of

$\big[0\,,\,\max\{\bar{t}\,,\,\hat{t}\}\big]$ allow us to conclude

$$\mathcal{E}(t) \leq \Lambda\gamma(t)^{-\nu}, \quad t \geq 0 \tag{2.95}$$

which complete the proof. ■

# Chapter 3

# Existence and uniform decay for a flexible satellite system under unknown distributed disturbance

In this chapter, we prove the well-posedness of a flexible satellite system with viscoelastic panels and under unknown distributed disturbances as well as we study the stabilization behavior, to this, we use the multiplier method to show the uniform stability of our system.

## 3.1 Introduction

This chapter is devoted to studying the uniform stability result for the problem of a flexible viscoelastic satellite subjected to an unknown distributed disturbance, for a good control $u$ applied at the center body of the satellite and a large class of kernels $\zeta$ satisfying the following condition (see [10])

$$\zeta'(t) \leq 0 \quad \text{and} \quad \gamma(t)\zeta(t) \in L^1(0,\infty)$$

where $\gamma$ is a non-negative function. Let us now consider a flexible viscoelastic satellite under unknown distributed disturbances during attitude maneuvering and we will work to improve its performance, we define the system of this problem by, for all $t \in [0, \infty)$,

$$\begin{cases} \rho A \omega_{tt}^{L}(x,t) + EI\omega_{xxxx}^{L}(x,t) - EI \int_{0}^{t} \zeta(t-s)\, \omega_{xxxx}^{L}(x,s)\, ds = f^{L}(x,t), \ \ x \in [0, l/2] \\ \rho A \omega_{tt}^{R}(x,t) + EI\omega_{xxxx}^{R}(x,t) - EI \int_{0}^{t} \zeta(t-s)\, \omega_{xxxx}^{R}(x,s)\, ds = f^{R}(x,t), \ \ x \in [l/2, l] \end{cases} \tag{3.1}$$

with the boundary conditions

$$\begin{cases} \omega_{x}^{L}(l/2,t) = \omega_{x}^{R}(l/2,t) = 0, \ \omega_{xx}^{L}(0,t) = \omega_{xx}^{R}(l,t) = 0, \ \omega_{xxx}^{L}(0,t) = \omega_{xxx}^{R}(l,t) = 0, \\ \omega^{L}(l/2,t) = \omega^{R}(l/2,t) = \omega(l/2,t), \\ m\omega_{tt}(l/2,t) = \ u(t) + EI\omega_{xxx}^{L}(l/2,t) - EI \int_{0}^{t} \zeta(t-s)\, \omega_{xxx}^{L}(l/2,s)\, ds \\ \qquad\qquad -EI\omega_{xxx}^{R}(l/2,t) + EI \int_{0}^{t} \zeta(t-s)\, \omega_{xxx}^{R}(l/2,s)\, ds + d(l/2,t), \end{cases} \tag{3.2}$$

and the initial data

$$\begin{cases} \omega^{L}(x,0) = \omega_{0}^{L}(x), \ \ \omega_{t}^{L}(x,0) = \omega_{1}^{L}(x), \quad x \in [0, l/2] \\ \omega^{R}(x,0) = \omega_{0}^{R}(x), \ \ \omega_{t}^{R}(x,0) = \omega_{1}^{R}(x), \quad x \in [l/2, l], \end{cases} \tag{3.3}$$

where $f^L$, $f^R$ and $d$ represent the distributed disturbance in the panels and in the centrebody, respectively and $\zeta(t)$ is the relaxation function.

**Remark 3.1** *An important question on vibration suppression of the flexible satellite has been raised by Ji and Liu [56]. They have been proved that the solutions of the closed-loop system decay exponentially to the equilibrium state provided that we have a viscous damping (i.e. $\omega_t^R(x,t)$ and $\omega_t^L(x,t)$) under the boundary control $u(t)$ applied on the center body of the satellite and the distributed control inputs. This damping is known as a frictional damping. In this work, we improve these results by establishing a simple control force $u(t)$ yielding the stability of the problem under weaker damping with unknown distributed disturbances, and without the need for distributed control inputs. Namely, a viscoelastic damping induced by the material itself.*

## 3.2 Preliminary Results

In this section, we introduce some lemmas and notations which will be utilized later. We use the same notations in [89, 90], for every measurable set $\mathcal{Z} \subset \mathbb{R}^{+}$, we define the probability measure $\widehat{\zeta}$ by

$$\widehat{\zeta}(\mathcal{Z}) = \frac{1}{\kappa} \int_{\mathcal{Z}} \zeta(s)\, ds. \tag{3.4}$$

The flatness set and the flatness rate of $\zeta$ are defined by

$$\chi_{\zeta} = \left\{ s \in \mathbb{R}^{+}\ \zeta(s) > 0 \text{ and } \zeta'(s) = 0 \right\} \tag{3.5}$$

and

$$\Re_{\zeta} = \widehat{\zeta}(\chi_{\zeta}), \tag{3.6}$$

respectively, we also have

$$\widetilde{\chi}_{\zeta} = \left\{ s \in \mathbb{R}^{+} : 0 \leqslant s \leqslant t,\ \zeta(t-s) > 0 \text{ and } \zeta'(t-s) = 0 \right\}. \tag{3.7}$$

Let $t_{\star} > 0$ be a real number such that $\int_0^{t_{\star}} \zeta(s)\, ds =: \zeta_{\star} > 0$.

We suppose that the kernel $\zeta(t)$ satisfies (see [104]):

$(H_1)$ For all $t \geqslant 0$, $\zeta(t) \geqslant 0$, $\zeta$ is a continuously differentiable function satisfying

$$0 < \kappa := \int_0^{+\infty} \zeta(s)\, ds < 1.$$

$(H_2)$ For all $t \geqslant 0$, $\zeta'(t) \leqslant 0$.

$(H_3)$ There exists a nondecreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \delta(t)$ is a decreasing function and

$$\gamma(t)\, \zeta(t) \in L^{1}(0, \infty).$$

The unknown distributed disturbances are assumed to fulfill the conditions:

$(H_4)$ The functions $f^L$, $f^R$ and $d$ are continuous in $t$ such that $f^L \in L^2(0,l/2)$, $f^R \in L^2(l/2,t)$ and there exist a positive constant $\overline{d}$ such that $d(l/2,t) \le \overline{d}$ for all $t \ge 0$.

To simplify, we denote

$$(\zeta\Box\omega)(t) := \int_0^t \zeta(t-s)\left(\omega(x,t)-\omega(x,s)\right)^2 ds, \quad t \geqslant 0.$$

and

$$(\zeta\Diamond\omega)(t) := \int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 ds, \quad t \geqslant 0.$$

The following lemma will be applied throughout this chapter.

**Lemma 3.1** *(See [1]). Let $Q(t), \sigma(t), \beta(t) \in C[0,\infty)$. If there exists a positive function $\Psi(t) \in C^1[0,\infty)$ such that*

$$0 \le \sigma(t) \le \frac{\Psi(t)}{2}\left(Q(t) - \frac{\Psi'(t)}{\Psi(t)}\right), \quad \beta(t) \le \frac{1}{2\Psi(t)}\left(Q(t) - \frac{\Psi'(t)}{\Psi(t)}\right), \quad t \ge 0,$$

*then a nonnegative solution $\vartheta$ of the following inequality*

$$\frac{d}{dt}\vartheta(t) \le -Q(t)\vartheta(t) + \sigma(t)\vartheta^2(t) + \beta(t), \quad t \ge 0$$

*where $\Psi(0)\vartheta(0) < 1$, satisfies the estimate*

$$\vartheta(t) < \frac{1}{\Psi(t)}, \quad t \ge 0.$$

**Lemma 3.2** *The energy functional associated to the problem* (3.1)–(3.3) *is defined by*

$$\begin{aligned}\mathcal{E}(t) =& \frac{\rho A}{2}\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t(l/2,t)\right]^2 + \frac{\rho A}{2}\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx \\ &+ \frac{EI}{2}\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx + \frac{EI}{2}\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx \\ &+ \frac{EI}{2}\left[1-\left(\int_0^t \zeta(s)\,ds\right)\right]\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx \right. \\ &\left. + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right)\end{aligned} \tag{3.8}$$

*satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = \frac{EI}{2}\left(\int_0^{l/2}(\zeta'\Box\omega_{xx}^L)(t)dx + \int_{l/2}^{l}(\zeta'\Box\omega_{xx}^R)(t)dx\right) - \frac{EI}{2}\zeta(t)\left(\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right) + \big(u(t) + d(l/2,t)\big)\omega_t(l/2,t)$$

$$+ \int_0^{l/2}\omega_t^L(x,t)\, f^L(x,t)dx + \int_{l/2}^{l}\omega_t^R(x,t)\, f^R(x,t)dx, \quad t \geqslant 0. \tag{3.9}$$

**Proof.** By multiplying the first equation of system (3.1) by $\omega_t^L(x,t)$ and integrating over $[0, l/2]$, and similarly, multiplying the second equation of (3.2) by $\omega_t^R(x,t)$ and integrating over $[l/2, l]$, then multiplying the last Eq. of (3.2) by $\omega_t(l/2,t)$, using the conditions of the borders and summarizing the findings, we obtain, $t \geq 0$

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho A}{2}\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2 dx + \frac{\rho A}{2}\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t(l/2,t)\right]^2\right.\\
&\qquad\left. + \frac{EI}{2}\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx + \frac{EI}{2}\int_{l/2}^{l}\left[\omega_{xx}^R(x,t)\right]^2 dx\right)\\
&= \int_0^{l/2}\omega_{txx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,dsdx + \int_{l/2}^{l}\omega_{txx}^R(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^R(x,s)\,dsdx\\
&\quad + \int_0^{l/2}\omega_t^L(x,t)\, f^L(x,t)dx + \int_{l/2}^{l}\omega_t^R(x,t)\, f^R(x,t)dx + \big(u(t) + d(l/2,t)\big)\omega_t(l/2,t).
\end{aligned} \tag{3.10}$$

Now, for $t \geq 0$, we have

$$\begin{aligned}
2\int_0^{l/2}\omega_{txx}^L(x,t)\int_0^t \zeta(t-s)\,\omega_{xx}^L(x,s)\,dsdx = &\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx - \zeta(t)\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx\\
&- \frac{d}{dt}\left[\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx\right.\\
&\left. - \left(\int_0^t \zeta(s)\,ds\right)\int_0^{l/2}\left[\omega_{xx}^L(x,t)\right]^2 dx\right]
\end{aligned} \tag{3.11}$$

and

$$2\int_{l/2}^{l}\omega_{txx}^{R}(x,t)\int_{0}^{t}\zeta(t-s)\,\omega_{xx}^{R}(x,s)\,dsdx=\int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^{R}\right)(t)\,dx-\zeta(t)\int_{l/2}^{l}\left[\omega_{xx}^{R}(x,t)\right]^{2}dx$$
$$-\frac{d}{dt}\left[\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^{R}\right)(t)\,dx-\left(\int_{0}^{t}\zeta(s)\,ds\right)\int_{l/2}^{l}\left[\omega_{xx}^{R}(x,t)\right]^{2}dx\right].\tag{3.12}$$

■

Therefore, using the relations (3.11) and (3.12) in (3.10), we get (3.9).

**Remark 3.2** *From the equality* (3.8)*, we can observe that the derivative of the modified energy functional is of an undefined sign. Now, we construct a Lyapunov functional K which plays an important role in the proof of our stability results.*

Let

$$K(t)=\mathcal{E}(t)+\sum_{i=1}^{5}\lambda_i\Xi_i(t),\ t\geqslant 0\tag{3.13}$$

where $\lambda_i$, $i=1,...,5$ are positive constants which will be selected later such that $\lambda_3=\lambda_4=1$, and

$$\Xi_1(t):=\rho A\int_{0}^{l/2}\omega^{L}(x,t)\,\omega_t^{L}(x,t)\,dx+\rho A\int_{l/2}^{l}\omega^{R}(x,t)\,\omega_t^{R}(x,t)\,dx+m\omega_t(l/2,t)\,\omega(l/2,t),$$

$$\begin{aligned}\Xi_2(t):=&-\rho A\int_{0}^{l/2}\omega_t^{L}(x,t)\int_{0}^{t}\zeta(t-s)\left(\omega^{L}(x,t)-\omega^{L}(x,s)\right)dsdx\\&-m\omega_t(l/2,t)\int_{0}^{t}\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\&-\rho A\int_{l/2}^{l}\omega_t^{R}(x,t)\int_{0}^{t}\zeta(t-s)\left(\omega^{R}(x,t)-\omega^{R}(x,s)\right)dsdx,\end{aligned}$$

$$\Xi_3(t):=\frac{1}{2}(\zeta\Diamond\omega)(t)+\frac{1}{2}\left[\omega(l/2,t)\right]^{2},$$

$$\Xi_4(t):=\int_{0}^{t}\varphi_\alpha(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^{2}ds,$$

$$\Xi_5(t):=EI\int_{0}^{t}\varphi_\gamma(t-s)\int_{0}^{l/2}\left[\omega_{xx}^{L}(x,s)\right]^{2}dxds+EI\int_{0}^{t}\varphi_\gamma(t-s)\int_{0}^{l/2}\left[\omega_{xx}^{R}(x,s)\right]^{2}dxds,$$

where

$$\varphi_\alpha(t) := e^{-\alpha t}\int_t^\infty \zeta(s)\, e^{\alpha s} ds, \quad \varphi_\gamma(t) := \gamma(t)^{-1}\int_t^\infty \zeta(s)\,\gamma(s)\, ds,$$

here $\alpha$ is a positive constant to be selected later, $t \geqslant 0$. Finally, in order to stabilize our problem, we propose the following control force $u(t)$

$$u(t) = -k_p\omega_t(l/2,t) - k_r\omega(l/2,t), \; t \geqslant 0, \tag{3.14}$$

where $k_p$ and $k_r$ are a positive "control gains".

## 3.3 Well posedness

Now, we state the existence result of the problem (3.1) – (3.3), which will be proven by using the Galerkin approximation method. For this end, we denote

$$\mathbb{V} = \Big\{(y^L, y^R) \in H^1(0,l/2)\times H^1(l/2,l), \quad y^L_x(l/2) = y^R_x(l/2) = 0, \\ y^L(l/2) = y^R(l/2) = y(l/2)\Big\},$$

and

$$\mathbb{W} = \Big\{(y^L, y^R) \in H^2(0,l/2)\times H^2(l/2,l)\cap\mathbb{V}, \quad y^L_{xx}(0) = y^R_{xx}(l) = 0, \quad y^L_{xxx}(0) = y^R_{xxx}(l) = 0\Big\}$$

where $H^1(0,l/2)$, $H^1(l/2,l)$, $H^2(0,l/2)$ and $H^2(l/2,l)$ are the usual Sobolev spaces.

**Theorem 3.1** *Let* $(\omega_0^L, \omega_0^R) \in \mathbb{V}$, $(\omega_1^L, \omega_1^R) \in L^2(0,l/2)\times L^2(l/2,l)$ *be given. Suppose that* $(H_1)$ *to* $(H_4)$ *holds. Then, under the external force* $u(t)$ *defined in* (3.14) *there exists a unique weak solution* $(\omega^L, \omega^R)$ *of the problem* (3.1) – (3.3) *which satisfies*

$$(\omega^L, \omega^R) \in C\big([0,T);\mathbb{V}\big), \; (\omega_t^L, \omega_t^R) \in C\big([0,T); L^2(0,l/2)\times L^2(l/2,l)\big),$$

*where* $T > 0$.

**Proof.** Let us solve the variational problem associated with (3.1) – (3.3), which is given by: find $\left(\omega^L,\omega^R\right)\in\mathbb{W}\times\mathbb{W}$ such that

$$\begin{aligned}
&\rho A\int_0^{l/2}\omega_{tt}^L(x,t)\,y^L(x)\,dx+\rho A\int_{l/2}^{l}\omega_{tt}^R(x,t)\,y^R(x)\,dx+EI\int_0^{l/2}\omega_{xx}^L(x,t)\,y_{xx}^L(x)\,dx\\
&+EI\int_{l/2}^{l}\omega_{xx}^R(x,t)\,y_{xx}^R(x)\,dx-EI\int_0^{l/2}y_{xx}^L(x)\int_0^t\zeta(t-s)\,\omega_{xx}^L(x,s)\,dsdx\\
&-EI\int_{l/2}^{l}y_{xx}^R(x)\int_0^t\zeta(t-s)\,\omega_{xx}^R(x,s)\,dsdx\\
&+y(l/2)\left[m\omega_{tt}(l/2,t)+k_p\omega_t(l/2,t)+k_r\omega(l/2,t)-d(l/2,t)\right]=\int_0^{l/2}f^L(x,t)\,y^L(x)\,dx\\
&\qquad\qquad+\int_{l/2}^{l}f^R(x,t)\,y^R(x)\,dx,
\end{aligned}$$

for any $\left(y^L,y^R\right)\in\mathbb{V}\times\mathbb{V}$.

Let $\left\{\left(y^{L0},y^{R0}\right),\left(y^{L1},y^{R1}\right),\left(y^{L2},y^{R2}\right),...\right\}$ be a complete orthogonal system of $\mathbb{W}$ for which

$$\left(\omega_0^L,\omega_0^R\right),\left(\omega_1^L,\omega_1^R\right)\in\operatorname{span}\left\{\left(y^{L0},y^{R0}\right),\left(y^{L1},y^{R1}\right)\right\}.$$

For each $m\in\mathbb{N}$, let us put

$$\mathbb{W}_m=\operatorname{span}\left\{\left(y^{L0},y^{R0}\right),\left(y^{L1},y^{R1}\right),\left(y^{L2},y^{R2}\right),...\right\}.$$

We seek solutions of the form

$$\begin{cases}\omega^{Lm}(x,t)=\sum\limits_{j=1}^{m}a_{j,m}(t)y^{Lj}(x), & x\in[0,l/2], \quad t\geq0\\ \omega^{Rm}(x,t)=\sum\limits_{j=1}^{m}b_{j,m}(t)y^{Rj}(x), & x\in[l/2,l], \quad t\geq0\end{cases}$$

for the following approximate problems in $\mathbb{W}_m$

$$\begin{cases} \rho A \int_0^{l/2} \omega_{tt}^{Lm}(x,t)\, y^L(x)\, dx + \rho A \int_{l/2}^{l} \omega_{tt}^{Rm}(x,t)\, y^R(x)\, dx + EI \int_0^{l/2} \omega_{xx}^{Lm}(x,t)\, y_{xx}^L(x)\, dx \\ + EI \int_{l/2}^{l} \omega_{xx}^{Rm}(x,t)\, y_{xx}^R(x)\, dx - EI \int_0^{l/2} y_{xx}^L(x) \int_0^t \zeta(t-s)\, \omega_{xx}^{Lm}(x,s)\, ds dx \\ - EI \int_{l/2}^{l} y_{xx}^R(x) \int_0^t \zeta(t-s)\, \omega_{xx}^{Rm}(x,s)\, ds dx \\ + y(l/2)\left[m\omega_{tt}^m(l/2,t) + k_p \omega_t^m(l/2,t) + k_r \omega^m(l/2,t) - d(l/2,t)\right] = \int_0^{l/2} f^L(x,t)\, y^L(x)\, dx \\ + \int_{l/2}^{l} f^R(x,t)\, y^R(x)\, dx, \\ \left(\omega^{Lm}(x,0) = \omega_0^{Lm}(x),\, \omega^{Rm}(x,0) = \omega_0^{Rm}(x)\right) \longrightarrow \left(\omega_0^L,\, \omega_0^R\right) \quad \text{in} \quad \mathbb{W} \\ \left(\omega_t^{Lm}(x,0) = \omega_1^{Lm}(x),\, \omega_t^{Rm}(x,0) = \omega_1^{Rm}(x)\right) \longrightarrow \left(\omega_1^L,\, \omega_1^R\right) \quad \text{in} \quad L^2(0,l/2) \times L^2(l/2,l). \end{cases} \tag{3.15}$$

Note that the equations (3.15) leads to a system of ODEs involving $m$ unknown functions $a_{j,m}$ and $b_{j,m}$. Standard ODE theory guarantees the existence of a unique $\left(\omega^L, \omega^R\right)$ on the maximal interval $[0, t_m)$, $t_m \in (0, T]$ for each $m \geq 1$. It remains to extend $t_m$ to $T$.

**A priori estimate.** Set $y^L = \omega_t^{Lm}$ and $y^R = \omega_t^{Rm}$ in (3.15) we get for any $t \geq 0$

$$\begin{aligned} &\frac{d}{dt}\Bigg( \frac{\rho A}{2} \int_0^{l/2} \left[\omega_t^{Lm}(x,t)\right]^2 dx + \frac{\rho A}{2} \int_{l/2}^{l} \left[\omega_t^{Rm}(x,t)\right]^2 dx + \frac{EI}{2} \int_0^{l/2} \left[\omega_{xx}^{Lm}(x,t)\right]^2 dx \\ &+ \frac{EI}{2} \int_{l/2}^{l} \left[\omega_{xx}^{Rm}(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t^m(l/2,t)\right]^2 \Bigg) = \omega_t^m(l/2,t) d(l/2,t) \\ &+ EI \int_0^{l/2} \omega_{txx}^{Lm}(x,t) \int_0^t \zeta(t-s)\, \omega_{xx}^{Lm}(x,s)\, ds dx - k_p \left[\omega_t^m(l/2,t)\right]^2 \\ &+ EI \int_{l/2}^{l} \omega_{txx}^{Rm}(x,t) \int_0^t \zeta(t-s)\, \omega_{xx}^{Rm}(x,s)\, ds dx - k_r \omega_t^m(l/2,t) \omega^m(l/2,t) \\ &+ \int_0^{l/2} f^L(x,t)\, \omega_t^{Lm}(x,t)\, dx + \int_{l/2}^{l} f^R(x,t)\, \omega_t^{Rm}(x,t)\, dx. \end{aligned} \tag{3.16}$$

Let

$$X^m(t) = \mathcal{E}^m(t) + \frac{k_r}{2}\left[\omega^m(l/2,t)\right]^2 \tag{3.17}$$

where

$$\mathcal{E}^m(t) = \frac{\rho A}{2}\int_0^{l/2}\left[\omega_t^{Lm}(x,t)\right]^2 dx + \frac{m}{2}\left[\omega_t^m(l/2,t)\right]^2 + \frac{\rho A}{2}\int_{l/2}^{l}\left[\omega_t^{Rm}(x,t)\right]^2 dx$$
$$+\frac{EI}{2}\int_0^{l/2}\left(\zeta\Box\omega_{xx}^{Lm}\right)(t)\,dx + \frac{EI}{2}\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^{Rm}\right)(t)\,dx$$
$$+\frac{EI}{2}\left[1-\left(\int_0^t \zeta(s)\,ds\right)\right]\left(\int_0^{l/2}\left[\omega_{xx}^{Lm}(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^{Rm}(x,t)\right]^2 dx\right) \quad (3.18)$$

A differentiation of $X^m(t)$, gives

$$\frac{d}{dt}X^m(t) = \frac{d}{dt}\mathcal{E}^m(t) + k_r\omega_t^m(l/2,t)\,\omega^m(l/2,t). \quad (3.19)$$

The equation (3.16), the relations (3.11) and (3.12) permits to obtain

$$\frac{d}{dt}\mathcal{E}^m(t) = \frac{EI}{2}\left(\int_0^{l/2}(\zeta'\Box\omega_{xx}^{Lm})(t)dx + \int_{l/2}^{l}(\zeta'\Box\omega_{xx}^{Rm})(t)dx\right) + \int_0^{l/2}\omega_t^{Lm}(x,t)\,f^L(x,t)dx$$
$$-\frac{EI}{2}\zeta(t)\left(\int_0^{l/2}\left[\omega_{xx}^{Lm}(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\omega_{xx}^{Rm}(x,t)\right]^2 dx\right) + \int_{l/2}^{l}\omega_t^{Rm}(x,t)\,f^R(x,t)dx$$
$$-k_r\omega_t^m(l/2,t)\omega^m(l/2,t) - k_p\left[\omega_t^m(l/2,t)\right]^2 + \omega_t^m(l/2,t)d(l/2,t). \quad (3.20)$$

Now, by using Lemma 1.4, we have

$$\omega_t^m(x,t)d(l/2,t) \le \eta\left[\omega_t^m(x,t)\right]^2 + \frac{1}{4\eta}d^2(l/2,t), \quad \eta = k_p \quad (3.21)$$

From (3.19), (3.20) and (3.21), we obtain

$$\frac{d}{dt}X^m(t) = \int_0^{l/2}\omega_t^{Lm}(x,t)\,f^L(x,t)dx + \int_{l/2}^{l}\omega_t^{Rm}(x,t)\,f^R(x,t)dx + \frac{1}{4k_p}d^2(l/2,t) \quad (3.22)$$

for any $t \in [0,t_n)$. Integrating (3.22) over $[0,t]$ using Hölder and Young's inequality and assumption $(H_2)$, result in

$$\begin{aligned} X^m(t) \leq & X^m(0) + \frac{1}{4k_p}\int_0^t \left[d\left(l/2,s\right)\right]^2 ds + \frac{1}{4\eta'}\int_0^t\int_0^{l/2}\left[f^L\left(x,s\right)\right]^2 dxds \\ & + \eta'\int_0^t\int_0^{l/2}\left[\omega_t^{Lm}\left(x,s\right)\right]^2 dxds + \frac{1}{4\eta'}\int_0^t\int_{l/2}^{l}\left[f^R\left(x,s\right)\right]^2 dxds \\ & + \eta'\int_0^t\int_{l/2}^{l}\left[\omega_t^{Rm}\left(x,s\right)\right]^2 dxds, \quad \eta' > 0. \end{aligned} \tag{3.23}$$

From (3.17), choosing $\eta'$ sufficiently small, taking into account assumption $(H_4)$ and $\left(\omega_0^{Lm}, \omega_0^{Rm}\right)$, $\left(\omega_1^{Lm}, \omega_1^{Rm}\right)$ are bounded in $\mathbb{W}$, and employing Gronwall's lemma, we infer

$$\mathcal{E}^m(t) \leq M \tag{3.24}$$

where $M$ is a positive constant independent of $t$ and $m$.

Therefore, using the fact that $(H_1)$ and the estimate (3.24) together with (3.18) give us, for all $m \in \mathbb{N}$, $t_m = T$, we deduce

$$\begin{aligned} & \left(\omega^{Lm}\right) \text{ and } \left(\omega^{Rm}\right) \text{ are bounded in } L^\infty\left((0,T),\ \mathbb{V}\right), \\ & \left(\omega_t^{Lm}\right) \text{ and } \left(\omega_t^{Rm}\right) \text{ are bounded in } L^\infty\left((0,T),\ L^2(0,l/2)\times L^2(l/2,l)\right). \end{aligned} \tag{3.25}$$

It follows from (3.25) that there exists a subsequence $\left(\omega^{L\nu}, \omega^{R\nu}\right)$ of $\left(\omega^{Lm}, \omega^{Rm}\right)$ such that

$$\begin{aligned} & \omega^{L\nu} \rightharpoonup \omega^L,\ \omega^{R\nu} \rightharpoonup \omega^R \quad \text{weakly star in } L^\infty\left((0,T),\mathbb{V}\right) \\ & \omega_t^{L\nu} \rightharpoonup \omega_t^L,\ \omega_t^{R\nu} \rightharpoonup \omega_t^R \quad \text{weakly star in } L^\infty\left((0,T),\ L^2(0,l/2)\times L^2(l/2,l)\right). \end{aligned} \tag{3.26}$$

Then from Aubin-Lions theorem [65], for any $T > 0$

$$\begin{aligned} & \omega^{L\nu} \longrightarrow \omega^L,\ \omega^{R\nu} \longrightarrow \omega^R \text{ strongly in } C\left((0,T),\mathbb{V}\right) \\ & \upsilon_t^{L\nu} \longrightarrow \upsilon_t^L,\ \upsilon_t^{R\nu} \longrightarrow \upsilon_t^R \text{ strongly in } C\left((0,T),\ L^2(0,l/2)\times L^2(l/2,l)\right). \end{aligned} \tag{3.27}$$

These results are sufficient to pass to the limit in (3.15) to get

$$\begin{aligned}
&\rho A\int_0^{l/2}\omega_{tt}^L(x,t)\,y^L(x)\,dx+\rho A\int_{l/2}^{l}\omega_{tt}^R(x,t)\,y^R(x)\,dx+EI\int_0^{l/2}\omega_{xx}^L(x,t)\,y_{xx}^L(x)\,dx\\
&+EI\int_{l/2}^{l}\omega_{xx}^R(x,t)\,y_{xx}^R(x)\,dx-EI\int_0^{l/2}y_{xx}^L(x)\int_0^t\zeta(t-s)\,\omega_{xx}^L(x,s)\,dsdx\\
&-EI\int_{l/2}^{l}y_{xx}^R(x)\int_0^t\zeta(t-s)\,\omega_{xx}^R(x,s)\,dsdx\\
&+y(l/2)\left[m\omega_{tt}(l/2,t)+k_p\omega_t(l/2,t)+k_r\omega(l/2,t)-d(l/2,t)\right]=\int_0^{l/2}f^L(x,t)\,y^L(x)\,dx\\
&+\int_{l/2}^{l}f^R(x,t)\,y^R(x)\,dx,
\end{aligned}$$

for any $\left(y^L,y^R\right)\in\mathbb{V}\times\mathbb{V}$.

For the uniqueness

Let $\left(\omega^{L_1},\omega^{R_1}\right)$ and $\left(\omega^{L_2},\omega^{R_2}\right)$ be two solutions of the problem $(3.1)-(3.3)$ which satisfy

$$\left(\omega^{L_i},\,\omega^{R_i}\right)\in C\big([0,T);\mathbb{V}\big),\;\left(\omega_t^{L_i},\,\omega_t^{R_i}\right)\in C\big([0,T);L^2(0,l/2)\times L^2(l/2,l)\big),$$

where $i=1,2$.

then $\left(z^L,z^R\right)=\left(\omega^{L_1}-\omega^{L_2},\omega^{R_1}-\omega^{R_2}\right)$ verifies for any $\left(y^L,y^R\right)\in\mathbb{V}\times\mathbb{V}$.

$$\begin{aligned}
&\rho A\int_0^{l/2}z_{tt}^L(x,t)\,y^L(x)\,dx+\rho A\int_{l/2}^{l}z_{tt}^R(x,t)\,y^R(x)\,dx+EI\int_0^{l/2}z_{xx}^L(x,t)\,y_{xx}^L(x)\,dx\\
&+EI\int_{l/2}^{l}z_{xx}^R(x,t)\,y_{xx}^R(x)\,dx-EI\int_0^{l/2}y_{xx}^L(x)\int_0^t\zeta(t-s)\,z_{xx}^L(x,s)\,dsdx\\
&-EI\int_{l/2}^{l}y_{xx}^R(x)\int_0^t\zeta(t-s)\,z_{xx}^R(x,s)\,dsdx \qquad (3.28)\\
&+y(l/2)\left[mz_{tt}(l/2,t)+k_pz_t(l/2,t)+k_rz(l/2,t)-d(l/2,t)\right]=\int_0^{l/2}f^L(x,t)\,y^L(x)\,dx\\
&\qquad+\int_{l/2}^{l}f^R(x,t)\,y^R(x)\,dx
\end{aligned}$$

By reasoning in the same way as for the case of the first a priori estimate, and using the Gronwall lemma again, we obtain directly. $\omega^{L_1}=\omega^{L_2},\omega^{R_1}=\omega^{R_2}$. ■

## 3.4 Technical lemmas

In this section, we establish some lemmas which will be used to have the main result (i.e. Theorem 3.2

**Lemma 3.3** *There exist two positive constants* $\delta_1$ *and* $\delta_2$*, such that*

$$\delta_1\left(\mathcal{E}(t)+\sum_{i=3}^{5}\Xi_i\left(t\right)\right)\leqslant K\left(t\right)\leqslant\delta_2\left(\mathcal{E}(t)+\sum_{i=3}^{5}\Xi_i\left(t\right)\right), \tag{3.29}$$

*for all* $t\geqslant 0$.

**Proof.** See chapter 3. ■

**Lemma 3.4** *Let* $(\omega^L,\,\omega^R)$ *be the solution of* (3.1)–(3.3). *Then, the functional* $\Xi_1\left(t\right)$ *satisfies for any positive* $\eta_1$ *and* $\varepsilon_1$*, the estimate*

$$\begin{aligned}\frac{d}{dt}\Xi_1\left(t\right)\leqslant&\rho A\left(\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx+\int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2dx\right)+\left(m+\frac{1}{4\eta_1}\right)\left[\omega_t\left(l/2,t\right)\right]^2\\&-\frac{EI}{2}\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)\left(x,t\right)dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)\left(x,t\right)dx\right)\\&+\frac{EI}{2}\left(\int_0^{t}\zeta\left(t-s\right)\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2dxds+\int_0^{t}\zeta\left(t-s\right)\int_{l/2}^{l}\left[\omega_{xx}^R\left(x,s\right)\right]^2dxds\right)\\&+EI\left[\frac{2\varepsilon_1 l^4}{EI}-\left(1-\frac{\kappa}{2}\right)\right]\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,t\right)\right]^2dx+\int_{l/2}^{l}\left[\omega_{xx}^R\left(x,t\right)\right]^2dx\right)\\&+\frac{1}{4\varepsilon_1}\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right)\\&+\left[\varepsilon_1(2l+1)+\eta_1k_p^2-k_r\right]\left[\omega\left(l/2,t\right)\right]^2\end{aligned} \tag{3.30}$$

*for* $t\geq 0$.

**Proof.** Direct computations, using (3.14), we get

$$\begin{aligned}\frac{d}{dt}\Xi_1\left(t\right)=&I_1(t)+I_2(t)+\rho A\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx+\rho A\int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2dx+m\left[\omega_t\left(l/2,t\right)\right]^2\\&-EI\omega\left(l/2,t\right)\int_0^{t}\zeta\left(t-s\right)\omega_{xxx}^L\left(l/2,s\right)ds+EI\omega\left(l/2,t\right)\int_0^{t}\zeta\left(t-s\right)\omega_{xxx}^R\left(l/2,s\right)ds\\&-k_p\omega\left(l/2,t\right)\omega_t\left(l/2,t\right)-k_r\left[\omega\left(l/2,t\right)\right]^2+d(l/2,t)\omega\left(l/2,t\right)\end{aligned} \tag{3.31}$$

$$+ EI\omega\left(l/2,t\right)\omega_{xxx}^{L}\left(l/2,t\right) - EI\omega\left(l/2,t\right)\omega_{xxx}^{R}\left(l/2,t\right)$$
$$+ \int_{0}^{l/2}\omega^{L}\left(x,t\right)f^{L}(x,t)dx + \int_{l/2}^{l}\omega^{R}\left(x,t\right)f^{R}(x,t)dx$$

where

$$I_{1}\left(t\right) = -EI\int_{0}^{l/2}\omega^{L}\left(x,t\right)\omega_{xxxx}^{L}\left(x,t\right)dx - EI\int_{l/2}^{l}\omega^{R}\left(x,t\right)\omega_{xxxx}^{R}\left(x,t\right)dx,$$

and

$$\begin{aligned} I_{2}\left(t\right) = & EI\int_{0}^{l/2}\omega^{L}\left(x,t\right)\int_{0}^{t}\zeta\left(t-s\right)\omega_{xxxx}^{L}\left(x,s\right)dsdx \\ & + EI\int_{l/2}^{l}\omega^{R}\left(x,t\right)\int_{0}^{t}\zeta\left(t-s\right)\omega_{xxxx}^{R}\left(x,s\right)dsdx. \end{aligned}$$

Integrating by parts in $I_i$, $i = 1, 2$, and in view of the boundary (3.2), we have

$$\begin{aligned} I_{1}\left(t\right) = & -EI\omega\left(l/2,t\right)\omega_{xxx}^{L}\left(l/2,t\right) - EI\int_{0}^{l/2}\left[\omega_{xx}^{L}\left(x,t\right)\right]^{2}dx + EI\omega\left(l/2,t\right)\omega_{xxx}^{R}\left(l/2,t\right) \\ & -EI\int_{l/2}^{l}\left[\omega_{xx}^{R}\left(x,t\right)\right]^{2}dx,\ t \geqslant 0 \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} I_{2}\left(t\right) = & EI\omega\left(l/2,t\right)\left(\int_{0}^{t}\zeta\left(t-s\right)\omega_{xxx}^{L}\left(l/2,s\right)ds - \int_{0}^{t}\zeta\left(t-s\right)\omega_{xxx}^{R}\left(l/2,s\right)ds\right) \\ & +EI\left(\int_{0}^{l/2}\omega_{xx}^{L}\left(x,t\right)\int_{0}^{t}\zeta\left(t-s\right)\omega_{xx}^{L}\left(x,s\right)dsdx\right. \\ & \left. + \int_{l/2}^{l}\omega_{xx}^{R}\left(x,t\right)\int_{0}^{t}\zeta\left(t-s\right)\omega_{xx}^{R}\left(x,s\right)dsdx\right),\ t \geqslant 0. \end{aligned} \tag{3.33}$$

Substituting the estimates (3.32) and (3.33) in (3.31) , we get

$$\begin{aligned} \frac{d}{dt}\Xi_{1}\left(t\right) = & \rho A\left(\int_{0}^{l/2}\left[\omega_{t}^{L}\left(x,t\right)\right]^{2}dx + \int_{l/2}^{l}\left[\omega_{t}^{R}\left(x,t\right)\right]^{2}dx\right) - EI\left(\int_{0}^{l/2}\left[\omega_{xx}^{L}\left(x,t\right)\right]^{2}dx\right. \\ & \left. + \int_{l/2}^{l}\left[\omega_{xx}^{R}\left(x,t\right)\right]^{2}dx\right) + \int_{l/2}^{l}\omega^{R}\left(x,t\right)f^{R}(x,t)dx + d(l/2,t)\omega\left(l/2,t\right) \\ & + EI\left(\int_{0}^{l/2}\omega_{xx}^{L}\left(x,t\right)\int_{0}^{t}\zeta\left(t-s\right)\omega_{xx}^{L}\left(x,s\right)dsdx\right. \end{aligned} \tag{3.34}$$

$$+\int_{l/2}^{l}\upsilon_{xx}^{R}(x,t)\int_{0}^{t}\zeta(t-s)\,\omega_{xx}^{R}(x,s)\,dsdx\Bigg)-k_{r}\left[\omega(l/2,t)\right]^{2}$$
$$+\,m\left[\omega_{t}(l/2,t)\right]^{2}-k_{p}\omega(l/2,t)\,\omega_{t}(l/2,t)+\int_{0}^{l/2}\omega^{L}(x,t)\,f^{L}(x,t)dx.$$

Now, we will estimate some terms in (3.34). For $\eta_1 > 0$ and $\varepsilon_1 > 0$, we have

$$k_{p}\omega(l/2,t)\,\omega_{t}(l/2,t)\leqslant\eta_{1}k_{p}^{2}\left[\omega(l/2,t)\right]^{2}+\frac{1}{4\eta_{1}}\left[\omega_{t}(l/2,t)\right]^{2},$$

$$\int_{0}^{l/2}\omega^{L}(x,t)\,f^{L}(x,t)dx\leq\varepsilon_{1}l\left[\omega(l/2,t)\right]^{2}+\varepsilon_{1}l^{4}\int_{0}^{l/2}\left[\omega_{xx}^{L}(x,t)\right]^{2}dx+\frac{1}{4\varepsilon_{1}}\int_{0}^{l/2}\left[f^{L}(x,t)\right]^{2}dx,\tag{3.35}$$

$$\int_{l/2}^{l}\omega^{R}(x,t)\,f^{R}(x,t)dx\leq\varepsilon_{1}l\left[\omega(l/2,t)\right]^{2}+\varepsilon_{1}l^{4}\int_{l/2}^{l}\left[\omega_{xx}^{R}(x,t)\right]^{2}dx+\frac{1}{4\varepsilon_{1}}\int_{l/2}^{l}\left[f^{R}(x,t)\right]^{2}dx\tag{3.36}$$

and

$$d(l/2,t)\omega(l/2,t)\leq\frac{1}{4\varepsilon_{1}}d^{2}(l/2,t)+\varepsilon_{1}\left[\omega(l/2,t)\right]^{2}\tag{3.37}$$

for all $t \geq 0$, and by using the Lemma 2.2, we obtain (3.30). ■

**Lemma 3.5** *Let* $(\omega^L, \omega^R)$ *be the solution of* (3.1)–(3.3). *Then, for some positive constants* $\eta_i$, $i = 2, 3, 4$ *the functional* $\Xi_2(t)$ *satisfies, the estimate*

$$\begin{aligned}\frac{d}{dt}\Xi_{2}(t)\leqslant{}&(\eta_{4}-\zeta_{\star})\,\rho A\left(\int_{0}^{l/2}\left[\omega_{t}^{L}(x,t)\right]^{2}dx+\int_{l/2}^{l}\left[\omega_{t}^{R}(x,t)\right]^{2}dx\right)+\left[\frac{k_{p}}{2}+m(\eta_{4}-\zeta_{\star})\right]\\&\times\left[\omega_{t}(l/2,t)\right]^{2}+\frac{EI}{2}(1-\zeta_{\star})\left(\int_{\chi_{t}}\zeta(t-s)\int_{0}^{l/2}\left[\omega_{xx}^{L}(x,s)\right]^{2}dxds\right.\\&\left.+\int_{\chi_{t}}\zeta(t-s)\int_{l/2}^{l}\left[\omega_{xx}^{R}(x,s)\right]^{2}dxds\right)-\left[\frac{\rho Al}{2\eta_{4}}+\frac{m}{4\eta_{4}}\right]\zeta(0)\,(\zeta'\Diamond\omega)(t)\\&+k_{r}\eta_{2}\left[\omega(l/2,t)\right]^{2}-\frac{\rho Al^{4}}{4\eta_{4}}\zeta(0)\left(\int_{0}^{l/2}\left(\zeta'\Box\omega_{xx}^{L}\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^{R}\right)(t)\,dx\right)\\&+EI(1-\zeta_{\star})\left[\eta_{3}+\frac{3}{2}\left(\int_{\chi_{t}}\zeta(t-s)\,ds\right)\right]\left(\int_{0}^{l/2}\left[\omega_{xx}^{L}(x,t)\right]^{2}dx+\int_{l/2}^{l}\left[\omega_{xx}^{R}(x,t)\right]^{2}dx\right)\\&+\left[\frac{EI(1-\zeta_{\star})}{4\eta_{3}}+2EI\right]\kappa\left(\int_{0}^{l/2}\int_{A_{t}}\zeta(t-s)\left(\omega_{xx}^{L}(t)-\omega_{xx}^{L}(s)\right)^{2}dsdx\right.\\&\left.+\int_{l/2}^{l}\int_{\mathcal{Z}_{t}}\zeta(t-s)\left(\omega_{xx}^{R}(t)-\omega_{xx}^{R}(s)\right)^{2}dsdx\right)\end{aligned}$$

$$+\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_4}\right)\kappa\int_{\mathcal{Z}_t}\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 dsdx$$
$$+2EI\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\left(\int_0^{l/2}\int_{\chi_t}\zeta(t-s)\left(\omega_{xx}^L(x,t)-\omega_{xx}^L(x,s)\right)^2 dsdx\right.$$
$$\left.+\int_{l/2}^{l}\int_{\chi_t}\zeta(t-s)\left(\omega_{xx}^R(x,t)-\omega_{xx}^R(x,s)\right)^2 dsdx\right)$$
$$+\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_2}\right)\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{\chi_t}\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 dsdx$$
$$+\varepsilon_2 l^4\kappa\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)(t)\,dx\right)+2\varepsilon_2 l\kappa\left(\zeta\Diamond\omega\right)(t)$$
$$+\frac{1}{4\varepsilon_1}\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx+d^2(l/2,t)\right),$$

*for all* $t\geq t_\star>0$ *where* $\varepsilon_2>0$.

**Proof.** The differentiation of the functional $\Xi_2(t)$, yields

$$\begin{aligned}
\frac{d}{dt}\Xi_2(t)=&-\rho A\int_0^{l/2}\omega_{tt}^L(x,t)\int_0^t\zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx\\
&-\rho A\int_0^{l/2}\omega_t^L(x,t)\int_0^t\zeta'(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx\\
&-\rho A\left(\int_0^t\zeta(s)\,ds\right)\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2dx-\rho A\left(\int_0^t\zeta(s)\,ds\right)\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2dx\\
&-\rho A\int_{l/2}^{l}\omega_{tt}^R(x,t)\int_0^t\zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx\\
&-m\omega_{tt}(l/2,t)\int_0^t\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\
&-\rho A\int_{l/2}^{l}\omega_t^R(x,t)\int_0^t\zeta'(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx\\
&-m\omega_t(l/2,t)\int_0^t\zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\
&-m\left(\int_0^t\zeta(s)\,ds\right)\left[\omega_t(l/2,t)\right]^2,\ t\geqslant 0.
\end{aligned}$$

By using the Eqs. of (3.1), boundary condition $(3.2)_3$ and (3.14) , we find

$$\begin{aligned}
\frac{d}{dt}\Xi_2(t) &= I_3(t)+I_4(t)+I_5(t)+I_6(t) - \int_0^{l/2} f^L(x,t)\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx \\
&- \int_{l/2}^{l} f^R(x,t)\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx \\
&- \rho A\int_0^{l/2} \omega_t^L(x,t)\int_0^t \zeta'(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx \\
&- \rho A\int_0^t \zeta(s)\,ds\int_0^{l/2}\left[\omega_t^L(x,t)\right]^2dx - \rho A\int_0^t \zeta(s)\,ds\int_{l/2}^{l}\left[\omega_t^R(x,t)\right]^2dx \\
&- \rho A\int_{l/2}^{l} \omega_t^R(x,t)\int_0^t \zeta'(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx \\
&- \left(EI\omega_{xxx}^L(l/2,t)+d(l/2,t)\right)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds \\
&+ EI\left(\int_0^t \zeta(t-s)\,\omega_{xxx}^L(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\right) \\
&- EI\left(\int_0^t \zeta(t-s)\,\omega_{xxx}^R(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\right) \\
&+ \left(EI\omega_{xxx}^R(l/2,t)+k_p\omega_t(l/2,t)+k_r\omega(l/2,t)\right)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds \\
&- m\omega_t(l/2,t)\int_0^t \zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds - m\left(\int_0^t \zeta(s)\,ds\right)\left[\omega_t(l/2,t)\right]^2
\end{aligned} \tag{3.38}$$

where

$$I_3(t) = EI\int_0^{l/2}\omega_{xxxx}^L(x,t)\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx,$$

$$I_4(t) = EI\int_{l/2}^{l}\omega_{xxxx}^R(x,t)\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx,$$

$$I_5(t) = -EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,\omega_{xxxx}^L(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)ds\right)dx,$$

and

$$I_6(t) = -EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\,\omega_{xxxx}^R(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)ds\right)dx.$$

Integrating by parts twice and using the boundary conditions (3.2), we get,

$$
\begin{aligned}
I_3(t) &= EI\omega^L_{xxx}(l/2,t)\int_0^t \zeta(t-s)\left(\omega^L(l/2,t)-\omega^L(l/2,s)\right)ds\\
&+EI\int_0^{l/2}\omega^L_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)dsdx,\\
I_4(t) &= -EI\omega^R_{xxx}(l/2,t)\int_0^t \zeta(t-s)\left(\omega^R(l/2,t)-\omega^R(l/2,s)\right)ds\\
&+EI\int_{l/2}^{l}\omega^R_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^R_{xx}(x,t)-\omega^R_{xx}(x,s)\right)dsdx,\\
I_5(t) &= -EI\left(\int_0^t \zeta(t-s)\,\omega^L_{xxx}(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^L(l/2,t)-\omega^L(l/2,s)\right)ds\right)\\
&-EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,\omega^L_{xx}(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)ds\right)dx\\
I_6(t) &= EI\left(\int_0^t \zeta(t-s)\,\omega^R_{xxx}(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^R(l/2,t)-\omega^R(l/2,s)\right)ds\right)\\
&-EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\,\omega^R_{xx}(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(\omega^R_{xx}(x,t)-\omega^R_{xx}(x,s)\right)ds\right)dx.
\end{aligned}
$$

Plug the estimates $I_3(t)$, $I_4(t)$, $I_5(t)$ and $I_6(t)$ into (3.38), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Xi_2(t) = &-\int_0^{l/2} f^L(x,t)\int_0^t \zeta(t-s)\left(\omega^L(x,t)-\omega^L(x,s)\right)dsdx\\
&-\int_{l/2}^{l} f^R(x,t)\int_0^t \zeta(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right)dsdx\\
&-d(l/2,t)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\
&+\left(k_p\omega_t(l/2,t)+k_r\omega(l/2,t)\right)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)ds\\
&+EI\left(1-\int_0^t \zeta(s)\,ds\right)\int_0^{l/2}\omega^L_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)dsdx\\
&+EI\left(1-\int_0^t \zeta(s)\,ds\right)\int_{l/2}^{l}\omega^R_{xx}(x,t)\int_0^t \zeta(t-s)\left(\omega^R_{xx}(x,t)-\omega^R_{xx}(x,s)\right)dsdx\\
&+EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\left(\omega^R_{xx}(x,t)-\omega^R_{xx}(x,s)\right)ds\right)^2 dx\\
&+EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\left(\omega^L_{xx}(x,t)-\omega^L_{xx}(x,s)\right)ds\right)^2 dx\\
&-\left(\int_0^t \zeta(s)\,ds\right)\left(\rho A\int_0^{l/2}\left[\omega^L_t(x,t)\right]^2dx+\rho A\int_{l/2}^{l}\left[\omega^R_t(x,t)\right]^2dx\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ m\left[\omega_t\left(l/2,t\right)\right]^2\Big) - \rho A \int_0^{l/2} \omega_t^L\left(x,t\right) \int_0^t \zeta'\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right) dsdx \\
&- \rho A \int_{l/2}^{l} \omega_t^R\left(x,t\right) \int_0^t \zeta'\left(t-s\right)\left(\omega^R\left(x,t\right)-\omega^R\left(x,s\right)\right) dsdx \\
&- m\omega_t\left(l/2,t\right) \int_0^t \zeta'\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right) ds, \quad t \geq 0.
\end{aligned} \tag{3.39}
$$

Now, using all of Lemma 1.4, Remark 1.10 and Young's, Cauchy-Schwarz inequalities we estimate the terms on the right-hand side of expression (3.4). First, we start with the 1st, 2nd, 3rd, 4th and the 5th terms, for $\varepsilon_2 > 0$, we have

$$
\begin{aligned}
&\int_0^{l/2} f(x,t) \int_0^t \zeta\left(t-s\right)\left(\omega^L\left(x,t\right)-\omega^L\left(x,s\right)\right) dsdx \leq \varepsilon_2 l\kappa \left(\zeta\Diamond\omega\right)(t) \\
&\qquad + \varepsilon_2 l^4 \kappa \int_0^{l/2} \left(\zeta\Box\omega_{xx}^L\right)(t)\, dx + \frac{1}{4\varepsilon_2}\int_0^{l/2} f^2(x,t)dx, \\
&\int_{l/2}^{l} f(x,t) \int_0^t \zeta\left(t-s\right)\left(\omega^R\left(x,t\right)-\omega^R\left(x,s\right)\right) dsdx \leq \varepsilon_2 l\kappa \left(\zeta\Diamond\omega\right)(t) \\
&\qquad + \varepsilon_2 l^4 \kappa \int_{l/2}^{l} \left(\zeta\Box\omega_{xx}^R\right)(t)\, dx + \frac{1}{4\varepsilon_2}\int_{l/2}^{l} f^2(x,t)dx
\end{aligned} \tag{3.40}
$$

and for all measurable sets $\mathcal{Z}$ and $\chi$ where $\mathcal{Z} = \mathbb{R}^+ \setminus \chi$, we have

$$
\begin{aligned}
d\left(l/2,t\right) \int_0^t \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right) ds &\leqslant \frac{1}{4\varepsilon_2} d^2\left(l/2,t\right) \\
&+ 2\kappa\varepsilon_2 \int_{\mathcal{Z}_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds \\
&+ 2\varepsilon_2 \left(\int_{\chi_t} \zeta\left(t-s\right) ds\right) \int_{\chi_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds, \\
\omega_t\left(l/2,t\right) \int_0^t \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right) ds &\leqslant \frac{1}{2}\left[\omega_t\left(l/2,t\right)\right]^2 \\
&+ \kappa \int_{\mathcal{Z}_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds \\
&+ \left(\int_{\chi_t} \zeta\left(t-s\right) ds\right) \int_{\chi_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds,
\end{aligned}
$$

$$
\begin{aligned}
\omega\left(l/2,t\right) \int_0^t \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right) ds &\leqslant \eta_2\left[\omega\left(l/2,t\right)\right]^2 \\
&+ \frac{\kappa}{2\eta_2} \int_{\mathcal{Z}_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds \\
&+ \frac{1}{2\eta_2}\left(\int_{\chi_t} \zeta\left(t-s\right) ds\right) \int_{\chi_t} \zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2 ds
\end{aligned}
$$

for $\eta_2 > 0$, and using the notation $\mathcal{Z}_t = \mathcal{Z} \cap [0,t]$.

Similar to Tatar [104], the 6th, 7th, 8th and 9th terms can be handled in the following manner

$$\begin{aligned} &\int_0^{l/2} \omega_{xx}^L(x,t) \int_0^t \zeta(t-s)\left(\omega_{xx}^L(x,t) - \omega_{xx}^L(x,s)\right) dsdx \\ &\leqslant \left(\eta_3 + \frac{3}{2}\int_{\chi_t} \zeta(t-s)\,ds\right)\int_0^{l/2} \left[\omega_{xx}^L(x,t)\right]^2 dx \\ &+ \frac{\kappa}{4\eta_3}\int_0^{l/2}\int_{\mathcal{Z}_t} \zeta(t-s)\left(\omega_{xx}^L(x,t) - \omega_{xx}^L(x,s)\right)^2 dsdx \\ &+ \frac{1}{2}\int_{\chi_t} \zeta(t-s)\int_0^{l/2}\left[\omega_{xx}^L(x,s)\right]^2 dxds, \end{aligned} \tag{3.41}$$

$$\begin{aligned} &\int_{l/2}^{l} \omega_{xx}^R(x,t) \int_0^t \zeta(t-s)\left(\omega_{xx}^R(x,t) - \omega_{xx}^R(x,s)\right) dsdx \\ &\leqslant \left(\eta_3 + \frac{3}{2}\int_{\chi_t} \zeta(t-s)\,ds\right)\int_{l/2}^{l} \left[\omega_{xx}^R(x,t)\right]^2 dx \\ &+ \frac{\kappa}{4\eta_3}\int_{l/2}^{l}\int_{\mathcal{Z}_t} \zeta(t-s)\left(\omega_{xx}^R(x,t) - \omega_{xx}^R(x,s)\right)^2 dsdx \\ &+ \frac{1}{2}\int_{\chi_t} \zeta(t-s)\int_{l/2}^{l}\left[\omega_{xx}^R(x,s)\right]^2 dxds, \quad \eta_2 > 0 \end{aligned} \tag{3.42}$$

$$\begin{aligned} &\int_0^{l/2}\left|\int_0^t \zeta(t-s)\left(\omega_{xx}^L(t) - \omega_{xx}^L(s)\right) ds\right|^2 dx \\ &\leqslant 2\kappa\int_0^{l/2}\int_{\mathcal{Z}_t} \zeta(t-s)\left(\omega_{xx}^L(t) - \omega_{xx}^L(s)\right)^2 dsdx \\ &+ 2\left(\int_{\chi_t} \zeta(t-s)\,ds\right)\int_0^{l/2}\int_{\chi_t} \zeta(t-s)\left(\omega_{xx}^L(t) - \omega_{xx}^L(s)\right)^2 dsdx \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} &\int_{l/2}^{l}\left|\int_0^t \zeta(t-s)\left(\omega_{xx}^L(t) - \omega_{xx}^R(s)\right) ds\right|^2 dx \\ &\leqslant 2\kappa\int_{l/2}^{l}\int_{\mathcal{Z}_t} \zeta(t-s)\left(\omega_{xx}^R(t) - \omega_{xx}^R(s)\right)^2 dsdx \\ &+ 2\left(\int_{\chi_t} \zeta(t-s)\,ds\right)\int_{l/2}^{l}\int_{\chi_t} \zeta(t-s)\left(\omega_{xx}^R(t) - \omega_{xx}^R(s)\right)^2 dsdx. \end{aligned} \tag{3.44}$$

Finally, using Lemma 1.4, hypothesis $(H_2)$ and Remark 1.10, the last 3 terms can be estimated as follows

$$\int_0^{l/2} \omega_t^L(x,t)\int_0^t \zeta'(t-s)\left(\omega^L(x,t) - \omega^L(x,s)\right) dsdx \tag{3.45}$$

$$\leqslant \eta_4 \int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx + \frac{1}{4\eta_4}\left(\int_0^t |\zeta'(s)|\, ds\right)\int_0^{l/2} \left(|\zeta'|\Box\omega^L\right)(t)\, dx$$
$$\leqslant \eta_4 \int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx - \frac{l}{4\eta_4}\zeta(0)\left(\zeta'\Diamond\omega\right)(t) - \frac{l^4}{4\eta_4}\zeta(0)\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)(t)\, dx,$$

$$\begin{aligned}&\int_{l/2}^{l} \omega_t^L(x,t)\int_0^t \zeta'(t-s)\left(\omega^R(x,t)-\omega^R(x,s)\right) ds dx \\ \leqslant\;& \eta_4 \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx + \frac{1}{4\eta_4}\left(\int_0^t |\zeta'(s)|\, ds\right)\int_{l/2}^{l} \left(|\zeta'|\Box\omega^R\right)(t)\, dx \\ \leqslant\;& \eta_4 \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx - \frac{l}{4\eta_4}\zeta(0)\left(\zeta'\Diamond\omega\right)(t) - \frac{l^4}{4\eta_4}\zeta(0)\int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)(t)\, dx,\end{aligned} \tag{3.46}$$

and

$$\begin{aligned}&\omega_t(l/2,t)\int_0^t \zeta'(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right) ds \\ \leqslant\;& \eta_4\left[\omega_t(l/2,t)\right]^2 + \frac{1}{4\eta_4}\left(\int_0^t |\zeta'(s)|\, ds\right)\int_0^t |\zeta'(t-s)|\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 ds \\ \leqslant\;& \eta_4\left[\omega_t(l/2,t)\right]^2 - \frac{1}{4\eta_4}\zeta(0)\left(\zeta'\Diamond\omega\right)(t),\ \ \eta_4>0 \quad t\geq 0.\end{aligned} \tag{3.47}$$

Plug the previous estimates (3.40) to (3.47) into (3.4), Lemma 3.5 is established. ■

**Lemma 3.6** *The derivative of the functional $\Xi_3(t)$ is estimated as follows*

$$\begin{aligned}\frac{d}{dt}\Xi_3(t) \leqslant\;& \frac{1}{2}\left(\zeta'\Diamond\omega\right)(t) + \eta_6\left[\omega(l/2,t)\right]^2 + 2\kappa\eta_5\int_{\mathcal{Z}_t}\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 ds \\ &+ 2\eta_5\left(\int_{\chi_t}\zeta(t-s)\, ds\right)\int_{\chi_t}\zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right)^2 ds \\ &+\left(\frac{1}{4\eta_5}+\frac{1}{4\eta_6}\right)\left[\omega_t(l/2,t)\right]^2,\end{aligned} \tag{3.48}$$

*for all $t \geqslant 0$ and some constants $\eta_5$, $\eta_6 > 0$.*

**Proof.** Clearly, we have

$$\begin{aligned}\frac{d}{dt}\Xi_3(t) =& \frac{1}{2}\left(\zeta'\Diamond\omega\right)(t) + \omega_t(l/2,t)\int_0^t \zeta(t-s)\left(\omega(l/2,t)-\omega(l/2,s)\right) ds \\ &+\omega_t(l/2,t)\,\omega(l/2,t),\quad t\geqslant 0\end{aligned} \tag{3.49}$$

then, using Lemma 1.4, we can write

$$\begin{aligned}\omega_t(l/2,t)\int_0^t \zeta(t-s)(\omega(l/2,t)-\omega(l/2,s))\,ds \leqslant \frac{1}{4\eta_5}[\omega_t(l/2,t)]^2\\ +2\kappa\eta_5\int_{\mathcal{Z}_t}\zeta(t-s)(\omega(l/2,t)-\omega(l/2,s))^2\,ds\\ +2\eta_5\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{\chi_t}\zeta(t-s)(\omega(l/2,t)-\omega(l/2,s))^2\,ds,\quad \eta_5>0,\ t\geqslant 0,\end{aligned}$$

$$\omega_t(l/2,t)\,\omega(l/2,t)\leqslant \frac{1}{4\eta_6}[\omega_t(l/2,t)]^2+\eta_6[\omega(l/2,t)]^2,\ \eta_6>0,\ t\geqslant 0. \tag{3.50}$$

This completes the proof. ■

**Lemma 3.7** *For all $t\geq 0$, we have*

$$\frac{d}{dt}\Xi_4(t)\leqslant(2\eta_7\overline{\varphi}_\alpha-\alpha)\,\Xi_4(t)-(\zeta\Diamond\omega)(t)+\frac{1}{2\eta_7}[\omega_t(l/2,t)]^2, \tag{3.51}$$

*where $\overline{\varphi}_\alpha=\int_0^\infty \varphi_\alpha(s)\,ds$ and $\eta_7>0$ is a constant.*

**Proof.** The differentiation of $\Xi_4(t)$, gives

$$\begin{aligned}\frac{d}{dt}\Xi_4(t)=-\alpha\Xi_4(t)-\int_0^t\zeta(t-s)(\omega(l/2,t)-\omega(l/2,s))^2\,ds\\ +2\omega_t(l/2,t)\int_0^t\varphi_\alpha(t-s)(\omega(l/2,t)-\omega(l/2,s))\,ds,\end{aligned} \tag{3.52}$$

then, for $t\geqslant 0$ the last term in the right-hand side of (3.52) will be estimated as follow

$$\omega_t(l/2,t)\int_0^t\varphi_\alpha(t-s)(\omega(l/2,t)-\omega(l/2,s))\,ds\leqslant\frac{1}{4\eta_7}[\omega_t(l/2,t)]^2+\eta_7\overline{\varphi}_\alpha\Xi_4(t),\quad \eta_7>0, \tag{3.53}$$

This completes the proof. ■

**Lemma 3.8** *(See [104]). For the functional $\Xi_5(t)$, we have*

$$\begin{aligned}\frac{d}{dt}\Xi_5(t)\leqslant &EI\varphi_\gamma(0)\left(\int_0^{l/2}\left[\omega^L_{xx}(x,t)\right]^2dx+\int_{l/2}^{l}\left[\omega^R_{xx}(x,t)\right]^2dx\right)-EI\delta(t)\,\Xi_5(t)\\ &-EI\int_0^t\zeta(t-s)\left(\int_0^{l/2}\left[\omega^L_{xx}(x,s)\right]^2dx+\int_{l/2}^{l}\left[\omega^R_{xx}(x,s)\right]^2dx\right)ds,\end{aligned} \tag{3.54}$$

*for all $t\geq 0$.*

## 3.5 Asymptotic behavior

In this section, we state and prove the uniform stability of the system (3.1)–(3.3) under a suitable control force $u(t)$ applied on the center body of the spacecraft. Now we are ready to state our main result.

**Theorem 3.2** *Assume that $(H_1)$ to $(H_4)$ holds. Under the control force $u(t)$ defined in (3.14), if $\Re_\zeta$ is sufficiently small and there exists a positive function $\Psi \in C^1[0,\infty)$ such that*

$$0 \leq \left( Q(t) - \frac{\Psi'(t)}{\Psi(t)} \right)$$

*and*

$$\int_0^{l/2} \left[\omega_t^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[\omega_t^R(x,t)\right]^2 dx + d^2(l/2,t) \leq \frac{B}{\Psi(t)} \left( Q(t) - \frac{\Psi'(t)}{\Psi(t)} \right), \quad t \geq 0,$$

*where $B$ is given in (3.61) below. Then, for some positive constant $C$*

$$\mathcal{E}(t) \leqslant \frac{C}{\Psi(t)}, \ t \geqslant 0,$$

*provide that $\Psi(0)K(0) < 1$ in the cases*

*(a) $\lim\limits_{t\to+\infty} \delta(t) = 0$ and $Q(t) = c_1\delta(t)$ ($c_1$ will be chosen in the proof), or*

*(b) $\lim\limits_{t\to+\infty} \delta(t) = \overline{\delta} \neq 0$ and $Q(t) = c_2$ ($c_2$ is as in (3.61)).*

**Proof.** Differentiating $K(t)$ with respect to $t$, gathering the estimates of the lemma 3.4 to lemma 3.7, making use of (3.14) in the expression (3.9) we get, for any $t \geqslant t_\star > 0$,

$$\begin{aligned}
\frac{d}{dt}K(t) \leqslant{}& B_1\left[\omega_t(l/2,t)\right]^2 + B_2\left[\omega(l/2,t)\right]^2 + (2\eta_7\overline{\varphi}_\alpha - \alpha)\,\Xi_4(t) - EI\lambda_5\delta(t)\,\Xi_5(t) \\
&+ \left(\frac{EI}{4} - \frac{\rho A l^4}{4\eta_4}\zeta(0)\lambda_2\right)\left(\int_0^{l/2} \left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx + \int_{l/2}^{l} \left(\zeta'\Box\omega_{xx}^R\right)(t)\,dx\right) \\
&+ \frac{EI}{4}\int_{l/2}^{l} \left(\zeta'\Box\omega_{xx}^R\right)(t)\,dx + \frac{1}{4}\left(\zeta'\Diamond\omega\right)(t) + \frac{EI}{4}\int_0^{l/2} \left(\zeta'\Box\omega_{xx}^L\right)(t)\,dx \\
&+ B_4\left(\int_0^{l/2} \left[\omega_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[\omega_{xx}^R(x,t)\right]^2 dx\right) + B_6\left(\zeta\Diamond\omega\right)(t)
\end{aligned} \tag{3.55}$$

$$
\begin{aligned}
&+\left[\frac{\lambda_1}{2}+\frac{\lambda_2}{2}\left(1-\zeta_\star\right)-\lambda_5\right]EI\int_0^t\zeta\left(t-s\right)\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,s\right)\right]^2dx\right.\\
&\left.+\int_{l/2}^l\left[\omega_{xx}^R\left(x,s\right)\right]^2dx\right)ds+\left[\frac{1}{4}-\lambda_2\left(\frac{\rho Al}{2\eta_4}+\frac{m}{4\eta_4}\right)\right]\zeta\left(0\right)\left(\zeta'\Diamond\omega\right)\left(t\right)\\
&+\left[\lambda_2\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_4}\right)+2\eta_5\right]\kappa\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2dsdx\\
&+\lambda_2\left(\frac{1-\zeta_\star}{4\eta_3}+2\right)EI\kappa\left(\int_0^{l/2}\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\omega_{xx}^L\left(t\right)-\omega_{xx}^L\left(s\right)\right)^2dsdx\right.\\
&\left.+\int_{l/2}^l\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\omega_{xx}^R\left(t\right)-\omega_{xx}^R\left(s\right)\right)^2dsdx\right)\\
&+B_5\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)\left(t\right)dx+\int_{l/2}^l\left(\zeta\Box\omega_{xx}^R\right)\left(t\right)dx\right)\\
&+B_7\left(\int_0^{l/2}\left[f^L\left(x,t\right)\right]^2dx+\int_{l/2}^l\left[f^R\left(x,t\right)\right]^2dx+d^2(l/2,t)\right),
\end{aligned}
\tag{3.56}
$$

where we have used the following estimates

$$
k_r\omega\left(l/2,t\right)\omega_t\left(l/2,t\right)\leqslant\frac{k_r^2}{4\eta_8}\left[\omega_t\left(l/2,t\right)\right]^2+\eta_8\left[\omega\left(l/2,t\right)\right]^2,\quad \eta_8>0,\ t\geqslant 0
$$

$$
\int_0^{l/2}\omega_t^L\left(x,t\right)f^L\left(x,t\right)dx\leqslant\varepsilon_3\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx+\frac{1}{4\varepsilon_3}\int_0^{l/2}\left[f^L\left(x,t\right)\right]^2dx,\ \varepsilon_3>0,
$$

$$
\int_{l/2}^l\omega_t^R\left(x,t\right)f^R\left(x,t\right)dx\leqslant\varepsilon_3\int_{l/2}^l\left[\omega_t^R\left(x,t\right)\right]^2dx+\frac{1}{4\varepsilon_3}\int_{l/2}^l\left[f^R\left(x,t\right)\right]^2dx
$$

and

$$
d(l/2,t)\omega_t\left(l/2,t\right)\leq\frac{1}{4\varepsilon_3}d^2(l/2,t)+\varepsilon_3\left[\omega_t\left(l/2,t\right)\right]^2,\quad \varepsilon_3>0
$$

such that

$$
B_1:=-k_p+\varepsilon_3+\frac{k_r^2}{4\eta_8}+\left(m+\frac{1}{4\eta_1}\right)\lambda_1+\left[\frac{k_p}{2}+m\left(\eta_4-\zeta_\star\right)\right]\lambda_2+\frac{1}{4\eta_5}+\frac{1}{4\eta_6}+\frac{1}{2\eta_7},
$$

$$
B_2:=\eta_8+\left(\varepsilon_1(2l+1)+\eta_1k_p^2-k_r\right)\lambda_1+k_r\eta_2\lambda_2+\eta_6,\quad B_3:=\varepsilon_3+\lambda_1\rho A+\lambda_2\left(\eta_4-\zeta_\star\right)\rho A
$$

$$
B_4:=+\left\{\lambda_2\left(1-\zeta_\star\right)\left(\eta_3+\frac{3}{2}\int_{\chi_t}\zeta\left(t-s\right)ds\right)+\lambda_5\varphi_\gamma\left(0\right)+\lambda_1\left[\frac{2\varepsilon_1l^4}{EI}-\left(1-\frac{\kappa}{2}\right)\right]\right\}EI,
$$

$$
B_5:=\left(2\lambda_2\int_{\chi_t}\zeta\left(t-s\right)ds-\frac{\lambda_1}{2}+\frac{\varepsilon_2l^4\kappa\lambda_2}{EI}\right)EI,
$$

$$B_6 := \left[\lambda_2\left(2\varepsilon_2 + k_p + \frac{k_r}{2\eta_2}\right) + 2\eta_5\right]\int_{\chi_t}\zeta(t-s)\,ds - 1 + 2\varepsilon_2 l\kappa\lambda_2$$

and

$$B_7 := \frac{1}{4\varepsilon_3} + \frac{\lambda_1}{4\varepsilon_1} + \frac{\lambda_2}{4\varepsilon_2}.$$

Now, as in [104], we introduce the sets

$$\mathcal{Z}_n := \left\{s \in \mathbb{R}^+ :\ n\zeta'(s) + \zeta(s) \leqslant 0\right\},\ n \in \mathbb{N}.$$

and

$$\widetilde{\mathcal{Z}}_{nt} := \left\{s \in \mathbb{R}^+,\ 0 \leqslant s \leqslant t,\ n\zeta'(t-s) + \zeta(t-s) \leqslant 0\right\},\ n \in \mathbb{N}.$$

We note that $\underset{n}{\cup}\mathcal{Z}_n = \mathbb{R}^+ \setminus \{\chi_\zeta \cup N_\zeta\}$, where $N_\zeta$ is the null set where $\zeta'$ is not defined. furthermore, if we denote that $\chi_n := \mathbb{R}^+ \setminus \mathcal{Z}_n$, then $\lim_{n\to+\infty}\widehat{\zeta}(\chi_n) = \widehat{\zeta}(\chi_g)$ because $\chi_{n+1} \subset \chi_n$ for all $n$ and $\underset{n}{\cap}\chi_n = \chi_\zeta \cup N_n$. We take $\mathcal{Z}_t := \widetilde{\mathcal{Z}}_{nt}$ and $\chi_t := \widetilde{\chi}_{nt}$ in (3.55). Therefore,

$$\frac{1}{4}\int_0^{l/2}\left(\zeta'\Box\omega^L_{xx}\right)dx \leqslant -\frac{1}{4n}\int_0^{l/2}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(\omega^L_{xx}(t) - \omega^L_{xx}(s)\right)^2 dsdx$$

$$\frac{1}{4}\int_{l/2}^{l}\left(\zeta'\Box\omega^R_{xx}\right)dx \leqslant -\frac{1}{4n}\int_{l/2}^{l}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(\omega^R_{xx}(t) - \omega^R_{xx}(s)\right)^2 dsdx$$

and

$$\frac{1}{4}\left(\zeta'\Diamond\omega\right)(t) \leqslant -\frac{1}{4n}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(\omega(l/2,t) - \omega(l/2,s)\right)^2 ds$$

For small $\varepsilon < \zeta_\star$, we select $\lambda_1 = (\zeta_\star - \varepsilon)\lambda_2$, $\lambda_5 = (1-\varepsilon)\lambda_2/2$, $\eta_4 = \varepsilon/2$, $\varepsilon_3 = \varepsilon\lambda_2\rho A/4$, $\eta_2 = (\zeta_\star - \varepsilon)/4$, $\eta_5 = 1/16\kappa n$, $\eta_6 = \eta_8 = k_r\lambda_1/8$, $\varepsilon_1 = k_r/2(2l-1)$ and $\eta_7 = \alpha/4\overline{\varphi}_\alpha$ for some positive constant $\alpha$.

Now, for notation convenience we shall write all the coefficients in the right hand side of (3.55) in the simple form, so for some $0 < \sigma < 1$, setting

$\tau_1 := -k_p + \frac{\varepsilon\rho A\lambda_2}{4} + \frac{2k_r}{(\zeta_\star-\varepsilon)\lambda_2} + \left(m + \frac{1}{4\eta_1}\right)(\zeta_\star-\varepsilon)\lambda_2 + \left[\frac{k_p}{2} + m\left(\frac{\varepsilon}{2} - \zeta_\star\right)\right]\lambda_2 + 4\kappa n + \frac{2}{k_r(\zeta_\star-\varepsilon)\lambda_2} + \frac{2\overline{\varphi}_\alpha}{\alpha}$,

$\tau_2 := \lambda_2(\zeta_\star - \varepsilon)\left[\varepsilon_1(2l+1) + \eta_1 k_p^2 - \frac{k_r}{2}\right]$, $\tau_3 := \frac{1}{4} - \frac{2\rho Al+m}{2\varepsilon}\zeta(0)\lambda_2$, $\tau_4 := \frac{EI}{4} - \frac{\rho Al^4}{2\varepsilon}\zeta(0)\lambda_2$,

$\tau_5 := \left[\lambda_2\left(2\varepsilon_2 + k_p + \frac{2k_r}{\zeta_\star-\varepsilon}\right) + \frac{1}{8\kappa n}\right]\kappa\widehat{\zeta}(\chi_n) - 1 + 2\varepsilon_2 l\kappa\lambda_2$,

$\tau_6 := \lambda_2\left\{(1-\zeta_\star)\left[\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}(\chi_n)\right] + \frac{(1-\varepsilon)}{2}\varphi_\gamma(0) + [(1-\sigma)+\sigma](\zeta_\star - \varepsilon)\left[\frac{2\varepsilon_1 l^4}{EI} - \left(1 - \frac{\kappa}{2}\right)\right]\right\}EI$,

$\tau_7 := \left[\lambda_2\left(\frac{1-\zeta_\star}{4\eta_3}+2\right)\kappa-\frac{1}{4n}\right]EI$, $\tau_8=\lambda_2\left(2\varepsilon_2+k_p+\frac{k_r}{\varepsilon}\right)\kappa-\frac{1}{8n}$,
$\tau_9 := \lambda_2\left(2\kappa\widehat{\zeta}\left(\chi_n\right)-\frac{\zeta_\star-\varepsilon}{2}+\frac{\varepsilon_2 l^4\kappa}{EI}\right)EI$ and $\tau_{10} := \frac{(\zeta_\star-\varepsilon)\lambda_2}{4\varepsilon_1}+\frac{\lambda_2}{\varepsilon_2}+\frac{1}{\varepsilon\lambda_2\rho A}$.

Therefore, for $t \geqslant t_\star > 0$, (3.55) becomes

$$\begin{aligned}
\frac{d}{dt}K(t) \leqslant\ & \tau_1\left[\omega_t\left(l/2,t\right)\right]^2+\tau_2\left[\omega\left(l/2,t\right)\right]^2-\frac{\alpha}{2}\Xi_4\left(t\right)-EI\delta\left(t\right)\Xi_5\left(t\right)\lambda_5+\tau_3\left(\zeta'\Diamond\omega\right)\left(t\right)\\
&+\tau_4\left(\int_0^{l/2}\left(\zeta'\Box\omega_{xx}^L\right)\left(t\right)dx+\int_{l/2}^{l}\left(\zeta'\Box\omega_{xx}^R\right)\left(t\right)dx\right)+\tau_5\left(\zeta\Diamond\omega\right)\left(t\right)\\
&+\tau_9\left(\int_0^{l/2}\left(\zeta\Box\omega_{xx}^L\right)\left(t\right)dx+\int_{l/2}^{l}\left(\zeta\Box\omega_{xx}^R\right)\left(t\right)dx\right)\\
&-\frac{\varepsilon}{2}\lambda_2\rho A\left(\int_0^{l/2}\left[\omega_t^L\left(x,t\right)\right]^2dx+\int_{l/2}^{l}\left[\omega_t^R\left(x,t\right)\right]^2dx\right)\\
&+\tau_6\left(\int_0^{l/2}\left[\omega_{xx}^L\left(x,t\right)\right]^2dx+\int_{l/2}^{l}\left[\omega_{xx}^R\left(x,t\right)\right]^2dx\right)\\
&+\tau_7\left(\int_0^{l/2}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta\left(t-s\right)\left(\omega_{xx}^L\left(x,t\right)-\omega_{xx}^L\left(x,s\right)\right)^2dsdx\right.\\
&\left.+\int_{l/2}^{l}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta\left(t-s\right)\left(\omega_{xx}^R\left(x,t\right)-\omega_{xx}^R\left(s,x\right)\right)^2dsdx\right)\\
&+\tau_8\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta\left(t-s\right)\left(\omega\left(l/2,t\right)-\omega\left(l/2,s\right)\right)^2dsdx\\
&+\tau_{10}\left(\int_0^{l/2}\left[f^L\left(x,t\right)\right]^2dx+\int_{l/2}^{l}\left[f^R\left(x,t\right)\right]^2dx+d^2(l/2,t)\right).
\end{aligned} \tag{3.57}$$

To achieve our goal, we start selecting the different parameters in the right hand side of (3.57) so that all the coefficients in the right-hand side (except $\tau_{10}$). For small $\varepsilon$ and large values of $n$ and $t_*$, if $\widehat{\zeta}(\chi_n)$ is sufficiently small, we have

$$\kappa\widehat{\zeta}(\chi_n)-\frac{\zeta_*-\varepsilon}{2}\leqslant 0$$

and

$$\frac{3}{2}\left(1-\zeta_\star\right)\kappa\widehat{\zeta}(\chi_n)<\sigma\left(\zeta_*-\varepsilon\right)\left(1-\frac{\kappa}{2}\right) \tag{3.58}$$

with

$$\sigma=\frac{3\kappa\left(1-\zeta_\star\right)}{4\zeta_\star\left(2-\kappa\right)}$$

Observe that, for $t_*$ large enough, we have $0<\sigma<1$. For the remaining $1-\sigma$, we require that $\varphi_\gamma(0)$ satisfies

$$\frac{(1-\varepsilon)}{2}\varphi_\gamma(0) < (1-\sigma)(\zeta_* - \varepsilon)\left(1-\frac{\kappa}{2}\right), \tag{3.59}$$

then (3.59) is satisfied if $\varphi_\gamma(0) < \frac{1}{4}[\zeta_*(8-\kappa) - 3\kappa]$ and $\zeta_* > 3\kappa/(8-\kappa)$. Consequently, combining all of (3.58), ( 3.59), select $\eta_3$ and $\varepsilon_1$ small enough, we obtain

$$(1-\zeta_\star)\left[\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}(\chi_n)\right] + \frac{(1-\varepsilon)}{2}\varphi_\gamma(0) + [(1-\sigma)+\sigma](\zeta_\star - \varepsilon)\left[\frac{2\varepsilon_1 l^4}{EI} - \left(1-\frac{\kappa}{2}\right)\right] < 0$$

Once $\eta_3$, $\varepsilon_1$, $n$, $t_\star$, and $\varepsilon$ are fixed, we pick $\eta_1$, $\varepsilon_2$, $\lambda_2$ small enough and we select $k_r$ large enough, lead to the following system

$$\begin{cases} \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3} + 2\right)\kappa - \frac{1}{4n} < 0, \\ \lambda_2\left(2\varepsilon_2 + k_p + \frac{k_r}{\varepsilon}\right)\kappa - \frac{1}{8n} < 0, \\ \frac{EI}{4} - \frac{\rho A l^4}{2\varepsilon}\zeta(0)\lambda_2 > 0, \\ \frac{1}{4} - \frac{2\rho A l + m}{2\varepsilon}\zeta(0)\lambda_2 > 0, \\ \left[\lambda_2\left(2\varepsilon_2 + k_p + \frac{2k_r}{\zeta_\star - \varepsilon}\right) + \frac{1}{8\kappa n}\right]\kappa\widehat{\zeta}(\chi_n) + 2\varepsilon_2 l\kappa\lambda_2 - 1 < 0 \\ 2\kappa\widehat{\zeta}(\chi_n) + \frac{\varepsilon_2 l^4\kappa}{EI} - \frac{\zeta_\star - \varepsilon}{2} < 0, \\ \varepsilon_1(2l+1) + \eta_1 k_p^2 - \frac{k_r}{2} < 0. \end{cases}$$

Finally, we select $k_p$ large enough, where

$$\begin{aligned} &-k_p + \frac{\varepsilon\rho A\lambda_2}{4} + \frac{2k_r}{(\zeta_\star - \varepsilon)\lambda_2} + \left(m + \frac{1}{4\eta_1}\right)(\zeta_\star - \varepsilon)\lambda_2 + \left[\frac{k_p}{2} + m\left(\frac{\varepsilon}{2} - \zeta_\star\right)\right]\lambda_2 \\ &+ 4\kappa n + \frac{2}{k_r(\zeta_\star - \varepsilon)\lambda_2} + \frac{2\overline{\varphi}_\alpha}{\alpha} < 0. \end{aligned}$$

Together with (3.57), those choices lead to

$$\begin{aligned} \frac{d}{dt}K(t) &\leqslant -c_0[\mathcal{E}(t) + \Xi_3(t) + \Xi_4(t)] - \lambda_5 EI\delta(t)\Xi_5(t) \\ &+ \varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx + d^2(l/2,t)\right), \quad t \geqslant t_\star > 0 \end{aligned}$$

For some positive constants $c_0$ and $\varrho$ and

if $\lim\limits_{t\to+\infty}\delta(t) = 0$ then, for this constant $c_0$, there exists a $\overline{t}(c_0) \geqslant t_*$ such that $\delta(t) \leqslant c_0$

for $t \geqslant \bar{t}\,(c_0)$. Therefore, from the right-hand side of the Lemma 3.3, we obtain

$$\frac{d}{dt}K(t) \leqslant -c_1\delta(t)K(t) + \varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx + d^2(l/2,t)\right),\ t \geqslant \widehat{t}, \tag{3.60}$$

for some positive constant $c_1$. Now, applying Lemma 3.1 with
$Q(t) = c_1\delta(t)$, $\sigma(t) = 0$ and $\beta(t) = \varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx + d^2(l/2,t)\right)$
we infer from that (3.60)

$$\mathcal{E}(t) \leqslant \frac{C}{\Psi(t)},\ t \geqslant 0$$

for some $C > 0$, on condition $\Psi(0)K(0) < 1$.

If $\lim_{t\to+\infty}\delta(t) = \overline{\delta} \neq 0$, then there exist a $\widehat{t} \geqslant t_*$ such that $\delta(t) \geqslant c_0$ for $\widehat{t} \geqslant t_*$. Hence,

$$\frac{d}{dt}K(t) \leqslant -c_2K(t) + \varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx + d^2(l/2,t)\right),\ t \geqslant \widehat{t}, \tag{3.61}$$

for $c_2 > 0$, consider
$Q(t) = c_2\delta(t)$, $\sigma(t) = 0$ and $\beta(t) = \varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2 dx + d^2(l/2,t)\right)$.
we conclude that

$$\mathcal{E}(t) \leqslant \frac{C}{\Psi(t)},\ t \geqslant 0$$

for some positive constant $C > 0$, provide that $\Psi(0)K(0) < 1$. ■

# Chapter 4

# Stabilization of a flexible satellite system with memory term under tension effects

In this chapter, we study a viscoelastic flexible satellite problem with unknown distributed disturbance and take into account the tension of the system. Based on the multiplier method, we prove the stability of the system for a large class of relaxation functions.

## 4.1 Introduction

We study a viscoelastic flexible satellite under unknown distributed disturbances during attitude maneuvering. The system composed of a rigid central hub that represents the spacecraft with two symmetrical viscoelastic Euler-Bernoulli beams and subject to undesirable vibrations under the tension effects. The problem can be modeled by a set of partial differential equations (PDEs) taking into account therefore the dynamic boundary condition. Namely, for a large class of the relaxation function $q(t)$ (see [101]) satisfying

$(A_1)$ We assume that the function $q$ is continuous, differential, non-negative and satisfies the condition $0 < \kappa = \int_0^{+\infty} q\,(s)\,ds < 1$.

$(A_2)$ We assume that $0 < q'\,(t) + \mu q\,(t) \leqslant \zeta\,(t)$ such that $\mu > O$ and $\zeta\,(t)$ is a non-negative function, for all $t > 0$.

Then, we consider our problem as follows

$$\begin{cases} \text{for } (x,t) \in [-l,0] \times [0,\infty) \\ \rho A w_{tt}^{L}(x,t) + EI w_{xxxx}^{L}(x,t) - T w_{xx}^{L}(x,t) - EI \int\limits_{0}^{t} q(t-s)\, w_{xxxx}^{L}(x,s)\, ds = h(x,t), \\ \text{for } (x,t) \in [0,l] \times [0,\infty) \\ \rho A w_{tt}^{R}(x,t) + EI w_{xxxx}^{R}(x,t) - T w_{xx}^{R}(x,t) - EI \int\limits_{0}^{t} q(t-s)\, w_{xxxx}^{R}(x,s)\, ds = h(x,t), \end{cases} \tag{4.1}$$

with the boundary conditions

$$\begin{cases} w_{x}^{L}(0,t) = w_{x}^{R}(0,t) = 0,\ w_{xx}^{L}(-l,t) = w_{xx}^{R}(l,t) = 0, \quad t \geq 0 \\ w^{L}(0,t) = w^{R}(0,t) = w(0,t), \quad t \geq 0 \\ EI w_{xxx}^{L}(-l,t) - EI \int\limits_{0}^{t} q(t-s)\, w_{xxx}^{L}(-l,s)\, ds = T w_{x}^{L}(-l,t), \quad t \geq 0 \\ EI w_{xxx}^{R}(l,t) - EI \int\limits_{0}^{t} q(t-s)\, w_{xxx}^{L}(l,s)\, ds = T w_{x}^{R}(l,t), \quad t \geq 0 \\ m w_{tt}(0,t) = \quad u(t) + EI w_{xxx}^{L}(0,t) - EI \int\limits_{0}^{t} q(t-s)\, w_{xxx}^{L}(0,s)\, ds \\ \qquad\qquad -EI w_{xxx}^{R}(0,t) + EI \int\limits_{0}^{t} q(t-s)\, w_{xxx}^{R}(0,s)\, ds + d(0,t), \quad t \geq 0 \end{cases} \tag{4.2}$$

and the initial data

$$\begin{cases} w^{L}(x,0) = w_{0}^{L}(x),\ \ u_{t}^{L}(x,0) = w_{1}^{L}(x), \quad x \in [-l,0] \\ w^{R}(x,0) = w_{0}^{R}(x),\ \ u_{t}^{R}(x,0) = w_{1}^{R}(x), \quad x \in [0,l], \end{cases} \tag{4.3}$$

where $T$ represents the tension of the satellite, we keep the same notation as in the previous chapters. In (4.1), the integral term describes the memory term.

Unknown distributed perturbations are assumed to satisfy the following conditions:

$(A_3)$ $h$, $d$ are a continuous functions in $t$ where $h \in L^2(-l,l)$ and there exists a positive constant $\bar{d}$ such that $d(0,t) \leqslant \bar{d}$ for all $t \geqslant 0$.

## 4.2 Preliminary Results

In this section, we introduce some lemmas and notations which will be utilized.

We show that the global existence and the uniqueness of the solution of our system can be established by using Faedo-Galerkin approximation, (we follow the same ideas presented in the previous chapter). To this end, we set

$$\mathbb{V} =\Big\{(\theta^L,\, \theta^R) \in H^2\left(-l,0\right) \times H^2\left(0,l\right), \quad \theta^L_x\left(0\right) = \theta^R_x\left(0\right) = 0, \quad \theta^L(0) = \theta^R(0) = \theta(0)\Big\},$$

and

$$\mathbb{W} = \Big\{(\theta^L,\, \theta^R) \in H^4\left(-l,0\right) \times H^4\left(0,l\right), \quad \theta^L_{xx}\left(-l\right) = \theta^R_{xx}\left(l\right) = 0\Big\}$$

where $H^2\left(-l,0\right)$, $H^2\left(0,l\right)$, $H^4\left(-l,0\right)$ and $H^4\left(0,l\right)$ are Sobolev spaces.

**Proposition 4.1** *Let* $\left(w_0^L, w_0^R\right) \in \mathbb{V}$, $\left(w_1^L, w_1^R\right) \in \mathbb{W}$ *be given. Suppose that* $(A_1)$ *to* $(A_4)$ *are satisfied. Then, under a suitable external force* $u\left(t\right)$ *set out in* (4.45), *the problem* $(4.1) - (4.3)$ *has a global weak unique solution* $(w^L,\, w^R)$ *in the sense that for any time* $t$, *satisfies*

$$(w^L,\, w^R) \in L^\infty\big([0,T];\mathbb{W}\big),\ (w^L_t,\, w^R_t) \in L^\infty\big([0,T];\mathbb{V}\big)$$

$$(w^L_{tt},\, w^R_{tt}) \in L^\infty\big([0,T]; L^2(-l,0) \times L^2(0,l)\big)$$

*where* $T > 0$.

**Lemma 4.1** *The classical energy functional associated to the problem* (4.1)–(4.3) *is given through*

$$\begin{aligned} E\left(t\right) =& \frac{\rho A}{2}\int_{-l}^{0}\left[w_t^L\left(x,t\right)\right]^2 dx + \frac{\rho A}{2}\int_{0}^{l}\left[w_t^R\left(x,t\right)\right]^2 dx + \frac{m}{2}\left[w_t\left(0,t\right)\right]^2 \\ &+ \frac{EI}{2}\int_{-l}^{0}\left[w_{xx}^L\left(x,t\right)\right]^2 dx + \frac{EI}{2}\int_{0}^{l}\left[w_{xx}^R\left(x,t\right)\right]^2 dx \\ &+ \frac{T}{2}\int_{-l}^{0}\left[w_{x}^L\left(x,t\right)\right]^2 dx + \frac{T}{2}\int_{0}^{l}\left[w_{x}^R\left(x,t\right)\right]^2 dx, \quad t \geq 0, \end{aligned} \tag{4.4}$$

*satisfies*

$$\frac{d}{dt}E(t) = \int_{-l}^{0} w_{txx}^{L}(x,t)\int_{0}^{t} q(t-s)\, w_{xx}^{L}(x,s)\, dsdx + \int_{0}^{l} w_{txx}^{R}(x,t)\int_{0}^{t} q(t-s)\, w_{xx}^{R}(x,s)\, dsdx$$
$$+ \int_{-l}^{0} w_{t}^{L}(x,t)\, h(x,t)dx + \int_{0}^{l} w_{t}^{R}(x,t)\, h(x,t)dx + \big(u(t) + d(0,t)\big) w_{t}(0,t) \tag{4.5}$$

*for all* $t \geq 0$.

**Proof.** Multiplying the first equation of the system (4.1) by $w_t^L(x,t)$ and integrating over $[-l,0]$, secondly, multiplying the second equation of (4.2) by $w_t^R(x,t)$ and integrating over $[0,l]$, also multiplying the last Eq. of (4.2) by $w_t(0,t)$, gathering the conditions of the borders and summarizing the results, we reach to (4.5). ■

**Remark 4.1** *For* $t \geq 0$, *we have*

$$2\int_{-l}^{0} w_{txx}^{L}(x,t)\int_{0}^{t} q(t-s)\, w_{xx}^{L}(x,s)\, dsdx = \int_{-l}^{0}\left(q'\Box w_{xx}^{L}\right)(t)\, dx - q(t)\int_{-l}^{0}\left[w_{xx}^{L}(x,t)\right]^2 dx$$
$$- \frac{d}{dt}\left[\int_{-l}^{0}\left(q\Box w_{xx}^{L}\right)(t)\, dx - \left(\int_{0}^{t} q(s)\, ds\right)\int_{-l}^{0}\left[w_{xx}^{L}(x,t)\right]^2 dx\right] \tag{4.6}$$

*and*

$$2\int_{0}^{l} w_{txx}^{R}(x,t)\int_{0}^{t} q(t-s)\, w_{xx}^{R}(x,s)\, dsdx = \int_{0}^{l}\left(q'\Box w_{xx}^{R}\right)(t)\, dx - q(t)\int_{0}^{l}\left[w_{xx}^{R}(x,t)\right]^2 dx$$
$$- \frac{d}{dt}\left[\int_{0}^{l}\left(q\Box w_{xx}^{R}\right)(t)\, dx - \left(\int_{0}^{t} q(s)\, ds\right)\int_{0}^{l}\left[w_{xx}^{R}(x,t)\right]^2 dx\right]. \tag{4.7}$$

Thus, in view of the above (4.6) and (4.7), we can find that the modified energy functional associated to problem (4.1)–(4.3) is defined by

$$\begin{aligned}\mathcal{E}(t) =& \frac{\rho A}{2}\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \frac{m}{2}\left[w_t(0,t)\right]^2 + \frac{\rho A}{2}\int_0^l \left[w_t^R(x,t)\right]^2 dx + \frac{EI}{2}\int_{-l}^{0}\left(q\square w_{xx}^L\right)(t)\,dx \\ &+ \frac{EI}{2}\left[1-\left(\int_0^t q(s)\,ds\right)\right]\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right) \\ &+ \frac{T}{2}\left(\int_{-l}^{0}\left[w_x^L(x,t)\right]^2 dx + \int_0^l \left[w_x^R(x,t)\right]^2 dx\right) + \frac{EI}{2}\int_0^l \left(q\square w_{xx}^R\right)(t)\,dx\end{aligned} \tag{4.8}$$

and satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) =& \frac{EI}{2}\left(\int_{-l}^{0}(q'\square w_{xx}^L)(t)dx + \int_0^l (q'\square w_{xx}^R)(t)dx\right) + \big(u(t)+d(0,t)\big)w_t(0,t) \\ &- \frac{EI}{2}q(t)\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right) \\ &+ \int_{-l}^{0} w_t^L(x,t)\,h(x,t)dx + \int_0^l w_t^R(x,t)\,h(x,t)dx, \quad t \geqslant 0.\end{aligned} \tag{4.9}$$

**Remark 4.2** *Through the equality* (4.9)*, clearly that the derivative of the modified energy functional is of an undefined sign. Now, we will build a Lyapunov functional $\mathcal{T}$ which plays an essential role in proving our stabilization results.*

Let

$$\mathcal{T}(t) = \mathcal{E}(t) + \sum_{i=1}^{5}\lambda_i \mathcal{F}_i(t)\,,\ t \geqslant 0 \tag{4.10}$$

where $\lambda_i$, $i = 1, ..., 5$ are positive constants that will be specified later. such that $\lambda_3 = \lambda_4 = 1$, and for $t \geqslant 0$

$$\begin{aligned}\mathcal{F}_1(t) :=& \rho A\int_{-l}^{0} w^L(x,t)\,w_t^L(x,t)\,dx + \rho A\int_0^l w^R(x,t)\,w_t^R(x,t)\,dx + m w_t(0,t)\,w(0,t)\,, \\ \mathcal{F}_2(t) :=& -\rho A\int_{-l}^{0} w_t^L(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\ &- m w_t(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\ &- \rho A\int_0^l w_t^R(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx,\end{aligned}$$

$$\mathcal{F}_3(t) := \frac{1}{2}(q\lozenge w)(t) + \frac{1}{2}\left[w(0,t)\right]^2,$$

$$\mathcal{F}_4(t) := \int_0^t \varphi_\beta(t-s) \int_{-l}^0 \left[w_{xx}^L(x,s)\right]^2 dxds + \int_0^t \varphi_\beta(t-s) \int_0^l \left[w_{xx}^R(x,s)\right]^2 dxds,$$
$$\mathcal{F}_5(t) := \int_0^t \varphi_\beta(t-s)\left[w(0,s)\right]^2 ds,$$

where $\varphi_\beta(t) = e^{-\beta t}\int_t^\infty \zeta(s) e^{\beta s} ds,\ t \geqslant 0$, such that $\beta$ is a positive constant to be precise later. Now, we prove that $\mathcal{T}(t)$ and $\mathcal{E}(t) + \sum_{i=3}^{5} \mathcal{F}_i(t)$ are equivalent .

**Proposition 4.2** *There exist two positive constants $\delta_1$ and $\delta_2$, such that, for all $t \geqslant 0$.*

$$\delta_1\left(\mathcal{E}(t) + \sum_{i=3}^{5} \mathcal{F}_i(t)\right) \leqslant \mathcal{T}(t) \leqslant \delta_2\left(\mathcal{E}(t) + \sum_{i=3}^{5} \mathcal{F}_i(t)\right), \tag{4.11}$$

**Proof.** From Cauchy-Schwarz, Young's and Poincarés inequalities, we get

$$\begin{aligned}\mathcal{F}_1(t) \leqslant & \frac{\rho A}{2}\int_{-l}^0 \left[w^L(x,t)\right]^2 dx + \frac{\rho A}{2}\int_{-l}^0 \left[w_t^L(x,t)\right]^2 dx + \frac{\rho A}{2}\int_0^l \left[w^R(x,t)\right]^2 dx \\ &+ \frac{\rho A}{2}\int_0^l \left[w_t^R(x,t)\right]^2 dx + \frac{m}{2}\left[w_t(0,t)\right]^2 + \frac{m}{2}\left[w(0,t)\right]^2,\end{aligned} \tag{4.12}$$

then applying Lemma 1.4 and Remark 1.10, it follows

$$\int_{-l}^0 \left[w^L(x,t)\right]^2 dx \leqslant 2l\left[w(0,t)\right]^2 + 16l^4 \int_{-l}^0 \left[w_{xx}^L(x,t)\right]^2 dx, \tag{4.13}$$

and

$$\int_0^l \left[w^R(x,t)\right]^2 dx \leqslant 2l\left[w(0,t)\right]^2 + 16l^4 \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx. \tag{4.14}$$

next, we obtain that

$$\begin{aligned}\mathcal{F}_1(t) \leqslant & 8\rho A l^4\left(\int_{-l}^0 \left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right) + \frac{m}{2}\left[w_t(0,t)\right]^2 \\ &+ \frac{\rho A}{2}\left(\int_{-l}^0 \left[w_t^L(x,t)\right]^2 dx + \int_0^l \left[w_t^R(x,t)\right]^2 dx\right) + \left(2\rho A l + \frac{m}{2}\right)\left[w(0,t)\right]^2,\ t \geqslant 0.\end{aligned} \tag{4.15}$$

For the functional $\Xi_2(t)$, we have

$$\begin{aligned}\mathcal{F}_2(t) \leqslant & \frac{\rho A}{2}\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \int_0^l \left[w_t^R(x,t)\right]^2 dx\right) + \frac{m}{2}\left[w_t(0,t)\right]^2 \\ & + \frac{\rho A}{2}\left(\int_0^t q(s)\,ds\right)\left(\int_{-l}^{0}\left(q\Box w^L\right)(t)\,dx + \int_0^l \left(q\Box w^R\right)(t)\,dx\right) \\ & + \frac{m}{2}\left(\int_0^t q(s)\,ds\right)(q\Diamond w)(t),\ t \geqslant 0.\end{aligned}$$

then by using the Remark 1.10 once more, we get

$$\int_{-l}^{0}\left(q\Box w^L\right)(t)\,dx \leqslant 2l\,(q\Diamond w)(t) + 16l^4 \int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx,\ t \geqslant 0$$

$$\int_0^l \left(q\Box w^R\right)(t)\,dx \leqslant 2l\,(q\Diamond w)(t) + 16l^4 \int_0^l \left(q\Box w_{xx}^R\right)(t)\,dx,\ t \geqslant 0.$$

So

$$\begin{aligned}\mathcal{F}_2(t) \leqslant & \frac{\rho A}{2}\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \int_0^l \left[w_t^R(x,t)\right]^2 dx\right) + \kappa\left(\frac{m}{2} + 2\rho A l\right)(q\Diamond w)(t) \\ & + 8\rho A l^4 \kappa\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx + \int_0^l \left(q\Box w_{xx}^R\right)(t)\,dx\right) + \frac{m}{2}\left[w_t(0,t)\right]^2.\end{aligned} \tag{4.16}$$

using (4.8), (4.15) and (4.16), we obtain

$$\begin{aligned}\mathcal{T}(t) \leqslant & \frac{1}{2}\rho A(1+\lambda_1+\lambda_2)\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \int_0^l \left[w_t^R(x,t)\right]^2 dx\right) \\ & + \left(\frac{EI}{2} + 8\lambda_2\rho A l^4\kappa\right)\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(x,t)\,dx + \int_0^l \left(q\Box w_{xx}^R\right)(x,t)\,dx\right) \\ & + \frac{1}{2}\left[\lambda_1(4\rho A l + m) + 1\right]\left[w(0,t)\right]^2 + \frac{1}{2}\left[\lambda_2\kappa(m + 4\rho A l) + 1\right](q\Diamond w)(t) \\ & + \left[\frac{EI}{2}\left(1 - \int_0^t q(s)\,ds\right) + 8\lambda_1\rho A l^4\right]\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right) \\ & + \frac{1}{2}m(1+\lambda_1+\lambda_2)\left[w_t(0,t)\right]^2 + \mathcal{F}_4(t) + \lambda_5\mathcal{F}_5(t).\end{aligned}$$

on the other hand, we have

$$\begin{aligned}
2\mathcal{T}(t) \geqslant{} & \rho A\left(1-\lambda_1-\lambda_2\right)\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx+\int_0^l\left[w_t^R(x,t)\right]^2 dx\right) \\
& +\left(EI-16\lambda_2\rho A l^4\kappa\right)\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(x,t)\,dx+\int_0^l\left(q\Box w_{xx}^R\right)(x,t)\,dx\right) \\
& +\left[EI(1-\kappa)-16\lambda_1\rho A l^4\right]\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx+\int_0^l\left[w_{xx}^R(x,t)\right]^2 dx\right) \\
& +\left[1-\lambda_2\kappa\left(m+4\rho A l\right)\right]\left(q\Diamond w\right)(t)+\left[1-\lambda_1\left(4\rho A l+m\right)\right]\left[w(0,t)\right]^2 \\
& +m\left(1-\lambda_1-\lambda_2\right)\left[w_t(0,t)\right]^2+2\mathcal{F}_4(t)+2\lambda_5\mathcal{F}_5(t).
\end{aligned}$$

Then, there exists $\delta_i > 0$, $i = 1, 2$ such that $\delta_1\Big(\mathcal{E}(t)+\sum\limits_{i=3}^{5}\mathcal{F}(t)\Big) \leqslant \mathcal{T}(t) \leqslant \delta_2\Big(\mathcal{E}(t)+\sum\limits_{i=3}^{5}\mathcal{F}_i(t)\Big)$, where $\lambda_1 < \min\left[1, \frac{1}{4\rho A l+m}, \frac{EI(1-\kappa)}{16\rho A l^4}\right]$ and $\lambda_2 < \min\left[1-\lambda_1, \frac{1}{(4\rho A l+m)\kappa}, \frac{EI}{16\rho A l^4\kappa}\right]$. ■

In the next sections, we will use the notations $\overline{\zeta} := \int_0^\infty \zeta(s)\,ds$ and $\overline{\zeta}_\beta := \int_0^\infty e^{\beta z}\zeta(z)\,dz$ and let $t_\star$ be a positive number such that $\int_0^{t_\star} q(s)\,ds := q_\star > 0$.

## 4.3 Technical lemmas

In this section, we will introduce some lemmas needed to help in the proof of our main result (i.e. Theorem 4.1).

**Lemma 4.2** *The derivative functional of $\mathcal{F}_1(t)$ along solutions of* (4.1) *to* (4.3) *satisfies, for any positive $\eta_1$, $\eta_2$ and $\varepsilon_1$*

$$\frac{d}{dt}\mathcal{F}_1(t) \leqslant \rho A\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx - T\left(\int_{-l}^{0}\left[w_x^L(x,t)\right]^2 dx + \int_0^l \left[w_x^R(x,t)\right]^2 dx\right) \quad (4.17)$$

$$\begin{aligned} &+\left[\frac{16\varepsilon_1 l^4}{EI} + \eta_1 - (1-k)\right] EI\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right)\\ &+\rho A\int_0^l \left[w_t^R(x,t)\right]^2 dx + \left(m + \frac{1}{4\eta_2}\right)\left[w_t(0,t)\right]^2\\ &+\frac{EI}{4\eta_1}\kappa\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx + \int_0^l \left(q\Box w_{xx}^R\right)(t)\,dx\right)\\ &+\left(\varepsilon_1(4l+1) + \eta_2 k_p^2 - k_r\right)\left[w(0,t)\right]^2 + \frac{1}{4\varepsilon_1}\left(\int_{-l}^{l}\left[h(x,t)\right]^2 dx + d^2(0,t)\right) \end{aligned} \quad (4.18)$$

*for $t \geq 0$.*

**Proof.** By simple calculations, a differentiation of $\mathcal{F}_1(t)$ with respect to $t$, making up of (4.45), we find

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_1(t) &= I_1(t) + I_2(t) + I_3(t) + \rho A\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \rho A\int_0^l \left[w_t^R(x,t)\right]^2 dx\\ &+ m\left[w_t(0,t)\right]^2 + EIw(0,t)\,w_{xxx}^L(0,t) - EIw(0,t)\,w_{xxx}^R(0,t) + d(0,t)w(0,t)\\ &- EIw(0,t)\int_0^t q(t-s)\,w_{xxx}^L(0,s)\,ds + EIw(0,t)\int_0^t q(t-s)\,w_{xxx}^R(0,s)\,ds\\ &- k_p w(0,t)\,w_t(0,t) - k_r\left[w(0,t)\right]^2 + \int_{-l}^{0} w^L(x,t)\,h(x,t)dx + \int_0^l w^R(x,t)\,h(x,t)dx \end{aligned} \quad (4.19)$$

where

$$I_1(t) = -EI\int_{-l}^{0} w^L(x,t)\, w^L_{xxxx}(x,t)\, dx - EI\int_{0}^{l} w^R(x,t)\, w^R_{xxxx}(x,t)\, dx,$$

$$I_2(t) = T\int_{-l}^{0} w^L(x,t)\, w^L_{xx}(x,t)\, dx + T\int_{0}^{l} w^R(x,t)\, w^R_{xx}(x,t)\, dx,$$

and

$$\begin{aligned} I_3(t) =& EI\int_{-l}^{0} w^L(x,t)\int_0^t q(t-s)\, w^L_{xxxx}(x,s)\, ds dx \\ &+ EI\int_{0}^{l} w^R(x,t)\int_0^t q(t-s)\, w^R_{xxxx}(x,s)\, ds dx. \end{aligned}$$

Integrating by parts $I_i$, $i = 1, 2, 3$ and using the boundary conditions, we have

$$\begin{aligned} I_1(t) &= -EIw(0,t)\, w^L_{xxx}(0,t) + EIw(-l,t)\, w^L_{xxx}(-l,t) \\ &\quad -EI\int_{-l}^{0}\left[w^L_{xx}(x,t)\right]^2 dx + EIw(0,t)\, w^R_{xxx}(0,t) \\ &\quad -EIw(l,t)\, w^R_{xxx}(l,t) - EI\int_0^l \left[w^R_{xx}(x,t)\right]^2 dx, \ t \geqslant 0 \end{aligned} \tag{4.20}$$

$$\begin{aligned} I_2(t) &= -Tw^L_x(-l,t)\, w^L(-l,t) + Tw^R_x(l,t)\, w^R(l,t) \\ &\quad -T\left(\int_{-l}^{0}\left[w^L_x(x,t)\right]^2 dx + \int_0^l \left[w^R_x(x,t)\right]^2 dx\right), \ t \geqslant 0 \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} I_3(t) &= EIw(0,t)\left(\int_0^t q(t-s)\, w^L_{xxx}(0,s)\, ds - \int_0^t q(t-s)\, w^R_{xxx}(0,s)\, ds\right) \\ &\quad +EI\int_{-l}^{0} w^L_{xx}(x,t)\int_0^t q(t-s)\, w^L_{xx}(x,s)\, ds dx \\ &\quad +EI\int_{0}^{l} w^R_{xx}(x,t)\int_0^t q(t-s)\, w^R_{xx}(x,s)\, ds dx \\ &\quad -EIw(-l,t)\int_0^t q(t-s)\, w^L_{xxx}(-l,s)\, ds \\ &\quad +EIw(l,t)\int_0^t q(t-s)\, w^R_{xxx}(l,s)\, ds, \ t \geqslant 0. \end{aligned} \tag{4.22}$$

Taking into account the above estimates in (4.19) , we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) &= \rho A\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2dx+\int_0^l\left[w_t^R(x,t)\right]^2dx\right)-EI\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2dx\right.\\&\left.+\int_0^l\left[w_{xx}^R(x,t)\right]^2dx\right)+\int_{-l}^{0}w^L(x,t)\,h(x,t)dx+\int_0^l w^R(x,t)\,h(x,t)dx\\&+EI\left(\int_{-l}^{0}w_{xx}^L(x,t)\int_0^t q(t-s)\,w_{xx}^L(x,s)\,dsdx\right.\\&\left.+\int_0^l w_{xx}^R(x,t)\int_0^t q(t-s)\,w_{xx}^R(x,s)\,dsdx\right)-k_r\left[w(0,t)\right]^2\\&+m\left[w_t(0,t)\right]^2-k_p w(0,t)\,w_t(0,t)+d(0,t)w(0,t)\\&-T\left(\int_{-l}^{0}\left[w_x^L(x,t)\right]^2dx+\int_0^l\left[w_x^R(x,t)\right]^2dx\right).\end{aligned}\tag{4.23}$$

Now, we will begin to estimate some terms in (4.23). For $\eta_1>0$, $\eta_2>0$ and $\varepsilon_1>0$, we have

$$\begin{aligned}EI\int_{-l}^{0}w_{xx}^L(x,t)\int_0^t q(t-s)\left(w_{xx}^L(x,s)-w_{xx}^L(x,t)\right)dsdx \leqslant&\eta_1 EI\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2dx\\&+\frac{EI}{4\eta_1}\kappa\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx,\end{aligned}\tag{4.24}$$

$$\begin{aligned}EI\int_0^l w_{xx}^R(x,t)\int_0^t q(t-s)\left(w_{xx}^R(x,s)-w_{xx}^R(x,t)\right)dsdx \leqslant&\eta_1 EI\int_0^l\left[w_{xx}^R(x,t)\right]^2dx\\&+\frac{EI}{4\eta_1}\kappa\int_0^l\left(q\Box w_{xx}^R\right)(t)\,dx\end{aligned}\tag{4.25}$$

$$k_p w(0,t)\,w_t(0,t)\leqslant \eta_2 k_p^2\left[w(0,t)\right]^2+\frac{1}{4\eta_2}\left[w_t(0,t)\right]^2,\tag{4.26}$$

and

$$\int_{-l}^{0}w^L(x,t)\,h(x,t)dx\le 2\varepsilon_1 l\left[w(0,t)\right]^2+16\varepsilon_1 l^4\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2dx+\frac{1}{4\varepsilon_1}\int_{-l}^{0}\left[h(x,t)\right]^2dx,\tag{4.27}$$

$$\int_0^l w^R(x,t)\,h(x,t)dx\le 2\varepsilon_1 l\left[w(0,t)\right]^2+16\varepsilon_1 l^4\int_0^l\left[w_{xx}^R(x,t)\right]^2dx+\frac{1}{4\varepsilon_1}\int_0^l\left[h(x,t)\right]^2dx\tag{4.28}$$

$$d(0,t)w(0,t)\le\frac{1}{4\varepsilon_1}d^2(0,t)+\varepsilon_1\left[w(0,t)\right]^2\tag{4.29}$$

for all $t \geq 0$, and by using the inequalities (4.24), (4.25), (4.26), (4.27), (4.28) and (4.29) in (4.23), we get (4.17). ■

**Lemma 4.3** *Suppose that* $(A_1)$ *and* $(A_2)$ *hold, let* $(w^L, w^R)$ *be a solution of* (4.1)–(4.3). *Then, for some positive constants* $\eta_i$, $i = 2,3,4,5,6,7$, *the estimate of* $\mathcal{F}_2'(t)$ *satisfies*

$$\begin{aligned}
\mathcal{F}_2'(t) \leqslant & \eta_3 EI\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx + \int_0^l \left[w_{xx}^R(x,t)\right]^2 dx\right) + \eta_5\left[w(0,t)\right]^2 \\
& + (\eta_6 + \eta_4\mu - q_\star)\left(\rho A\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \rho A\int_0^l \left[w_t^R(x,t)\right] dx\right) \\
& + \left(16\varepsilon_2 l^4 + \frac{EI}{4\eta_3} + EI + \frac{4\rho A\mu l^4}{\eta_4} + \frac{Tl^2}{\eta_7}\right)\kappa\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx\right. \\
& \left. + \int_0^l \left(q\Box w_{xx}^R\right)(t)\,dx\right) + \left((4l+1)\varepsilon_2 + \frac{k_p + m\mu}{4\eta_4} + \frac{k_r^2}{4\eta_5} + \frac{\rho A\mu l}{\eta_4}\right)\kappa\left(q\Diamond w\right)(t) \\
& + \frac{4\rho A l^4}{\eta_6}\left(\int_0^t \zeta(s)\,ds\right)\left(\int_{-l}^{0}\left(\zeta\Box w_{xx}^L\right)(t)\,dx + \int_0^l \left(\zeta\Box w_{xx}^R\right)(t)\,dx\right) \\
& + \frac{1}{\eta_6}\left(\rho A l + \frac{m}{4}\right)\left(\int_0^t \zeta(s)\,ds\right)\left(\zeta\Diamond w\right)(t) + \left[\eta_6 m + (k_p + m\mu)\eta_4 - m q_\star\right]\left[w_t(0,t)\right]^2 \\
& + \eta_7 T\left(\int_{-l}^{0}\left[w_x^L(x,t)\right]^2 dx + \int_0^l \left[w_x^R(x,t)\right]^2 dx\right) + \frac{1}{4\varepsilon_2}\left(\int_{-l}^{l}\left[h(x,t)\right]^2 dx + d^2(0,t)\right),
\end{aligned}$$

*for all* $t \geq t_\star > 0$ *where* $\varepsilon_2 > 0$.

**Proof.** Differentiating the functional $\mathcal{F}_2(t)$ , we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_2(t) = & -\rho A\int_{-l}^{0} w_{tt}^L(x,t)\int_0^t q(t-s)\left(w^L(x,t) - w^L(x,s)\right) ds dx \\
& -\rho A\left(\int_0^t q(s)\,ds\right)\int_0^l \left[w_t^L(x,t)\right]^2 dx - \rho A\left(\int_0^t q(s)\,ds\right)\int_0^l \left[w_t^R(x,t)\right]^2 dx \\
& -\rho A\int_{-l}^{0} w_t^L(x,t)\int_0^t q'(t-s)\left(w^L(x,t) - w^L(x,s)\right) ds dx \\
& -\rho A\int_0^l w_{tt}^R(x,t)\int_0^t q(t-s)\left(w^R(x,t) - w^R(x,s)\right) ds dx \\
& - m w_{tt}(0,t)\int_0^t q(t-s)\left(w(0,t) - w(0,s)\right) ds - m\left(\int_0^t q(s)\,ds\right)\left[w_t(0,t)\right]^2 \\
& -\rho A\int_0^l w_t^R(x,t)\int_0^t q'(t-s)\left(w^R(x,t) - w^R(x,s)\right) ds dx \\
& - m w_t(0,t)\int_0^t q'(t-s)\left(w(0,t) - w(0,s)\right) ds,\ t \geqslant 0.
\end{aligned}$$

By utilizing the Eqs. of (4.1), boundary condition (4.2) and (4.45) , result in

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_2(t) =& I_3(t)+I_4(t)+I_5(t)+I_6(t)+I_7(t)+I_8(t) \\
&-\rho A\int_{-l}^{0} w_t^L(x,t)\int_0^t q'(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\
&-\rho A\int_0^t q(s)\,ds\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx-\rho A\int_0^t q(s)\,ds\int_0^l\left[w_t^R(x,t)\right]^2 dx \\
&-\rho A\int_0^l w_t^R(x,t)\int_0^t q'(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx \\
&-\left(EIw_{xxx}^L(0,t)+d(0,t)\right)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\
&+EI\left(\int_0^t q(t-s)\,w_{xxx}^L(0,s)\,ds\right)\left(\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds\right) \\
&-EI\left(\int_0^t q(t-s)\,w_{xxx}^R(0,s)\,ds\right)\left(\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds\right) \\
&+\left(EIw_{xxx}^R(0,t)+k_p w_t(0,t)+k_r w(0,t)\right)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\
&-mw_t(0,t)\int_0^t q'(t-s)\left(w(0,t)-w(0,s)\right)ds-m\left(\int_0^t q(s)\,ds\right)\left[w_t(0,t)\right]^2
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&-\int_{-l}^{0} h(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\
&-\int_0^l h(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
I_3(t) &= EI\int_{-l}^{0} w_{xxxx}^L(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx, \\
I_4(t) &= EI\int_0^l w_{xxxx}^R(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx, \\
I_5(t) &= -T\int_{-l}^{0} w_{xx}^L(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx, \\
I_6(t) &= -T\int_0^l w_{xx}^L(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx, \\
I_7(t) &= -EI\int_{-l}^{0}\left(\int_0^t q(t-s)\,w_{xxxx}^L(x,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)ds\right)dx,
\end{aligned} \tag{4.32}$$

and

$$I_8(t) = -EI\int_0^l\left(\int_0^t q(t-s)\,w_{xxxx}^R(x,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)ds\right)dx.$$

Integrating by parts twice and using the boundary conditions, we obtain, for $t \geq 0$

$$
\begin{aligned}
&I_3(t) = EIw^L_{xxx}(0,t)\int_0^t q(t-s)\left(w^L(0,t)-w^L(0,s)\right)ds\\
&-EIw^L_{xxx}(-l,t)\int_0^t q(t-s)\left(w^L(-l,t)-w^L(-l,s)\right)ds\\
&+EI\int_{-l}^0 w^L_{xx}(x,t)\int_0^t q(t-s)\left(w^L_{xx}(x,t)-w^L_{xx}(x,s)\right)dsdx,\\
&I_4(t) = -EIw^R_{xxx}(0,t)\int_0^t q(t-s)\left(w^R(0,t)-w^R(0,s)\right)ds\\
&+EIu^R_{xxx}(l,t)\int_0^t q(t-s)\left(w^R(l,t)-w^R(l,s)\right)ds\\
&+EI\int_0^l w^R_{xx}(x,t)\int_0^t q(t-s)\left(w^R_{xx}(x,t)-w^R_{xx}(x,s)\right)dsdx,\\
&I_5(t) = Tw^L_x(-l,t)\int_0^t q(t-s)\left(w^L(-l,t)-w^L(-l,s)\right)ds\\
&+T\int_{-l}^0 w^L_x(x,t)\int_0^t q(t-s)\left(w^L_x(x,t)-w^L_x(x,s)\right)dsdx,\\
&I_6(t) = -Tw^R_x(l,t)\int_0^t q(t-s)\left(w^R(l,t)-w^R(l,s)\right)ds\\
&+T\int_0^l w^R_x(x,t)\int_0^t q(t-s)\left(w^R_x(x,t)-w^R_x(x,s)\right)dsdx,
\end{aligned}
$$

$$
\begin{aligned}
I_7(t) \;=\; & -EI\left(\int_0^t q(t-s)\,w^L_{xxx}(0,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^L(0,t)-w^L(0,s)\right)ds\right)\\
&+EI\left(\int_0^t q(t-s)\,w^L_{xxx}(-l,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^L(-l,t)-w^L(-l,s)\right)ds\right)\\
&-EI\int_{-l}^0\left(\int_0^t q(t-s)\,w^L_{xx}(x,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^L_{xx}(x,t)-w^L_{xx}(x,s)\right)ds\right)dx,
\end{aligned}
$$

and

$$
\begin{aligned}
I_8(t) \;=\; & EI\left(\int_0^t q(t-s)\,w^R_{xxx}(0,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^R(0,t)-w^R(0,s)\right)ds\right)\\
&-EI\left(\int_0^t q(t-s)\,w^R_{xxx}(l,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^R(l,t)-w^R(l,s)\right)ds\right)\\
&-EI\int_0^l\left(\int_0^t q(t-s)\,w^R_{xx}(x,s)\,ds\right)\left(\int_0^t q(t-s)\left(w^R_{xx}(x,t)-w^R_{xx}(x,s)\right)ds\right)dx,
\end{aligned}
$$

According to the estimates $I_3(t)$, $I_4(t)$, $I_5(t)$ , $I_6(t)$, $I_7(t)$ and $I_8(t)$, (4.30) becomes

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_2(t) = &-\int_{-l}^{0} h(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\
&-\int_0^l h(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx \\
&-d(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\
&+k_r w(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\
&+EI\left(1-\int_0^t q(s)\,ds\right)\int_{-l}^{0} w_{xx}^L(x,t)\int_0^t q(t-s)\left(w_{xx}^L(x,t)-w_{xx}^L(x,s)\right)dsdx \\
&+EI\left(1-\int_0^t q(s)\,ds\right)\int_0^l w_{xx}^R(x,t)\int_0^t q(t-s)\left(w_{xx}^R(x,t)-w_{xx}^R(x,s)\right)dsdx \\
&+EI\int_{-l}^{0}\left(\int_0^t q(t-s)\left(w_{xx}^L(x,t)-w_{xx}^L(x,s)\right)ds\right)^2 dx \\
&+EI\int_0^l\left(\int_0^t q(t-s)\left(w_{xx}^R(x,t)-w_{xx}^R(x,s)\right)ds\right)^2 dx
\end{aligned}$$

$$\begin{aligned}
&+T\int_{-l}^{0} w_x^L(x,t)\int_0^t q(t-s)\left(w_x^L(x,t)-w_x^L(x,s)\right)dsdx \\
&+T\int_0^l w_x^R(x,t)\int_0^t q(t-s)\left(w_x^R(x,t)-w_x^R(x,s)\right)dsdx \\
&-\left(\int_0^t q(s)\,ds\right)\left(\rho A\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx+\rho A\int_0^l\left[w_t^R(x,t)\right]^2 dx+m\left[w_t(0,t)\right]^2\right) \\
&-\rho A\int_{-l}^{0} w_t^L(x,t)\int_0^t\left(q'(t-s)+\mu\, q(t-s)\right)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\
&-\rho A\int_0^l w_t^R(x,t)\int_0^t\left(q'(t-s)+\mu\, q(t-s)\right)\left(w^R(x,t)-w^R(x,s)\right)dsdx \\
&+\rho A\mu\int_{-l}^{0} w_t^L(x,t)\int_0^t q(t-s)\left(w^L(x,t)-w^L(x,s)\right)dsdx \\
&+\rho A\mu\int_0^l w_t^R(x,t)\int_0^t q(t-s)\left(w^R(x,t)-w^R(x,s)\right)dsdx \\
&+\left(k_p+m\mu\right)w_t(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \\
&-mw_t(0,t)\int_0^t\left(q'(t-s)+\mu\, q(t-s)\right)\left(w(0,t)-w(0,s)\right)ds, \quad t\geq 0.
\end{aligned} \tag{4.33}$$

Next, using Lemma 1.4 and Remark 1.10, Young's and Cauchy-Schwarz inequalities, we can estimate the terms on the right-hand side of (4.33). Thus we have for $\varepsilon_2 > 0$

$$\begin{aligned}\int_{-l}^{0} h(x,t)\int_{0}^{t} q(t-s)\left(w^{L}(x,t)-w^{L}(x,s)\right)dsdx &\leq 2\varepsilon_2 l\kappa\,(q\Diamond w)(t)\\ &+16\varepsilon_2 l^4\kappa\int_{-l}^{0}\left(q\Box w_{xx}^{L}\right)(t)\,dx+\frac{1}{4\varepsilon_2}\int_{-l}^{0}h^2(x,t)dx,\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_{0}^{l} h(x,t)\int_{0}^{t} q(t-s)\left(w^{R}(x,t)-w^{R}(x,s)\right)dsdx &\leq 2\varepsilon_2 l\kappa\,(q\Diamond w)(t)\\ &++16\varepsilon_2 l^4\kappa\int_{0}^{l}\left(q\Box w_{xx}^{R}\right)(t)\,dx+\frac{1}{4\varepsilon_2}\int_{0}^{l}h^2(x,t)dx\end{aligned} \tag{4.35}$$

$$d(0,t)\int_{0}^{t} q(t-s)\left(w(0,t)-w(0,s)\right)dsdx \leq \frac{1}{4\varepsilon_2}d^2(0,t)dx+\varepsilon_2\kappa\,(q\Diamond w)(t) \tag{4.36}$$

and

$$\begin{aligned}\int_{-l}^{0} w_{xx}^{L}(x,t)\int_{0}^{t} q(t-s)\left(w_{xx}^{L}(x,t)-w_{xx}^{L}(x,s)\right)dsdx &\leqslant \eta_3\int_{-l}^{0}\left[w_{xx}^{L}(x,t)\right]^2dx\\ &+\frac{\kappa}{4\eta_3}\int_{-l}^{0}\left(q\Box w_{xx}^{L}\right)(t)\,dx,\ \ \eta_3>0\end{aligned}$$

$$\begin{aligned}\int_{0}^{l} w_{xx}^{R}(x,t)\int_{0}^{t} q(t-s)\left(w_{xx}^{R}(x,t)-w_{xx}^{R}(x,s)\right)dsdx &\leqslant \eta_3\int_{0}^{l}\left[w_{xx}^{R}(x,t)\right]^2dx\\ &+\frac{\kappa}{4\eta_3}\int_{0}^{l}\left(q\Box w_{xx}^{R}\right)(t)\,dx,\ \ \eta_3>0\end{aligned}$$

$$\int_{-l}^{0}\left(\int_{0}^{t} q(t-s)\left(w_{xx}^{L}(x,t)-w_{xx}^{L}(s)\right)ds\right)^2dx \leqslant \kappa\int_{-l}^{0}\left(q\Box w_{xx}^{L}\right)(t)\,dx,$$

$$\int_{0}^{l}\left(\int_{0}^{t} q(t-s)\left(w_{xx}^{R}(x,t)-w_{xx}^{R}(s)\right)ds\right)^2dx \leqslant \kappa\int_{0}^{l}\left(q\Box w_{xx}^{R}\right)(t)\,dx,$$

$$\begin{aligned}&(k_p+m\mu)\,w_t(0,t)\int_{0}^{t} q(t-s)\left(w(0,t)-w(0,s)\right)ds\\ \leqslant\ &(k_p+m\mu)\,\eta_4\left[w_t(0,t)\right]^2+\frac{(k_p+m\mu)}{4\eta_4}\kappa\,(q\Diamond w)(t)\,,\ \ \eta_4>0\end{aligned}$$

$$\begin{aligned}\int_{-l}^{0} w_{t}^{L}(x,t)\int_{0}^{t} q(t-s)\left(w^{L}(x,t)-w^{L}(x,s)\right)dsdx \leqslant&\eta_4\int_{-l}^{0}\left[q_{t}^{L}(x,t)\right]^2dx+\frac{\kappa l}{2\eta_4}(q\Diamond w)(t)\\ &+\frac{4\kappa l^4}{\eta_4}\int_{-l}^{0}\left(q\Box w_{xx}^{L}\right)(t)\,dx,\end{aligned}$$

$$\int_0^l w_t^R(x,t)\int_0^t q(t-s)(w_R(x,t)-w_R(x,s))\,dsdx \leqslant \eta_4\int_0^l \left[w_t^R(x,t)\right]^2 dx+\frac{\kappa l}{2\eta_4}(q\Diamond w)(t)$$
$$+\frac{4\kappa l^4}{\eta_4}\int_0^l \left(q\Box w_{xx}^R\right)(t)\,dx$$

and

$$k_r w(0,t)\int_0^t q(t-s)(w(0,t)-w(0,s))\,ds \leqslant \eta_5[w(0,t)]^2+\frac{k_r^2}{4\eta_5}\kappa(q\Diamond w)(t),\ \ \eta_5>0,$$

for all $t \geq 0$. Next, by Lemma 1.4, assumptions $(A_2)$ and Remark 1.10, leads to, for $\eta_6 > 0$

$$\begin{aligned}&\int_{-l}^0 w_t^L(x,t)\int_0^t (q'(t-s)+\mu q(t-s))\left(w^L(x,t)-w^L(x,s)\right)dsdx\\&\leqslant \eta_6\int_{-l}^0\left[w_t^L(x,t)\right]^2dx+\frac{l}{2\eta_6}\left(\int_0^t\zeta(s)\,ds\right)(\zeta\Diamond w)(t)\\&+\frac{4l^4}{\eta_6}\left(\int_0^t\zeta(s)\,ds\right)\int_{-l}^0\left(\zeta\Box w_{xx}^L\right)(t)\,dx,\end{aligned} \tag{4.37}$$

$$\begin{aligned}&\int_0^l w_t^R(x,t)\int_0^t (q'(t-s)+\mu q(t-s))\left(w^R(x,t)-w^R(x,s)\right)dsdx\\&\leqslant \eta_6\int_0^l\left[w_t^R(x,t)\right]^2dx+\frac{l}{2\eta_6}\left(\int_0^t\zeta(s)\,ds\right)(\zeta\Diamond w)(t)\\&+\frac{4l^4}{\eta_6}\left(\int_0^t\zeta(s)\,ds\right)\int_0^l\left(\zeta\Box w_{xx}^R\right)(t)\,dx\end{aligned} \tag{4.38}$$

and

$$w_t(0,t)\int_0^t (q'(t-s)+\mu q(t-s))(w(0,t)-w(0,s))\,ds \leqslant \eta_6[w_t(0,t)]^2$$
$$+\frac{1}{4\eta_6}\left(\int_0^t\zeta(s)\,ds\right)(\zeta\Diamond w)(t),$$

Due to Young's inequality and Lemma 1.6, we obtain, for all $\eta_7>0$ the following estimate

$$\begin{aligned}\int_{-l}^0 w_x^L(x,t)\int_0^t q(t-s)\left(w_x^L(x,t)-w_x^L(x,s)\right)dsdx &\leqslant \eta_7\int_{-l}^0\left[w_x^L(x,t)\right]^2dx\\&+\frac{\kappa l^2}{\eta_7}\int_{-l}^0\left(q\Box w_{xx}^L\right)(t)\,dx,\end{aligned} \tag{4.39}$$

and

$$\int_0^l w_x^R(x,t)\int_0^t q(t-s)\left(w_x^R(x,t)-w_x^R(x,s)\right)dsdx \leqslant \eta_7\int_0^l \left[w_x^R(x,t)\right]^2 dx + \frac{\kappa l^2}{\eta_7}\int_0^l \left(q\square w_{xx}^L\right)(t)\,dx \tag{4.40}$$

Combining the last estimates in (4.33), the assertion of the Lemma 4.3 is presented. ■

**Lemma 4.4** *The estimate derivative of the functional $\mathcal{F}_3(t)$ is defined as follows, for all $\eta_8, \eta_9 > 0$*

$$\mathcal{F}_3'(t) \leqslant \frac{1}{2}\left(q'\Diamond w\right)(t) + \left(\frac{1}{4\eta_8}+\frac{1}{4\eta_9}\right)\left[w_t(0,t)\right]^2 + \eta_8\kappa\left(q\Diamond w\right)(t) + \eta_9\left[w(0,t)\right]^2, \tag{4.41}$$

*for all $t \geqslant 0$.*

**Proof.** It is clear to see that, for $t \geqslant 0$.

$$\mathcal{F}_3'(t) = \frac{1}{2}\left(q'\Diamond w\right)(t) + w_t(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds + w_t(0,t)\,w(0,t), \tag{4.42}$$

Hence, applying Lemma 1.6, leads to

$$w_t(0,t)\int_0^t q(t-s)\left(w(0,t)-w(0,s)\right)ds \leqslant \frac{1}{4\eta_8}\left[w_t(0,t)\right]^2 + \eta_8\kappa\left(q\Diamond w\right)(t),\ \eta_8 > 0. \tag{4.43}$$

$$w_t(0,t)\,w(0,t) \leqslant \frac{1}{4\eta_9}\left[w_t(0,t)\right]^2 + \eta_9\left[w(0,t)\right]^2,\ \eta_9 > 0, \tag{4.44}$$

consequently, we obtain (4.41). ■

**Lemma 4.5** *For all $t \geq 0$, the functional $\mathcal{F}_4'$ can be estimated as follow*

$$\begin{aligned}\mathcal{F}_4'(t) \ &\leqslant\ -\beta\mathcal{F}_4(t) - \int_0^t \zeta(t-s)\int_{-l}^0 \left[w_{xx}^L(s)\right]^2 dxds - \int_0^t \zeta(t-s)\int_0^l \left[w_{xx}^R(s)\right]^2 dxds \\ &\quad + \overline{\zeta}_\beta\int_{-l}^0 \left[w_{xx}^L(t)\right]^2 dx + \overline{\zeta}_\beta\int_0^l \left[w_{xx}^R(t)\right]^2 dx\end{aligned}$$

*for all $t \geqslant 0$*

**Lemma 4.6** *For the functional $\mathcal{F}_5(t)$, we have*

$$\mathcal{F}_5'(t) \leqslant -\beta\mathcal{F}_5(t) - \int_0^t \zeta(t-s)[w(0,s)]^2 ds + \overline{\zeta}_\beta [w(0,t)]^2,$$

*for all $t \geq 0$.*

## 4.4 Asymptotic behavior

In this section, we state and illustrate the uniform stability of the system (4.1)–(4.3) under a suitable control force $u(t)$ applied on the center body of the spacecraft.

$$u(t) = -k_p w_t(0,t) - k_r w(0,t), \ t \geqslant 0, \tag{4.45}$$

where $k_p$ and $k_r$ are a positive "control gains".

The stabilization result reads as follows.

**Theorem 4.1** *Assume that $(A_1)$, $(A_2)$ and $\int_0^\infty e^{\beta s}\zeta(s)\,ds < \infty$ are satisfied. Under the control force $u(t)$ defined in* (4.45)*, if there exists a positive function $\Phi(t) \in C^1[0,\infty)$ such that*

$$0 \leq \left(Q(t) - \frac{\Phi'(t)}{\Phi(t)}\right), \quad \int_{-l}^{l} h^2(x,t)dx + d^2(0,t) \leq \frac{B}{\Phi(t)}\left(Q(t) - \frac{\Phi'(t)}{\Phi(t)}\right), \quad t \geq 0,$$

*note that $B$ is given in* (4.51)*. Then for some positive constant $C$*

$$\mathcal{E}(t) \leqslant \frac{C}{\Phi(t)}, \ t \geqslant 0,$$

*provided that $\overline{\zeta}_\beta = \int_0^\infty e^{\beta s}\zeta(s)\,ds$ is sufficiently small and $\Phi(0)\mathcal{T}(0) < 1$.*

**Proof.** Differentiation of $\mathcal{T}(t)$, collecting Lemma 4.2 to Lemma 4.6 (see Appendix A for more detailed derivations) and plug (4.45) into (4.9) and using the estimates as follows

$$k_r w(0,t)\, w_t(0,t) \leqslant \frac{k_r^2}{4\eta_{10}} [w_t(0,t)]^2 + \eta_{10}[w(0,t)]^2, \ \eta_{10} > 0,$$

and

$$\int_{-l}^{0} w_t^L(x,t)\, h(x,t)dx \le \varepsilon_3 \int_{-l}^{0} \left[w_t^L(x,t)\right]^2 dx + \frac{1}{4\varepsilon_3}\int_{-l}^{0} h^2(x,t)dx, \quad \varepsilon_3 > 0$$

$$\int_{0}^{l} w_t^R(x,t)\, h(x,t)dx \le \varepsilon_3 \int_{0}^{l} \left[w_t^R(x,t)\right]^2 dx + \frac{1}{4\varepsilon_3}\int_{0}^{l} h^2(x,t)dx, \quad \varepsilon_3 > 0$$

$$d(0,t)\, w_t(0,t) \le \frac{1}{4\varepsilon_3} d^2(0,t) + \varepsilon_3 \left[w_t(0,t)\right]^2, \quad \varepsilon_3 > 0$$

we obtain

$$\begin{aligned}
\mathcal{T}'(t) \leqslant & \frac{EI}{2}\left(\int_{-l}^{0}\left(q'\Box w_{xx}^L\right)(t)\,dx + \int_{0}^{l}\left(q'\Box w_{xx}^R\right)(t)\,dx\right) + 2\mathcal{F}_5'(t) + \frac{1}{2}\left(q'\Diamond w\right)(t) \\
& + \left\{\lambda_1\left[\frac{16\varepsilon_1 l^4}{EI} + \eta_1 - (1-\kappa)\right] + \lambda_2\eta_3\right\} EI\left(\int_{-l}^{0}\left[w_{xx}^L(x,t)\right]^2 dx\right. \\
& \left. + \int_{0}^{l}\left[w_{xx}^R(x,t)\right]^2 dx\right) + \frac{1}{4}\left(\frac{1}{\varepsilon_3} + \frac{\lambda_1}{\varepsilon_1} + \frac{\lambda_2}{\varepsilon_2}\right)\left(\int_{-l}^{l} h^2(x,t)dx + d^2(0,t)\right) \\
& + \left[\lambda_1\frac{EI}{4\eta_1} + \lambda_2\left(16\varepsilon_2 l^4 + \frac{EI}{4\eta_3} + EI + \frac{4\rho A\mu l^4}{\eta_4} + \frac{Tl^2}{\eta_7}\right)\right]\kappa\left(\int_{-l}^{0}\left(q\Box w_{xx}^L\right)(t)\,dx\right. \\
& \left. + \int_{0}^{l}\left(q\Box w_{xx}^R\right)(t)\,dx\right) + 2EI\mathcal{F}_4'(t) + \frac{\lambda_2}{\eta_6}\left(\rho A l + \frac{m}{4}\right)\left(\int_{0}^{t}\zeta(s)\,ds\right)\left(\zeta\Diamond w\right)(t) \\
& + \left\{\varepsilon_3 + \lambda_1\left(m + \frac{1}{4\eta_2}\right) + \lambda_2\Big[\eta_6 m + (k_p + m\mu)\,\eta_4 - m q_\star\Big] + \frac{1}{4\eta_8} + \frac{1}{4\eta_9} + \frac{k_r^2}{4\eta_{10}}\right. \\
& \left. - k_p\right\}\left[w_t(0,t)\right]^2 + \Big[\lambda_1\left(\varepsilon_1(4l+1) + \eta_2 k_p^2 - k_r\right) + \lambda_2\eta_5 + \eta_9 + \eta_{10}\Big]\left[w(0,t)\right]^2 \\
& + \Big[\varepsilon_3 + \lambda_1\rho A + \lambda_2\left(\eta_6 + \eta_4\mu - g_\star\right)\rho A\Big]\left(\int_{-l}^{0}\left[w_t^L(x,t)\right]^2 dx + \int_{0}^{l}\left[w_t^R(x,t)\right]^2 dx\right) \\
& + \left[\lambda_2\left(\frac{(k_p + m\mu)}{4\eta_4} + \frac{k_r^2}{4\eta_5} + \frac{\rho A\mu l}{\eta_4} + (4l+1)\,\varepsilon_2\right) + \eta_8\right]\kappa\left(q\Diamond w\right)(t) \\
& + \lambda_2\frac{4\rho A l^4}{\eta_6}\left(\int_{0}^{t}\zeta(s)\,ds\right)\left(\int_{0}^{l}\left(\zeta\Box w_{xx}^L\right)(t)\,dx + \int_{0}^{l}\left(\zeta\Box w_{xx}^R\right)(t)\,dx\right) \\
& + \left(-\lambda_1 + \eta_7\lambda_2\right)T\left(\int_{-l}^{0}\left[w_x^L(x,t)\right]^2 dx + \int_{0}^{l}\left[w_x^R(x,t)\right]^2 dx\right) \qquad (4.46)
\end{aligned}$$

for all $t \geqslant t_* > 0$.

Now, according to the hypothesis $(A_2)$, we can write

$$\begin{aligned}
&\frac{1}{2}\left(q'\Diamond w\right)(t)+\left[\lambda_2\left(\frac{(k_p+m\mu)}{4\eta_4}+\frac{k_r^2}{4\eta_5}+\frac{\rho A\mu l}{\eta_4}+(4l+1)\,\varepsilon_2\right)+\eta_8\right]\kappa\left(q\Diamond w\right)(t)\\
&+\frac{\lambda_2}{\eta_6}\left(\rho Al+\frac{m}{4}\right)\left(\int_0^t\zeta(s)\,ds\right)\left(\zeta\Diamond u\right)(t) \qquad (4.47)\\
&\leqslant\left\{\left[\lambda_2\left(\frac{(k_p+m\mu)}{4\eta_4}+\frac{k_r^2}{4\eta_5}+\frac{\rho A\mu l}{\eta_4}+(4l+1)\,\varepsilon_2\right)+\eta_8\right]\kappa-\frac{\mu}{2}\right\}\left(q\Diamond w\right)(t)
\end{aligned}$$

$$+2\overline{\zeta}\left[\frac{1}{2}+\frac{\lambda_2}{\eta_6}\left(\rho Al+\frac{m}{4}\right)\overline{\zeta}\right][w(0,t)]^2+2\left[\frac{1}{2}+\frac{\lambda_2}{\eta_6}\left(\rho Al+\frac{m}{4}\right)\overline{\zeta}\right]\int_0^t\zeta(t-s)\,[w(0,s)]^2\,ds,$$

and

$$\begin{aligned}
&\left[\lambda_1\frac{EI}{4\eta_1}+\lambda_2\left(16\varepsilon_2l^4+\frac{EI}{4\eta_3}+EI+\frac{4\rho A\mu l^4}{\eta_4}+\frac{Tl^2}{\eta_7}\right)\right]\kappa\int_{-l}^0\left(q\Box w_{xx}^L\right)(t)\\
&\lambda_2\frac{4\rho Al^4}{\eta_6}\left(\int_0^t\zeta(s)\,ds\right)\int_{-l}^0\left(\zeta\Box w_{xx}^L\right)(t)\,dx+\frac{EI}{2}\int_{-l}^0\left(q'\Box w_{xx}^L\right)(t)\,dx\leqslant\\
&\left\{\left[\lambda_1\frac{EI}{4\eta_1}+\lambda_2\left(16\varepsilon_2l^4+\frac{EI}{4\eta_3}+EI+\frac{4\rho A\mu l^4}{\eta_4}+\frac{Tl^2}{\eta_7}\right)\right]\kappa-\frac{EI}{2}\mu\right\}\int_{-l}^0\left(q\Box w_{xx}^L\right)(t)\,dx\\
&+2\overline{\zeta}\left(\frac{EI}{2}+\lambda_2\frac{4\rho Al^4}{\eta_6}\overline{\zeta}\right)\int_{-l}^0\left[w_{xx}^L(x,t)\right]^2dx\\
&+2\left(\frac{EI}{2}+\lambda_2\frac{4\rho Al^4}{\eta_6}\overline{\zeta}\right)\int_{-l}^0\int_0^t\zeta(t-s)\left[w_{xx}^L(x,s)\right]^2dsdx.
\end{aligned} \qquad (4.48)$$

hence, we get for all $t\geqslant t_*>0$

$$\begin{aligned}
\mathcal{T}'(t)\leqslant\;&\alpha_1\left(\int_{-l}^0\left[w_t^L(x,t)\right]^2dx+\int_0^l\left[w_t^R(x,t)\right]^2dx\right)+\alpha_7\int_0^t\zeta(t-s)\,[w(0,s)]^2\,ds\\
&+\alpha_2EI\left(\int_{-l}^0\left[w_{xx}^L(x,t)\right]^2dx+\int_0^l\left[w_{xx}^R(x,t)\right]^2dx\right)+\alpha_6\,[w(0,t)]^2\\
&+\alpha_3\,[w_t(0,t)]^2+\alpha_4\left(q\Diamond w\right)(t)+\alpha_5\left(\int_{-l}^0\left(q\Box w_{xx}^L\right)(t)\,dx+\int_0^l\left(q\Box w_{xx}^R\right)(t)\,dx\right)\\
&+\alpha_8\left(\int_{-l}^0\int_0^t\zeta(t-s)\left[w_{xx}^L(x,s)\right]^2dsdx+\int_0^l\int_0^t\zeta(t-s)\left[w_{xx}^R(x,s)\right]^2dsdx\right)\\
&-2\beta\mathcal{F}_5(t)-2\beta EI\mathcal{F}_4(t)+\alpha_9T\left(\int_{-l}^0\left[w_x^L(x,t)\right]^2dx+\int_0^l\left[w_x^R(x,t)\right]^2dx\right)\\
&+\frac{1}{4}\left(\frac{1}{\varepsilon_3}+\frac{\lambda_1}{\varepsilon_1}+\frac{\lambda_2}{\varepsilon_2}\right)\left(\int_{-l}^l h^2(x,t)dx+d^2(0,t)\right)
\end{aligned} \qquad (4.49)$$

where

$$\alpha_1 := \varepsilon_3 + \lambda_1 \rho A + \lambda_2 \left(\eta_6 + \eta_4 \mu - q_\star\right) \rho A,$$

$$\alpha_2 := \lambda_1 \left[\frac{16\varepsilon_1 l^4}{EI} + \eta_1 - (1-\kappa)\right] + \lambda_2 \eta_3 + \overline{\zeta}\left(1 + \lambda_2 \frac{8\rho A l^4}{\eta_6 EI}\overline{\zeta}\right) + 2\overline{\zeta}_\beta$$

$$\alpha_3 := \varepsilon_3 + \lambda_1 \left(m + \frac{1}{4\eta_2}\right) + \lambda_2 \left[\eta_6 m + (k_p + m\mu)\,\eta_4 - m q_\star\right] + \frac{1}{4\eta_8} + \frac{1}{4\eta_9} + \frac{k_r^2}{4\eta_{10}} - k_p,$$

$$\alpha_4 := \left[\lambda_2 \left(\frac{(k_p + m\mu)}{4\eta_4} + \frac{k_r^2}{4\eta_5} + \frac{\rho A \mu l}{\eta_4} + (4l+1)\,\varepsilon_2\right) + \eta_8\right]\kappa - \frac{\mu}{2},$$

$$\alpha_5 := \left[\lambda_1 \frac{EI}{4\eta_1} + \lambda_2 \left(16\varepsilon_2 l^4 + \frac{EI}{4\eta_3} + EI + \frac{4\rho A \mu l^4}{\eta_4} + \frac{Tl^2}{\eta_7}\right)\right]\kappa - \frac{EI}{2}\mu$$

$$\alpha_6 := \lambda_1 \left(\varepsilon_1 (4l+1) + \eta_2 k_p^2 - k_r\right) + \lambda_2 \eta_5 + \eta_9 + \eta_{10} + \overline{\zeta}\left[1 + \frac{2\lambda_2}{\eta_6}\left(\rho A l + \frac{m}{4}\right)\overline{\zeta}\right] + 2\overline{\zeta}_\beta,$$

$$\alpha_7 := \frac{\lambda_2}{\eta_6}\left(\rho A l + \frac{m}{2}\right)\overline{\zeta} - 1, \quad \alpha_8 := \lambda_2 \frac{\rho A l^4}{2\eta_6}\overline{\zeta} - EI.$$

and

$$\alpha_9 := -\lambda_1 + \eta_7 \lambda_2$$

Now, we select the remaining constants with attentively thus that all the coefficients in the right hand side of (4.49) will be negative exception the last one. First, we choose $\lambda_1 = \frac{q_\star}{8}\lambda_2$, $\eta_1 = \frac{1-\kappa}{2}$, $\eta_2 = \frac{k_r}{2k_p^2}$, $\eta_3 = \frac{q_\star(1-\kappa)}{32}$, $\eta_4 = \frac{q_\star}{2\mu}$, $\eta_5 = \frac{q_\star k_r}{64}$, $\eta_6 = \frac{q_\star}{4}$,$\eta_7 = q_\star/16$, ,$\eta_8 = \frac{\mu}{4\kappa}$, $\eta_9 = \eta_{10} = \frac{\lambda_1 k_r}{8}$, $\varepsilon_3 = \frac{q_\star \rho A}{16}\lambda_2$. Then, we determine $\varepsilon_1$, $\varepsilon_2$ and$\lambda_2$ sufficiently small so that

$$\lambda_2 \left[\frac{q_\star EI\kappa}{16(1-\kappa)} + \kappa\left(16\varepsilon_2 l^4 + \frac{8EI}{q_\star(1-\kappa)} + EI + \frac{8\rho A \mu^2 l^4}{q_\star} + \frac{16Tl^2}{q_\star}\right)\right] - \frac{EI}{2}\mu < 0,$$

$$\lambda_2 \kappa \left(\frac{(k_p + m\mu)\,\mu}{2q_\star} + \frac{16k_r}{q_\star} + \frac{\rho A \mu^2 l}{2q_\star} + (4l+1)\,\varepsilon_2\right) - \frac{\mu}{4} < 0,$$

$$\lambda_2 \frac{2\rho A l^4}{q_\star}\overline{\zeta} - EI < 0,$$

and

$$\frac{4\lambda_2}{q_\star}\left(\rho A l + \frac{m}{2}\right)\overline{\zeta} - 1 < 0.$$

Once $\lambda_2$ is fixed, we then take $k_r$ large enough and $\overline{\zeta}$ and $\overline{\zeta}_\beta$ small enough (i.e. $\overline{\zeta} < \overline{\zeta}_\beta$

)that

$$-\lambda_2 \frac{q_\star (1-\kappa)}{32} + \lambda_2 \frac{2\varepsilon_1 l^4 q_\star}{EI} + \overline{\zeta}\left(1 + \lambda_2 \frac{32\rho A l^4}{q_\star EI}\overline{\zeta}\right) + 2\overline{\zeta}_\beta < 0,$$

$$-\lambda_2 \frac{q_\star k_r}{64} + \lambda_2 \frac{q_\star \varepsilon_1 (4l+1)}{8} + \overline{\zeta}\left[1 + \frac{8\lambda_2}{q_\star}\left(\rho A l + \frac{m}{2}\right)\overline{h}\right] + 2\overline{\zeta}_\beta < 0,$$

Finally, we select $k_p$ sufficiently large, so that

$$-\frac{q_\star}{8}\lambda_2 m - k_p + \frac{q_\star k_p^2}{16k_r}\lambda_2 + \frac{k_p q_\star}{2\mu}\lambda_2 + \frac{q_\star \rho A}{16}\lambda_2 + \frac{16k_r}{q_\star \lambda_2} + \frac{16}{q_\star \lambda_2 k_r} + \frac{\kappa}{\mu} < 0.$$

thus, we have all the coefficients in the right-hand side of (4.49) negative, and we deduce that

$$\begin{aligned} \mathcal{T}'(t) \leqslant & -C\left(\mathcal{E}(t) + \mathcal{F}_3(t) + \mathcal{F}_4(t) + \mathcal{F}_5(t)\right) \\ & + \frac{1}{4}\left(\frac{1}{\varepsilon_3} + \frac{\lambda_1}{\varepsilon_1} + \frac{\lambda_2}{\varepsilon_2}\right)\left(\int_{-l}^{l} h^2(x,t)dx + d^2(0,t)\right),\ t \geqslant t_\star > 0, \end{aligned} \tag{4.50}$$

where $C > 0$. Furthermore, in virtue of the right-hand side of Prop. 4.2, we get

$$\mathcal{T}'(t) \leqslant \frac{-C}{\delta_2}\mathcal{T}(t) + \frac{1}{4}\left(\frac{1}{\varepsilon_3} + \frac{\lambda_1}{\varepsilon_1} + \frac{\lambda_2}{\varepsilon_2}\right)\left(\int_{-l}^{l} h^2(x,t)dx + d^2(0,t)\right), t \geqslant t_\star > 0. \tag{4.51}$$

Now, using Lemma 3.1 with

$$Q(t) = \frac{C}{\delta_2}, \quad \sigma(t) = 0$$

and

$$\beta(t) = \frac{1}{4}\left(\frac{1}{\varepsilon_3} + \frac{\lambda_1}{\varepsilon_1} + \frac{\lambda_2}{\varepsilon_2}\right)\left(\int_{-l}^{l} h^2(x,t)dx + d^2(0,t)\right),$$

and with the aid of the left-hand side in Prop 4.2, we conclude that

$$\mathcal{E}(t) \leqslant \frac{C}{\Phi(t)},\ t \geqslant 0,$$

provide that $\Phi(0)\mathcal{T}(0) < 1$ for some positive constant $C$ . This completes the proof of Theorem 4.1. ■

# Appendix

In this chapter, we will outline the fundamental procedures for deriving the governing constitutive equations of motion for a flexible satellite system.

## 4.5 Introduction

We consider a flexible satellite system with a rigid hub and long flexible solar panels subject to unknown distributed disturbances and some undesirable vibrations. The left and the right panels are modeled as two viscoelastic Euler-Bernoulli beams as shown in the following image

In Fig 4.1, the functions $\omega^L(x,t)$, $\omega^R(x,t)$: represent the transverse displacements of the left and right panels at the position x for the time t, and $\omega(l/2,t)$ is the transverse displacements of a lumped mass. Also, $f_L$, $f_R$ and $d$ describe the unknown distributed disturbances during attitude maneuvering. We define the classical energy of a flexible

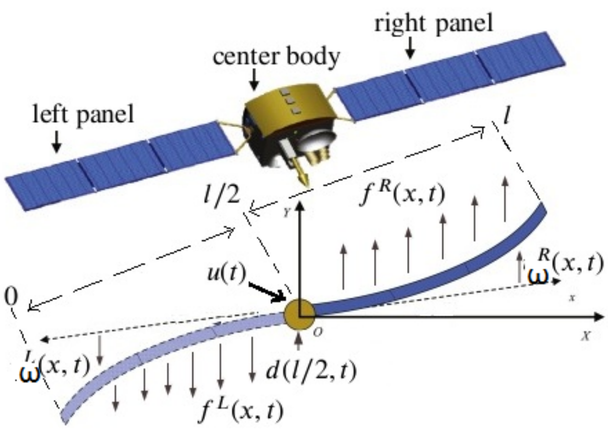


Figure 4.1: Flexible satellite system with distributed disturbance

satellite by

$$E\left(t\right)=E_{k}\left(t\right)+E_{p}\left(t\right),\ t\geqslant 0,$$

where the kinetic energy $E_k(t)$ of the beam can be defined by

$$E_{k}\left(t\right)=\frac{1}{2}\int\int_{V}\int\rho\left(\frac{\partial\omega\left(x,t\right)}{\partial t}\right)^{2}dV=\frac{1}{2}\int_{0}^{l}\rho A\left(\frac{\partial\omega\left(x,t\right)}{\partial t}\right)^{2}dx, \tag{4.52}$$

then from (4.52), we find

$$E_{k}\left(t\right)=\frac{1}{2}\rho A\int_{0}^{l/2}\left(\frac{\partial\omega^{L}\left(x,t\right)}{\partial t}\right)^{2}dx+\frac{1}{2}m\left(\frac{\partial\omega\left(x,t\right)}{\partial t}\right)^{2}\Bigg|_{x=l/2} \tag{4.53}$$

$$+\frac{1}{2}\rho A\int_{l/2}^{l}\left(\frac{\partial\omega^{R}\left(x,t\right)}{\partial t}\right)^{2}dx. \tag{4.54}$$

The potential energy $E_p(t)$ is given by

$$E_{p}\left(t\right)=\frac{1}{2}K\int_{0}^{l}\left(\frac{\partial^{2}\omega\left(x,t\right)}{\partial x^{2}}\right)^{2}dx,$$

where $K=EI$, the independent variables for space and time are denoted by $x$ and $t$, respectively. The source of the potential energy $E_p(t)$ resulting from the bending, then, we can write $E_p(t)$ as

$$E_{p}\left(t\right)=\frac{1}{2}EI\int_{0}^{l/2}\left(\frac{\partial^{2}\omega^{L}\left(x,t\right)}{\partial x^{2}}\right)^{2}dx+\frac{1}{2}EI\int_{l/2}^{l}\left(\frac{\partial^{2}\omega^{R}\left(x,t\right)}{\partial x^{2}}\right)^{2}dx, \tag{4.55}$$

and the total virtual work done by the external forces on the system is described by

$$\delta W\left(t\right)=\delta W_{u}\left(t\right)+\delta W_{d}\left(t\right),$$

where $W_u$ and $W_d$ represent: the virtual work created by control force $u(t)$ and the virtual work created the unknown distributed disturbances during attitude maneuvering, respectively. Therefore we set $\delta W_u(t)=u(t)\,\delta\omega\left(/2,t\right)$ and

$$\delta W_{d}\left(t\right)=d(l/2,t)\delta\omega\left(l/2,t\right)+\int_{0}^{l/2}f^{L}(x,t)\delta\omega^{L}\left(x,t\right)dx+\int_{l/2}^{l}f^{R}(x,t)\delta\omega^{R}\left(x,t\right)dx.$$

## 4.6 Mathematical modeling of a flexible satellite

To derive the governing equations of the dynamic motion of the flexible satellite system, we need to use Lagrangian for the system and applying the generalized Hamilton's principle [74]. The Lagrangian for the system is defined by $L = E_k(t) - E_p(t) + W(t)$ . Considering the associated functional of Hamilton's principle

$$B = \int_{t_1}^{t_2} \delta L dt,$$

where $\delta$ is the variational operator, $t_1$ and $t_2$ represent two time instants and $t_1 < t < t_2$ represent the operating interval.

The minimizing statement considered by Hamilton's principle is

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \delta E_k(t)\, dt - \int_{t_1}^{t_2} \delta E_p(t) + \int_{t_1}^{t_2} \delta W(t) = 0.$$

Then; applying the variational operator and integrating over $(t_0;\ t_1)$ in (4.53), (4.55), we get

$$\int_{t_1}^{t_2} \delta E_k(t)\, dt = \rho A \int_{t_1}^{t_2} \int_0^{l/2} \omega_t^L(x,t) \frac{\partial}{\partial t} \delta\omega^L(x,t)\, dxdt + m \int_{t_1}^{t_2} \omega_t(x,t) \frac{\partial}{\partial t} \delta\omega(x,t) \bigg|_{x=l/2} dt$$
$$+ \rho A \int_{t_1}^{t_2} \int_{l/2}^{l} \omega_t^R(x,t) \frac{\partial}{\partial t} \delta\omega^R(x,t)\, dxdt.$$

Then, using the following conditions

$$\begin{cases} \delta\omega^L(x,t_0) = \delta\omega^R(x,t_1) = 0, \\ \delta\omega^R(x,t_0) = \delta\omega^R(x,t_1) = 0, \\ \delta\omega(l/2,t_0) = \delta\omega(l/2,t_1) = 0, \end{cases} \tag{4.56}$$

we obtain

$$\int_{t_1}^{t_2} \delta E_k(t)\, dt = -\rho A \int_{t_1}^{t_2} \int_0^{l/2} \omega_t^L \omega_{tt}^L(x,t)\, \delta\omega^L(x,t)\, dxdt$$
$$- \rho A \int_{t_1}^{t_2} \int_{l/2}^{l} \omega_{tt}^R(x,t)\, \delta\omega^R(x,t)\, dxdt - m \int_{t_1}^{t_2} \omega_{tt}(l/2,t)\, \delta\omega(l/2,t)\, dt.$$

Moreover

$$\int_{t_1}^{t_2} \delta E_p(t)\, dt = EI \int_{t_1}^{t_2} \int_{0}^{l/2} \omega_{xx}^{L}(x,t) \frac{\partial^2}{\partial x^2} \delta\omega^{L}(x,t)\, dxdt + EI \int_{t_1}^{t_2} \int_{l/2}^{l} \omega_{xx}^{R}(x,t) \frac{\partial^2}{\partial x^2} \delta\omega^{R}(x,t)\, dxdt$$

and if we set

$$\begin{cases} \delta\omega^{L}(l/2,t) = \delta\omega^{R}(l/2,t) = 0, \\ \delta\omega_{xx}^{R}(0,t) = \delta\omega_{xx}^{R}(l,t) = 0, \\ \delta\omega_{xxx}^{R}(0,t) = \delta\omega_{xxx}^{R}(l,t) = 0, \end{cases} \tag{4.57}$$

we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \delta E_p(t)\, dt = {} & EI \int_{t_1}^{t_2} \int_{0}^{/2} \omega_{xxxx}^{L}(x,t)\, \delta\omega^{L}(x,t)\, dxdt \\ & + EI \int_{t_1}^{t_2} \int_{l/2}^{l} \omega_{xxxx}^{R}(x,t)\, \delta\omega^{R}(x,t)\, dxdt \\ & + EI \int_{t_1}^{t_2} \Big[ EI\omega_{xxx}^{L}(l/2,t) - EI\omega_{xxx}^{R}(l/2,t) \Big] \delta\omega(/2,t)\, dt. \end{aligned}$$

Now, applying Hamilton's principle with $\omega^{L}(l/2,t) = \omega^{R}(l/2,t) = \omega(l/2,t)$, to obtain

$$\begin{aligned} & -\int_{t_1}^{t_2} \int_{0}^{l/2} \Big[ \rho A\omega_{tt}^{L}(x,t) + EI\omega_{xxxx}^{L}(x,t) - f^{L}(x,t)(x,t) \Big] \delta\omega^{L}(x,t)\, dsdt \\ & -\int_{t_1}^{t_2} \int_{l/2}^{l} \Big[ \rho A\omega_{tt}^{R}(x,t) + EI\omega_{xxxx}^{R}(x,t) - f^{R}(x,t)(x,t) \Big] \delta\omega^{R}(x,t)\, dxdt \\ & + \int_{t_1}^{t_2} \left[ EI\omega_{xxx}^{L}(l/2,t) - EI\omega_{xxx}^{R}(l/2,t) - m\omega_{tt}^{L}(l/2,t) + u(t) \right] \delta\omega(l/2,t)\, dt = 0. \end{aligned}$$

Finally, since $\omega^{L}(x,t) \neq 0$, $\omega^{R}(x,t) \neq 0$ and $\omega(l/2,t) \neq 0$, we can summarize the dynamic of this system as follows

$$\begin{cases} \rho A\omega_{tt}^{L}(x,t) + EI\omega_{xxxx}^{L}(x,t) = f^{L}(x,t), \quad (x,t) \in [0,l/2] \times [0,\infty), \\ \rho A\omega_{tt}^{R}(x,t) + EI\omega_{xxxx}^{R}(x,t) = f^{R}(x,t), \quad (x,t) \in [l/2,l] \times [0,\infty), \end{cases} \tag{4.58}$$

with the boundary conditions

$$\begin{cases} \omega_x^L(l/2,t) = \omega_x^R(l/2,t) = 0,\ \omega_{xx}^L(0,t) = \omega_{xx}^R(l,t) = 0,\ \omega_{xxx}^L(0,t) = \omega_{xxx}^R(l,t) = 0, \\ \omega^L(l/2,t) = \omega^R(l/2,t) = \omega(l/2,t), \\ m\omega_{tt}(l/2,t) = \quad u(t) + EI\omega_{xxx}^L(l/2,t) - EI\omega_{xxx}^R(l/2,t). \end{cases} \tag{4.59}$$

# Conclusion

The thesis addressed the stability analysis of a flexible satellite system with viscoelastic damping. A precise mathematical model was designed to describe the system behavior, considering the effects of viscoelastic damping and structural flexibility. Using Lyapunov's method, the stability of the flexible system was proved under certain assumptions, considering viscoelastic damping properties.

In general, this research highlighted the significance of employing control and damping techniques to mitigate excessive vibrations in flexible satellite systems while ensuring the stability of system in the presence of structural flexibility and viscoelastic damping. Such study contributes to enhancing the performance of satellite systems and to achieve a practical mission objectives with high efficiency. For instance, research in the field of satellite quality is still ongoing, we mention of them the work [64].

In conclusion, the use of viscoelastic materials is a critical component in reducing and suppressing unwanted vibrations in satellite system. By controlling these vibrations and suggesting a simple control force $u(t)$, the study demonstrates the stability of the problem under weaker damping with or without unknown distributed disturbances, these materials help to ensure the safe and successful operation of these advanced technologies, enabling us to benefit from the many services they provide.

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