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MELIK Ammar

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Under the direction of

Prof. BERBICHE Mohamed , Prof, University of Biskra

Members of the jury :

Dr. LAKHDARI Imad Eddine	MCA.	University of Biskra	**President**
Dr. BERBICHE Mohamed	Prof	University of Biskra	**Supervisor**
Dr. HOUAS Amrane	MCA.	University of Biskra	**Examiner**
Dr. ZARAI Abdderrahmane	Prof.	University of Tebessa	**Examiner**
Dr. AKROUT Kamel	Prof.	University of Tebessa	**Examiner**

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Dedication

I dedicate this thesis

to my dear mother

who watches me constantly with her prayers and recommendations. O God

save my mother and grant her a long and better life.

And to the soul of my dear father, may God have mercy on him.

And to my dear wife

who prays for me in every prostration and every moment of her life

may God protect her for me.

And to my dear brothers and sisters.

And for all my family. And to my dear friends.

Ammar MELIK

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Abstract

In this thesis, we consider the cauchy problem for weakly coupled systems of fractional semilinear Volterra integro differential equations of pseudo-parabolic type with a memory term in multi-dimensional space $\mathbb{R}^n$ $(n \geq 1)$, under small initial data and the conditions on the convolution kernel k which are weaker than the classical differential inequalities, we establish new results for exponential decay of solutions for single equation of the systems in the Fourier space, and we prove the global existence and uniqueness of solutions for weakly coupled systems where data are supposed to belong to different classes of regularity by introducing a set of time-weighted Sobolev spaces and applying the contracting mapping theorem.

Keywords

parabolic equation, viscoelasticity, critical Fujita exponent, global existence, energy estimate, decay estimates, exponential stability.

Résumé

Dans cette thèse, nous considérons le problème de cauchy pour les systèmes faiblement couplés d'équations intégro-différentielles fractionnaires semi-linéaires de type Volterra pseudoparabolique avec un terme mémoire dans l'espace multidimensionnel $\mathbb{R}^n$ $(n \geq 1)$, avec petites données initiales et les conditions sur le noyau de convolution k qui sont plus faibles que les inégalités différentielles classiques, nous établissons de nouveaux résultats pour la décroissance exponentielle des solutions pour une seule équation des systèmes dans l'espace de Fourier, et nous prouvons l'existence globale et l'unicité des solutions pour systèmes faiblement couplés où les données sont supposées appartenir à différentes classes de régularité en introduisant un ensemble d'espaces de Sobolev pondérés dans le temps et en appliquant le théorème d'application de contraction.

Mots clés équation parabolique, viscoélasticité, exposant critique de Fujita, existence globale, estimation d'énergie, estimations de décroissance, stabilité exponentielle.

ملخص
في هذه الأطروحة ، نأخذ في الاعتبار مسألة كوشي للأنظمة ضعيفة الاقتران من معادلات فولتيراالتفاضليةالمتكاملة نصف الخطية من النوع شبه المكافئ مع الحد المسمى بـالذاكرة في الفضاء متعدد الأبعاد $\mathbb{R}^n$ $(n \geq 1)$ في ظل شروط إبتدائية صغيرة و شروط خاصة على النواة g والتي هي أضعف من المتباينات التفاضلية التقليدية ، فإننا بذلك نؤسس لنتائج جديدة للانحلال الأسي لحلول معادلة واحدة للأنظمة في فضاء فورييه ، ونثبت وحدانية والوجود الشامل للحلول للأنظمة ضعيفة الاقتران حيث من المفترض أن تكون الشروط الإبتدائية في مراتب مختلفة من الانتظام وذلك من خلال تقديم مجموعة من فضاءات سوبولاف المتعلقة بالزمن وتطبيق نظرية النقطة الثابتة .

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Symbols and Abbreviations

Sets:

$\mathbb{R}^n$ the real Euclidean space of dimension $n \geq 1$.

Functions and functions spaces:

$F[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix.\xi} f(x) dx$ The Fourier transform of f.

F^{-1} Inverse of Fourier transform.

$\mathcal{L}[f](\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t) dt$ The laplace transform of f.

$\mathcal{L}^{-1}$ Inverse of laplace transform.

$C([0,T], X)$	The space of continuous functions on $[0,T]$ to values in X.
$L^p(\mathbb{R}^n)$	The space of measurable functions u on $\mathbb{R}^n$ such that $\lvert u\rvert^p$ is integrable.
$L^p([0,T], X)$	The space of measurable functions u on $[0,T]$ to values in X such that $\lvert\lvert u\rvert\rvert_X^p$ is integrable $(1 \leq p < \infty)$.
$L^\infty(\mathbb{R}^n)$	The space of measurable functions u on $\mathbb{R}^n$ such that there exists k such that $\lvert u(x)\rvert \leq k$ for almost every $x \in \mathbb{R}^n$.
$W^{m,p}(\mathbb{R}^n)$	The usual Sobolev space.
$H^m(\mathbb{R}^n)$	$W^{m,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n), D^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha \in N^n$ such that $\lvert\alpha\rvert \leq m\}$.

Norms:

$||u||_p := \left(\int_{\mathbb{R}^n} |u|^p\right)^{1/p}$ for $u \in L^p(\mathbb{R}^n)$.

$||u||_\infty := \inf\{k > 0, |u(x)| < k$ almost every where$\}$, for $u \in L^\infty(\mathbb{R}^n)$.

$||u||_{H^s} = \left[\int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}|^2 d\xi\right]^{1/2}$ where ξ is the Fourier transform variable of $\hat{u}$.

$$||u||_{H^m} \quad := \quad \left(\sum_{\alpha \leq m} (||D^\alpha u||_{L^P} \;)^2 \right)^{\frac{1}{2}} \text{ for } u \in H^m\left(\mathbb{R}^n\right). \text{ such that}$$

$$D^\alpha := \frac{\partial^\alpha}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_N} x_n}, \alpha = \left(\alpha_1, ..., \alpha_n\right), |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Mathematical operators:

$*$	The convolution product.
$\|.\|$	Absolute value.
Δ	The classical Laplace operator: $\Delta u(t,x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(t,x)$.
$a \lesssim b$	$i.e$ $a \leq Cb$.such that C is a positive constant.

Finally, we denote every positive constant by the same symbol C or c without confusion.

General Introduction

The theory of parabolic integro-differential equations of Volterra type (PVIDE's) is an exciting branch of applied mathematics. It is a mixture of parabolic differential eqs and integral equations and is still one of the actively developing branch of the theory of differential equations. We cite a few monographs which are the classical source of fundamental facts and approaches in this field [111, 37, 71, 103, 18]. Over the last 50 years or so the theory of PVIDE's has been revealed as a very powerful and important tool in the study of linear or non linear phenomena. This kind of eqs have a great importance in mathematical modelling in which many physical phenomenon and engineering problems are governed by different types of PVIDE's such as viscoelasticity, thermodynamics of phase transition, image processing, control theory, theory of heat conduction with memory, compression of viscoelastic media and in the theory of nuclear dynamics, atomic energy, biology, bi-dimensional gravity chemistry, differential geometry, economy, engineering techniques, fluid mechanics, information theory, Jacobi fields, medicine, population dynamics and many others, (see e,g [114, 145, 146]).

In particular, there has been a great deal of interest on purely time dependent systems with memory delay, and on reaction-diffusion systems containing terms which involve time memory delays. Many authors have proved results on global convergence in some rather general settings (e.g. Pozio [121] 1983; Yamada [145] 1984). In most of these works the nonlinear term forcing is written as $f\left(u(t,x), \int_0^t g(t,x,s,u(s,x)ds\right), t > 0, x \in \Omega$. A particular case widely encountered in various thermal problems, in which the "hereditary" term takes the form of a convolution with a kernel, are the equations of the type

$$\partial_t u - \beta \triangle u - \int_0^t k\,(t-s)\,\triangle u\,(s)\,ds = g(u) + h(t,x), \text{ in } \Omega \times \mathbb{R}^+, \tag{1}$$

supplemented with Dirichlet boundary conditions

$$u(x,t) = 0, \qquad \text{on } \partial\Omega \times \mathbb{R}^+$$

and by the initial condition

$$u(x,0) = u_0\,(x) \text{ in } \Omega,$$

where Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary occupied by a rigid heat conductor $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the temperature variation field relative to the equilibrium reference value, $k : \mathbb{R}^+ \to \mathbb{R}$ is the heat flux memory kernel, the constant $\beta > 0$ denotes the instantaneous conductivity. A wide range of physical, chemical, and biological phenomena in porous media are modeled by these equations that arise from various studies such as the anomalous diffusion/transport in the heterogeneous media, the thermal conduction in materials with memory and so on. A prototype of (1) can be derived from the following constitutive equations due to Coleman-Gurtin thermal law's

$$\begin{aligned} e(x,t) &= e_0 + c_0 T(x,t) \\ q(x,t) &= -k_0 \nabla T(x,t) - \textstyle\int_{-\infty}^t k\,(t-s)\,\nabla T(x,s)ds \end{aligned} \tag{2}$$

where $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is the *temperature variation field* relative to the equilibrium reference value, $k : \mathbb{R}_+ \to \mathbb{R}$ is the heat flux memory kernel, whose properties will be specified later, and the constants e_0, c_0 and k_0 denote the internal energy at equilibrium, the specific heat and the instantaneous conductivity, respectively.

Eqs (2) together with the energy balance law of heat conduction

$$e_t + \nabla . q = r \tag{3}$$

and with assumption that a nonlinear temperature dependent heat source r is involved, namely,

$$r(x,t) = h(t,x) - g\left(T(t,x)\right)$$

that, after substitution in (3), leads to (1).

There are considerable amount of works devoted to the mathematical analysis of model (1) with homogeneous Dirichlet boundary condition for u. Existence, uniqueness and stability of the linear problem case corresponding to (1) (i.e., with $g \equiv 0, h \equiv 0$), we refer to (e.g., Grabmüller [67], Miller [104], Nunziato [110], Slemrod [130]). More recently, Gentili & Giorgi [64] revised the subject on the basis of thermodynamical arguments, and Colli, Grasselli and coworkers [30] extended such results to include phase transition phenomena.

Existence of strong solution of the nonlinear problem has been proved by Barbu & Malik [10], Crandall, Londen & Nohel [38], and Londen & Nohel [97], assuming nonlinear terms of the form of maximal monotone (possibly multivalued) operators. In particular, in [10] it is also proved the uniqueness of the solution.

It is well known that the solution of (1) in general does not exist for all times, due to the nonlinear interaction in the model and may be blows up in a finite time, see e.g. Messaoudi [101],[102], Tian [129] and [57]. In particular the authors studied the blow-up in finite time with positive or negative initial energy of the solutions to the initial boundary value problem corresponding to (1), with $g(u) = |u|^{p-1}u$, $p > 2$. By using the convexity method and some differential inequality techniques, they obtained the finite time blow-up results under certain suitable conditions on g provided that the initial energy satisfies different conditions.

The question of global existence and large time behavior of solutions of integro-partial differential equation arising from the theory of heat conduction with memory has inspired a wide research, it consists in determining the asymptotic behavior of the energy $E(t)$.

The principal object is to study its limit when tends to $+\infty$ also to determine whether this limit is zero, and to give in a unified way the decay rates of the energy if this limit is zero.

Decay properties of the semigroup generated by a linear parabolic integro-differential equation with memory functions in a Hilbert space arising from heat conduction with memory has been studied by [22],[23],[28],[100].

Stability and boundedness results of the solutions of the homogeneous part of abstract forms of (1) in Banach space have been investigated widely; see, for instance ([20, 22, 23, 40, 41, 20, 122, 144]) by means of representing the solution in terms of the resolvent operator.

The existence of global and exponential attractors with some of the previously enumerated properties or another type of volterra integro-differential equations in a bounded domain have been made (see [28],[27],[45],[61],[54],[66][70],[100]) and the reference therein), where the authors adopt the theory of infinite-dimensional dynamical systems, they followed an approach based on an idea introduced by Dafermos in his pioneering paper [43], and then developed by several of authors in the context of dynamical systems. This idea consists of introducing an additional variable, usually called the past history accounting for all the past values of the unknown function, which is given by the following

$$\eta^t(x,s) = \int_{t-s}^{t} u(x,\tau)\, d\tau, s \geq 0.$$

Then we can reformulate the original boundary and initial value problem as the following integro-partial differential system in the variables (u, η^t)

$$\begin{cases} \partial_t u - \beta \triangle u - \int_0^\infty k(s) \triangle \eta^t(x,s)\, ds = g(u) + h(t,x), & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_t \eta^t(x,s) = -\partial_s \eta^t(x,s) + u & \text{in } \Omega \times \mathbb{R}^+ \end{cases} \tag{4}$$

with the initial and boundary values Problem (4) has been studied by many authors, both for the sake of well-posedness and stability issues, and it is well known (see, e.g., [26, 55, 70]) that, for every $u_0 \in H$, it possesses a unique weak solution $u \in C([0,\infty), H)$, with H is a real Hilbert space $(H, \langle ., . \rangle, ||.||)$.

Energy decay results have already been achieved in linear viscoelasticity with memory, both for the infinite delay [116] and for the Volterra [33] equation, under very general assumptions on the memory kernel.

The decay of the solutions to (4) has attracted the attention of several authors in the past years. Exponential stability has been subsequently studied in many other works (see e.g. Refs. [7, 28, 56, 65, 105, 115]), and it is now well known that $E(t)$ is exponentially stable if the kernel μ satisfies the differential inequality. In [66] it is shown that, whenever $k(s)$ is smooth and $\mu(s) = -k'(s)$ fulfills the differential inequality

$$\mu'(s) + \delta\mu(s) \le 0, s > 0, \tag{5}$$

for some positive δ, then the semigroup is exponentially stable. It is immediate to see that (5) is equivalent to

$$\mu(t+s) \le e^{-\delta t}\mu(s), s > 0, t > 0. \tag{6}$$

On the other hand, (6) is weakened in [26], where the assumption (first introduced in [116])

$$\mu(t+s) \le Ce^{-\delta t}\mu(s), t > 0, \tag{7}$$

for some $C \ge 1$, $\delta > 0$ and for a.e. $s > 0$, is shown to be necessary and sufficient. Since

(7) implies the existence of $\gamma > 0$ such that

$$k'(s) + \gamma k(s) \leq 0, \tag{8}$$

for a.e. $s > 0$, we have a necessary condition for the exponential decay in terms of the kernel k.

The dissipativity of (4) is witnessed by the fact that $E(t)$ is a decreasing function. The loss of energy is due to the presence of dissipation given by the memory term $k * \triangle u$. This dissipative mechanism could be reflected from the decay of solutions. For more studies on various aspects of dissipation, we refer to [15, 17, 107]. Also, as for the study of decay properties memory-type dissipation, we refer to [49, 53, 91, 105].

In particular E. Mainini and G. Mola [100], studied the problem (1) with $g \equiv 0, h \equiv 0$, they extended the solution u to negative times by zero, and then introduced the auxiliary *past history variable* η^t in which the problem reduced to the system with the history framework (1). The authors proved exponential (resp. polynomial) decay results of the associated semi-group for a broader class of kernels satisfying

$$\int_0^\infty e^{\delta s} k(s) ds < \infty, \left(\text{resp. } \int_0^\infty (1+s)^r k(s) ds < \infty, r > 0\right). \tag{9}$$

for some $\delta > 0$. In particular, assumptions (5) and (6) recovered by condition in (9) and there is a gap between them. A natural extension of model for heat flow in materials with memory, is that when adding the term $-\beta_0 \triangle \partial_t u$ to the Eq.(1) for some positive constant β_0, we obtain the so called nonclassical-diffusion equations with memory

$$\begin{cases} \partial_t u - \beta \triangle u - \beta_0 \triangle \partial_t u - \int_0^t k(s) \triangle u(x, t-s) ds = g(u) + h(t, x), \text{ in } \Omega \times \mathbb{R}^+, \\ u(0) = u_0 \\ u(x)|_{\partial\Omega} = 0. \end{cases} \tag{10}$$

which contain viscosity, elasticity and pressure, that describes physical phenomena, such as non-Newtonian flows, soil mechanics, and heat conduction theory (see, e.g.[138]), when supposing that the diffusing species persist in a linear viscous fluid form, which leads to their inclusion velocity gradient in the constitutive laws [119, 141], and this is specific to certain classes of which we mention such as polymers and highly viscous liquids. In addition, the convolution term takes into account the effect of the past history of u on its future evolution, providing a more precise description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers (see, e.g., [81]). From the mathematical point of view, the study of qualitative properties of evolution equations with memory damping term are also important as such systems occur in various problems of applied science, and have attracted some of attention of many mathematicians. Many authors have broadly studied the long-time behaviour of solutions to nonclassical diffusion equations for both autonomous and non-autonomous cases [143, 142, 127, 4, 3, 134, 135]. Anh et al in [6] studied the existence and long-time behaviour in terms of existence of global attractors of weak solutions to a class of the equation and the existence of solutions and long-time behaviour of solutions to non-classical diffusion equations with memory and their behaviour for a long time in the case of time-independent memory kernel [5, 34]. The comprehensive time-dependent global attractor for the nonclassical diffusion equations was investigated in [95, 99]. In Sun et al [136], the authors proved the results of global existence and finite time blow-up for the solutions and obtained the upper bound for the blow-up time of the semilinear problem (10) with $g(u) = |u|^{p-2} u$.

The limiting case, when k is equal to the Dirac mass at 0^+ Eq. (10) reduces to the so-called pseudo parabolic equation

$$\partial_t u - \beta \triangle u - \beta_0 \triangle \partial_t u = g(u) + h(t, x), \text{ in } \Omega \times \mathbb{R}^+,$$

which describes a variety of important physical processes, such as the seepage of homogeneous fluids through a fissured rock [11], the heat conduction involving two temperatures [21], the unidirectional propagation of nonlinear, dispersive, long waves [12], two-phase porous media flow models with dynamic capillarity or hysteresis [76], phase field-type models for unsaturated porous media flows [39] and the aggregation of populations [113].

Pseudo-parabolic equations can also viewed as a Sobolev-type equation or a Sobolev-Galpern-type equation; one can see the papers Aifantis [1]) and [98] or the book [2].

We are briefly going to mention some of them that motivate our thesis. The authors in [128, 139] investigated the initial-boundary value problem and the Cauchy problem for the linear pseudo-parabolic equation and established the existence and uniqueness of solutions. After those precursory results, there are many papers (see, for example, [82, 83, 89, 123]) studied the nonlinear pseudo-parabolic equations, like semilinear pseudo-parabolic equations, quasilinear pseudo-parabolic equations, and even singular and degenerate pseudo-parabolic equations.

Chapter 1

Preliminary Concepts

In this chapter, we introduce some propositions and important lemmas of functional analysis which will be needed in this study of semilinear evolution equations.

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1.1 *Let Ω be an open, connected set (domain) in $\mathbb{R}^n$ and $1 \leq p$ be a real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω for which $\int_\Omega |u(x)|^p dx < \infty$. $L^p(\Omega)$ is a Banach space with the norm*

$$\|u\|_{L^p} := \int_\Omega |u(x)|^p dx < \infty.$$

In particular, for $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) := \int_\Omega u(x)v(x)dx,$$

and the norm

$$\|u\| := \int_\Omega |u(x)|^2 dx,$$

for $u, v \in L^2(\Omega)$.

For $p = \infty, L^\infty(\Omega)$ *is a Banach space with the norm*

$$\|u\|_{L^\infty} = ess \sup_{x\in\Omega} |u(x)|.$$

Theorem 1.1 (Young's inequality) *Let* $p, q, r \in [1, \infty]$, *such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

For $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, *we have* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

1.1.2 Spaces of test functions and distributions

Definition 1.2 *Let* $\Omega \subset \mathbb{R}^n$ *be an open, connected set and let* $\Phi : \Omega \to \mathbb{R}$ *be a function. Support of* Φ *can be defined as*

$$supp(\Phi) = \overline{\{x \in \Omega : \Phi(x) \neq 0\}}.$$

Definition 1.3 *Let* $\Omega \subset \mathbb{R}^n$ *be an open, connected set. A function* $u : \Omega \to \mathbb{R}$ *is said to be locally integrable if for every compact set* $K \subset \Omega$,

$$\int_K |u(x)|dx < \infty.$$

We denote by $L^1_{loc}(\Omega)$ *the space of locally integrable functions defined on* Ω.

Definition 1.4 *Let* $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$ *is the space of infinitely differentiable functions with* *compact support. This space is also known as the space of all test functions defined on* Ω*.*

Definition 1.5 *Let* $x \in \mathbb{R}^n$ *with coordinates* $x = (x_1, \ldots, x_n)$*. A multi-index is an* n $-$ *tuple,* $\alpha = (\alpha_1, \ldots, \alpha_n)\,(\alpha_i \in \mathbb{N})$*. If we set*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

then

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}},$$

represents the α*-th order partial differentiation operator.*

Definition 1.6 *Let* $\Omega \subset \mathbb{R}^n$*, connected set. Suppose* $u, v \in L^1_{loc}(\Omega)$*, and* α *is a multi-index. We say that* v *is the* α^{th}*-weak partial derivative of* u*, written*

$$D^{\alpha} u = v,$$

provided

$$\int_{\Omega} u D^{\alpha} \phi dx = (-1)^{|\alpha|} \int_{\Omega} v \phi dx,$$

for all test function $\phi \in \mathcal{C}_0^{\infty}(\Omega)$*.*

Definition 1.7 *A sequence of functions* ϕ_m *in* $\mathcal{C}_0^{\infty}(\Omega)$ *is said to converge to* 0 *if there exists a fixed compact set* $K \subset \Omega$ *such that* $supp(\phi_m) \subset K$ *for all* m *and* ϕ_m *and all its derivatives converge uniformly to zero on* K*.*

Definition 1.8 *A linear functional* T *from* $\mathcal{C}_0^{\infty}(\Omega)$ *to* $\mathbb{R}$ *is said to be a distribution, or generalized function if whenever* $\phi_m \to 0$ *in* $\mathcal{C}_0^{\infty}(\Omega)$*, we have* $T(\phi_m) \to 0$*.*

Definition 1.9 *We denote by* $\mathcal{D}'(\Omega)$ *the space of all distributions defined on the space* $\mathcal{D}\,(\Omega)$*.*

Definition 1.10 *The space $C([0,T],X)$ consists of all continuous functions $u:[0;T]\to X$ such that*

$$||u||_{C([0,T],X)} = \max_{0\leq t\leq T} ||u(t,.)||_X < \infty.$$

Proposition 1.1 *The space $\mathcal{D}(\Omega)$ is contained in and is dense in $L^p(\Omega), 1\leqslant p<+\infty$ and every convergent sequence in $\mathcal{D}$ converges in L^p.*

Definition 1.11 *If $T\in\mathcal{D}'(\Omega)$ the derivative $D_iT, i=1,\ldots,n,$ of T is the distribution on Ω defined by:*

$$\langle D_iT,\varphi\rangle = -\langle T, D_i\varphi\rangle, \quad i=1,2,\ldots,n.$$

If $\alpha\in\mathbb{N}^n$

$$\langle D^\alpha T,\varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle \ \textit{for all } \varphi\in\mathcal{D}(\Omega).$$

Proposition 1.1 *Let $T\in\mathcal{D}'(\mathbb{R}^n)$ and let φ and ψ belong to $\mathcal{D}(\mathbb{R}^n)$.*

(a) $T*\varphi\in\mathcal{C}^{\infty}(\mathbb{R}^n)$ and $D^\alpha(T*\varphi)=(D^\alpha T)*\varphi=T*D^\alpha\varphi$.

(b) $supp(T*\varphi)\subset suppT+supp\varphi$.

(c) $T*(\varphi*\psi)=(T*\varphi)*\psi=(T*\psi)*\varphi$.

1.2 Sobolev spaces

Definition 1.12 *Let k be a non-negative integer and let $1\leq p\leq\infty$. Then we define $W^{k,p}(\Omega)$ to be set of all distributions $u\in L^p(\Omega)$ such that $D^\alpha u\in L^p(\Omega)$ for $|\alpha|\leq k$. In $W^{k,p}(\Omega)$, we define a norm by*

$$||u||_{W^{k,p}} = \left(\sum_{|\alpha|\leq k} ||u||^p_{L^P}\right)^{1/p} \qquad if \quad 1\leq p\leq\infty,$$

and

$$||u||_{W^{k,\infty}} = \max_{0\leq|\alpha|\leq k} ||D^\alpha u||_{L^\infty}, \qquad if \qquad p=\infty.$$

For $p = 2$ we define an inner product by

$$(u, v)_k := \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

We also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $L^2(\Omega)$ for $W^{0,2}(\Omega)$.

Definition 1.13 *By $W_0^{k,p}(\Omega)$ we denote the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. This means that $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in \mathcal{C}_0^\infty(\Omega)$ such that $u_m \to u \in W_0^{k,p}(\Omega)$.*

1.3 Fourier Transform

Definition 1.14 *(Fourier Transform of L^1-Functions)*

For $u \in L^1$, we may define the Fourier transform of u by the formula:

$$\hat{u}(y) = \mathcal{F}u(y) = \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x) \mathrm{d}x, \quad y \in \mathbb{R}^n. \tag{1.1}$$

The mapping $u \to \hat{u}$ defined by (1.1) is obviously linear. From the inequality:

$$|\hat{u}(y)| \leqslant \int_{\mathbb{R}^n} |u(x)| \mathrm{d}x \text{ for all } y \in \mathbb{R}^n,$$

we deduce: if $u \in L^1, \hat{u}$ is a bounded continuous function on $\mathbb{R}^n$ with $\|\hat{u}\|_{L^\infty} \leqslant \|u\|_{L^1}$.

Remark 1.1 *We can also define the Fourier transform by the formula*

$$\mathcal{F}u(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x) \mathrm{d}x,$$

(then we have $\mathcal{F}^{-1} = \overline{\mathcal{F}}$) or again by

$$\mathcal{F}u(y) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot y} u(x) \mathrm{d}x.$$

Definition 1.15 *We denote by: $\tau_a f$ the translate of the function f of amplitude $a \in \mathbb{R}^n$ defined by $\tau_a f(x) = f(x-a)$ for all $x \in \mathbb{R}^n, a \in \mathbb{R}^n$,*
and we denote by $\check{f}$ the symmetries of the function f defined by:
$\check{f}(x) = f(-x)$ for all $x \in \mathbb{R}^n$.

Again for $x \in \mathbb{R}^n, \alpha \in \mathbb{N}$, we put $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$; we have:

Proposition 1.2 *For $u, v \in L^1$, we have*

(i) $\overline{\hat{u}} = \overline{\mathcal{F}(u)} = \overline{F}(\bar{u}); \quad \overline{\mathcal{F}}(\check{u}) = \mathcal{F}(u) = \hat{u},$

(ii) $\int_{\mathbb{R}^n} \hat{u}(x) v(x) \mathrm{d}x = \int_{\mathbb{R}^n} u(y) \hat{v}(y) \mathrm{d}y,$

(iii) $\begin{cases} \widehat{\tau_a u}(y) = e^{-ia.y} \hat{u}(y), a \in \mathbb{R}^n, y \in \mathbb{R}^n, \\ \mathcal{F}\left(e^{-ia \cdot x} \cdot u\right) = \tau_{-a} \hat{u}. \end{cases}$

Proposition 1.2 *(a) If $x^\alpha u \in L^1, |\alpha| \leqslant k$, then $\hat{u} \in \mathcal{C}^k$ and we have $D^\alpha(\hat{u}) = (-i)^{|\alpha|} \widehat{x^\alpha u}$.*
(b) If $u \in \mathcal{C}^k$ with $D^\beta u \in L^1$ for all $\beta \in \mathbb{N}^n, |\beta| \leqslant k$, then $y^\beta \hat{u} \in L^\infty$ and

$$i^{|\beta|} y^\beta \hat{u} = \widehat{D^\beta u}.$$

Proposition 1.3 *The Fourier transform of a function of L^2 will thus be a function of L^2 satisfying :*

$$\begin{cases} (u, v) = \frac{1}{(2\pi)^n} (\hat{u}, \hat{v}) \quad (\textit{ Parseval's formula}) , \\ \|u\| = \frac{1}{(2\pi)^{n/2}} \|\hat{u}\| \qquad (\textit{Plancherel's formula}). \end{cases}$$

Remark 1.2 *The integral $\int_{\mathbb{R}^n} u(x) e^{-ix \cdot y} \, \mathrm{d}x$ does not have a meaning in so far as the Lebesgue integral $\int_{\mathbb{R}^n} |u(x)| \mathrm{d}x$ diverges for $u \in L^2$ with $u \notin L^1$.*

1.3.1 The Space $\mathcal{S}(\mathbb{R}^n)$

Definition 1.16 *We put*

$$\mathcal{S}(\mathbb{R}^n) = \mathcal{S} = \left\{u \in \mathcal{C}^\infty(\mathbb{R}^n); \forall \alpha, \beta \in \mathbb{N}^n, x^\alpha D^\beta u \to 0 \text{ as } |x| \to +\infty\right\};$$

$\mathcal{S}$ is the space of functions of class $\mathcal{C}^\infty$ of rapid decay at infinity, which is not a normed space, but of which the topology can be defined by the (denumerable) sequence of semi-norms:

$$u \to \sup_{x \in \mathbb{R}^n} \left|x^\alpha D^\beta u(x)\right| = d_{\alpha,\beta}(u),$$

and we have:

$$\begin{cases} \widehat{D^\alpha u} = (iy)^\alpha \hat{u}, \\ D^\beta \hat{u} = \widehat{(-ix)^\beta u} \text{ for all } u \in \mathcal{S} \text{ and for all } \alpha, \beta \in \mathbb{N}^n, \end{cases}$$

and

$$\begin{cases} (u, v) = \frac{1}{(2\pi)^n}(\hat{u}, \hat{v}) \quad (\text{ Parseval's formula}), \\ \|u\| = \frac{1}{(2\pi)^{n/2}}\|\hat{u}\| \qquad (\text{Plancherel's formula}). \end{cases}$$

1.3.2 Fourier transform of tempered distributions

Definition 1.17 *We say that $T : \mathcal{S} \to \mathbb{C}$ defines a tempered distribution if*

(1) T is linear

(2) T is continuous, i.e. if $\varphi_j \to 0$ as $j \to \infty$, then the numerical sequence $\langle T, \varphi_j \rangle \to 0$ as $j \to \infty$.

(The bracket $\langle , \rangle$, denotes here the duality between $\mathcal{S}'$ and $\mathcal{S}$.)

Definition 1.18 *The space $\mathcal{S}'$ will be furnished with the following notion of convergence: a sequence $\{T_p\}_{p \in \mathbb{N}}$ of tempered distributions tends to T in $\mathcal{S}'$ if, for all $\varphi \in \mathcal{S}, \langle T_p, \varphi \rangle \to \langle T, \varphi \rangle$ in $\mathbb{C}$ when $p \to +\infty$.*

Definition 1.19 *(The Fourier transform for tempered distributions)*

for each $T \in \mathcal{S}'$, the Fourier transform of T, denoted by $\hat{T}$ or $\mathcal{F}(T)$ will be the tempered distribution defined by:

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \text{ for all } \varphi \in \mathcal{S}.$$

Proposition 1.3 *(a) The Fourier transform F and $\bar{F}$ are two isomorphisms of $\mathcal{S}'$ onto $\mathcal{S}'$.*

(b) For all $a \in \mathbb{R}^n$, we have:

$$\begin{cases} \mathcal{F}(\tau_a T) = e^{-ia.y} \cdot \mathcal{F}(T), \\ \mathcal{F}(e^{-ia.x} \cdot T) = \tau_{-a} \mathcal{F}(T). \end{cases}$$

Proposition 1.4 *(c) For all $\alpha, \beta \in \mathbb{N}^n$, we have*

$$\begin{cases} \mathcal{F}(D^\alpha T) = (iy)^\alpha \mathcal{F}(T), \\ D^\beta \mathcal{F}(T) = \mathcal{F}\left[(-ix)^\beta T\right]. \end{cases}$$

Proposition 1.4 *Let m be a positive integer or zero, then:*

(i) $H^m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, (the space of tempered distributions).

(ii) $H^m(\mathbb{R}^n)$ coincides with the space of tempered distributions u such that

$$\left(1 + |\xi|^2\right)^{m/2} \hat{u} \in L^2(\mathbb{R}^n),$$

(iii) The norm $\|u\|_{L^m(\mathbb{R}^n)}$ is equivalent to

$$||u||_{L^m(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^m |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

Definition 1.20 *For $s \in \mathbb{R}, H^s(\mathbb{R}^n)$ is the space of tempered distributions u, such that*

$$\left(1 + |\xi|^2\right)^{s/2} \cdot \hat{u} \in L^2(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

We furnish $H^s(\mathbb{R}^n)$ with the scalar product

$$(u,v)_s = \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \hat{u}(\xi)\cdot\overline{\hat{v}(\xi)}\mathrm{d}\xi,$$

and the associated norm is

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2}.$$

Remark 1.3 *For $s = m \in \mathbb{N}$, the space $H^s(\mathbb{R}^n)$ coincides with the space $H^m(\mathbb{R}^n)$.*

Proposition 1.5 *For each real number s, the space $H^s(\mathbb{R}^n)$ satisfies the following properties:*

(i) $H^s(\mathbb{R}^n)$ is a Hilbert space.

(ii) If $s_1 \geqslant s_2$ then $H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n)$ and the injection is continuous.

Proposition 1.6 *For all $s_1 \geqslant s_2 \geqslant 0$, we have*

$$\begin{aligned}\mathcal{S}(\mathbb{R}^n) \subset H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \subset \cdots \subset H^0(\mathbb{R}^n)&\\ = L^2(\mathbb{R}^n) \subset H^{-s_2}(\mathbb{R}^n) \subset H^{-s_1}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)&.\end{aligned}$$

Lemma 1.1 *For all $s \in \mathbb{R}$, the dual of $H^s(\mathbb{R}^n)$ coincides in $\mathcal{D}'(\mathbb{R}^n)$ (algebraically and topologically) with $H^{-s}(\mathbb{R}^n)$.*

1.4 Some important inequalities

1.4.1 Classical Gagliardo-Nirenberg inequality

Proposition 1.5 *(See [59]) Let $j, m \in \mathbb{N}$ with $j < m$, and let $f \in \mathcal{C}_0^m$. Let $\beta = \beta_{j,m} \in \left[\frac{j}{m}, 1\right]$ with $p, q, r \in [1, \infty]$ satisfy*

$$j - \frac{n}{q} = \left(m - \frac{n}{r}\right)\beta - \frac{n}{p}(1-\beta).$$

Then, we have the following inequality:

$$\left\|D^j f\right\|_{L^q} \lesssim \|f\|_{L^p}^{1-\beta} \|D^m f\|_{L^r}^{\beta},$$

provided that $\left(m - \frac{n}{r}\right) - j \notin \mathbb{N}$. *If* $\left(m - \frac{n}{r}\right) - j \in \mathbb{N}$, *then the classical Gagliardo-Nirenberg inequality holds provided that* $\beta \in \left[\frac{j}{m}, 1\right)$.

The proof of the classical Gagliardo-Nirenberg inequality can be found in [29, 68, 69, 72, 84, 85].

1.4.2 Fractional Gagliardo-Nirenberg inequality

Proposition 1.6 *(See[75]) Let* p , p_0 , $p_1 \in (1, \infty)$ *and* $\kappa \in (0, s)$ *with* $s > 0$. *Then, for all* $f \in L^{p_0} \cap \dot{H}^s_{p_1}$ *the following inequality holds:*

$$\|f\|_{\dot{H}^\kappa_p} \lesssim \|f\|_{L^{p_0}}^{1-\beta} \|f\|_{\dot{H}^s_{p_1}}^{\beta},$$

where $\beta = \beta_{\kappa,s} = \left(\frac{1}{p_0} - \frac{1}{p} + \frac{\kappa}{n}\right) / \left(\frac{1}{p_0} - \frac{1}{p_1} + \frac{s}{n}\right)$ *and* $\beta \in \left[\frac{\kappa}{s}, 1\right]$.

1.4.3 Fractional Leibniz rule

Proposition 1.7 *(See [69]) Let* $s > 0$ *and* $1 \leq r \leq \infty, 1 < p_1, p_2, q_1, q_2 \leq \infty$ *satisfy the relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, for all $f \in \dot{H}^s_{p_1} \cap L^{q_1}$ *and* $g \in \dot{H}^s_{q_2} \cap L^{q_2}$ *the following inequality holds:*

$$\|fg\|_{\dot{H}^s_r} \lesssim \|f\|_{\dot{H}^s_{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|g\|_{\dot{H}^s_{q_2}}.$$

1.4.4 Fractional chain rule

Proposition 1.8 *(See [112]) Let $s > 0, p > \lceil s \rceil$ and $1 < r, r_1, r_2 < \infty$ satisfy the relation*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Then, for all $f \in \dot{H}^s_{r_2} \cap L^{r_1}$ the following inequality holds:

$$\left\| \pm f|f|^{p-1} \right\|_{\dot{H}^s_r} + \| |f|^p \|_{\dot{H}^s_r} \lesssim \|f\|^{p-1}_{L^{r_1}} \|f\|_{\dot{H}^s_{r_2}}.$$

1.4.5 Fractional powers

Proposition 1.9 *Let $r \in (1, \infty), p > 1$ and $s \in (0, p)$. Then, for all $f \in \dot{H}^s_r \cap L^\infty$ the following inequality holds:*

$$\left\| \pm f|f|^{p-1} \right\|_{\dot{H}^s_r} + \| |f|^p \|_{\dot{H}^s_r} \lesssim \|f\|_{\dot{H}^s_r} \|f\|^{p-1}_{L^\infty}.$$

Proposition 1.10 *(See [125]) Let $r \in (1, \infty)$ and $s > 0$. Then, for all $f, g \in \dot{H}^s_r \cap L^\infty$ the following inequality holds:*

$$\|fg\|_{\dot{H}^s_r} \lesssim \|f\|_{\dot{H}^s_r} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{H}^s_r}.$$

Proposition 1.11 *(See [42]) Let $0 < 2s^* < n < 2s$. Then for any function $f \in \dot{H}^{s^*} \cap \dot{H}^s$ one has the estimate*

$$\|f\|_{L^\infty} \leq \|f\|_{\dot{H}^{s^*}} + \|f\|_{\dot{H}^s}.$$

1.4.6 Gronwall's inequality

Lemma 1.1 *If the function α is non-decreasing, β is non-negative, and if f satisfies the integral inequality*

$$f(t) \leq \alpha(t) + \int_0^t \beta(s) f(s)\, ds, \text{ for all } t \in [0, T].$$

Then, $f(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)\, ds\right)$ *for all* $t \in [0, T]$.

1.4.7 Banach Contraction-Mapping Principle

Theorem 1.2 *Let (X, d) be a complete metric space and $G : X \to X$ a map such that there exists $\theta \in [0, 1)$ satisfying $d(G(x), G(y)) \leq \theta d(x, y)$ for all $x, y \in X$. Then, there exists a unique $x_0 \in X$ such that $G(x_0) = x_0$.*

Chapter 2

Global existence and decay estimates for the semilinear heat equation with memory in $\mathbb{R}^n$.

In this chapter we consider the initial value problem of the following semi-linear Volterra integro-differential equations of the first order posed in the whole space $\mathbb{R}^n$

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + g * (-\Delta)^{\theta} u = f(u),\ t>0,\ \ x\in\mathbb{R}^n, \\ u(0,x) = u_0\left(x\right),\ \ x\in\mathbb{R}^n, \end{cases} \tag{2.1}$$

here $u = u(t,x)$ is an unknown real valued function of $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $t > 0$, $u_0\left(x\right)$ is a given initial data and the function f is an external nonlinear force. The fractional Laplace operator $(-\Delta)^{\theta}$ may be defined through its Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by

$$(-\Delta)^{\theta} h(x) = \mathcal{F}^{-1}\left(|\xi|^{2\theta}\, \mathcal{F}\left(h\right)\left(\xi\right)\right)\left(x\right),\ x\in\mathbb{R}^n,$$

or by its representation $(-\Delta)^{\theta} h(x) = C(n,\theta)\displaystyle\int_{\mathbb{R}^n} \frac{h(x)-h(y)}{|x-y|^{n+2\theta}} dy$, with $0<\theta<1$. In the limit $\theta\to 1$ the standard Laplace operator, $-\Delta$, is recovered (see Section 4 of [50]).

The convolution $g * (-\Delta)^{\theta} u := \int_0^t g(t-s)(-\Delta)^{\theta} u(s)ds$ corresponds to the memory term g, that satisfies the following assumptions :

a) The kernel g is a nonnegative summable function having the explicit form

$$g\left(t\right)=\int_{t}^{+\infty}\mu\left(s\right)ds,$$

for some (nonnegative) non-increasing piecewise absolutely continuous function $\mu \in L^1(\mathbb{R}^+)$ of total mass

$$g=\int_0^{\infty}\mu(s)ds<\infty.$$

b) Moreover, we require that

$$g(s)\leq K\mu(s)$$

for some positive constant K and every $s > 0$. As shown in [25], this is equivalent to the requirement that there exist $C \geq 1$ and $\delta > 0$ such that for any $t \geq 0$ and almost every $s > 0$

$$\mu(t+s)\leq Ce^{-\delta t}\mu(s).$$

In particular, the kernel μ is allowed to exhibit (infinitely many) jumps. For example, a typical kernels μ considered in the papers [25],[28],[32],[100] where the authors assumed that the set of jump points of μ is a strictly increasing sequence $\{s_i\}$, with $s_0 = 0$, either finite (possibly reduced to s_0 only) or converging to $s_\infty \in (0, \infty]$ such that, for all $i \geq 1$, μ has jumps at $s = s_i$, and it is absolutely continuous on each interval $I_i = (s_{i-1}, s_i)$ and on the interval $I_\infty = (s_\infty, \infty)$, unless I_∞ is not defined. If $s_\infty < \infty$, then μ may or may not have a jump at $s = s_\infty$. Thus, μ may be singular at $s = 0$, and μ' exists almost everywhere .

Assumption on f Assume that $f \in C^\infty(\mathbb{R})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1 + \frac{2}{n}$, $n = 1, 2$, and α is assumed to be an integer for $n \geq 3$.

2.1 Mild solution formula

In this section we try to obtain the solution formula for the problem (2.1), in order to do so, we consider the following linear problem corresponding (2.1)

$$\begin{cases} \partial_t u - \Delta u + g * (-\Delta)^{\theta} u = 0, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

By applying Fourier transform and Laplace transform to Eq. (2.2), we obtain the solution $\bar{u}$ which expressed in terms of H, by $\bar{u} = H(t) * u_0$ where H is a fundamental solution to the following problem,

$$\begin{cases} \partial_t H - \Delta H + g * (-\Delta)^{\theta} H = 0, \ x \in \mathbb{R}^n, \ t > 0, \\ H(0,x) = \delta, \ x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

Here δ is the Dirac distribution in $x = 0$ with respective to the spatial variables.

The Fourier transform of the fundamental solution of (2.3) is given formally through Laplace inverse transform by

$$\hat{H}(t,\xi) = \hat{H}_0(\xi)\, \mathcal{L}^{-1}[\frac{1}{\lambda + \beta|\xi|^2 + |\xi|^{2\theta}\mathcal{L}[g]}](t,\xi). \tag{2.4}$$

Lemma 2.1 *The fundamental solution $\hat{H}(t,\xi)$ given by* (2.4) *exists.*

Proof. Denote $F(\lambda) := \lambda + |\xi|^2 + |\xi|^{2\theta}\mathcal{L}[g](\lambda)$. To prove $\mathcal{L}^{-1}[\frac{1}{F(\lambda)}]$ exists, we need to consider the zero points of $F(\lambda)$. Denote $\lambda = \sigma + i\nu, \ \ \sigma > -\frac{\delta}{C}$, then $\mathcal{L}[g](\lambda)$ exists.

Assume that $\lambda_1 = \sigma_1 + i\nu_1$ is a zero point of $F(\lambda)$ and $\sigma_1 > -\frac{\delta}{C}$, then σ_1 and ν_1 satisfy

$$\begin{cases} \Re F(\lambda_1) = \sigma_1 + |\xi|^2 + |\xi|^{2\theta} \int_0^{\infty} \cos(\nu_1 t) e^{-\sigma_1 t} g(t) dt = 0, \\ \Im F(\lambda_1) = \nu_1 - |\xi|^{2\theta} \int_0^{\infty} \sin(\nu_1 t) e^{-\sigma_1 t} g(t) dt = 0. \end{cases} \tag{2.5}$$

In order to show that $F(\lambda)$ does not vanish in the region $\{\lambda \in \mathbb{C}; \Re(\lambda) \geq \bar{g}\}$, we distinguish to cases

case 1 : If $|\xi| < 1$, we assume that $\sigma_1 \geq \sqrt{\bar{g}}$, then

$$\begin{aligned}\Re F(\lambda_1) &= \sigma_1 + |\xi|^2 + |\xi|^{2\theta} \int_0^\infty e^{-\sigma_1 t} g(t)dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta} \bar{g} \int_0^\infty e^{-\sigma_1 t} dt,\end{aligned}$$

$$\Re F(\lambda_1) \geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta} \frac{\bar{g}}{\sigma_1},$$

which implies

$$\Re F(\lambda_1) \geq \frac{\bar{g}}{\sigma_1} + |\xi|^2 - |\xi|^{2\theta} \frac{\bar{g}}{\sigma_1}.$$

Consequently

$$\Re F(\lambda_1) \geq |\xi|^2 + \frac{\bar{g}}{\sigma_1}(1 - |\xi|^{2\theta}) > 0.$$

It yields contradiction with $(2.5)_1$. Then $\sigma_1 < \sqrt{\bar{g}}$.

case 2 : In $|\xi| \geq 1$ we assume that $\sigma_1 \geq \bar{g}$, then we have

$$\begin{aligned}\Re F(\lambda_1) &= \sigma_1 + |\xi|^2 - |\xi|^{2\theta} \int_0^\infty e^{-\sigma_1 t} g(t)dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta} \bar{g} \int_0^\infty e^{-\sigma_1 t} dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta} \frac{\bar{g}}{\sigma_1} \\ &\geq \sigma_1 + \frac{\bar{g}}{\sigma_1}|\xi|^2 - |\xi|^{2\theta} \frac{\bar{g}}{\sigma_1},\end{aligned}$$

which gives

$$\Re F(\lambda_1) \geq \sigma_1 + \frac{\bar{g}}{\sigma_1}(|\xi|^2 - |\xi|^{2\theta}) \geq \sigma_1 > 0,$$

which it yields a contradiction with $(2.5)_1$. Therefore $\sigma_1 < \bar{g}$.

Combining the two cases, we know that $\frac{1}{F(\lambda)}$ is analytic in $\{\lambda \in \mathbb{C}; \Re(\lambda) \geq \sqrt{\bar{g}}\}$ if $|\xi| < 1$ and in $\{\lambda \in \mathbb{C}; \Re(\lambda) \geq \bar{g}\}$ if $|\xi| \geq 1$.

Take $\lambda = \sigma + i\nu, \quad \sigma > \max\{\Re\lambda_s\}$, here $\{\lambda_s\}$ is the set of all the singular points of $F(\lambda)$, then we have that

$$\begin{aligned}\mathcal{L}^{-1}[\frac{1}{F(\lambda)}](t) &= \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\lambda t}}{F(\lambda)}d\lambda=\int_{-\infty}^{+\infty}i\frac{e^{(\sigma+i\nu)t}}{F(\sigma+i\nu)}d\nu\\ &= \left(\int_{\{\nu;|\nu|\le R\}}+\int_{\{\nu;|\nu|>R\}}\right)\left(i\frac{e^{(\sigma+i\nu)t}}{F(\sigma+i\nu)}d\nu\right)\\ &=: J_1+J_2.\end{aligned}$$

The integral J_1 converges, so we only need to consider J_2. Notice that $\frac{1}{F(\lambda)}=\frac{1}{\lambda}-\frac{|\xi|^2+|\xi|^{2\theta}\mathcal{L}[g](\lambda)}{\lambda F(\lambda)}$ and $|\mathcal{L}[g](\lambda)|\le C$, then it is not difficult to prove that J_2 converges, then we proved that J_2 converges, so far we complete the proof. ■

By Duhamel principle, the solution to the problem (2.1) could be expressed as following :

$$u(t)=H(t)*u_0+\int_0^t H(t-\tau)*f(u)(\tau)d\tau. \tag{2.6}$$

We denote

$$\bar{u}(t):=H(t)*u_0, \tag{2.7}$$

then $\bar{u}(t)$ is the solution to the linear problem (2.3).

2.2 Decay properties of solution operators

We look at (2.1) as an ordinary differential equation in a proper Hilbert space accounting for the past history of the variable u. Extending the solution to (2.1) for all times, by setting $u(t)=0$ when $t<0$, and considering for $t\ge 0$ the auxiliary variable

$$\eta^t(s,x)=\int_{t-s}^t u(r,x)dr, \qquad t\ge 0,\quad s>0.$$

Note immediately that $\eta^0(s)=0$ for all $s>0$, the integro-differential equation of problem (2.2) reads

$$\partial_t u(t)-\Delta u(t)+\int_0^\infty \mu(s)\left(-\Delta\right)^\theta\eta^t(s)ds=0,\ \ t>0. \tag{2.8}$$

The past history variable η is the unique mild solution (in the sense of [118]) of an abstract Cauchy problem in the μ-weighted space $\mathcal{M} = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$, that is,

$$\begin{cases} \partial_t \eta^t = T\eta^t + u(t), \qquad t > 0, \\ \eta^0 = 0, \end{cases} \tag{2.9}$$

where, as a consequence of the basic assumption (see [70]), the linear operator T is the infinitesimal generator of the right-translation C_0-semigroup on $\mathcal{M}$,defined as

$$T\eta = -\eta', \;\text{ with domain } \mathcal{D}(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\; \eta(0) = 0\}.$$

Here the prime "$'$"symbol denotes the distributional derivative with respect to the internal variable s.

Applying the Fourier transform to (2.8) and (2.9), we obtain, for every $\xi \in \mathbb{R}^n$, the following system

$$\begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} + |\xi|^{2\theta} \displaystyle\int_0^\infty \mu(s)\hat{\eta}^t(s)ds = 0, \quad t > 0, \\ \partial_t \hat{\eta}^t = T\hat{\eta}^t + \hat{u}(t), \qquad t > 0, \\ \hat{u}(0) = \hat{u}_0, \quad \hat{\eta}^0 = 0, \end{cases} \tag{2.10}$$

in the transformed variables $\hat{u}(t,\xi)$ and $\hat{\eta}^t(t,\xi)$, where now T is the infinitesimal generator of the right-translation semigroup on $L^2_\mu(\mathbb{R}^+;\mathbb{R}^n)$, and $|.|$ stands for the standard euclidian norm in $\mathbb{R}^n$.

The energy density function is given by

$$\mathcal{E}(t,\xi) = |\hat{u}(t,\xi)|^2 + |\xi|^{2\theta} \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds. \tag{2.11}$$

In particular,

$$\mathcal{E}(0,\xi) = |\hat{u}_0(\xi)|^2.$$

Moreover, by the Plancherel theorem, we have a relation between the energy and the density

$$E(t) = \int_{\mathbb{R}^n} \mathcal{E}(t,\xi)d\xi.$$

Performing standard multiplication the first equation in (2.10) by $\overline{\hat{u}}$, using the second equation and then taking real parts, it can be seen as in [28] that the functional density satisfies for every fixed $\xi \in \mathbb{R}^n$ the following differential equality

$$\frac{d}{dt}\mathcal{E}(t,\xi)+2|\xi|^2|\hat{u}(t,\xi)|^2-|\xi|^{2\theta}\int_0^{\infty}\mu'(s)|\hat{\eta}^t(s,\xi)|^2ds+|\xi|^{2\theta}\sum_{i\geq 1}(\mu(s_i^-)-\mu(s_i^+))|\hat{\eta}^t(s_i,\xi)|^2 = 0. \tag{2.12}$$

where the sum includes the value $i = \infty$ if $s_\infty < \infty$. We notice that

$$-|\xi|^{2\theta}\int_0^{\infty}\mu'(s)|\hat{\eta}^t(s,\xi)|^2ds + |\xi|^{2\theta}\sum_{i\geq 1}(\mu(s_i^-) - \mu(s_i^+))|\hat{\eta}^t(s_i,\xi)|^2 \geq 0.$$

This means that the functional density $\mathcal{E}(t,\xi)$ is a non-increasing function of t.

Theorem 2.1 *Let μ satisfy the assumptions a), b), and let $\theta \in [0,1]$. There exists a positive constant $C > 0$, such that the solution u of (2.3) satisfies the following pointwise estimate in the Fourier space:*

$$\mathcal{E}(t,\xi) \leq C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t}, \textit{ for any } t > 0,$$

with $\rho(\xi) = \dfrac{|\xi|^2}{1+|\xi|^2}$.

We begin by introducing the following functionals :

$$\Upsilon(t,\xi) = |\xi|^{2\theta}\int_0^{\infty} g(s)|\hat{\eta}^t(s,\xi)|^2ds,$$

$$\Theta(t,\xi) = |\xi|^{2\theta}\int_0^{\infty} \mu(s)|\hat{\eta}^t(s,\xi)|^2ds.$$

Lemma 2.2 *The functionals* Υ *and* Θ *are well defined and fulfill the following inequality*

$$\Upsilon(t,\xi) \leq K\Theta(t,\xi). \tag{2.13}$$

Proof. Inequality (2.13) follows directly by assumption b). Moreover, by using (2.11) we have

$$\Theta(t,\xi) \leq \mathcal{E}(t,\xi) \leq \mathcal{E}(0,\xi) < \infty,$$

the well-definedness of Θ is achieved whereas the one for Υ is a consequence of (2.13). ■

Lemma 2.3 *The following differential inequality holds:*

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \leq 2K^2\bar{g}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2, \tag{2.14}$$

Proof. By means of the equation for the past history variable, a direct calculation leads to the following differential equality for $\Upsilon(t,\xi)$:

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds = 2\Re|\xi|^{2\theta}\int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds. \tag{2.15}$$

Concerning the right-hand side of (2.15), we have

$$\Re\int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds \leq \nu(\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds)^2 + \frac{1}{4\nu}|\hat{u}(t,\xi)|^2,$$

for some $\nu > 0$. Using assumption b) and Cauchy-Shwarz's inequality, we get

$$\begin{aligned}
\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds &\leq K\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|ds \\
&\leq K(\int_0^\infty \mu(s)ds)^{\frac{1}{2}}(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds)^{\frac{1}{2}} \\
&\leq K\sqrt{\bar{g}}(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$2\Re|\xi|^{2\theta}\int_0^\infty k(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds \leq 2\nu K^2\bar{g}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds + \frac{1}{2\nu}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2. \tag{2.16}$$

Now, we go back to equality (2.15), we infer from (2.16)

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \leq 2\nu K^2\bar{g}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds + \frac{1}{2\nu}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2,$$

taking $\nu = \frac{1}{4K^2\bar{g}}$, we obtain the desired inequality

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \leq 2K^2\bar{g}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2.$$

■

Proof of Theorem2.1. We define the additional functional

$$\mathcal{L}(t,\xi) = \mathcal{E}(t,\xi) + \beta\rho(\xi)\Upsilon(t,\xi),$$

for some $\beta > 0$ and $0 \leq \rho(\xi) \leq 1$ to be determined later.

Clearly we have $\mathcal{L}(t,\xi) \geq \mathcal{E}(t,\xi)$. On the other hand, we have

$$\begin{aligned}
\mathcal{L}(t,\xi) &= \mathcal{E}(t,\xi) + \beta\rho(\xi)\Upsilon(t,\xi) \\
&\leq \mathcal{E}(t,\xi) + \beta\Upsilon(t,\xi) \\
&\leq \mathcal{E}(t,\xi) + \beta|\xi|^{2\theta}\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|^2ds \\
&\leq \mathcal{E}(t,\xi) + \beta K|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \\
&\leq (1+\beta K)\mathcal{E}(t,\xi).
\end{aligned}$$

From (2.12) and (2.14), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t,\xi) &= \frac{d}{dt}\mathcal{E}(t,\xi)+\beta\rho(\xi)\frac{d}{dt}\Upsilon(t,\xi)\\ &\leq -2|\xi|^2|\hat{u}(t,\xi)|^2+|\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2ds\\ &\quad +\beta\rho(\xi)\left(2K^2\bar{g}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2-\frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\right).\end{aligned}$$

From which, it follows

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}(t,\xi)+2|\xi|^2|\hat{u}(t,\xi)|^2-|\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2ds+\frac{\beta}{2}\rho(\xi)|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\\ \leq\;& 2K^2\bar{g}\beta\rho(\xi)|\xi|^{2\theta}|\hat{u}(t,\xi)|^2.\end{aligned}$$

Therefore

$$\frac{d}{dt}\mathcal{L}(t,\xi)+2\left(|\xi|^2-K^2\bar{g}\beta\rho(\xi)|\xi|^{2\theta}\right)|\hat{u}(t,\xi)|^2+\frac{\beta}{2}\rho(\xi)|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\leq 0.$$

We choose $\beta\leq 1/\left(\frac{1}{4}+K^2\bar{g}\right)$ so that $2\left(|\xi|^2-K^2\bar{g}\beta\rho(\xi)|\xi|^{2\theta}\right)\geq\rho(\xi)\frac{\beta}{2}$. In fact by substituting $\rho(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$ and using that $\frac{|\xi|^{2\theta}}{1+|\xi|^2}\leq 1$, we obtain that

$$2\left(|\xi|^2-K^2\bar{g}\beta\rho(\xi)|\xi|^{2\theta}\right)\geq 2\left(1-K^2\bar{g}\beta\right)|\xi|^2\geq\frac{\beta}{2}|\xi|^2\geq\rho(\xi)\frac{\beta}{2},\text{ for any }\xi\in\mathbb{R}^n.$$

Hence, we arrive at

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2}\rho(\xi)|\hat{u}(t,\xi)|^2+\frac{\beta}{2}\rho(\xi)|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\leq 0.$$

From the definition of $\mathcal{E}$, we obtain

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2}\rho(\xi)\mathcal{E}(t,\xi)\leq 0. \tag{2.17}$$

Making use the equivalence between functionals $\mathcal{E}$ and $\mathcal{L}$ in (2.17), we infer

$$\frac{d}{dt}\mathcal{L}(t,\xi) + \frac{\beta}{2\,(1+\beta K)}\rho\,(\xi)\,\mathcal{L}(t,\xi) \le 0,$$

with $\rho(\xi) = \frac{|\xi|^2}{|\xi|^2+1}$. By invoking the Gronwall Lemma, we get

$$\mathcal{E}(t,\xi) \le \mathcal{L}(t,\xi) \le C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t}, \tag{2.18}$$

which concludes the proof. ■

Now we study the decay estimates of solutions to the linear problem (2.2).

Theorem 2.2 (Energy estimate for linear problem) *Let $s \ge 1$ be an integer, and $\theta \in [0,1]$. Assume that $u_0 \in H^s(\mathbb{R}^n)$, and put*

$$I_0 := ||u_0||_{H^s(\mathbb{R}^n)}.$$

Then the solution $\bar{u}$ to the problem (2.2) *given by* (2.7) *satisfies*

$$\bar{u} \in C^0([0,+\infty); H^s(\mathbb{R}^n)),$$

and the following energy estimate :

$$||\bar{u}(t)||^2_{H^s} + \int_0^t ||\partial_x \bar{u}(t)||^2_{H^{s-1}} d\tau \le cI_0^2$$

Proof. From (2.17) we have that

$$\frac{d}{dt}\mathcal{L}(t,\xi) + c\rho(\xi)\mathcal{E}(t,\xi) \le 0.$$

Integrate the previous inequality with respect to t and appeal to (2.18), then we obtain

$$\mathcal{E}(t,\xi) + c\int_0^t \rho(\xi)\mathcal{E}(\tau,\xi)d\tau \leq c\mathcal{E}(0,\xi). \tag{2.19}$$

Multiply (2.19) by $(1+|\xi|^2)^s$ and integrate the resulting inequality with respect to $\xi \in \mathbb{R}^n$, then we have that

$$||\bar{u}(t)||_{H^s}^2 + \int_0^t ||\partial_x \bar{u}(t)||_{H^{s-1}}^2 d\tau \leq cI_0^2 \tag{2.20}$$

(2.20) guarantees the regularity of the solution (2.7). So far we complete the proof of Theorem 2.2. ■

Lemma 2.4 *The fundamental solution $H(t,x)$ satisfies :*

$$|\hat{H}(t,\xi)| \leq Ce^{-c\rho(\xi)t}$$

where $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proof. From the representation formula of solution to the linear problem, we have

$$\bar{u}(t,x) := H(t,.) * u_0\,(x)\,.$$

Using the expression of $\bar{u}$ in Theorem 2.1, we find

$$|\widehat{\bar{u}}(t,\xi)|^2 = \left|\hat{H}(t,\xi)\right|^2 |\hat{u}_0(\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2,$$

which gives the desired estimate. ■

Lemma 2.5 (Pointwise estimate) *Assume $\bar{u}$ is the solution of* (2.2) *and if $\theta \in [0,1]$, then it satisfies the following point-wise estimate in the Fourier space:*

$$|\widehat{\bar{u}}(t,\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2, \tag{2.21}$$

where $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proof. we have

$$\bar{u}(t,x) := H(t,.) * u_0\,(x)\,.$$

Then by Fourier transform property and from Lemma 2.4, we get

$$|\widehat{\bar{u}}(t,\xi)|^2 = |\hat{H}(t,\xi)|^2|\hat{u}_0(\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2.$$

■

Proposition 2.1 *Let $s \geq 1$ be an integer and $1 \leq p \leq 2$. Let $\varphi \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. If $\theta \in [0,1]$ then the following estimates hold :*

$$||\partial_x^k H(t) * \varphi||_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||\varphi||_{L^p} + Ce^{-ct}||\partial_x^k \varphi||_{L^2}, \tag{2.22}$$

for $0 \leq k \leq s$.

Proof. In view of Lemma 2.5 we have that

$$\begin{aligned} ||\partial_x^k H(t) * \varphi||_{L^2}^2 &\leq C\int_{\mathbb{R}^n} |\xi|^{2k} e^{-2c\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi \\ &\leq C\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2k} e^{-c|\xi|^2 t}|\hat{\varphi}|^2 d\xi + C\int_{\{\xi,|\xi|\geq 1\}} |\xi|^{2k} e^{-2ct}|\hat{\varphi}|^2 d\xi \leq k_1 + k_2. \end{aligned}$$

Assume that p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then by Hausdorff–Young's inequality, we obtain

$$k_1 \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}||\hat{\varphi}||_{p'}^2 \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}||\varphi||_p^2.$$

On the other hand

$$k_2 \leq Ce^{-2ct}||\partial_x^k \varphi||_{L^2}^2, \text{ for } 0 \leq k \leq s,$$

then

$$||\partial_x^k H(t) * \varphi||_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||\varphi||_p + Ce^{-ct}||\partial_x^k \varphi||_{L^2}, \text{ for } 0 \leq k \leq s.$$

■

By using Proposition 2.1 with $p = 2$, we obtain the following decay estimates of $\bar{u}$ given by (2.7), if initial data $u_0 \in H^s(\mathbb{R}^n)$.

2.3 Decay estimates for linear problem

Theorem 2.3 (Decay estimates for linear problem) *Under the same assumptions as in Theorem 2.2, the solution $\bar{u}$ given by (2.7) satisfies the decay estimates:*

$$||\partial_x^k \bar{u}(t)||_{H^{s-k}} \leq cI_0(1+t)^{-\frac{k}{2}}$$

for $0 \leq k \leq s$.

Also, if initial data $u \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ then by using (2.22) we have that sharp decay estimates of the solution $\bar{u}$ to (2.2) . Therefore the theorem 2.3 can be stated as follows.

Theorem 2.4 (Sharp decay estimates for linear problem) *Let $s \geq 1$ be an integer and $\theta \in [0,1]$. Assume that $u_0 \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and put $I_p := ||u_0||_{H^s} + ||u_0||_{L^p}$ with $1 \leq p < 2$. Then the solution $\bar{u}$ to (2.2) given by (2.7) satisfies the following decay estimates :*

$$||\partial_x^k \bar{u}(t)||_{H^{s-k}} \leq cI_p(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}, \tag{2.23}$$

for $0 \leq k \leq s$.

Since proof of Theorem 2.3 and Theorem 2.4 are similar, here we only prove Theorem 2.4.

Proof. Let $k \geq 0$, $m \geq 0$ are an integers. In view of (2.7), by using (2.22), we have that

$$\begin{aligned} ||\partial_x^{k+m}\bar{u}(t)||_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||u_0||_{L^p} + Ce^{-ct}||\partial_x^{k+m}u_0||_{L^2} \\ &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}(||u_0||_{L^p} + ||\partial_x^{k+m}u_0||_{L^2}) \end{aligned} \tag{2.24}$$

for $k+m \leq s$ then $m \leq s-k$ we have that

$$||\partial_x^k\bar{u}(t)||_{H^{s-k}} \leq CJ_p(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}, \text{ for } 0 \leq k \leq s.$$

Thus (2.23) is proved. ■

Remark 2.1 *The estimates in Theorem 2.3 and Theorem 2.4 indicate that the decay structure of solutions to the linear problem* (2.2) *is not of regularity-loss type.*

2.4 Global existence and decay estimates for semi-linear problem

In this section we will first introduce a set of time-weighted Sobolev spaces and employ the contraction mapping theorem to prove the global existence and optimal decay of solution to the semi-linear problem.

Recall Assumption of f, we know that $f \in C^\infty(\mathbb{R}\backslash\{0\})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1 + \frac{2}{n}$, $n \geq 1$, and α is assumed to be an integer for obtain the following result about the global existence and optimal decay estimates of solution to the semi-linear problems (2.1).

Theorem 2.5 (existence and decay estimates for semi-linear problem) *Let s be an integer such that $s \geq [n/2]+1$ for $n \geq 1$ and $\theta \in [0,1]$. Let $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put*

$$I_0 := ||u_0||_{H^s} + ||u_0||_{L^1}.$$

If I_0 is suitably small, then there exists a unique solution $u(t,x) \in C^0([0,\infty); H^s(\mathbb{R}^n))$ of (2.1) *satisfying the following decay estimates:*

$$||\partial_x^k u(t)||_{H^{s-k}} \leq cI_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \text{ for } 0 \leq k \leq s. \tag{2.25}$$

Proof. Let us define the following space

$$X := \{u \in C([0,\infty), H^s(\mathbb{R}^n));\ ||u||_X < \infty\},$$

where

$$||u||_X := \sum_{k\leq s} \sup_{t\geq 0}(1+t)^{\frac{n}{4}+\frac{k}{2}}||\partial_x^k u(t)||_{H^{s-k}}.$$

We introduce the closed ball

$$B_R := \{u \in X; ||u||_X \leq R\},\ R > 0;$$

Denote

$$\Phi[u](t) := \Phi_{\text{lin}}(t) + \int_0^t H(t-\tau) * f(u)(\tau)d\tau,$$

where

$$\Phi_{\text{lin}}(t) := H(t) * u_0.$$

We have

$$\forall v, w \in X,\ \Phi[v](t) - \Phi[w](t) = \int_0^t H(t-\tau) * (f(v) - f(w))(\tau)d\tau.$$

Noticing that $f(v) = O(|v|^\alpha)$ and using Proposition 1.9 and Proposition 1.10. We have the following inequalities for $k \geq 0$

$$
\begin{aligned}
||\partial_x^k(f(v)-f(w))(\tau)||_{L^1} \;\leq\; & C||(v,w)(\tau)||_{L^\infty}^{\alpha-2} \\
& (||(v,w)(\tau)||_{L^2}||\partial_x^k(v-w)(\tau)||_{L^2} \\
& +||\partial_x^k(v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^2}), \qquad (2.26)
\end{aligned}
$$

and

$$
\begin{aligned}
||\partial_x^k(f(v)-f(w))(\tau)||_{L^2} \;& \leq C||(v,w)(\tau)||_{L^\infty}^{\alpha-2} \\
& (||(v,w)(\tau)||_{L^\infty}||\partial_x^k(v-w)(\tau)||_{L^2} \\
& +||\partial_x^k(v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^\infty}). \qquad (2.27)
\end{aligned}
$$

Also, if $v \in X$, then the following estimate holds

$$
||v(\tau)||_{L^\infty} \leq C||v||_X(1+\tau)^{-\frac{n}{4}}. \qquad (2.28)
$$

In fact, by using the Gagliardo-Nirenberg inequality, we get

$$
||v(\tau)||_{L^\infty} \leq ||v(\tau)||_{L^2}^{1-\lambda} \left\|\partial_x^{s_0} v(\tau)\right\|_{L^2}^{\lambda},
$$

where $s_0 = \left[\frac{n}{2}\right]+1$, $\lambda = \frac{n}{2s_0}$. We have for any $n \geq 1$,

$$
||v(\tau)||_{L^2} \leq C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X
$$

and since $s \geq s_0$, we obtain

$$
\left\|\partial_x^{s_0} v(\tau)\right\|_{L^2} \leq \left\|v(\tau)\right\|_{H^{s_0}} \leq \left\|v(\tau)\right\|_{H^s} \leq C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X.
$$

Hence, we deduce

$$\|v(\tau)\|_{L^\infty} \le C(1+\tau)^{-\frac{n}{4}(1-\lambda)-\frac{n}{4}\lambda}||v(\tau)||_X = C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X.$$

Now, we prove the estimate

$$||\partial_x^k(\Phi[v]-\Phi[w])(t)||_{H^{s-k}} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{2.29}$$

for $0 \le k \le s$.

Assume that k, m are non-negative integers, such that $k \le s$. By applying ∂_x^{k+m} to $\Phi[v]-\Phi[w]$, we have that

$$\begin{aligned} ||\partial_x^{k+m}(\Phi[v]-\Phi[w])(t)||_{L^2} &\le (\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t)||\partial_x^{k+m}H(t-\tau)*(f(v)-f(w))(\tau)||_{L^2}d\tau \\ &=: I_1+I_2. \end{aligned} \tag{2.30}$$

By virtue of (2.22) with $p=1$, we have

$$\begin{aligned} I_1 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}}||(f(v)-f(w))(\tau)||_{L^1}d\tau \\ &\quad +C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2}d\tau \\ &=: I_{11}+I_{12}. \end{aligned} \tag{2.31}$$

By using (2.26) with $k=0$ and (2.28), we have that

$$||(f(v)-f(w))(\tau)||_{L^1} \le C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{4}(\alpha-1)-\frac{n}{4}}. \tag{2.32}$$

Since $\alpha > \alpha_n = 1 + \frac{2}{n}$ for $n = 1, 2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (2.32), we have

$$I_{11} \leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}} (1+\tau)^{-\frac{n}{4}(\alpha-1)-\frac{n}{4}} d\tau ||(v,w)||_X^{\alpha-1} ||(v-w)||_X$$

then

$$I_{11} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} ||(v,w)||_X^{\alpha-1} ||v-w||_X. \tag{2.33}$$

If $k+m \leq s$, by virtue of (2.27) with k replaced by $k+m$ and (2.28), it yields that

$$||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}}. \tag{2.34}$$

By appealing to (2.34) and notice that $\alpha \geq 2$, we obtain

$$I_{12} \leq C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}} d\tau ||(v,w)||_X^{\alpha-1} ||(v-w)||_X.$$

Hence

$$I_{12} \leq Ce^{-ct} ||(v,w)||_X^{\alpha-1} ||v-w||_X,$$

Therefore by putting estimates I_{11} and I_{12} into (2.31), we get

$$I_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} ||(v,w)||_X^{\alpha-1} ||v-w||_X + Ce^{-ct} ||(v,w)||_X^{\alpha-1} ||v-w||_X. \tag{2.35}$$

Thus

$$I_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} ||(v,w)||_X^{\alpha-1} ||v-w||_X \tag{2.36}$$

with $0 \leq m \leq s-k$. Also by employing (2.22) with $p = 1$ to the term I_2, we obtain

$$\begin{aligned} I_2 \leq\ & C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}} ||\partial_x^k(f(v)-f(w))(\tau)||_{L^1} d\tau \\ & + C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} ||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2} d\tau \\ :=\ & I_{21} + I_{22}. \end{aligned} \tag{2.37}$$

In view of (2.26) and (2.28), we have that

$$||\partial_x^k(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{k}{2}}, \tag{2.38}$$

for $0 \leq k \leq s$.

Since $\alpha > \alpha_n = 1 + \frac{2}{n}$ for $n = 1, 2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (2.38), we have

$$I_{21} \leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}}(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{k}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X$$

then

$$I_{21} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $0 \leq k \leq s$. We use (2.34) to have that

$$I_{22} \leq C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X.$$

Therefore

$$I_{22} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $0 \leq m \leq s-k$. We substitute the estimates for I_{21} and I_{22} into (2.37), we infer

$$I_2 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{2.39}$$

with $0 \leq m \leq s-k$.

Using estimates (2.36) and (2.39) into (2.30) and take the sum with respect to m, $0 \leq m \leq s-k$, we get the estimate (2.29). Hence, we obtain that

$$||\Phi[v]-\Phi[w]||_X \leq C||(v,w)||_X^{\alpha-1}||v-w||_X.$$

So far we proved that $||\Phi[v]-\Phi[w]||_X \leq C_1R^{\alpha-1}||v-w||_X$ for $v, w \in B_R$.

On the other hand $\Phi[0](t) = \Phi_{\text{lin}}(t) = \bar{u}(t)$ and from Theorem 2.4 we know that $||\Phi_{\text{lin}}||_X \leq C_2 I_0$ if I_0 suitably small. We put $R = 2C_2I_0$. Now, if I_0 suitably small such that $R < 1$ and $C_1 R \leq \frac{1}{2}$, then we infer that

$$||\Phi[v] - \Phi[w]||_X \leq \frac{1}{2}||v - w||_X.$$

So, it yields for $v \in B_R$ that

$$||\Phi[v]||_X \leq ||\Phi_0||_X + \frac{1}{2}||v||_X \leq C_2 I_0 + \frac{1}{2}R = R,$$

i.e. $\Phi[v] \in B_R$. Thus $v \to \Phi[v]$ is a contraction mapping on B_R, which implies that there exists a unique $u \in B_R$ satisfying $\Phi[u] = u$, and it is a solution to the semi-linear problem (2.1) satisfying the decay estimate (2.25). So we complete the proof of Theorem 2.5. ■

Exemple 2.1 *Consider the following initial value problem*

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + g * (-\Delta)^{1/2} u = (\sin t)\, e^{-u^2} u^3, \; t > 0, \; x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \; x \in \mathbb{R}^n, \end{cases} \tag{2.40}$$

here, $\theta = 1/2$, $g(t) = \int_t^{+\infty} \mu(s)\, ds$ *where* $\mu(t) = t^{-\gamma} e^{-\beta t} \chi_{(0,s_0]} + \sum_{i=1}^{+\infty} a_i \chi_{[s_{i-1},s_i]}(t)$, *with* $s_0 = 1$, $0 \leq \gamma < 1$, $\beta \geq 0$ *and where* $\{a_i\}$ *is a strictly decreasing positive sequence such that* $a_1 \leq e^{-\beta}$. *Note that* $\mu'(t) \leq 0$ *for almost everywhere* $t > 0$, *and* $g = g(0) = \int_0^1 s^{-\gamma} e^{-\beta s} ds + \sum_{i=1}^{+\infty} a_i (s_i - s_{i-1})$. *Now, we check that* $g(t) \leq K\mu(t)$ *holds for any* $t > 0$

and for some $K > 0$. In fact, for $t < 1$, we have

$$\begin{aligned} g(t) &= \int_t^{+\infty} \mu(s)\, ds = \int_t^1 s^{-\gamma} e^{-\beta s} ds + \sum_{i=1}^{+\infty} a_i (s_i - s_{i-1}) \\ &= \int_t^1 s^{-\gamma} e^{-\beta s} ds + \left[\sum_{i=1}^{+\infty} a_i (s_i - s_{i-1}) \right] t^{\gamma} e^{\beta t} t^{-\gamma} e^{-\beta t} \\ &\leq (1-t)\, t^{-\gamma} e^{-\beta t} + g e^{\beta} t^{-\gamma} e^{-\beta t} \\ &\leq \left(1 + g e^{\beta}\right) t^{-\gamma} e^{-\beta t} = K \mu(t)\,. \end{aligned}$$

When $t \geq 1$, there exists $i_0 \geq 1$ such that $t \in [s_{i_0 - 1}, s_{i_0}]$, so we have

$$\begin{aligned} g(t) &= \int_t^{+\infty} \mu(s)\, ds = a_{i_0} (s_{i_0} - t) + \sum_{i=i_0+1}^{+\infty} a_i (s_i - s_{i-1}) \\ &\leq a_{i_0} \left(s_{i_0} - 1 + \frac{g}{a_{i_0}} \right) = K \mu(t)\,. \end{aligned}$$

On the other hand the function $f(u) = (\sin t)\, e^{-u^2} u^3 \in C^{\infty}(\mathbb{R})$ satisfies $f(u) = O\left(|u|^3\right)$ as $u \to 0$. Thus, all the assumptions in Theorem 2.5 satisfied, and, hence, the initial value problem (2.40) has a unique global solution $u(t,x) \in C([0,\infty); H^s(\mathbb{R}^n))$, $s > n/2$.

Chapter 3

Global existence and decay estimates for the semilinear nonclassical-diffusion equations with memory in $\mathbb{R}^n$.

In this chapter, we consider the initial value problem for the following semi-linear Volterra integro-differential equations of the first order posed in the whole space $\mathbb{R}^n$ $(n \geq 1)$

$$\begin{cases} \partial_t u(t,x) - \Delta \partial_t u(t,x) + (-\Delta)^{\theta} u(t,x) - g * \Delta u = f(u),\ t > 0,\ \ x \in \mathbb{R}^n, \\ u(0,x) = u_0\left(x\right),\ \ x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Here $u = u(t,x)$ is an unknown real valued function of $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $t > 0$, $u_0\left(x\right)$ is a given initial data and the function f is an external nonlinear force. The fractional Laplace operator $(-\Delta)^{\theta}$ may be defined through its Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by

$$(-\Delta)^{\theta} h(x) = \mathcal{F}^{-1}\left(|\xi|^{2\theta} \mathcal{F}\left(h\right)\left(\xi\right)\right)(x)\,,\ x \in \mathbb{R}^n,$$

or by its representation $(-\Delta)^{\theta}h(x) = C(n,\theta)\displaystyle\int_{\mathbb{R}^n}\frac{h(x)-h(y)}{|x-y|^{n+2\theta}}dy$, with $0<\theta<1$.
In the limit $\theta \to 1$ the standard Laplace operator $-\Delta$, is recovered (see Section 4 of [50]).

The convolution $-g*\Delta u := -\displaystyle\int_0^t g(t-s)\Delta u(s)ds$ corresponds to the memory term, and both g and f satisfies the same assumptions of the chapter 03.

The equation (3.1) can be viewed as an approximation of non-classical diffusion equation

$$\partial_t u(t,x) - \Delta\partial_t u(t,x) + (-\Delta)^{\theta}u(t,x) - \Delta u = f(u),\ t>0,\ x\in\mathbb{R}^n, \tag{3.2}$$

subject to initial conditions, when g is equal to the Dirac mass at 0^+, which arises in several diffusion processes [19],[119].

3.1 Mild solution formula

In this section, our aim is to derive the solution formula to the problem (3.1). Let $H(t,x)$ be a solution of the following problem,

$$\begin{cases} \partial_t H(t,x) - \Delta\partial_t H(t,x) + (-\Delta)^{\theta}H(t,x) - g*\Delta H(t,x) = 0, & t>0,\ x\in\mathbb{R}^n, \\ H(0,x) = H_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{3.3}$$

We apply Fourier transform and Laplace transform to (3.3), then we have formally that

$$\hat{H}(t,\xi) = \hat{H}_0\mathcal{L}^{-1}\Big[\frac{1}{\lambda + (\frac{|\xi|^{2\theta}}{1+|\xi|^2}) + (\frac{|\xi|^2}{1+|\xi|^2})\mathcal{L}[g]}\Big](t,\xi).$$

Lemma 3.1 *The function $\hat{H}(t,\xi)$ exists.*

Proof. Denote $F(\lambda) := \lambda + \frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{|\xi|^2}{1+|\xi|^2}\mathcal{L}[g]$. To prove $\mathcal{L}^{-1}[\frac{1}{F(\lambda)}]$ exists, we need to consider the zero points of $F(\lambda)$. Denote $\lambda = \sigma + i\nu$, $\sigma > -\frac{\delta}{C}$, then $\mathcal{L}[g](\lambda)$ exists.

Assume that $\lambda_1 = \sigma_1 + i\nu_1$ is a zero point of $F(\lambda)$ and $\sigma_1 > -\frac{\delta}{C}$, then σ_1 and ν_1 satisfy

$$\begin{cases} \Re(F)(\lambda_1) = \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} + (\frac{|\xi|^2}{1+|\xi|^2}) \int_0^\infty \cos(\nu_1 t) e^{-\sigma_1 t} g(t)dt = 0, \\ \Im(F)(\lambda_1) = \nu_1 - \frac{|\xi|^2}{1+|\xi|^2} \int_0^\infty \sin(\nu_1 t) e^{-\sigma_1 t} g(t)dt = 0. \end{cases} \tag{3.4}$$

We claim that for any ξ there exists $C > 0$, such that $\Re(F)(\lambda)$ does not vanish for $\Re(\lambda) \geq C$.

case 1 : If $\nu_1 = 0$, we have $\Im(F)(\lambda_1) = 0$. We assume that $\sigma_1 > 0$, then we obtain for any $\xi \in \mathbb{R}^n$

$$\Re(F)(\lambda_1) = \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{|\xi|^2}{1+|\xi|^2} \int_0^\infty e^{-\sigma_1 t} g(t)dt > 0.$$

Thus, it yields a contradiction with $(3.4)_1$. Therefore $\sigma_1 \leq 0$.

Case 2 : $\nu_1 \neq 0$, if $|\xi| \leq 1$, we have for $\sigma_1 \geq \bar{g}$

$$\begin{aligned} \Re(F)(\lambda_1) &= \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{|\xi|^2}{1+|\xi|^2} \int_0^\infty \cos(\nu_1 t) e^{-\sigma_1 t} g(t)dt \\ &\geq \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} - \frac{|\xi|^2}{1+|\xi|^2} \bar{g} \int_0^\infty e^{-\sigma_1 t} dt \\ &= \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} - \frac{|\xi|^2}{1+|\xi|^2} \frac{\bar{g}}{\sigma_1} > 0. \end{aligned}$$

Then $\sigma_1 < \frac{\bar{g}}{\beta}$.

If $|\xi| \geq 1$, we assume that $\sigma_1 \geq \sqrt{\bar{g}}$. Then

$$\begin{aligned} \Re(F)(\lambda_1) &= \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{|\xi|^2}{1+|\xi|^2} \int_0^\infty \cos(\nu_1 t) e^{-\sigma_1 t} g(t)dt \\ &\geq \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} - \frac{|\xi|^2}{1+|\xi|^2} \bar{g} \int_0^\infty e^{-\sigma_1 t} dt \end{aligned}$$

which gives

$$\Re(F)(\lambda_1) \geq \sigma_1 + \frac{|\xi|^{2\theta}}{1+|\xi|^2} - \frac{|\xi|^2}{1+|\xi|^2} \frac{\bar{g}}{\sigma_1},$$

and since $\sigma_1 \geq \sqrt{\bar{g}} \geq \frac{\bar{g}}{\sigma_1}$, we get

$$\Re(F)(\lambda_1) \geq \frac{\bar{g}}{\sigma_1} + \frac{|\xi|^{2\theta}}{1+|\xi|^2} - \frac{|\xi|^2}{1+|\xi|^2}\frac{\bar{g}}{\sigma_1}.$$

So, we get

$$\Re(F)(\lambda_1) \geq \frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{\bar{g}}{\sigma_1}(1 - \frac{|\xi|^2}{1+|\xi|^2}) > 0,$$

which is again a contradiction with $(3.4)_1$. Therefore $\sigma_1 < \sqrt{\bar{g}}$.

Combining the two cases, we know that $\frac{1}{F(\lambda)}$ is analytic in $\{\lambda \in \mathbb{C}; \Re(\lambda) \geq \bar{g}\}$ if $|\xi| < 1$ and in $\{\lambda \in \mathbb{C}; \Re(\lambda) \geq \sqrt{\bar{g}}\}$ if $|\xi| \geq 1$. Take $\lambda = \sigma + i\nu$, $\sigma > C \geq \max\{\sqrt{\bar{g}}, \bar{g}\}$, therefore $\{\lambda_s\}$ is the set of all the singular points of $F(\lambda)$ does not lie in $\Re(\lambda) > C$. Thus we have that

$$\mathcal{L}^{-1}[\frac{1}{F(\lambda)}](t) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\lambda t}}{F(\lambda)} d\lambda = \int_{-\infty}^{+\infty} i\frac{e^{(\sigma+i\nu)t}}{F((\sigma+i\nu))} d\nu$$

$$\int_{\{\nu;|\nu|\leq R\}} + \int_{\{\nu;|\nu|>R\}} =: J_1 + J_2.$$

Note that the first integral converges, so we only need to consider J_2.

We note that

$$\frac{1}{F(\lambda)} = \frac{1}{\lambda} - \frac{\frac{|\xi|^{2\theta}}{1+|\xi|^2} + \frac{|\xi|^2}{1+|\xi|^2}\mathcal{L}[g](\lambda)}{\lambda F(\lambda)}$$

and $|\mathcal{L}[g](\lambda)| \leq C$, so it is not difficult to prove that J_2 converges, then we proved that J_2 converges, So far we complete the proof. ■

By Duhamel principle, the solution to the problem (3.1) could be expressed as following :

$$u(t) = H(t) * u_0 + \int_0^t (1-\Delta)^{-1} H(t-\tau) * f(u(\tau)) d\tau. \tag{3.5}$$

We denote

$$\bar{u}(t) := H(t) * u_0. \tag{3.6}$$

Then $\bar{u}(t)$ is the solution to the linear problem corresponding (3.3).

3.2 Decay Property

We look at (3.1) as an ordinary differential equation in a proper Hilbert space accounting for the past history of the variable u. Extending the solution to (3.1) for all times, by setting $u(t) = 0$ when $t < 0$, and considering for $t \geq 0$ the auxiliary variable

$$\eta^t(s,x) = \int_{t-s}^{t} u(r,x)dr, \qquad t \geq 0,\ s > 0.$$

Note immediately that $\eta^0(s) = 0$ for all $s > 0$, the integro-differential equation of problem (3.3) reads

$$\partial_t u + A\partial_t u + A^\theta u + \int_0^\infty \mu(s)A\eta^t(s)ds = 0, \qquad t > 0, \tag{3.7}$$

where $A = -\Delta$. The past history variable η is the unique mild solution (in the sense of [118]) of an abstract Cauchy problem in the μ-weighted space $\mathcal{M} = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$, that is,

$$\begin{cases} \partial_t \eta^t = T\eta^t + u(t), & t > 0, \\ \eta^0 = 0, \end{cases} \tag{3.8}$$

where, as a consequence of the basic assumption (see [70]), the linear operator T is the infinitesimal generator of the right-translation C_0-semigroup on $\mathcal{M}$,defined as

$$T\eta = -\eta'(s)\,, \quad \text{with domain } \mathcal{D}(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M}, \eta(0) = 0\},$$

where the prime stands for the distributional derivative with respect to $s \in \mathbb{R}^+$, and $\eta(0) = \lim_{s\to 0}\eta(s)$ in $H^1(\mathbb{R}^n)$. Applying the Fourier transform to (3.7) and (3.8), we obtain, for every $\xi \in \mathbb{R}^n$, system

$$\begin{cases} \partial_t\hat{u} + |\xi|^2\partial_t\hat{u} + |\xi|^{2\theta}\hat{u} + |\xi|^2\int_0^\infty \mu(s)\hat{\eta}^t(s)ds = 0, & t > 0, \\ \partial_t\hat{\eta}^t(s) = T\hat{\eta}^t(s) + \hat{u}(t), & s,t > 0, \\ \hat{u}(0) = \hat{u}_0, \quad \hat{\eta}^0 = 0 \end{cases} \tag{3.9}$$

in the transformed variables $\hat{u}(t,\xi)$ and $\hat{\eta}^t(s,\xi)$, where now T is the infinitesimal generator of the right-translation semigroup on $L^2_\mu(\mathbb{R}^+;\mathbb{R}^n)$, and $|.|$ stands for the standard euclidian norm in $\mathbb{R}^n$.

The energy density function is given by

$$\mathcal{E}(t,\xi) = (1+|\xi|^2)|\hat{u}(t,\xi)|^2 + |\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds. \tag{3.10}$$

In particular,

$$\mathcal{E}(0,\xi) = (1+|\xi|^2)|\hat{u}_0(\xi)|^2.$$

Moreover, by the Plancherel theorem, we have

$$E(t) = \int_{\mathbb{R}^n} \mathcal{E}(t,\xi) d\xi.$$

Lemma 3.2 *The energy density function $\mathcal{E}(t,\xi)$ satisfies for every fixed $\xi \in \mathbb{R}^n$ the differential equality*

$$\frac{d}{dt}\mathcal{E}(t,\xi)+2|\xi|^{2\theta}|\hat{u}(t,\xi)|^2-|\xi|^2\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2 ds+|\xi|^2\sum_{i\geq 1}(\mu(s_i^-)-\mu(s_i^+))|\hat{\eta}^t(s_i,\xi)|^2 = 0,$$

where the sum includes the value $i=\infty$ if $s_\infty < \infty$. In particular, $\mathcal{E}(t,\xi)$ is a decreasing function of t.

Proof. Arguing as in [117], by performing a standard multiplication in (3.9) with $\overline{\hat{u}}$ and then taking the real part of the resulting identities one has

$$\Re\left(\partial_t \hat{u}.\overline{\hat{u}}\right) + |\xi|^2\Re\left(\partial_t \hat{u}.\overline{\hat{u}}\right) + |\xi|^{2\theta}\hat{u}.\overline{\hat{u}} + |\xi|^2\Re\int_0^\infty \mu(s)\hat{\eta}^t(s).\overline{\hat{u}} ds = 0,$$

which is equivalent to

$$\frac{1}{2}\frac{d}{dt}\left|\overline{\hat{u}}\right|^2 + \frac{1}{2}\frac{d}{dt}|\xi|^2\left|\overline{\hat{u}}\right|^2 + |\xi|^{2\theta}\left|\overline{\hat{u}}\right|^2 + |\xi|^2\Re\int_0^\infty \mu(s)\hat{\eta}^t(s).\overline{\hat{u}} ds = 0. \tag{3.11}$$

Using the second equation from (3.9) in (3.11), we get

$$\frac{1}{2}\frac{d}{dt}\left|\overline{\hat{u}}\right|^2+\frac{1}{2}\frac{d}{dt}|\xi|^2\left|\overline{\hat{u}}\right|^2+|\xi|^{2\theta}\left|\overline{\hat{u}}\right|^2+|\xi|^2\Re\int_0^\infty\mu(s)\hat{\eta}^t(s).\left(\partial_t\overline{\hat{\eta}^t(s)}+\partial_s\overline{\hat{\eta}^t(s)}\right)ds=0.$$

From which, we infer

$$\frac{1}{2}\frac{d}{dt}\left(\left|\overline{\hat{u}}\right|^2+|\xi|^2\left|\overline{\hat{u}}\right|^2+|\xi|^2\int_0^\infty\mu(s)\left|\hat{\eta}^t(s)\right|^2ds\right)+|\xi|^{2\theta}\left|\overline{\hat{u}}\right|^2=-|\xi|^2\Re\int_0^\infty\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds.$$

Now, we analyze the term $|\xi|^2\Re\int_0^\infty\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds$, since μ contains infinitely jumps terms

$$|\xi|^2\Re\int_0^\infty\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds=|\xi|^2\Re\int_0^{s_\infty}\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds+|\xi|^2\Re\int_{s_\infty}^{S_\infty}\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds,$$

where $S_\infty=\sup\{s\in\mathbb{R}^+:\mu(s)>0\}$. We note that the last term is different from zero only if $s_\infty<S_\infty$.

The first term can be written as,

$$|\xi|^2\Re\int_0^{s_\infty}\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds=\frac{1}{2}|\xi|^2\lim_{N\to+\infty}\sum_{i=1}^N\int_{s_{i-1}}^{s_i}\mu(s)\frac{d}{ds}\left|\hat{\eta}^t(s)\right|^2ds,$$

so, integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}|\xi|^2\sum_{i=1}^N\int_{s_{i-1}}^{s_i}\mu(s)\frac{d}{ds}\left|\hat{\eta}^t(s)\right|^2ds &= \frac{1}{2}|\xi|^2\lim_{y\to0}\left(\mu(y)\left|\hat{\eta}^t(y)\right|^2+\int_y^{s_1}\mu'(s)\left|\hat{\eta}^t(s)\right|^2ds\right)\\ &+\frac{1}{2}|\xi|^2\mu(s_1^-)\left|\hat{\eta}^t(s_1)\right|^2+\frac{1}{2}|\xi|^2\sum_{i=2}^N\left[\mu(s)\left|\hat{\eta}^t(s)\right|^2\right]_{s_{i-1}}^{s_i}\\ &-\frac{1}{2}|\xi|^2\sum_{i=2}^N\int_{s_{i-1}}^{s_i}\mu'(s)\left|\hat{\eta}^t(s)\right|^2ds.\end{aligned}$$

Using the fact that the norm of $\hat{\eta}^t(0)$ is zero, we get

$$\begin{aligned}
\lim_{y\to 0}\mu(y)\left|\hat{\eta}^t(y)\right|^2 &\le \limsup_{y\to 0}\mu(y)\left(\int_0^y \left|\left(\hat{\eta}^t\right)'(\zeta)\right| d\zeta\right)^2 \\
&\le \limsup_{y\to 0}\left(\int_0^y \sqrt{\mu(\zeta)}\left|\left(\hat{\eta}^t\right)'(\zeta)\right| d\zeta\right)^2 \\
&\le \limsup_{y\to 0}\left(y\int_0^y \mu(\zeta)\left|\left(\hat{\eta}^t\right)'(\zeta)\right|^2 d\zeta\right) = 0,
\end{aligned}$$

since $(\hat{\eta}^t)' \in L^2_\mu(\mathbb{R}^+;\mathbb{R}^n)$. By rearranging the terms of the series, we can then write

$$\begin{aligned}
|\xi|^2\Re\int_0^{s_\infty}\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds &= -\frac{1}{2}|\xi|^2\int_0^{s_\infty}\mu'(s)\left|\hat{\eta}^t(s)\right|^2 ds \\
&+ \frac{1}{2}|\xi|^2\sum_{i=1}^{+\infty}\mu_i\left|\hat{\eta}^t(s_i)\right|^2 + \frac{1}{2}|\xi|^2\lim_{s\to s_\infty}\mu(s)\left|\hat{\eta}^t(s)\right|^2.
\end{aligned}$$

We see that the last expression is nonnegative and finite since the right-hand side is so. We obtain the convergence of the integral and of the series, and consequently the existence of the limit. In particular, since $\mu'(s)\left|\hat{\eta}^t(s)\right|^2$ is integrable on $\mathbb{R}^+$, the limit is zero if $s_\infty = \infty$; if instead $s_\infty < \infty$, using the continuity of $\left|\hat{\eta}^t(s)\right|^2$ in s_∞, we find

$$\lim_{s\to s_\infty}\mu(s)\left|\hat{\eta}^t(s)\right|^2 = \mu(s_\infty^-)\left|\hat{\eta}^t(s_\infty)\right|^2$$

when $s_\infty < S_\infty$, we evaluate the remaining integral. We have,

$$\begin{aligned}
|\xi|^2\Re\int_{s_\infty}^{S_\infty}\mu(s)\hat{\eta}^t(s).\partial_s\overline{\hat{\eta}^t(s)}ds &= \frac{1}{2}|\xi|^2\int_{s_\infty}^{S_\infty}\mu(s)\frac{d}{ds}\left|\hat{\eta}^t(s)\right|^2 ds \\
&= -\frac{1}{2}|\xi|^2\left[\mu\left(s_\infty^+\right)\left|\hat{\eta}^t(s_\infty)\right|^2 - \lim_{y\to S_\infty}\mu(y)\left|\hat{\eta}^t(y)\right|^2\right] \\
&\quad -\frac{1}{2}|\xi|^2\int_{s_\infty}^{S_\infty}\mu'(s)\left|\hat{\eta}^t(s)\right|^2 ds,
\end{aligned}$$

from which we obtain that the limit in the second member exists and finite; on the other hand, $\lim\limits_{s\to S_\infty}\mu(s)=0$, and therefore,

$$\begin{aligned}\lim_{y\to S_\infty}\mu(y)\left|\hat{\eta}^t(y)\right|^2 &= \lim_{y\to S_\infty}\mu(y)\left(\left|\hat{\eta}^t(s_\infty)\right|+\int_{s_\infty}^{y}\left|\left(\hat{\eta}^t\right)'(s)\right|ds\right)^2\\ &\le 2\limsup_{y\to S_\infty}\mu(y)\left|\hat{\eta}^t(s_\infty)\right|^2+2\limsup_{y\to S_\infty}\left(\int_{s_\infty}^{S_\infty}\chi_{(s_\infty,y)}\sqrt{\mu(y)}\left|\left(\hat{\eta}^t\right)'(s)\right|ds\right)^2=0,\end{aligned}$$

where we have used the Lebesgue dominated convergence theorem to the last term. Then put

$$J\left[\hat{\eta}^t\right]=\sum_i\mu_i\left|\hat{\eta}^t(s_i)\right|^2,$$

where $\mu_i=\mu(s_i^-)-\mu(s_i^+)$ and the sum over all the values of i at the jump points s_i, including s_∞ if $s_\infty<\infty$, we derive the desired inequality

$$\frac{1}{2}\frac{d}{dt}\left(\left(1+|\xi|^2\right)\left|\bar{\hat{u}}\right|^2+|\xi|^2\int_0^\infty\mu(s)\left|\hat{\eta}^t\left(s\right)\right|^2ds\right)+|\xi|^{2\theta}\left|\bar{\hat{u}}\right|^2=\frac{1}{2}\int_0^\infty\mu'(s)\left|\hat{\eta}^t\left(s\right)\right|^2ds-\frac{1}{2}J\left[\hat{\eta}^t\right]\le 0.$$

■

Theorem 3.1 *(Pointwise estimates in the Fourier space) Let g satisfies the assumptions a), b), and $\theta\in[0,1]$. Let u be the solution to the linear problem* (3.9)*. Then its Fourier image $\hat{u}$ verifies the pointwise estimates*

$$\mathcal{E}(t,\xi)\le C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t},$$

for $\xi\in\mathbb{R}^n$ and $t\ge 0$, where $\rho\left(\xi\right)=\frac{|\xi|^{2\theta}}{1+|\xi|^2}$.

We begin by introducing the following functional:

$$\Upsilon(t,\xi)=|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|^2ds.$$

Lemma 3.3 *The following differential inequality holds:*

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \le 2\Theta^2 \bar{g}|\xi|^2|\hat{u}(t,\xi)|^2. \tag{3.12}$$

Proof. By means of the equation for the past history variable, a direct calculation leads to the following differential equality for $\Upsilon(t,\xi)$:

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds = 2|\xi|^2 \Re \int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)} ds. \tag{3.13}$$

Concerning the right-hand side of (3.13), we have

$$\Re \int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)} ds \le \nu\Big(\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds\Big)^2 + \frac{1}{4\nu}|\hat{u}(t,\xi)|^2,$$

for any $\nu > 0$. Using assumption b) and Cauchy-Shwarz's inequality, we get

$$\begin{aligned}
\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds &\le \Theta \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|ds \\
&\le \Theta\Big(\int_0^\infty \mu(s)ds\Big)^{\frac{1}{2}}\Big(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds\Big)^{\frac{1}{2}} \\
&= \Theta\sqrt{\bar{g}}\Big(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$2|\xi|^2 \Re \int_0^\infty k(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)} ds \le 2\nu\Theta^2\bar{g}|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds + \frac{1}{2\nu}|\xi|^2|\hat{u}(t,\xi)|^2. \tag{3.14}$$

Now, we go back to equality (3.13), we infer from (3.14)

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \le 2\nu\Theta^2\bar{g}|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds + \frac{1}{2\nu}|\xi|^2|\hat{u}(t,\xi)|^2,$$

taking $\nu = \frac{1}{4\Theta^2\bar{g}}$, we obtain the desired inequality

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \leq 2\Theta^2\bar{g}|\xi|^2|\hat{u}(t,\xi)|^2.$$

■

Proof of Theorem 3.1. We define the additional functional

$$\mathcal{L}(t,\xi) = \mathcal{E}(t,\xi) + \beta\rho(\xi)\,\Upsilon(t,\xi),$$

for some $\beta > 0$ and $0 \leq \rho(\xi) \leq 1$ to be determined later.

Clearly we have $\mathcal{L}(t,\xi) \geq \mathcal{E}(t,\xi)$. On the other hand, we have

$$\begin{aligned}
\mathcal{L}(t,\xi) &= \mathcal{E}(t,\xi) + \beta\rho(\xi)\,\Upsilon(t,\xi) \\
&\leq \mathcal{E}(t,\xi) + \beta\Upsilon(t,\xi) \\
&\leq \mathcal{E}(t,\xi) + \beta|\xi|^2 \int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|^2 ds \\
&\leq \mathcal{E}(t,\xi) + \beta\Theta|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \\
&\leq (1+\beta\Theta)\,\mathcal{E}(t,\xi).
\end{aligned}$$

From Lemma 3.2 and (3.12), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t,\xi) &= \frac{d}{dt}\mathcal{E}(t,\xi) + \beta\rho(\xi)\,\frac{d}{dt}\Upsilon(t,\xi) \\
&\leq -2|\xi|^{2\theta}|\hat{u}(t,\xi)|^2 + |\xi|^2 \int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2 ds \\
&\quad +\beta\rho(\xi)\left(2\Theta^2\bar{g}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2 - \frac{1}{2}|\xi|^2 \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds\right).
\end{aligned}$$

From which, it follows

$$\frac{d}{dt}\mathcal{L}(t,\xi)+2|\xi|^{2\theta}|\hat{u}(t,\xi)|^2-|\xi|^2\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2ds \quad + \quad \frac{\beta}{2}\rho(\xi)|\xi|^2\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \leq 2\Theta^2\bar{g}\beta\rho(\xi)|\xi|^2|\hat{u}(t,\xi)|^2.$$

Therefore

$$\frac{d}{dt}\mathcal{L}(t,\xi)+2\left(|\xi|^{2\theta}-\Theta^2\bar{g}\beta\rho(\xi)|\xi|^2\right)|\hat{u}(t,\xi)|^2+\frac{\beta}{2}\rho(\xi)|\xi|^2\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\leq 0.$$

Now, we choose $\beta\leq 1/\left(\frac{1}{4}+\Theta^2\bar{g}\right)$ so that $2\left(|\xi|^{2\theta}-\Theta^2\bar{g}\beta\rho(\xi)|\xi|^2\right)\geq\frac{\beta}{2}\rho(\xi)(1+|\xi|^2)$. In fact by substituting $\rho(\xi)=\frac{|\xi|^{2\theta}}{1+|\xi|^2}$ and using that $\frac{|\xi|^2}{1+|\xi|^2}\leq 1$, we obtain

$$2\left(|\xi|^{2\theta}-\Theta^2\bar{g}\beta\rho(\xi)|\xi|^2\right)\geq 2\left(1-\Theta^2\bar{g}\beta\right)|\xi|^{2\theta}\geq\frac{\beta}{2}|\xi|^{2\theta}=\frac{\beta}{2}\rho(\xi)\left(1+|\xi|^2\right), \text{ for any } \xi\in\mathbb{R}^n.$$

Hence we arrive at

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2}\rho(\xi)\left(1+|\xi|^2\right)|\hat{u}(t,\xi)|^2+\frac{\beta}{2}\rho(\xi)|\xi|^2\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\leq 0.$$

From the definition of $\mathcal{E}$, we obtain

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2}\rho(\xi)\mathcal{E}(t,\xi)\leq 0. \tag{3.15}$$

Making use the equivalence between functionals $\mathcal{E}$ and $\mathcal{L}$ in (3.15), we obtain

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2(1+\beta\Theta)}\rho(\xi)\mathcal{L}(t,\xi)\leq 0, \tag{3.16}$$

with $\rho(\xi)=\frac{|\xi|^2}{|\xi|^2+1}$. By invoking the Gronwall Lemma, we get

$$\mathcal{E}(t,\xi)\leq\mathcal{L}(t,\xi)\leq C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t}, \tag{3.17}$$

which concludes the proof. ■

Now we study the decay estimates of solution to the linear problem (3.3).

Theorem 3.2 *(Energy estimate for linear problem). Let $s \geq 1$ be a real number, and $\theta \in [0,1]$. Assume that $u_0 \in H^s(\mathbb{R}^n)$, and put*

$$I_0 := ||u_0||_{H^s(\mathbb{R}^n)}.$$

Then the solution $\bar{u}$ to the problem (3.3) *given by* (3.6) *satisfies*

$$\bar{u} \in C^0([0,+\infty); H^s(\mathbb{R}^n)),$$

and the following energy estimate :

$$||\bar{u}(t)||^2_{H^s} + \int_0^t |||\nabla|^\theta \bar{u}(t)||^2_{H^{s-1}} d\tau \leq c I_0^2.$$

Proof. From (3.16) we have that

$$\frac{d\mathcal{L}(t,\xi)}{dt} + c\rho(\xi)\mathcal{L}(t,\xi) \leq 0.$$

By integrating the previous inequality with respect to t and appealing to (3.17), we obtain

$$\mathcal{E}(t,\xi) + \int_0^t \rho(\xi)\mathcal{E}(\tau,\xi) d\tau \leq c\mathcal{E}(0,\xi). \tag{3.18}$$

Multiplying equation (3.18) by $(1+|\xi|^2)^{s-1}$ and integrating the resulting inequality with respect to $\xi \in \mathbb{R}^n$, then we have that

$$||\bar{u}(t)||^2_{H^s} + \int_0^t |||\nabla|^\theta \bar{u}(t)||^2_{H^{s-1}} d\tau \leq c I_0^2.$$

The last inequality guarantees the regularity of the solution (3.6). So far we complete the proof of Theorem 3.2. ■

Lemma 3.4 *(Pointwise estimate). Assume u is the solution of* (3.3) *and if $\theta \in [0,1]$ then it satisfies the following pointwise estimate in the Fourier space:*

$$|\hat{u}(t,\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2, \tag{3.19}$$

here $\rho(\xi) = \frac{|\xi|^{2\theta}}{1+|\xi|^2}$.

Lemma 3.5 *The Fourier transform of the function $H(t,x)$ satisfies :*

$$|\hat{H}(t,\xi)| \leq ce^{-c\rho(\xi)t},$$

where $\rho(\xi) = \frac{|\xi|^{2\theta}}{1+|\xi|^2}$.

Proof. We have

$$\bar{u}(t) := H(t) * u_0,$$

and by the pointwise estimate we have that

$$|\widehat{\bar{u}}(t,\xi)|^2 = |\hat{H}(t,\xi)|^2|\hat{u}_0(\xi)|^2 \leq ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2.$$

Then

$$|\hat{H}(t,\xi)| \leq ce^{-c\rho(\xi)t}.$$

■

Proposition 3.1 *Let $s \geq 0$ be a real number. We have*

Case 1: *if $\theta \in (0,1)$, $\varphi \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), 1 \leq p \leq 2$, then the following estimates hold:*

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq C(1+t)^{-\frac{n(2-p)+2pr}{4\theta p}}||\varphi||_{L^p} + C(1+t)^{-\frac{l}{2(1-\theta)}}||\varphi||_{H^{r+m+l}}. \tag{3.20}$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r + m + l \leq s$.

Case 2 : *if $\theta = 0$, $\varphi \in H^s(\mathbb{R}^n)$, then the following estimates hold :*

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq Ce^{-ct}||\varphi||_{H^{r+m}} + C(1+t)^{-\frac{l}{2}}||\varphi||_{H^{r+m+l}}. \tag{3.21}$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r + m + l \leq s$.

Case 3 : *if $\theta = 1$, $\varphi \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), 1 \leq p \leq 2$, then the following estimates hold :*

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq C(1+t)^{-\frac{n(2-p)+2pr}{4p}}||\varphi||_{L^p} + Ce^{-ct}||\varphi||_{H^{r+m}}. \tag{3.22}$$

for $r \geq 0, m \geq 0$ and $r + m \leq s$.

Proof. In view of Lemma 3.5 we have that

Case 1 : if $\theta \in (0, 1)$

$$\begin{aligned} |||\nabla|^r H(t) * \varphi||^2_{L^2} &\leq c\int_{\mathbb{R}^n} |\xi|^{2r} e^{-c\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi \\ &\leq c\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2r} e^{-\frac{c}{2}|\xi|^2 t}|\hat{\varphi}|^2 d\xi + c\int_{\{\xi,|\xi|\geq 1\}} |\xi|^{2r} e^{-\frac{c}{2}\frac{1}{|\xi|^{2(1-\theta)}}t}|\hat{\varphi}|^2 d\xi \\ :&= k_1 + k_2. \end{aligned}$$

Assume that p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$k_1 \leq c(1+t)^{-\frac{n(2-p)+2pr}{2\theta p}}||\hat{\varphi}||^2_{p'} \leq c(1+t)^{-\frac{n(2-p)+2pr}{2\theta p}}||\varphi||^2_p.$$

On the other hand, for r replaced by $r + m$ we have that

$$k_2 \leq c(1+t)^{-\frac{l}{1-\theta}}||\varphi||^2_{H^{r+m+l}},$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$. Hence

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq c(1+t)^{-\frac{n(2-p)+2pr}{4\theta p}}||\varphi||_p + c(1+t)^{-\frac{l}{2(1-\theta)}}||\varphi||_{H^{r+m+l}},$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$.

Case 2 : if $\theta = 0$, we have

$$\begin{aligned} |||\nabla|^r H(t) * \varphi||_{L^2}^2 &\leq c\int_{\mathbb{R}^n} |\xi|^{2r} e^{-c\rho(\xi)t} |\hat{\varphi}(\xi)|^2 d\xi \\ &\leq c\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2r} e^{-\frac{c}{2}t} |\hat{\varphi}\,(\xi)\,|^2 d\xi + c\int_{\{\xi,|\xi|\geq 1\}} |\xi|^{2r} e^{-\frac{c}{2}\frac{1}{|\xi|^2}t} |\hat{\varphi}\,(\xi)\,|^2 d\xi \\ : &= J_1 + J_2. \end{aligned}$$

For the first term, we have

$$J_1 \leq ce^{-ct}\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2r} |\hat{\varphi}|^2 d\xi,$$

which gives

$$J_1 \leq ce^{-ct}|||\nabla|^r \varphi||_{L^2}^2,$$

for r replaced by $r+m$ we have that

$$J_1 \leq ce^{-ct}||\varphi||_{H^{r+m}}^2.$$

On the other hand,

$$J_2 \leq c(1+t)^{-l}||\varphi||_{H^{r+m+l}}^2,$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$. Then, we obtain

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq ce^{-ct}||\varphi||_{H^{r+m}} + c(1+t)^{-\frac{l}{2}}||\varphi||_{H^{r+m+l}},$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$.

Case 3 : if $\theta = 1$**,** we have

$$\begin{aligned} |||\nabla|^r H(t) * \varphi||_{L^2}^2 &\leq c\int_{\mathbb{R}^n} |\xi|^{2r} e^{-c\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi \\ &\leq c\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2r} e^{-\frac{c}{2}|\xi|^2 t}|\hat{\varphi}|^2 d\xi + c\int_{\{\xi,|\xi|\geq 1\}} |\xi|^{2r} e^{-\frac{c}{2}t}|\hat{\varphi}|^2 d\xi \\ : &= J_1 + J_2. \end{aligned}$$

Assume that p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$J_1 \leq c(1+t)^{-\frac{n(2-p)+2pr}{2p}} ||\hat{\varphi}||_{p'}^2 \leq c(1+t)^{-\frac{n(2-p)+2pr}{2p}} ||\varphi||_p^2.$$

On the other hand, for r replaced by $r+m$ we have that

$$J_2 \leq ce^{-ct}||\varphi||_{H^{r+m}}^2,$$

for $r \geq 0, m \geq 0$ and $r+m \leq s$.

Therefore

$$|||\nabla|^r H(t) * \varphi||_{H^m} \leq c(1+t)^{-\frac{n(2-p)+2pr}{4p}} ||\varphi||_p + ce^{-ct}||\varphi||_{H^{r+m}},$$

for $r \geq 0, m \geq 0$ and $r+m \leq s$. ■

3.3 Decay estimates for linear problem

Also, by using Proposition (3.1) we have that sharp decay estimates of the solution $\bar{u}$ to (3.3).

Denote for $\theta \in (0,1)$

$$\sigma_\theta(r,n) = r + (1-\theta)(\frac{n}{2\theta} + \frac{r}{\theta}), \qquad n \geq 1, \tag{3.23}$$

and

$$\sigma_0(r,n) = 2r + \frac{n}{2}, \qquad n \geq 1. \tag{3.24}$$

Then the theorem can be stated as follows.

Theorem 3.3 *(Sharp decay estimates for linear problem) Let $s \geq 0$ be a real number. Set $I_1 := ||u_0||_{H^s} + ||u_0||_{L^1}$.*

Case 1 : *$\theta \in (0,1)$. Assume that $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the solution $\bar{u}$ to (3.3) given by (3.6) satisfies the following decay estimates :*

$$|||\nabla|^r \bar{u}(t)||_{H^{s-\sigma_\theta(k,n)}} \leq cI_1(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}, \tag{3.25}$$

for $r \geq 0$ be a real number and $\sigma_\theta(r,n) \leq s$.

Case 2 : *$\theta = 0$. Assume that $u_0 \in H^s(\mathbb{R}^n)$, then the solution $\bar{u}$ to (3.3) given by (3.6) satisfies the following decay estimates :*

$$|||\nabla|^r \bar{u}(t)||_{H^{s-\sigma_0(r,n)}} \leq cI_0(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{3.26}$$

for $r \geq 0$ be a real number and $\sigma_0(r,n) \leq s$.

Case 3 : *$\theta = 1$. Assume that $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the solution $\bar{u}$ to (3.3) given by (3.6) satisfies the following decay estimates:*

$$|||\nabla|^r \bar{u}(t)||_{H^{s-k}} \leq cI_1(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{3.27}$$

for $r \geq 0$ be a real number and $r \leq s$.

Proof. Let $s \geq 0$, $m \geq 0$ be real numbers.

Case 1 : if $\theta \in (0,1)$. In view of (3.6), by using (3.20) with $p = 1$, we have that

$$|||\nabla|^r \bar{u}(t)||_{H^m} \leq c(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||u_0||_{L_1} + c(1+t)^{-\frac{l}{2(1-\theta)}}||u_0||_{H^{r+m+l}}, \tag{3.28}$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$. We choose $l = 2(1-\theta)(\frac{n}{4\theta} + \frac{r}{2\theta})$ for $r+m+l \leq s$, we have that

$$|||\nabla|^r \bar{u}(t)||_{H^{s-\sigma_\theta(r,n)}} \leq cI_1(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}},$$

for $r \geq 0$ and $\sigma_\theta(r,n) \leq s$.

Thus (3.25) is proved.

Case 2 : if $\theta = 0$. In view of (3.6), by using (3.21), we have that

$$|||\nabla|^r \bar{u}(t)||_{H^m} \leq ce^{-ct}||u_0||_{H^{r+m}} + c(1+t)^{-\frac{l}{2}}||u_0||_{H^{r+m+l}}, \tag{3.29}$$

for $r \geq 0, m \geq 0, l \geq 0$ and $r+m+l \leq s$. So by taking $l = \frac{n}{2} + r$ in (3.29), we obtain

$$|||\nabla|^r \bar{u}(t)||_{H^m} \leq ce^{-ct}||u_0||_{H^s} + c(1+t)^{-\frac{n}{4}-\frac{r}{2}}||u_0||_{H^s}.$$

This shows that

$$|||\nabla|^r \bar{u}(t)||_{H^{s-\sigma_0(r,n)}} \leq cI_0(1+t)^{-\frac{n}{4}-\frac{r}{2}},$$

for $r \geq 0$ and $\sigma_0(r,n) \leq s$. Thus (3.26) is proved.

Case 3 : if $\theta = 1$. In view of (3.6), by using (3.22) with $p = 1$, we have that

$$|||\nabla|^r \bar{u}(t)||_{H^m} \leq c(1+t)^{-\frac{n}{4}-\frac{r}{2}}||u_0||_{L_1} + ce^{-ct}||u_0||_{H^{r+m}}, \tag{3.30}$$

for $r \geq 0, m \geq 0$ and $r+m \leq s$.

Then

$$|||\nabla|^r \bar{u}(t)||_{H^{s-r}} \leq cI_1(1+t)^{-\frac{n}{4}-\frac{r}{2}},$$

for $0 \leq r \leq s$. Thus (3.27) is proved. ■

Remark 3.1 *The estimates in Theorem* (3.3) *indicate that the decay structure of solutions to the linear problem* (3.3) *is of regularity-loss type.*

3.4 Global existence and decay for semi-linear problem

In this section we will first introduce a set of time-weighted Sobolev spaces and employ the contraction mapping theorem to prove the global existence and optimal decay of solution to the semi-linear problem.

Recall Assumption of f, we know that $f \in C^\infty(\mathbb{R}\backslash\{0\})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ for $n \geq 1$ and

$$\alpha_n := \begin{cases} 1 + \frac{2\theta}{n}, & \theta \in (0,1), \\ 1 + \frac{2}{n}, & \theta = 0, 1, \end{cases}$$

with α is assumed to be an integer for obtaining the following result about the global existence and optimal decay estimates of solution to the semi-linear problems (3.1).

Theorem 3.4 *(Existence and decay estimates for semi-linear problem). Let $s > \frac{n}{2}$ be a real number. Let $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. If I_1 is suitably small, then there exists a unique solution $u(t,x)$ of (3.1) in*

$$u \in C([0,\infty); H^s(\mathbb{R}^n)),$$

satisfying the following decay estimates :

Case 1 : $\theta \in (0,1)$

$$|||\nabla|^r u(t)||_{H^{s-\sigma_\theta(r,n)}} \leq cI_1(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}, \tag{3.31}$$

for $r \geq 0$ and $\sigma_\theta(r,n) \leq s$.

Case 2 : $\theta = 0$

$$|||\nabla|^r u(t)||_{H^{s-\sigma_0(k,n)}} \leq cI_1(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{3.32}$$

for $r \geq 0$ and $\sigma_0(r,n) \leq s$

Case 3 : $\theta = 1$

$$|||\nabla|^r u(t)||_{H^{s-r}} \leq cI_1(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{3.33}$$

for $r \geq 0$ *and* $r \leq s$.

Proof. Define

$$X := \{u \in C([0,\infty), H^s(\mathbb{R}^n)); ||u||_X < \infty\},$$

here

$$||u||_X := \begin{cases} \sup\limits_{t\geq 0} \|u(t)\|_{H^s} + \sup\limits_{\{r,\sigma_\theta(r,n)\leq s\}} \sup\limits_{t\geq 0} (1+t)^{\frac{n}{4\theta}+\frac{r}{2\theta}} |||\nabla|^r u(t)||_{H^{s-\sigma_\theta(r,n)}} & \text{for} \quad \theta \in (0,1), \\ \sup\limits_{t\geq 0} \|u(t)\|_{H^s} + \sup\limits_{\{r,\sigma_0(r,n)\leq s\}} \sup\limits_{t\geq 0} (1+t)^{\frac{n}{4}+\frac{r}{2}} |||\nabla|^r u(t)||_{H^{s-\sigma_0(r,n)}} & \text{for} \quad \theta = 0, \\ \sup\limits_{r\leq s} \sup\limits_{t\geq 0} (1+t)^{\frac{n}{4}+\frac{r}{2}} |||\nabla|^r u(t)||_{H^{s-r}} & \text{for} \quad \theta = 1. \end{cases}$$

We denote by S_R, the closed ball

$$S_R := \{u \in X; ||u||_X \leq R\}, \ R > 0;$$

Consider the mapping

$$\Phi[u](t) := \Phi_0(t) + \int_0^t H(t-\tau) * (1-\Delta)^{-1} f(u)(\tau) d\tau,$$

where

$$\Phi_0(t) := H(t) * u_0.$$

First, we show that Φ is a contraction on S_R. We have

$$\forall v, w \in X, \Phi[v](t) - \Phi[w](t) = \int_0^t H(t-\tau) * (1-\Delta)^{-1} (f(v) - f(w))(\tau) d\tau.$$

Noticing that $f(v) = O(|v|^\alpha)$ and using Proposition 1.7 and Proposition 1.9, we have the following inequalities for a given real number $r \geq 0$:

$$\begin{aligned} ||\partial_x^{[r]}(f(v)-f(w))(\tau)||_{L^1} &\leq C||(v,w)(\tau)||_{L^\infty}^{\alpha-2}\left(||(v,w)(\tau)||_{L^2}||\partial_x^{[r]}(v-w)(\tau)||_{L^2}\right. \\ &\quad \left.+||\partial_x^{[r]}(v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^2}\right), \qquad (3.34) \end{aligned}$$

$$\begin{aligned} |||\nabla|^r f(v)-f(w)(\tau)||_{L^2} &\leq C||(v,w)(\tau)||_{L^\infty}^{\alpha-2}\left(||(v,w)(\tau)||_{L^\infty}|||\nabla|^r(v-w)(\tau)||_{L^2}\right. \\ &\quad \left.+\,|||\nabla|^r(v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^\infty}\right). \qquad (3.35) \end{aligned}$$

From the Gagliardo-Nirenberg inequality, we have

$$||v(\tau)||_{L^\infty} \leq C||v||_X(1+\tau)^{-d_n}, \qquad (3.36)$$

with

$$d_n = \begin{cases} (1+\tau)^{-\frac{n}{2\theta}}, & \theta \in (0,1)\,,\ n=1, \\ (1+\tau)^{-\frac{n}{4\theta}}, & \theta \in (0,1)\,,\ n \geq 2, \\ (1+\tau)^{-\frac{n}{4}}, & \theta = 1,\ n \geq 1, \\ (1+\tau)^{-\frac{n}{4}}, & \theta = 0, \quad n \geq 1. \end{cases}$$

Indeed, by the Gagliardo-Nirenberg inequality, we have

$$||v(\tau)||_{L^\infty} \leq \|v\|_{L^2}^{1-\lambda}\, \||\nabla|^{s_0} v\|_{L^2}^{\lambda},$$

here $s_0 = \frac{n}{2} + \varepsilon_0$ with $\varepsilon_0 \in \left(0, \left(s+\frac{1}{2}-\frac{1}{\theta}\right)\theta\right]$ fixed and $\lambda = \frac{n}{2s_0}$.

When $n=1$, since $s \geq \frac{1}{2\theta}+1$, we have $s-\sigma_\theta(0,1) \geq 0$, thus $||v(\tau)||_{L^2} \leq (1+t)^{-\frac{1}{4\theta}}\|v\|_X$. Similarly, since $s > \frac{1}{\theta}-\frac{1}{2}$, we have from the choice of s_0 that $s-\sigma_\theta(s_0,1) \geq 0$ we have $\||\nabla|^{s_0} v\|_{L^2} \leq (t+1)^{-\frac{1}{4\theta}-\frac{s_0}{2\theta}}\|v\|_X$. Then $d_n = (1-\lambda)\frac{1}{4\theta}+\lambda\left(\frac{1}{4\theta}+\frac{s_0}{2\theta}\right) = \frac{1}{2\theta}$, it yields (3.36) with $n=1$.

When $n \geq 2$, is analogous to the first case, from $s \geq \frac{n}{2\theta}+1$, we get $s-\sigma_\theta(0,n) \geq s_0$.

Then $\|v\|_{L^2} \leq (t+1)^{-\frac{n}{4\theta}} \|v\|_X$ and $\||\nabla|^{s_0} v\|_{L^2} \leq (t+1)^{-\frac{n}{4\theta}} \|v\|_X$. Consequently we obtain $d_n = (1-\lambda)\frac{1}{4\theta} + \lambda\left(\frac{1}{4\theta}\right) = \frac{1}{4\theta}$ and $||v(\tau)||_{L^\infty} \leq (t+1)^{-\frac{1}{4\theta}} \|v\|_X$.

The case $\theta = 1$, by a similar calculation, we have for $n \geq 1$ and $s \geq s_0$ we have $||v(\tau)||_{L^\infty} \leq (t+1)^{-\frac{n}{4}} \|v\|_X$.

In fact for $n \geq 1$, we have $\|v\|_{L^2} \leq (t+1)^{-\frac{n}{4}} \|v\|_X$ and $\||\nabla|^{s_0} v\|_{L^2} \leq \|v\|_{H^{s_0}} \leq \|v\|_{H^s} \leq (t+1)^{-\frac{n}{4}} \|v\|_X$.

When $\theta = 0$, for $n = 1$ and $s \geq \frac{1}{2}$, we have $s - \sigma_0(0,1) \geq 0$, which implies $\|v\|_{L^2} \leq (t+1)^{-\frac{1}{4}} \|v\|_X$, and for $s > \frac{3}{2}$, we choose $\varepsilon_0 < \left(s - \frac{3}{2}\right)/2$ such that $s - \sigma_0(s_0, 1) \geq 0$, so $\||\nabla|^{s_0} v\|_{L^2} \leq (t+1)^{-\frac{1}{4}-\frac{s_0}{2}} \|v\|_X$.

For $n \geq 2$ and $s > n$, we have $s - \sigma_0(0,n) \geq s_0$, which gives $||v(\tau)||_{L^\infty} \leq (t+1)^{-\frac{n}{4}} \|v\|_X$.

Step 1 : $\theta \in (0,1)$

Now we prove the estimate:

$$|||\nabla|^r(\Phi[v] - \Phi[w])(t)||_{H^{s-\sigma_\theta(r,n)}} \leq C(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{3.37}$$

with $r \geq 0$ and $\sigma_\theta(r,n) \leq s$.

Let $r \geq 0$ be a real number satisfying $\sigma_\theta(r,n) \leq s$, and $m = s - \sigma_\theta(r,n)$, then we have

$$\begin{aligned} |||\nabla|^r(\Phi[v] - \Phi[w])(t)||_{H^{s-\sigma_\theta(r,n)}} &\leq \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)|||\nabla|^r H(t-\tau) * (f(v) - f(w))(\tau)||_{H^{s-\sigma_\theta(r,n)-2}} d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{3.38}$$

By virtue of (3.20) with $p = 1$, we have

$$\begin{aligned} I_1 &\leq C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(f(v) - f(w))(\tau)||_{L^1} d\tau \\ &\quad + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{l}{2(1-\theta)}}||(f(v) - f(w))(\tau)||_{H^{k+s-\sigma_\theta(k,n)+l-2}} d\tau \\ &=: I_{11} + I_{12}. \end{aligned} \tag{3.39}$$

By using (3.34) with $r = 0$ and (3.36) ,we have that

$$||(f(v) - f(w))(\tau)||_{L^1} \leq C||(v, w)||_X^{\alpha-1}||(v - w)||_X(1 + \tau)^{-d_n(\alpha-2)-\frac{n}{2\theta}}. \tag{3.40}$$

Since $\alpha > \alpha_n = 1 + \frac{2\theta}{n}$ for $n = 1, 2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (3.40), we have

$$I_{11} \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}(1 + \tau)^{-d_n(\alpha-2)-\frac{n}{2\theta}} d\tau||(v, w)||_X^{\alpha-1}||(v - w)||_X,$$

then

$$I_{11} \leq C(1 + t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v, w)||_X^{\alpha-1}||v - w||_X. \tag{3.41}$$

If $r + m + l - 2 \leq s$, taking $l = 2 + (1 - \theta)(\frac{n}{2\theta} + \frac{r}{\theta})$ by virtue of (3.35) and (3.36), it yields that

$$||(f(v) - f(w))(\tau)||_{H^s} \leq C||(v, w)||_X^{\alpha-1}||(v - w)||_X(1 + \tau)^{-d_n(\alpha-1)}. \tag{3.42}$$

We appeal to (3.42) and notice that $\alpha \geq 2$, then we have that

$$I_{12} \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{4\theta}-\frac{r}{2\theta}-\frac{1}{1-\theta}}(1 + \tau)^{-d_n(\alpha-1)} d\tau||(v, w)||_X^{\alpha-1}||(v - w)||_X.$$

Then, we get

$$I_{12} \leq C(1 + t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v, w)||_X^{\alpha-1}||v - w||_X,$$

Inserting the estimates I_{11} and I_{12} in (3.39), we obtain

$$I_1 \leq C(1 + t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v, w)||_X^{\alpha-1}||v - w||_X. \tag{3.43}$$

Also by employing (3.20) with $l=2$ and $p=1$ to the term I_2, we have

$$\begin{aligned} I_2 &\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4\theta}-\frac{r-[r]}{2\theta}}||\partial_x^{[r]}(f(v)-f(w))(\tau)||_{L^1}d\tau \\ &\quad +C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{1-\theta}}||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_\theta(r,n)}}d\tau \\ :&= I_{21}+I_{22}. \end{aligned} \tag{3.44}$$

In view of (3.34) and (3.36), we have that

$$||\partial_x^{[r]}(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2\theta}-\frac{[r]}{2\theta}}, \tag{3.45}$$

for $\sigma_\theta(r,n)\leq s$. Since $\alpha>\alpha_n=1+\frac{2\theta}{n}$ for $n=1,2$ and $\alpha\geq 2$ for $n\geq 3$, appealing to (3.45), we have

$$I_{21}\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4\theta}-\frac{r-[r]}{2\theta}}(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2\theta}-\frac{[r]}{2\theta}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X,$$

with $\sigma_\theta(r,n)\leq s$. Then

$$I_{21}\leq C(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $\sigma_\theta(r,n)\leq s$.

By applying (3.35) and (3.36) with r replaced by $r+m$ we get that

$$||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_\theta(r,n)}} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-d_n(\alpha-1)-\frac{n}{4\theta}-\frac{r}{2\theta}}. \tag{3.46}$$

It yields that

$$
\begin{aligned}
I_{22} &\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{1-\theta}}(1+\tau)^{-d_n(\alpha-1)-\frac{n}{4\theta}-\frac{r}{2\theta}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X,\\
&\leq C(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v,w)||_X^{\alpha-1}||v-w||_X.
\end{aligned}
$$

Using the estimates for I_{21} and I_{22} in (3.44), we find

$$
I_2 \leq C(1+t)^{-\frac{n}{4\theta}-\frac{r}{2\theta}}||(v,w)||_X^{\alpha-1}||v-w||_X. \tag{3.47}
$$

Putting the estimates (3.43) and (3.47) into (3.38), then we have (3.37).

Step 2 : $\theta = 0$

Now we prove the estimate:

$$
|||\nabla|^r(\Phi[v]-\Phi[w])(t)||_{H^{s-\sigma_0(r,n)}} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{3.48}
$$

with $r \geq 0$ and $\sigma_0(r,n) \leq s$.

Let $r \geq 0$ be a real number satisfying $\sigma_0(r,n) \leq s$, and $m = s-\sigma_0(r,n)$, then we have

$$
\begin{aligned}
|||\nabla|^r(\Phi[v]-\Phi[w])(t)||_{H^{s-\sigma_0(r,n)}} &\leq (\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^{t})|||\nabla|^r H(t-\tau)*(f(v)-f(w))(\tau)||_{H^{s-\sigma_0(r,n)-2}}d\tau\\
&=: I_1 + I_2.
\end{aligned} \tag{3.49}
$$

By virtue of (3.21), we have

$$
\begin{aligned}
I_1 &\leq C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)-2}}d\tau\\
&\quad +C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l}{2}}||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)+l-2}}d\tau\\
&=: I_{11}+I_{12}.
\end{aligned} \tag{3.50}
$$

By using (3.35) and (3.36), we have that

$$||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)-2}} \le C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)}. \qquad (3.51)$$

Since $\alpha > \alpha_n = 1+\frac{2}{n}$ for $n=1,2$ and $\alpha \ge 2$ for $n \ge 3$, appealing to (3.51), we have

$$I_{11} \le C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X$$

then

$$I_{11} \le Ce^{-ct}||(v,w)||_X^{\alpha-1}||v-w||_X. \qquad (3.52)$$

If $r+m+l-2 \le s$, taking $l=\frac{n}{2}+r+2$ by virtue of (3.35) and (3.36), it yields that

$$||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)+l-2}} \le C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)}. \qquad (3.53)$$

We appeal to (3.53) and notice that $\alpha \ge 2$, to have that

$$I_{12} \le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r}{2}-1}(1+\tau)^{-\frac{n}{2}(\alpha-1)}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X.$$

So

$$I_{12} \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X.$$

We put the estimates for I_{11} and I_{12} in (3.50), then we obtain

$$I_1 \le Ce^{-ct}||(v,w)||_X^{\alpha-1}||v-w||_X + C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X.$$

This shows that

$$I_1 \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X. \qquad (3.54)$$

Also by employing (3.21), we have

$$\begin{aligned} I_2 &\leq C\int_{\frac{t}{2}}^{t} e^{-c(t-\tau)}||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)-2}}d\tau \\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{l}{2}}||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)+l-2}}d\tau, \\ : &= I_{21}+I_{22}. \end{aligned} \tag{3.55}$$

In view of (3.35) and (3.36), we have that

$$||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)-2}} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{r}{2}}, \tag{3.56}$$

for $\sigma_0(r,n)\leq s$.

Since $\alpha>\alpha_n=1+\frac{2}{n}$ for $n=1,2$ and $\alpha\geq 2$ for $n\geq 3$, appealing to (3.56), we have

$$I_{21}\leq C\int_{\frac{t}{2}}^{t} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{r}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X,$$

with $\sigma_0(r,n)\leq s$, then

$$I_{21}\leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $\sigma_0(r,n)\leq s$.

By applying (3.35) with r replaced by $r+m$ and (3.36) we have that

$$||(f(v)-f(w))(\tau)||_{H^{r+s-\sigma_0(r,n)+l-2}} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{r}{2}}. \tag{3.57}$$

We choose $l=2$ we have that

$$I_{22}\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-1}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{r}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X.$$

Then

$$I_{22} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X.$$

Inserting the estimates for I_{21} and I_{22} in (3.55), we find

$$I_2 \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X. \tag{3.58}$$

We use the estimates (3.54) and (3.58) into (3.49), then we have (3.48).

Step 3 : $\theta = 1$**,** we prove the estimate:

$$|||\nabla|^k(\Phi[v]-\Phi[w])(t)||_{H^{s-r}} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{3.59}$$

with $0 \leq r \leq s$.

Let $r \geq 0$ be a real number satisfying $r \leq s$, and $m = s-r$, then we have

$$\begin{aligned} |||\nabla|^r(\Phi[v]-\Phi[w])(t)||_{H^{s-r}} &\leq (\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)|||\nabla|^r H(t-\tau)*(f(v)-f(w))(\tau)||_{H^{s-r-2}}d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{3.60}$$

By virtue of (3.22) with $p=1$, we have

$$\begin{aligned} I_1 &\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r}{2}}||(f(v)-f(w))(\tau)||_{L^1}d\tau \\ &\quad +C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}||(f(v)-f(w))(\tau)||_{H^{s-2}}d\tau, \\ &= : I_{11} + I_{12}. \end{aligned} \tag{3.61}$$

By using (3.34) with $r=0$ and (3.36), we have that

$$||(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)}. \tag{3.62}$$

Since $\alpha > \alpha_n = 1 + \frac{2}{n}$ for $n = 1, 2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (3.62), we have

$$I_{11} \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{4} - \frac{r}{2}} (1 + \tau)^{-\frac{n}{2}(\alpha-1)} d\tau ||(v, w)||_X^{\alpha-1} ||(v - w)||_X,$$

which implies

$$I_{11} \leq C(1 + t)^{-\frac{n}{4} - \frac{r}{2}} ||(v, w)||_X^{\alpha-1} ||v - w||_X. \tag{3.63}$$

By virtue of (3.35) and (3.36), it yields that

$$||(f(v) - f(w))(\tau)||_{H^{s-2}} \leq C||(v, w)||_X^{\alpha-1} ||(v - w)||_X (1 + \tau)^{-\frac{n}{2}(\alpha-1) - \frac{n}{4} - \frac{r}{2}}. \tag{3.64}$$

We appeal to (3.64) and notice that $\alpha \geq 2$, we get

$$I_{12} \leq C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1 + \tau)^{-\frac{n}{2}(\alpha-1) - \frac{n}{4} - \frac{r}{2}} d\tau ||(v, w)||_X^{\alpha-1} ||(v - w)||_X.$$

Therefore

$$I_{12} \leq Ce^{-ct} ||(v, w)||_X^{\alpha-1} ||v - w||_X.$$

Putting the estimates for I_{11} and I_{12} in (3.61), we obtain

$$I_1 \leq C(1 + t)^{-\frac{n}{4} - \frac{r}{2}} ||(v, w)||_X^{\alpha-1} ||v - w||_X + Ce^{-ct} ||(v, w)||_X^{\alpha-1} ||v - w||_X. \tag{3.65}$$

Hence

$$I_1 \leq C(1 + t)^{-\frac{n}{4} - \frac{r}{2}} ||(v, w)||_X^{\alpha-1} ||v - w||_X. \tag{3.66}$$

Also by employing (3.22) with $p = 1$ to the term I_2, we have

$$\begin{aligned} I_2 \leq\ & C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{n}{4} - \frac{r-[r]}{2}} ||\partial_x^{[r]} (f(v) - f(w))(\tau)||_{L^1} d\tau \\ & + C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} ||(f(v) - f(w))(\tau)||_{H^{s-2}} d\tau \\ :=\ & I_{21} + I_{22}. \end{aligned} \tag{3.67}$$

In view of (3.34) and (3.36), we have that

$$||\partial_x^{[r]}(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}, \tag{3.68}$$

for $0 \leq r \leq s$.

Since $\alpha > \alpha_n = 1+\frac{2}{n}$ for $n=1,2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (3.68), we have

$$I_{21} \leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[k]}{2}}(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X,$$

with $r \leq s$. Then

$$I_{21} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $r \leq s$.

Using (3.64) we have

$$I_{22} \leq C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{r}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X.$$

This yields

$$I_{22} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X.$$

Using the estimates I_{21} and I_{22} in (3.67), we get

$$I_2 \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X. \tag{3.69}$$

We put the estimates (3.66) and (3.69) into (3.60), then we have (3.59).

Step 4 : Combining the estimates (3.37) for $\theta \in (0,1)$, (3.48) for $\theta = 0$ and (3.59) for $\theta = 1$ we obtain that

$$||\Phi[v]-\Phi[w]||_X \leq C||(v,w)||_X^{\alpha-1}||v-w||_X.$$

So far we proved that $||\Phi[v] - \Phi[w]||_X \le C_1 R^{\alpha-1} ||v - w||_X$. if $v, w \in S_R$. On the other hand, $\Phi[0](t) = \Phi_0(t) = \bar{u}(t)$, and from Theorem 3.3 we know that $||\Phi_0||_X \le C_2 I_1$ if I_1 is suitably small. Take $R = 2C_2 I_1$. If I_1 is suitably small such that $R < 1$ and $C_1 R \le \frac{1}{2}$, then we have that

$$||\Phi[v] - \Phi[w]||_X \le \frac{1}{2} ||v - w||_X.$$

It yields that, for $v \in S_R$

$$||\Phi[v]||_X \le ||\Phi_0||_X + \frac{1}{2} ||v||_X \le C_2 I_1 + \frac{1}{2} R = R,$$

i.e. $\Phi[v] \in S_R$. Thus $v \to \Phi[v]$ is a contraction mapping on S_R, so there exists a unique $u \in S_R$ satisfying $\Phi[u] = u$, and it is the solution to the semi-linear problem (3.1) satisfying the decay estimate (3.31), (3.32) and (3.33). So we complete the proof of Theorem 3.4.

■

Chapter 4

Global existence and decay estimates of solutions for a system of semi-linear heat equations with memory involving the fractional Laplacian.

In this chapter, we study the global existence of small data solutions to the Cauchy problem associated to the coupled systems of semi-linear heat equations with dissipation of memory-type

$$\begin{cases} \partial_t u - \beta\Delta u + \displaystyle\int_0^t g(t-s)(-\Delta)^\theta u(s)ds = |v|^p, & t>0,\ x\in\mathbb{R}^n, \\ \partial_t v - \beta\Delta v + \displaystyle\int_0^t g(t-s)(-\Delta)^\theta v(s)ds = |u|^q, & t>0,\ x\in\mathbb{R}^n, \\ (u(0),v(0)) = (u_0(x), v_0(x)), & x\in\mathbb{R}^n, \end{cases} \tag{4.1}$$

where $n \geq 1$, $\beta > 0, \theta \in [0,1]$, $p > 1$, $q > 1$. Here u, v are real-valued unknown functions, $u_0(x), v_0(x)$ are the given initial data and $(-\Delta)^\theta$ is the fractional Laplacian operator which is defined in the whole space $\mathbb{R}^n$ through the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by

$$(-\Delta)^{\theta}h(x) = \mathcal{F}^{-1}\left(|\xi|^{2\theta}\,\mathcal{F}\,(h)\,(\xi)\right)(x)\,,\; x \in \mathbb{R}^n.$$

The function term g corresponds to the memory term, that satisfies the same assumptions of the chapter 03

4.0.1 Linear cauchy problem with memory-type dissipation

Now, let us consider the linear Cauchy problem with memory term, namely

$$\begin{cases} \partial_t u\,(t,x) - \Delta u\,(t,x) + g * (-\Delta)^{\theta} u\,(t,x) = 0, & t > 0,\; x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

The solution of problem (4.2) given via the Laplace and Fourier inverse transforms by

$$u(t,x) = u_0\,(x) * H(t,x), \tag{4.3}$$

such that H is the fundamental solution corresponding Cauchy problem (4.2), where

$$H(t,x) = \mathcal{F}^{-1}\left(\mathcal{L}^{-1}\left[\frac{1}{\lambda + \beta|\xi|^2 + |\xi|^{2\theta}\mathcal{L}[g]}\right]\right)(t,x).$$

Therefore, from Theorem 2.1, we can easily prove the next Theorem

Theorem 4.1 *Let $s \geq 0$, $\theta \in [0,1], n \geq 1$ and $m \in [1,2]$. Assume that $u_0 \in \mathcal{D}^s_m(\mathbb{R}^n)$. Then, we have the following estimate:*

$$|||D|^s u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}}||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}.$$

On the other hand, if we assume $u_0 \in L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ for $m \in [1,2]$, then the following *estimate holds:*

$$||u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u_0||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}. \tag{4.4}$$

Corollary 4.1 *Let us consider the Cauchy problem* (4.2) *with* $\theta \in [0,1]$ *and initial data* u_0 *is from* $\dot{H}^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ *for* $s \geq 1, m \in [1,2)$*. Then, we obtain the following estimate:*

$$|||D|^s u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}} ||u_0||_{\dot{H}^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}.$$

4.1 Semi-linear Cauchy problem with memory-type dissipation

This section is devoted to the study of global existence results for the following Cauchy problem

$$\begin{cases} \partial_t u\,(t,x) - \Delta u\,(t,x) + g * (-\Delta)^\theta u\,(t,x) = |u|^p\,, & t>0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.5}$$

4.1.1 Main results

Let us now denote by $p_{Fuj}\left(\dfrac{n}{m}\right)$ to $1+\dfrac{2m}{n}$ for $n \geq 1, m \in [1,2)$. Then, we can distinguish the following classification of regularity: data from $\mathcal{D}^s_m$ with $s=1$ and data from $\mathcal{D}^s_m$ with $s > \frac{n}{2}$.

Theorem 4.2 *Let* $n \leq \dfrac{4}{2-m}$ *and* $n < \dfrac{2m}{m-1}$ *where* $m \in (1,2)$*. Assume that* $u_0 \in \mathcal{D}^1_m(\mathbb{R}^n)$ *for* $m \in (1,2)$*. The exponent* p *satisfies*

$$p > p_{Fuj}\left(\frac{n}{m}\right), \tag{4.6}$$

and

$$\begin{aligned} &\tfrac{2}{m} \leq p < \infty \qquad &\text{if } n \leq 2, \\ &\tfrac{2}{m} \leq p \leq \frac{n}{n-2} \qquad &\text{if } n \geq 3. \end{aligned} \tag{4.7}$$

Then, there exists a small constant $\varepsilon_0 > 0$ *such that if* $||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} \leq \varepsilon_0$ *then there exists a uniquely determined globally (in time) solution to* (4.5) *such that*

$$u \in \mathcal{C}([0,\infty), H^1(\mathbb{R}^n)).$$

Moreover, the solution satisfies the estimates

$$||u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)},$$

$$||\nabla u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}} ||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)}.$$

Now, we are interested in the case when initial conditions having a large regularity such that they belong to $L^\infty(\mathbb{R}^n)$.

Theorem 4.3 *Let* $n \geq 5$ *and* $s > \frac{n}{2}$. *Let* $u_0 \in \mathcal{D}^s_m(\mathbb{R}^n)$ *for* $m \in [1,2)$. *Let us assume the exponent* p *satisfies*

$$1 + s < p.$$

Then, there exists a constant $\varepsilon_0 > 0$ *such that if* $||u_0||_{\mathcal{D}^s_m} \leq \varepsilon_0$, *then there exists a uniquely globally in time solution to* (4.5) *such that*

$$u \in \mathcal{C}([0,\infty), H^s(\mathbb{R}^n)).$$

Moreover, the solution satisfies the estimates

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)},$$
$$\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}} ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}.$$

4.1.2 Proof of Theorem 4.2

Assume first that N is an operator defined as follows:

$$N : u \in X(T) \to Nu = u^{\text{ln}}(t,x) + u^{\text{nl}}(t,x),$$

and we put:

$$u^{\text{ln}}(t,x) := H(t,x) * u_0(x),$$

$$u^{\text{nl}}(t,x) := \int_0^t H(t-\tau,x) * |u(\tau,x)|^p d\tau, \qquad fort > \tau, x \in \mathbb{R}^n,$$

such that H is the fundamental solution to the cauchy problem (4.2). Let us introduce the solutions space $X(T)$ for all $T > 0$ by:

$$X(T) = \mathcal{C}([0,T], H^1(\mathbb{R}^n)),$$

and the corresponding norm of $X(T)$ has the form

$$||u||_{X(T)} = \sup_{0 \le t \le T} \big((1+t)^{\frac{n(2-m)}{4m}} ||u(t,.)||_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2m}{4m}} ||\nabla u(t,.)||_{L^2(\mathbb{R}^n)} \big).$$

We are now interested the proof of following inequalities:

$$||Nu||_{X(T)} \lesssim ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)} + ||u||^p_{X(T)}, \tag{4.8}$$

and also, the Lipschitz condition

$$||Nu - N\tilde{u}||_{X(T)} \lesssim ||u - \tilde{u}||_{X(T)} (||u||^{p-1}_{X(T)} + ||\tilde{u}||^{p-1}_{X(T)}). \tag{4.9}$$

The previous inequality (4.8) implies for fixed point $u = u(t,x)$ and for small u_0 the following estimate

$$||u||_{X(T)} \lesssim ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)},$$

we can find from this inequality the desired estimates for the solution. We use the explicit formula of the norm of the solution space $X(T)$, we deduce

$$||u^{\text{ln}}||_{X(T)} \lesssim ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}.$$

Now, our aim is to show

$$||u^{\text{nl}}||_{X(T)} \lesssim ||u||^p_{X(T)}. \tag{4.10}$$

Firstly, we present the following Lemma

Lemma 4.1 *Let $t > 0$ and for all $\tau > 0$, such that $0 < \tau < t$, and p satisfies the condition* (4.7)*, then we have*

$$\||u(\tau,\cdot)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{4}} \|u\|^p_{X(t)}, \tag{4.11}$$

$$\||u(\tau,\cdot)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}} \|u\|^p_{X(t)}, \tag{4.12}$$

$$\||u(\tau,\cdot)|^p - |\tilde{u}(\tau,\cdot)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{4}} \times \|u-\tilde{u}\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|\tilde{u}\|^{p-1}_{X(t)}\right), \tag{4.13}$$

moreover, we have

$$\||u(\tau,\cdot)|^p - |\tilde{u}(\tau,\cdot)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}} \times \|u-\tilde{u}\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} + \|\tilde{u}\|^{p-1}_{X(t)}\right), \tag{4.14}$$

Proof. We start the proof of this Lemma by the inequality (4.12). We use the Gagliardo-Nirenberg inequality, we get

$$\begin{aligned}
&\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \\
&= \left(\int_{\mathbb{R}^n} |u(\tau,x)|^{mp} dx\right)^{\frac{1}{mp}p} = \|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim \|u(\tau,\cdot)\|^{(1-\theta_1)p}_{L^2(\mathbb{R}^n)} \||D|^s u(\tau,\cdot)\|^{\theta_1 p}_{L^2(\mathbb{R}^n)},
\end{aligned}$$

noting by $\theta_1 = \frac{n}{s}\left(\frac{1}{2} - \frac{1}{mp}\right)$ from $[0,1]$.

From explicit formula of the norm of the space $X(t)$, we have for $0 \leq \tau \leq t$

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+\tau)^{(1-\theta_1)p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta_1 p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)} \|u\|^p_{X(t)},$$

then, we deduce that,

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}} \|u\|^p_{X(t)}.$$

If we put $m=2$, then we can deduce (4.11). On the other hand, we use the Hölder's inequality for $0 \leq \tau \leq t$, we obtain

$$\||u(\tau,x)|^p - |\tilde{u}(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim \|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\|_{L^{mp}(\mathbb{R}^n)} \left(\|u(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)} + \|\tilde{u}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}\right).$$

Next, we show that by using the explicit formula of the norm of the space $X(t)$ and using the Gagliardo-Nirenberg inequality to estimate the norms $\|u(\tau,\cdot)- \tilde{u}(\tau,\cdot)\|_{L^{mp}(\mathbb{R}^n)}$, $\| u(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}$ and $\|\tilde{u}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}$ in a similar way as (4.12) we get the desired estimates. Finally, by setting $m=2$ we conclude (4.13). ■ We now return to the proof of Theorem 4.2. Firstly, to prove inequality (4.10), we start by estimating $||u^{\mathrm{nl}}||_{L^2(\mathbb{R}^n)}$ and $||\nabla_x u^{\mathrm{nl}}||_{L^2(\mathbb{R}^n)}$. From definition of nonlinear part, we obtain via Theorem 4.1

$$\begin{aligned}
||u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n(2-m)}{4m}} |||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)} d\tau + \int_{t/2}^{t} |||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)} d\tau \\
&\lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u||^p_{X(t)} \int_0^{t/2} (1+\tau)^{-\frac{n(p-1)}{2m}} d\tau + (1+t)^{1-\frac{n(2p-m)}{4m}} ||u||^p_{X(t)} \\
&\lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u||^p_{X(t)},
\end{aligned}$$

such that $p > p_{Fuj}(\frac{n}{m})$. It follows that

$$||u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u||^p_{X(t)}. \tag{4.15}$$

Similarly, by using Corollary 4.1, with $\dot{H}^1 - H^1 \cap L^m$ estimates for the integral over the interval $[0, t/2]$ and $\dot{H}^1 - L^2$ estimates for the integral over the interval $[t/2, t]$ to obtain

$$\begin{aligned}
||\nabla u^{\text{nl}}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim & \int_0^{t/2} (1+t-\tau)^{-\frac{n(2-m)+2m}{4m}} |||u(\tau,x)|^p||_{\dot{H}^1(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)} d\tau \\
& + \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} |||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)} d\tau \\
\lesssim & \int_0^{t/2} (1+t-\tau)^{-\frac{n(2-m)+2m}{4m}} (1+\tau)^{-\frac{n(p-1)}{2m}} ||u||^p_{X(t)} d\tau \\
& + \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{n(2p-m)}{4m}} ||u||^p_{X(t)} d\tau \\
\lesssim & (1+t)^{-\frac{n(2-m)+2m}{4m}} ||u||^p_{X(t)} \int_0^{t/2} (1+\tau)^{-\frac{n(p-1)}{2m}} d\tau \\
& + (1+t)^{-\frac{n(2p-m)}{4m}} ||u||^p_{X(t)} \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} d\tau,
\end{aligned}$$

such that $p > p_{Fuj}\left(\dfrac{n}{m}\right)$. We find

$$||\nabla u^{\text{nl}}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}} ||u||^p_{X(t)}. \tag{4.16}$$

Thus, by combining (4.15) with (4.16) we obtain the desired estimate (4.10). Next, we prove the inequality (4.9) of Lipschitz condition. We suppose that $u = u(t,x)$ and $\tilde{u} = \tilde{u}(t,x)$ are two elements in the space $X(T)$. It follows that

$$Nu - N\tilde{u} = \int_0^t H(t-\tau,x) * (|u(\tau,x)|^p - |\tilde{u}(\tau,x)|^p) d\tau.$$

Finally, in a similar way to that for (4.15) and (4.16) yields

$$||(Nu - N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u-\tilde{u}||_{X(t)} (||u||^{p-1}_{X(t)} + ||\tilde{u}||^{p-1}_{X(t)}),$$

and also

$$||\nabla_x(Nu - N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}} ||u-\tilde{u}||_{X(t)}(||u||^{p-1}_{X(t)} + ||\tilde{u}||^{p-1}_{X(t)}).$$

In this way, the proof is completed.

4.1.3 Proof of Theorem 4.3

We start the proof by defining the Banach space $X(T)$ for all $T>0$ by:

$$X(T) = \mathcal{C}([0,T], H^s(\mathbb{R}^n)),$$

the corresponding norm of space $X(T)$ is

$$||u||_{X(T)} = \sup_{0\le t\le T}((1+t)^{\frac{n(2-m)}{4m}}||u(t,.)||_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2ms}{4m}}|||D|^s u(t,.)||_{L^2(\mathbb{R}^n)}).$$

We devoted to estimate all norms making up the norm $||u||_{X(T)}$ which are $||u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)}$ and $|||D|^s u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)}$.

We now return to the proof and we start by estimating $|||D|^s u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)}$. Using the definition of nonlinear part of the operator N and applying Corollary 4.1, we obtain

$$\begin{aligned}
|||D|^s u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p||_{H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}d\tau, \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}d\tau \\
&+ \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)}d\tau, \\
&\equiv I_1 + I_2.
\end{aligned} \tag{4.17}$$

Make $m=2$, for all $\tau \in [t/2, t]$, it follows that

$$
\begin{aligned}
I_1 &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}d\tau \\
&\quad + \int_{t/2}^{t}(1+t-\tau)^{-\frac{s}{2}}|||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)}d\tau \\
&\equiv I_{11}+I_{12}.
\end{aligned}
$$

On the other hand, for $m=2$, for $\tau\in[t/2,t]$, we get

$$
\begin{aligned}
I_2 &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)}d\tau + \int_{t/2}^{t}(1+t-\tau)^{-\frac{s}{2}}|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)}d\tau \\
&\equiv I_{21}+I_{22}.
\end{aligned}
$$

Using the fractional Gagliardo-Nirenberg inequality, we get for $0\le\tau\le t$

$$
|||u(\tau,x)|^p||_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n(p-1)}{2m}}||u||^p_{X(t)}, \tag{4.18}
$$

such that

$$
\begin{aligned}
&\tfrac{2}{m}\le p<\infty \qquad \text{if } n\le 2s,\\
&\tfrac{2}{m}\le p\le \tfrac{2n}{m(n-2s)} \quad \text{if } n>2s.
\end{aligned} \tag{4.19}
$$

Therefore

$$
\begin{aligned}
I_{11} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}(1+\tau)^{-\frac{n(p-1)}{2m}}d\tau||u||^p_X \\
&\lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}\int_0^{t/2}(1+\tau)^{-\frac{n(p-1)}{2m}}d\tau||u||^p_X \\
&\lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}||u||^p_X,
\end{aligned}
$$

such that $p>p_{Fuj}(\frac{n}{m})$. Moreover, by applying the fractional Gagliardo-Nirenberg inequality, it follows that for $0\le\tau\le t$,

$$
|||u(\tau,x)|^p||_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n(2p-m)}{4m}}||u||^p_{X(t)}, \tag{4.20}
$$

with

$$\begin{aligned} &1 \le p < \infty && \text{if } n \le 2s, \\ &1 \le p \le \tfrac{n}{n-2s} && \text{if } n > 2s. \end{aligned} \tag{4.21}$$

We use both inqualities (4.18) and (4.20) into the expression of I_{12}, it follows that

$$\begin{aligned} I_{12} &\lesssim \int_{t/2}^{t} (1+t-\tau)^{-\frac{s}{2}} (1+\tau)^{-\frac{n(2p-m)}{4m}} d\tau ||u||_X^p \\ &\lesssim (1+t)^{-\frac{n(2p-m)}{4m}} \int_{t/2}^{t} (1+t-\tau)^{-\frac{s}{2}} d\tau ||u||_X^p \\ &\lesssim (1+t)^{-\frac{n(2p-m)}{4m}-\frac{s}{2}+1} ||u||_X^p, \end{aligned}$$

it is clear that for $p > p_{Fuj}(\frac{n}{m})$. we can deduce

$$I_{12} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} ||u||_X^p.$$

Gathering the estimates for I_{11} and I_{12}, yields

$$I_1 \le (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} ||u||_X^p.$$

On the other hand, we deal with I_2. Let us start with an estimate for the term I_{21}. We use the fractional powers rule, we get

$$|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)} \lesssim ||u(\tau,x)||_{L^\infty(\mathbb{R}^n)}^{p-1} ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}$$

Using Proposition 1.11, yields

$$\begin{aligned} |||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)} &\lesssim \Big[||u(\tau,x)||_{\dot{H}^{s^*}(\mathbb{R}^n)} + ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}\Big]^{p-1} ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)} \\ &\lesssim ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}^p + ||u(\tau,x)||_{H^{s^*}(\mathbb{R}^n)}^{p-1} ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}. \end{aligned}$$

such that $s^* < \frac{n}{2} < s$, by the definition of the space $X(t)$, we have

$$\begin{aligned}
|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)} &\lesssim \left[(1+\tau)^{-\frac{n(2-m)+2ms}{4m}p} + (1+\tau)^{-\frac{n(2-m)+2ms^*}{4m}(p-1)-\frac{n(2-m)+2ms}{4m}}\right]||u||_X^p \\
&\lesssim \left[(1+\tau)^{-\frac{n(2-m)+2ms}{4m}p} + (1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}\right]||u||_X^p.
\end{aligned}$$

Thus, we get for $0 \le \tau \le t$

$$|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}||u||_X^p. \tag{4.22}$$

Therefore, the estimate of I_{21} is obtained as follows

$$I_{21} \lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}(1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}d\tau||u||_X^p,$$

such that

$$\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m} > 1.$$

We deduce that

$$I_{21} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}||u||_X^p,$$

such that

$$p > \frac{2+s^*-s}{n(\frac{2-m}{2m})+s^*}.$$

In a similar was as for I_{21}, we can estimate I_{22}. Indeed, from (4.22) we get

$$\begin{aligned}
I_{22} &\equiv \int_{t/2}^{t}(1+t-\tau)^{-\frac{s}{2}}|||u(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)}d\tau \\
&\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{s}{2}}(1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}||u||_X^p d\tau \\
&\lesssim (1+t)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}||u||_X^p\int_{t/2}^{t}(1+t-\tau)^{-\frac{s}{2}}d\tau,
\end{aligned}$$

since $n > 4$ and we have $1 < \frac{n}{4} < \frac{s}{2}$,

$$I_{22} \lesssim (1+t)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}} ||u||_X^p,$$

such that $\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m} > \frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) + \frac{s}{2}$, we get

$$I_{22} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} ||u||_X^p.$$

We conclude that

$$I_2 \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} ||u||_X^p.$$

Consequently, we get

$$|||D|^s u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} ||u||_X^p. \tag{4.23}$$

In a similar way, we use (4.18) and (4.20) yields

$$||u^{\mathrm{nl}}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||u||_{X(t)}^p. \tag{4.24}$$

We deduce from the estimates (4.23) and (4.24), the desired estimate (4.10).

Finally, we derive the Lipschitz condition. It is easy to show that

$$Nu - N\tilde{u} = \int_0^t H(t-\tau,x) * (|u(\tau,x)|^p - |\tilde{u}(\tau,x)|^p) d\tau,$$

such that $u = u(t,x)$ and $\tilde{u} = \tilde{u}(t,x)$ are belong the function space $X(T)$.

Next, we have to control all norms making up $||(Nu - N\tilde{u})(t,.)||_{X(t)}$, which are $|||D|^s(Nu - N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)}$ and $||Nu - N\tilde{u}||_{L^2(\mathbb{R}^n)}$.

Similarly, we use the same argument as in the above, to get

$$\begin{aligned} |||D|^s(Nu-N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)} &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p||_{H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)}d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}|||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)}d\tau. \end{aligned} \tag{4.25}$$

We use the fractional powers rule and proposition 1.11 for $s^* < \frac{n}{2}$, we find for $0 \leq \tau \leq t$:

$$|||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p||_{\dot{H}^s(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{2np-m(p-1)+2ms^*(p-1)+2ms}{4m}}||u-\tilde{u}||_{X(t)}(||u||^{p-1}_{X(t)}+||\tilde{u}||^{p-1}_{X(t)}). \tag{4.26}$$

Consequently, we have

$$|||D|^s(Nu-N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}}||u-\tilde{u}||_{X(t)}(||u||^{p-1}_{X(t)}+||\tilde{u}||^{p-1}_{X(t)}).$$

Similarly, we can prove

$$||(Nu-N\tilde{u})(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}}||u-\tilde{u}||_{X(t)}(||u||^{p-1}_{X(t)}+||\tilde{u}||^{p-1}_{X(t)}).$$

The proof is complete.

4.2 Global existence and decay estimates of solutions for a weakly coupled system of semi-linear cauchy problem with memory-type dissipation

4.2.1 Main results

In this section we apply results of the previous section to study the system of weakly coupled cauchy problem with memory-type

Initial data from the energy space

We start by a result of global existence with a loss of decay and the exponents p, q satisfying the following condition :

$$\alpha_{\max}(m) = m\left(\frac{\max\{p;q\}+1}{pq-1}\right) < \frac{n}{2}.$$

We have

Theorem 4.4 *Let $m \in [1,2)$, $n \leq \frac{2m^2}{2-m}$, $n < \frac{2m}{m-1}$, and assume that $(u_0, v_0) \in \mathcal{D}^1_m(\mathbb{R}^n) \times \mathcal{D}^1_m(\mathbb{R}^n)$. If the exponents p and q satisfy*

$$\begin{array}{lr} \frac{2}{m} \leq \min\{p;q\} < p_{Fuj}\left(\frac{n}{m}\right) < \max\{p;q\} < \infty & \text{if } n \leq 2, \\ \frac{2}{m} \leq \min\{p;q\} < p_{Fuj}\left(\frac{n}{m}\right) < \max\{p;q\} \leq \frac{n}{n-2} & \text{if } n \geq 3, \end{array} \tag{4.27}$$

and

$$\alpha_{\max}(m) = m\left(\frac{\max\{p;q\}+1}{pq-1}\right) < \frac{n}{2}. \tag{4.28}$$

Then there exists a small constant ε_0 such that if $||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} + ||v_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} \leq \varepsilon_0$, then there exists a uniquely global mild solution to (4.1) such that

$$(u, v) \in (\mathcal{C}([0,\infty), H^1(\mathbb{R}^n)))^2.$$

Moreover, the following estimates hold

$$||u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}+[\gamma_{n,m}(p)]^+}(||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} + ||v_0||_{\mathcal{D}^1_m(\mathbb{R}^n)}),$$

$$||\nabla_x u(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}+[\gamma_{n,m}(p)]^+}(||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} + ||v_0||_{\mathcal{D}^1_m(\mathbb{R}^n)}),$$

$$||v(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}+[\gamma_{n,m}(q)]^+}(||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} + ||v_0||_{\mathcal{D}^1_m(\mathbb{R}^n)}),$$

$$||\nabla_x v(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}+[\gamma_{n,m}(q)]^+}(||u_0||_{\mathcal{D}^1_m(\mathbb{R}^n)} + ||v_0||_{\mathcal{D}^1_m(\mathbb{R}^n)}).$$

such that

$$\gamma_{n,m}(p) = -\frac{n}{2m}(p-1)+1 \text{ or } \gamma_{n,m}(q) = -\frac{n}{2m}(q-1)+1,$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution u or v to the linear Cauchy problem with vanishing right hand-side.

4.2.2 Initial data from Sobolev spaces

This case has been classified to benefit from embedding in $L^\infty(\mathbb{R}^n)$, where the data are supposed to have a high regularity.

Theorem 4.5 *Let $n \geq 5$, let $(u_0, v_0) \in \mathcal{D}^s_m(\mathbb{R}^n)\times \mathcal{D}^s_m(\mathbb{R}^n)$ with $m \in [1,2)$ and $s > \frac{n}{2}+1$. Moreover, we assume for the exponents p and q the conditions*

$$p > s \quad \text{and} \quad q > s,$$

Then, there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} + \|v_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} \leq \varepsilon_0$, then there exists a uniquely globally solution to (4.1) such that

$$(u,v) \in \mathcal{C}\left([0,\infty), H^s(\mathbb{R}^n)\right) \times \mathcal{C}\left([0,\infty), H^s(\mathbb{R}^n)\right).$$

Moreover, the solution satisfies the estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}} \left(\|u_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} + \|v_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)}\right), \\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}} \left(\|u_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} + \|v_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)}\right), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}} \left(\|u_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} + \|v_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)}\right), \\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}} \left(\|u_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)} + \|v_0\|_{\mathcal{D}^s_m(\mathbb{R}^n)}\right).
\end{aligned}$$

4.2.3 Proof of Theorem 4.4

We define first the norm of the solution space $X(T)$ as follows

$$||(u,v)||_{X(T)} = \sup_{0\leq t\leq T}(M_1(t,u)+M_2(t,v)),$$

such that $M_1(t,u)$ and $M_2(t,v)$ are suitably chosen for each Theorem. Assume that N is an operator defined by

$$N:(u,v)\in X(T)\rightarrow N(u,v)=(u^{\text{ln}}+u^{\text{nl}},v^{\text{ln}}+v^{\text{nl}}),$$

such that

$$u^{\text{ln}}(t,x):=H(t,x)*u_0(x),\quad u^{\text{nl}}(t,x):=\int_0^t H(t-\tau,x)*|v(\tau,x)|^p d\tau,$$

$$v^{\text{ln}}(t,x):=H(t,x)*v_0(x),\quad v^{\text{nl}}(t,x):=\int_0^t H(t-\tau,x)*|u(\tau,x)|^q d\tau.$$

We are interested the proof of following inequalities which imply among other things the global existence of small data solutions:

$$||N(u,v)||_{X(T)}\lesssim ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}+||v_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}+||(u,v)||^p_{X(T)}+||(u,v)||^q_{X(T)}, \tag{4.29}$$

$$\begin{aligned}||N(u,v)-N(\tilde{u},\tilde{v}||_{X(T)} \lesssim\ & ||(u,v)-(\tilde{u},\tilde{v})||_{X(T)}\\ &\times(||(u,v)||^{p-1}_{X(T)}+||(\tilde{u},\tilde{v})||^{p-1}_{X(T)}+||(u,v)||^{q-1}_{X(T)}+||(\tilde{u},\tilde{v})||^{q-1}_{X(T)}).\end{aligned} \tag{4.30}$$

We use explicit formula from definition of the norm of the space $X(T)$, we deduce

$$||(u^{\text{ln}},v^{\text{ln}})||_{X(T)}\lesssim ||u_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}+||v_0||_{\mathcal{D}^s_m(\mathbb{R}^n)}.$$

Next, we complete the proof of all results separately by showing the inequality

$$||(u^{\mathrm{nl}}, v^{\mathrm{nl}})||_{X(T)} \lesssim ||(u,v)||^p_{X(T)} + ||(u,v)||^q_{X(T)}, \tag{4.31}$$

then, we can find from this inequality the desired estimate (4.29). Without loss of generality, we assume in the proof only the case $p < q$.

We assume $p < p_{Fuj}(\frac{n}{m}) < q$. Let $s = 1$ and the solution space

$$X(T) = \left(\mathcal{C}\left([0,T], H^1(\mathbb{R}^n)\right)\right)^2,$$

and we put

$$M_1(t,u) = (1+t)^{-\gamma_{n,m}(p)}((1+t)^{\frac{n(2-m)}{4m}}||u(t,.)||_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2m}{4m}}||\nabla_x u(t,.)||_{L^2(\mathbb{R}^n)}),$$

with

$$M_2(t,v) = (1+t)^{\frac{n(2-m)}{4m}}||v(t,.)||_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2m}{4m}}||\nabla_x v(t,.)||_{L^2(\mathbb{R}^n)}.$$

Lemma 4.2 *For $0 \le \tau \le t$, the following estimates*

$$\||v(\tau,\cdot)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{4}}\,\|(u,v)\|^p_{X(t)}, \tag{4.32}$$

$$\||v(\tau,\cdot)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}}\,\|(u,v)\|^p_{X(t)}, \tag{4.33}$$

$$\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m}(p)q}\,\|(u,v)\|^q_{X(t)}, \tag{4.34}$$

$$\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q}\,\|(u,v)\|^q_{X(t)}, \tag{4.35}$$

$$\||v(\tau,\cdot)|^p - |\tilde{v}(\tau,\cdot)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{4}} \times \|v-\tilde{v}\|_{X(t)}\left(\|v\|^{p-1}_{X(t)} + \|\tilde{v}\|^{p-1}_{X(t)}\right), \tag{4.36}$$

$$\||v(\tau,\cdot)|^p - |\tilde{v}(\tau,\cdot)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}} \times \|v-\tilde{v}\|_{X(t)}\left(\|v\|^{p-1}_{X(t)} + \|\tilde{v}\|^{p-1}_{X(t)}\right), \tag{4.37}$$

$$\left\| |u(\tau,\cdot)|^q - |\tilde{u}(\tau,\cdot)|^q \right\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m}(p)q} \times \|u-\tilde{u}\|_{X(t)} \left(\|u\|^{q-1}_{X(t)} + \|\tilde{u}\|^{q-1}_{X(t)} \right), \tag{4.38}$$

and

$$\left\| |u(\tau,\cdot)|^q - |\tilde{u}(\tau,\cdot)|^q \right\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q} \times \|u-\tilde{u}\|_{X(t)} \left(\|u\|^{q-1}_{X(t)} + \|\tilde{u}\|^{q-1}_{X(t)} \right), \tag{4.39}$$

hold provided that the condition (4.27) *is satisfied.*

We start by the proof of (4.33). We use the Gagliardo-Nirenberg inequality, we get

$$\left\| |v(\tau,x)|^p \right\|_{L^m(\mathbb{R}^n)} = \|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim \|v(\tau,\cdot)\|^{(1-\theta)p}_{L^2(\mathbb{R}^n)} \left\| |D|^s v(\tau,\cdot) \right\|^{\theta p}_{L^2(\mathbb{R}^n)},$$

such that $\theta = \frac{n}{s}\left(\frac{1}{2} - \frac{1}{mp}\right) \in [0,1]$.

We use for $0 \le \tau \le t$ the definition of the norm of the solution space $X(t)$ we find

$$\|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+\tau)^{(1-\theta)p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)} \|(u,v)\|^p_{X(t)}.$$

Therefore

$$\|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{2m}p+\frac{n}{2m}} \|(u,v)\|^p_{X(t)}.$$

Similarly, we can prove (4.35). Let $m=2$ we deduce (4.32) and (4.34).

For (4.37), using the mean value theorem and Hölder's inequality, it follows that

$$\begin{aligned} &\left\| |v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p \right\|_{L^m(\mathbb{R}^n)} \\ &\quad\lesssim \|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\|_{L^{mp}(\mathbb{R}^n)} \left(\|v(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)} + \|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)} \right). \end{aligned}$$

A repeated application of the Gagliardo-Nirenberg inequality to the terms $\|v(\tau,\cdot) - \tilde{v}(\tau,\cdot)\|_{L^{mp}(\mathbb{R}^n)}$, $\| v(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}$ and $\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}$ with an argument similar to the one used to show estimate (4.33), we conclude (4.37), which leads to the desired estimates. Following the same ideas we obtain (4.39). Finally, let $m=2$ one may conclude (4.36) and (4.38).

We come back to the proof of Theorem 4.4. To prove (4.31) we have to control all components of the norm with respect to $u^{\rm nl} = u^{\rm nl}(t,x)$ and $v^{\rm nl} = v^{\rm nl}(t,x)$. For $u^{\rm nl}$ we use the estimates, which are proved in Theorem 4.1 with $m = 2$ for the integral over $[t/2,t]$, we have

$$||u^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n(2-m)}{4m}} |||v(\tau,x)|^p||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)} d\tau + \int_{t/2}^t |||v(\tau,x)|^p||_{L^2(\mathbb{R}^n)} d\tau. \tag{4.40}$$

Inserting these estimates into (4.40), it implies

$$\begin{aligned} ||u^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}} ||(u,v)||^p_{X(t)} \int_0^{t/2} (1+\tau)^{-\frac{n(p-1)}{2m}} d\tau \\ &\quad + (1+t)^{1-\frac{n(2p-m)}{4m}} ||(u,v)||^p_{X(t)} \\ &\lesssim (1+t)^{-\frac{n(2-m)}{4m} + \gamma_{n,m}(p)} ||(u,v)||^p_{X(t)}. \end{aligned}$$

Then

$$||u^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m} + \gamma_{n,m}(p)} ||(u,v)||^p_{X(t)}. \tag{4.41}$$

Similarly, we find

$$||\nabla_x u^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m} + \gamma_{n,m}(p)} ||(u,v)||^p_{X(t)}. \tag{4.42}$$

For $v^{\rm nl}$ we use also the estimates proved in Theorem 4.1 with $m = 2$ for the integral over $[t/2,t]$ to obtain

$$||v^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n(2-m)}{4m}} |||u(\tau,x)|^q||_{L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)} d\tau + \int_{t/2}^t |||u(\tau,x)|^q||_{L^2(\mathbb{R}^n)} d\tau.$$

Therefore, we get

$$\begin{aligned}||v^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}}||(u,v)||^q_{X(t)}\int_0^{t/2}(1+\tau)^{-\frac{n(q-1)}{2m}+q(\gamma_{n,m}(p))}d\tau\\ &\quad +(1+t)^{1-\frac{n(2q-m)}{4m}+q(\gamma_{n,m}(p))}||(u,v)||^q_{X(t)}\\ &\lesssim (1+t)^{-\frac{n(2-m)}{4m}}||(u,v)||^q_{X(t)},\end{aligned}$$

such that $-\frac{n(q-1)}{2m}+q(\gamma_{n,m}(p))<-1$ which is equivalent to (4.28), then, we have

$$||v^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)}{4m}}||(u,v)||^q_{X(t)}. \tag{4.43}$$

Similarly, we prove

$$||\nabla_x v^{\rm nl}(t,.)||_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n(2-m)+2m}{4m}}||(u,v)||^q_{X(t)}. \tag{4.44}$$

By using all inequalities from (4.41) to (4.44) we have the desired inequality (4.31). The proof (4.30) is completely analogous to the proof of (4.31) by using the same steps from the proof of (4.9). Therefore, we complete the proof.

4.2.4 Proof of Theorem 4.5

We start the proof by defining the Banach space $X(T)$ for all $T>0$, by :

$$X(T)=\mathcal{C}\left([0,T],H^s\right)\times\mathcal{C}\left([0,T],H^s\right),$$

the corresponding norm of space $X(T)$ is

$$\|(u,v)\|_{X(t)}=\sup_{t\geq 0}\left(M_1(t,u)+M_2(t,v)\right),$$

such that

$$M_1(t,u) = (1+t)^{\frac{n(2-m)}{4m}}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2ms}{4m}}|||D|^s u(t,.)||_{L^2(\mathbb{R}^n)},$$
$$M_2(t,v) = (1+t)^{\frac{n(2-m)}{4m}}\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+t)^{\frac{n(2-m)+2ms}{4m}}|||D|^s v(t,.)||_{L^2(\mathbb{R}^n)}.$$

Firstly, we shall estimate all terms making up of the norm $||(u^{\mathrm{nl}}, v^{\mathrm{nl}})||_{X(T)}$. We use the Gagliardo-Nirenberg inequality analogously to (4.18) we obtain for $0 \le \tau \le t$

$$\begin{aligned} |||v(\tau,x)|^p||_{L^m(\mathbb{R}^n)} &\lesssim (1+\tau)^{-\frac{n(p-1)}{2m}}||(u,v)||^p_{X(t)}, \\ |||u(\tau,x)|^q||_{L^m(\mathbb{R}^n)} &\lesssim (1+\tau)^{-\frac{n(q-1)}{2m}}||(u,v)||^q_{X(t)}. \end{aligned} \tag{4.45}$$

Now, we estimate the nonlinear terms in homogeneous Sobolev spaces with high regularity. In order to see this we use the fractional powers rule with $2s^* < n < 2s$, to obtain for $0 \le \tau \le t$

$$\begin{aligned} |||v(\tau,x)|^p||_{\dot H^s(\mathbb{R}^n)} &\lesssim ||v(\tau,x)||^{p-1}_{L^\infty(\mathbb{R}^n)}||v(\tau,x)||_{\dot H^s(\mathbb{R}^n)} \\ &\lesssim \Big[||v(\tau,x)||_{\dot H^{s^*}(\mathbb{R}^n)} + ||v(\tau,x)||_{\dot H^s(\mathbb{R}^n)}\Big]^{p-1}||v(\tau,x)||_{\dot H^s(\mathbb{R}^n)} \\ &\lesssim ||v(\tau,x)||^p_{\dot H^s(\mathbb{R}^n)} + ||v(\tau,x)||^{p-1}_{\dot H^{s^*}(\mathbb{R}^n)}||v(\tau,x)||_{\dot H^s(\mathbb{R}^n)}. \end{aligned}$$

From definition of the space $X(t)$, we have

$$\begin{aligned} |||v(\tau,x)|^p||_{\dot H^s(\mathbb{R}^n)} &\lesssim \left((1+\tau)^{-\frac{n(2-m)+2ms}{4m}p} + (1+\tau)^{-\frac{n(2-m)+2ms^*}{4m}(p-1)-\frac{n(2-m)+2ms}{4m}}\right)\|(u,v)\|^p_{X(t)} \\ &\lesssim \left[(1+\tau)^{-\frac{n(2-m)+2ms}{4m}p} + (1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}\right]\|(u,v)\|^p_{X(t)}. \end{aligned}$$

Consequently, we get

$$|||v(\tau,x)|^p||_{\dot H^s} \lesssim (1+\tau)^{-\frac{n(2-m)p+2ms^*(p-1)+2ms}{4m}}\|(u,v)\|^p_{X(t)}. \tag{4.46}$$

Moreover, we repeat the same steps of (4.46), but we use the following estimates by the fractional powers and $2s^* < n$, we get for $0 \le \tau \le t$

$$\begin{aligned}
|||u(\tau,x)|^q||_{\dot{H}^s(\mathbb{R}^n)} &\lesssim ||u(\tau,x)||^{q-1}_{L^\infty(\mathbb{R}^n)}||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}\\
&\lesssim [||u(\tau,x)||_{\dot{H}^{s^*}(\mathbb{R}^n)} + ||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}]^{q-1}||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}\\
&\lesssim ||u(\tau,x)||^q_{\dot{H}^s(\mathbb{R}^n)} + ||u(\tau,x)||^{q-1}_{H^{s^*}(\mathbb{R}^n)}||u(\tau,x)||_{\dot{H}^s(\mathbb{R}^n)}\\
&\lesssim (1+\tau)^{-\frac{n(2-m)+2ms}{4m}q}\|(u,v)\|^q_{X(t)}\\
&\quad +(1+\tau)^{-\frac{n(2-m)+2ms^*}{4m}(q-1)}(1+\tau)^{-\frac{n(2-m)+2ms}{4m}}\|(u,v)\|^q_{X(t)}.
\end{aligned}$$

It follows that

$$|||u(\tau,x)|^q||_{\dot{H}^s(\mathbb{R}^n)} \lesssim \left[(1+\tau)^{-\frac{n(2-m)+2ms}{4m}q} + (1+\tau)^{-\frac{n(2-m)q+2ms^*(q-1)+2ms}{4m}}\right]\|(u,v)\|^q_{X(t)}.$$

Consequently, we find

$$|||u(\tau,x)|^q||_{\dot{H}^s} \lesssim (1+\tau)^{-\frac{n(2-m)q+2ms^*(q-1)+2ms}{4m}}\|(u,v)\|^q_{X(t)}. \tag{4.47}$$

Plugging these estimates in the inequalities derived from Corollary 4.1, we get similarly to (4.23) and (4.24) for $0 \le \tau \le t$ the following estimates

$$\begin{aligned}
\|u^{\rm nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}}||(u,v)||^p_{X(t)},\\
|||D|^s u^{\rm nl}(t,\cdot)||_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}}||(u,v)||^p_{X(t)},\\
\|v^{\rm nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)}{4m}}||(u,v)||^q_{X(t)},\\
|||D|^s v^{\rm nl}(t,\cdot)||_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n(2-m)+2ms}{4m}}||(u,v)||^q_{X(t)}.
\end{aligned}$$

This completes the proof.

Conclusion

In this thesis, we study in one hand the decay estimates of solutions to the cauchy problem for weakly coupled systems of fractional semilinear pseudoparabolic viscoelastic equations of Volterra integro-differential type and we established a new decay results of the systems, and on the other hand, the global existence and uniqueness of solutions for weakly coupled systems is proved with conditions imposed on the parameters p and q of the systems. We should also note the importance of studying the global existence of our problem because it is closest to reality and considered to be a model to many problems in many areas for example in applied sciences.

Many questions remain unresolved and deserve closer consideration, including

- The blow up case of the cauchy problem for weakly coupled systems of fractional semilinear pseudoparabolic viscoelastic equations of Volterra integro-differential type.
- The study global existence in the case ($\theta = \theta_1$ in the first equation and $\theta = \theta_2$ in the second equation with $\theta_1 \neq \theta_2$) of the cauchy problem for weakly coupled systems of fractional semilinear pseudoparabolic viscoelastic equations of Volterra integro-differential type.

Bibliography

[1] E.C. Aifantis : *On the problem of diffusion in solids*. Acta Mech. 37 (1980), 265-296.

[2] A.B. Al'shin, M.O. Korpusov, A.G. Sveshnikov : *Blow-up in nonlinear Sobolev type equations*. De Gruyter Series in Nonlinear Analysis and Applications (2011).

[3] C.T. Anh and T.Q. Bao : *Pullback attractors for a class of non-autonomous nonclassical diffusion equations*. Nonlinear Anal. 73 (2010) 399-412.

[4] C. T. Anh and T. Q. Bao : *Dynamics of non-autonomous nonclassical diffusion equations on* $\mathbb{R}^n$. Communications on Pure and Applied Analysis, vol. 11 (2012) 1231-1252.

[5] C.T. Anh, D.T.P. Thanh and N.D. Toan : *Averaging of noncassical diffusion equations with memory and singularly oscillating forces*. Z. Anal. Anwend. 37 (2018) 299-314.

[6] C.T. Anh, D.T.P. Thanh and N.D. Toan : *Global attractors for nonclassical diffusion equations with hereditary memory and a new class of nonlinearities*. Ann. Polon. Math. 119 1-21.

[7] J. A. D. Appleby, M. Fabrizio, B. Lazzari and D. W. Reynolds : *On exponential asymptotic stability in linear viscoelasticity* . Math. Mod. Meths. Appl. Sci. 16 (2006) 1677-1694.

[8] D.G. Aronson, H. Weinberger : *Multidimensional nonlinear diffusion arising in population genetics*. Adv. in Math. 30 (1978) 33-76.

[9] V. Barbu : *A semigroup approach to an infinite delay equation in Hilbert space, Abstract Cauchy problems and functional differential equations*. F. Kappel and W. Schappacher, eds., Research Notes in Math, Pitman, Boston, 48 (1981) 1-25.

[10] V. Barbu, M.A. Malik : *Semilinear integro-differential equations in Hilbert space.* J. Math. Anal. Appl. 67 (1979) 452-475.

[11] G. I. Barenblatt, I. P. Zheltov, I. N. Kochina : *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]*. Journal of applied mathematics and mechanics, 24(5) 1286-1303.

[12] T. B. Benjamin, J. L. Bona, J. J. Mahony : *Model equations for long waves in nonlinear dispersive systems*. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 272(1220) (1972) 47-78.

[13] M.Berbiche, A. Melik : *Global existence and decay estimates for the semilinear heat equation with memory in* $\mathbb{R}^n$. Int. J. Nonlinear Anal. Appl. 13 2 (2022) 2271-2285.

[14] M.Berbiche, A. Melik : *Global existence and decay estimates for the semilinear nonclassical-diffusion equations with memory in* $\mathbb{R}^n$. Euro-Tbilisi Mathematical Journal 15(2) (2022) 29-53.

[15] M.E. Bradley and S. Lenhart : *Bilinear spatial control of the velocity term in a Kirchhoff plate equation*. Electronic J. Differential Equations, (2001) 1-15.

[16] H. Brill : *A semilinear Sobolev evolution equation in a Banach space*. J. Differ. Equ. 24(3)(1977) 412-425.

[17] C. Buriol : *Energy decay rates for the Timoshenko system of thermoelastic plates* . Nonlinear Analysis, 64 (2006) 92-108.

[18] T.A. Burton : *Volterra integral and differential equations* . Mathematics in Science and Engineering, 202 (2005).

[19] R. W. Carroll and R. E. Showalter : *Singular and degenerate Cauchy problems*. Math. Sci. Engrg. 127, Academic Press, New York (1976).

[20] Y.K. Chang, R. Ponce : *Uniform exponential stability and its applications to bounded solutions of integrodifferential equations in Banach spaces*. J. Integral Equations Appl. 30 (2018) 347-369.

[21] Chen Peter, J. GURTIN, E. Morton : *On a theory of heat conduction involving two temperatures*. Zeitschrift für angewandte Mathematik und Physik ZAMP, vol. 19, no 4 (1968) 614-627.

[22] J. Chen, T. Xiao and J. Liang : *Uniform exponential stability of solutions to abstract Volterra equations.* J. Evol. Equ. 4 (2009), no. 9, 661-674.

[23] J. Chen, J. Liang, T. Xiao : *Stability of solutions to integro-differential equations in Hilbert spaces*. Bull. Belg. Math. Soc. 18 (2011) 781-792.

[24] Wenhui Chen, Abdelhamid Mohammed Djaouti : *Weakly coupled systems ofsemi-linear wave equations with memory-type dissipation, different power nonlinearities and different regularity of data*. arXiv:1807.06393v1 [math.AP] (2018).

[25] V.V. Chepyzhov, E. Mainini and V. Pata : *Stability of abstract linear semigroups arising from heat conduction with memory*. Asymptot. Anal. 46 (2006) 251273.

[26] V. V. Chepyzhov, E. Mainini, V. Pata : *Stability of abstract linear semigroups arising from heat conduction with memory.* Asymptot. Anal. 50 (2006) 269-291.

[27] V.V. Chepyzhov and A. Miranville : *On trajectory and global attactors for semilinear heat equations with fadding memory.* Indiana Univ. Math. J. 55 (2006) 119-168.

[28] V. V. Chepyzhov and V. Pata : *Some remarks on stability of semigroups arising from linear viscoelasticity* . Asympt. Anal. 46 (2006) 251-273.

[29] F. Christ, M. Weinstein : *Dispersion of Small-Amplitude Solutions of the Generalized Korteweg-de Vries Equation.* J. Funct. Anal. 100 (1991) 87-109.

[30] P. Colli, M. Grasseli : *Phase transition problems in materials with memory.* J. Integral Equations Appl. 5 (1993) 1-22.

[31] B. D. Coleman, M. E. Gurtin : *Equipresence and constitutive equations for rigid heat conductors.* Z. Angew. Math. Phys. 18 (1967) 199-208.

[32] M. Conti, S. Gatti, V. Pata : *Decay rates of Volterra equation on* $\mathbb{R}^n$. Central Journal of Mathematics volume 5 (4) (2007) 720-732.

[33] M. Conti, S. Gatti, V. Pata : *Uniform decay properties of linear Volterra integro-differential equations.* Mathematical models and methods in applied siences, vol.18 No.1 (2008) 21-45.

[34] M. Conti, E.M. Marchini and V. Pata : *Nonclassical diffusion with memory.* Math. Appl. Sci. 38 (2015) 948-958.

[35] M. Conti and E. M. Marchini : *A remark on nonclassical diffusion equations with memory.* Appl. Math. Optim. 73 (2016) 1-21.

[36] M. Conti, V. Danese and V. Pata : *Viscoelasticity with time-dependent memory kernels.* Part II: Asymptotic behavior of solutions, American Journal of Mathematics V140, No 4 (2018).

[37] C. Corduneanu : *Integral equations and stability of feedback systems.* Mathematics in Science and Engineering, 104, Academic Press, New York (1973).

[38] M. G. Crandall, S. O. Londen, J. A. Nohel : *An abstract nonlinear Volterra integrodifferential equation.* Journal of Mathematical Analysis and Applications, 64(3) (1978) 701-735.

[39] L. Cueto-Felgueroso, R. Juanes : *A phase field model of unsaturated flow.* Water resources research 45.10 (2009).

[40] G. Da Prato, M. Iannelli : *Existence and regularity for a class of integro-differential equations of parabolic type.* J. Math. Anal. Appl. 112 , no. 1 (1985) 36-55.

[41] G. Da Prato, A. Lunardi : *Solvability on the real line of a class of linear Volterra integro-differential equations of parabolic type.* Ann. Mat. Pura Appl. 150 no. 4 (1988) 67-117.

[42] M. D'Abbicco, M.R. Ebert, S. Lucente : *Self-similar asymptotic profile of the solution to a nonlinear evolution equation with critical dissipation.* Math. Methods Appl. Sci. 40 (18) (2017) 6480-6494.

[43] C. M. Dafermos : *Asymptotic stability in viscoelasticity.* Archive for rational mechanics and analysis, 37(4) (1970) 297-308.

[44] C. M. Dafermos, M. Slemrod : *Asymptotic behavior of nonlinear contraction semigroups.* Journal of Functional Analysis, 13(1) (1973) 97-106.

[45] V. Danese, P.G. Geredeli and V. Pata : *Exponential attractors for abstract equations with memory and applications to viscoelasticity*. J. Discrete Contin. Dyn. Syst. 35 (2015) 2881-2904.

[46] C. David, W. Jet : *Asymptotic behaviour of the fundamental solution to the equation of heat conduction in two temperatures*. J. Math. Anal. Appl. 69 (2) (1979) 411-418.

[47] R. Dautray, J. Louis Lions: *Mathematical analysis and numerical methods for science and technology*. Springer-Verlag Berlin Heidelberg v. 2 (1988).

[48] W. Desch, R. K. Miller : *Exponential stabilization of Volterra integrodifferential equations in Hilbert space*. Journal of differential equations, 70(3) (1987) 366-389.

[49] P. M. N. Dharmawardane, J. E. Munoz Rivera and S. Kawashima : *Decay property for second order hyperbolic systems of viscoelastic materials*. J. Math. Anal. Appl. 366 (2010) 621-635.

[50] E. Di Nezza, G. Palatucci and E. Valdinoci : *Hitchhikers guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 no. 5 (2012) 521573.

[51] A.D. Drozdov and V.B. Kolmanovskii : *Stability in viscoelasticity* . North-Holland Series in Applied Mathematics and Mechanics, 38, North-Holland Publishing Co., Amsterdam, (1994).

[52] M. Escobedo, M.A. Herrero : *Boundedness and blow up for a semilinear reaction-diffusion system* . J.Differ. Equ. 89 (1991) 176-202.

[53] M. Fabrizio and B. Lazzari : *On the existence and the asymptotic stability of solutions for linear viscoelastic solids* . Arch. Rational Mech. Anal. 116 (1991) 139-152.

[54] M. Fabrizio and A. Morro : *Mathematical Problems in Linear Viscoelasticity*, SIAM Studies in Applied Mathematics, Vol. 12 SIAM (1992).

[55] M. Fabrizio, G. Gentili, D.W. Reynolds : *On rigid linear heat condution with memory*. Int. J. Eng. Sci. 36 (1998) 765-782.

[56] M. Fabrizio and S. Polidoro : *Asymptotic decay for some differential systems with fading memory* . Appl. Anal. 81 (2002) 1245-1264.

[57] Fang, Zhong Bo, and Lu Sun : *Blow up of solutions with positive initial energy for the nonlocal semilinear heat equation.* Journal of the Korean Society for Industrial and Applied Mathematics 16.4 (2012) 235-242.

[58] A. Favini, A. Yagi : *Degenerate Differential Equations in Banach Spaces* . Pure Appl. Math. (N. Y.), vol. 215, Marcel Dekker, New York, (1999).

[59] A. Friedamn : *Partial differential Equations* . Corrected reprint of the original edition, Robert E. Krieger Publishing Co., Huntington, N. Y. (1976).

[60] H. Fujita : *On the blowup of solutions of the Cauchy problem for* $u_t = \triangle u + u^{1+\alpha}$. J.Fac. Sci., Univ. Tokyo, Sect. I. 13 (1966) 109-124.

[61] S. Gatti, A. Miranville, V. Pata and S. Zelik : *Attractors for semilinear equations of viscoelasticity with very low dissipation*. Rocky Mountain J. Math. 38 (2008) 1117-1138.

[62] G. Gentili : *Dissipativity conditions and variational principles for the heat flux equation with memory*. Differential and Integral Equations, 4(5)(1991) 977-989.

[63] J. M. Ghidaglia, A. Marzocchi : *Longtime behaviour of strongly damped wave equations, global attractors and their dimension*. SIAM journal on mathematical analysis, 22(4) (1991) 879-895.

[64] C. Giorgi, G. Gentili : *Thermodynamic properties and stability for the heat flux equation with linear memory*. Quarterly of applied mathematics, 51(2) (1993) 343-362.

[65] C. Giorgi and B. Lazzari : *On the stability for linear viscoelastic solids* . Quart. Appl. Math. 55 (1997) 659-675.

[66] C. Giorgi, M.G. Naso and V. Pata : *Exponential stability in linear heat conduction with memory*. A semigroup approach, Comm. Appl. Anal. 5 (2001) 121-134.

[67] H. Grabmüller : *On linear theory of heat conduction in materials with memory. Existence and uniqueness theorems for the final value problem*. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 76(2) (1977) 119-137.

[68] L. Grafakos : *Classical and modern Fourier analysis* . Prentice Hall, (2004) .

[69] L. Grafakos, S. Oh : *The Kato-Ponce inequality* . Comm. Partial Differential Equations 39 (6) (2014) 1128-1157.

[70] M. Grasselli, V. Pata : *Uniform attractors of nonautonomous systems with memory,in Evolution Equations,Semigroups and Functional Analysis* . Nonlinear Differential Equations Appl. no. 50, Birkauser, Boston (2002).

[71] G. Gripenberg, S.Q. Londen, O. Staffans : *Volterra integral and functional equations*. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge (1990).

[72] A. Gulisashvili, M. Kon : *Exact smoothing properties of Schrondinger semigroups* . Amer. J. Math. 118 (1996) 1215-1248.

[73] B. Guo, Y. Han, J.Xin : *Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrodinger equation*. Appl. Math. Comput. 204 (2008) 468-477.

[74] M. E. Gurtin and A. C. Pipkin : *A general theory of heat conduction with finite wave speeds*. Arch. Rational Mech. Anal. 31 (1968) 113-126.

[75] H. Hajaiej, L. Molinet, T. Ozawa, B. Wang : *Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations* . Harmonic Analysis and Nonlinear Partial Differential Equations RIMS KÃkyuroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto (2011) 159-175.

[76] Hassanizadeh, S. Majid, and William G. Gray : *Thermodynamic basis of capillary pressure in porous media*. Water resources research 29.10 (1993) 3389-3405.

[77] T. Hosono and S. Kawashima: *Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system* . Math. Models Meth. Appl. Sci. 16 (2006) 1839-1859.

[78] K. Ide and S. Kawashima : *Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system* . Math. Models Meth. Appl. Sci. 18 (2008) 1001-1025.

[79] R. Ikehata, K. Tanizawa : *Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^n$ with noncompactly supported initial data*. Nonlinear Anal. 61 (7) (2005) 1189-1208.

[80] Ikehata : *Energy decay estimates for wave equations with a fractional damping* . Diff. Int. Eqns 25 (9-10) (2012) 939-956.

[81] J. Jackle : *Heat conduction and relaxation in liquids of high viscosity*. Physic, Statistical Mechanics and its Applications, 162(3)(1990) 377-404.

[82] E.I. Kaikina, P.I. Naumkin, I.A. Shishmarev : *The Cauchy problem for a Sobolev type equation with power like nonlinearity*. Izv. Math. 69(1) (2005) 59-111.

[83] G. Karch : *Asymptotic behaviour of solutions to some pesudo-parabolic equations.* Math. Methods Appl. Sci. 20(3) (1997) 271-289.

[84] T. Kato, G. Ponce : *Well-posedness and scattering results for the generalized Kortewegde-Vries equation via the contraction principle* . Comm. Pure Appl. Math. 46(4) (1993) 527-620.

[85] C.E. Kenig, G. Ponce, L. Vega : *Commutator estimates and the Euler and NavierStokes equations* . Comm. Pure Appl. Math. 41 (1988) 891-907.

[86] A.A.A. Kilbas, H.M. Srivastava, J.J. Trujillo : *Theory and Applications of Fractional Differential Equations* . North-Holland Mathematics Studies, vol. 204 (2006).

[87] K. Kobayashi, T. Sirao, H. Tanaka : *On the growing up problem for semilinear heat equations* . J. Math. Soc. Japan. 29 (1977) 407-429.

[88] M.O. Korpusov, A.G. Sveshnikov : *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics.* Comput. Math. Math. Phys. 43 (12) (2003) 1765-1797.

[89] K.H. Kwek, C.C. Qu : *Alternative principle for pseudo-parabolic equations.* Dyn. Syst. Appl. 5(2) (1996) 211-217.

[90] F. Li, F. Rong : *Decay of solutions to fractal parabolic conservation laws with large initial data* : Commun. Pure Appl. Anal. 12 (2012) 973-984.

[91] Z. Liu and S. Zheng : *On the exponential stability of linear viscoelasticity and thermoviscoelasticity* . Quart. Appl. Math. 54 (1996) 21-31.

[92] Z. Liu and S. Zheng : *Semigroups associated with dissipative systems* . Chapman and Hall/CRC Research Notes in Mathematics, Vol. 398 (1999).

[93] Y. Liu, S. Kawashima : *Decay property for a plate equation with memory-type dissipation* . Kinet. Relat. Models 4 (2011) 531-547.

[94] Y. Liu, S. Kawashima : *Global Existence and Asymptotic Behavior of Solution for Quasi-linear Dissipation Plate Equation* . Discrete Contin. Dyn. Syst. 29 (2011) 1113-1139.

[95] Y. Liu : *Time-dependent global attractor for the nonclassical diffusion equations.* Appl. Anal. 94 (2015) 1439-1449.

[96] H.A. Levine : *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$. Arch. Ration. Mech. Anal. 51 (1973) 371-386.

[97] Londen, S-O. and John A. Nohel : *Nonlinear Volterra integrodifferential equation occurring in heat flow.* The Journal of Integral Equations (1984) 11-50.

[98] S. L'vovich Sobolev : *On a new problem of mathematical physics.* Izv. Akad. Nauk SSSR Ser. Mat. 18 (1954) 3-50.

[99] Q. Ma, X. Wang, L. Xu : *Existenceand regularity of time-dependent global attractors for the nonclassical reaction-diffusion equations with lower forcing term.* Bound. Value Probl. Paper No. 10, 11 (2016).

[100] E. Mainini,G. Mola : *Exponential and polyomial decay for first order linear Volterra evolution equations.* Quarterly of Applied Mathematics volume LXVII (2009) 93-111.

[101] S. A. Messaoudi : *Blow-up of solutions of a semilinear heat equation with a visco-elastic term.* Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl., vol. 64, Birkhäuser, Basel, (2005) 351-356.

[102] S. A. Messaoudi : *Blow-up of solutions of a semilinear heat equation with a memory term*. Abstract and Applied Analysis, no. 2 (2005) 87-94.

[103] R.K. Miller : *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park, CA (1971).

[104] R. K. Miller : *An integrodifferential equation for rigid heat conductors with memory. Journal of Mathematical Analysis and Applications*. 66(2) (1978) 313-332.

[105] J. E. Munoz Rivera : *: Asymptotic behaviour in linear viscoelasticity* . Quart. Appl. Math. 52 (1994) 629-648.

[106] J.E. Munoz Rivera and R. Racke : *Global stability for damped Timoshenko systems*. Discrete Contin. Dyn. Syst. 9 (2003) 1625-1639.

[107] J.E. Munoz Rivera, M.G. Naso and F.M. Vegni : *Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory* . J. Math. Anal. Appl. 286 (2003) 692-704.

[108] J. E. Munoz Rivera and M. G. Naso : *On the decay of the energy for systems with memory and indefinite dissipation*. Asympt. Anal. 49 (2006) 189-204.

[109] K. Nishihara, Y. Wakasugi : *Critical exponent for the Cauchy problem to the weakly coupled damped wave system* . Nonlinear Anal. 108 (2014) 249-259.

[110] J. W. Nunziato : *On heat conduction in materials with memory*. Quarterly of Applied Mathematics, 29(2) (1971) 187-204.

[111] B. G. Pachpatte : *On a nonlinear diffusion system arising in reactor dynamics* . J.Math. Anal. Appl., 94 (1983) 501-508.

[112] A. Palmieri, M. Reissig : *Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation* . II, Math. Nach. (2018) 1-34.

[113] V. Padrón : *Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation*. Transactions of the American Mathematical Society, 356(7) (2004) 2739-2756.

[114] C. V. Pao : *Bifurcation analysis of a nonlinear diffusion system in reactor dynamics*. Appl. Anal. 9 (1979) 107-125.

[115] V. Pata, A. Zucchi : *Attractors for a damped hyperbolic equation with linear memory*. Adv. Math. Sci. Appl. 11 (2001) 505-529.

[116] V. Pata : *Exponential stability in linear viscoelasticity* . vol LXIV, (2006) 499-513.

[117] V. Pata : *Stability and exponential stability in linear viscoelasticity*. Milan J. Math. 77 (2009) 333-360.

[118] A. Pazy : *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York (1983).

[119] J.C. Peter, M.E. Gurtin : *On a theory of heat conduction involving two temperatures*. Z. Angew. Math. Phys. 19 (1968) 614-627.

[120] H. Petzeltová : *Solution semigroup and invariant manifolds for functional equations with infinite delay*. Mathematica Bohemica, 118(2) (1993) 175-193.

[121] M. A. Pozio : *Some conditions for global asymptotic stability of equilibria of integrodifferential equations*. J. Math. Analysis Applic. 95 (1983) 501-527.

[122] J. PrÂ¨uss : *Evolutionary integral equations and applications*. Monographs in Mathematics, vol. 87, BirkhÂ¨auser Verlag, Basel (1993).

[123] M. Ptashnyk : *Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities* . Nonlinear Anal. 66 (12) (2007) 2653-2675.

[124] X. Pu, B. Guo, J. Zhang : *Global weak solutions to the 1-D fractional Landau a Lifshitz equation*. Discrete Contin. Dyn. Syst., Ser. B 14 (2012) 199-207.

[125] T. Runst, W. Sickel : *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. De Gruyter Series in Nonlinear Analysis and Applications. 3, Walter de Gruter & Co., Berlin (1996).

[126] B. Said-Houri, S. A. Messaoudi : *General decay estimates for a cauchy viscoelastic wave problem* . Pure and applied analysis, volume 13, Number 4 (2014) 1541-1551.

[127] Y. Shang and B. Guo : *Exponential attractor for a class of nonclassical diffusion equation*. Journal of Partial Differential Equations, vol. 16 (2003) 289-298.

[128] R.E. Showalter, T.W. Ting : *Pseudoparabolic partial differential equations*. SIAM J. Math. Anal. 88(1) (1970) 1-26.

[129] Shuying Tian : *Bounds for blow-up time in a semilinear parabolic problem with viscoelastic term*. Computers Mathematics with Applications, Volume 74, Issue 4 (2017) 736-743.

[130] M. Slemrod : *A hereditary partial differential equation with applications in the theory of simple fluids*. Archive for Rational Mechanics and Analysis, 62(4) (1976) 303-321.

[131] O. J. Staffans : *Semigroups generated by a convolution equation. In Infinite-dimensional systems*. Springer, Berlin, Heidelberg (1984) 209-226.

[132] Y. Sugitani, S. Kawashima : *Decay estimates of solutions to a semi-linear dissipative plate equation*. J. Hyperbolic Differential Equations, 7 (2010) 471-501.

[133] F. Sun, M. Wang : *Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping*. Nonlinear Anal. 66 (12) (2007) 2889-2910.

[134] C. Sun, S. Wang and C.K. Zhong : *Global attractors for a nonclassical diffusion equation*. Acta Math. Appl. Sin. Engl. Ser. 23 (2007) 1271-1280.

[135] C. Sun and M. Yang : *Dynamics of the nonclassical diffusion equations*. Asymptot. Anal. 59 (2009) 51-81.

[136] Sun Fenglong, Lishan Liu, and Yonghong Wu : *Global existence and finite time blow-up of solutions for the semilinear pseudo-parabolic equation with a memory term*. Applicable Analysis 98.4 (2019) 735-755.

[137] G.A. Sviridyuk,V.E. Fedorov : *Analytic semigroups with kernels, and linear equations of Sobolev type* . Sibirsk. Mat. Zh. 36 (5) (1995) 1130-1145.

[138] TW Ting : *Certain non-steady flows of second-order fluids*. Arch Ration Mech Anal. 14 (1963) 1-26.

[139] T.W. Ting : *Parabolic and pseudo-parabolic partial differential equations*. J. Math. Soc. Jpn. 21 (1969) 440-453.

[140] G. Todorova, B. Yordanov : *Critical exponent for a nonlinear wave equation with damping*. J. DifferentialEquations 174 (2) (2001) 464-489.

[141] C. Truesdell, W. Noll : *The Nonlinear Field Theories of Mechanics*. Encyclomedia of Physics, Springer, Berlin (1995).

[142] Y. Wang, P. Li, and Y. Qin : *Upper semicontinuity of uniform attractors for nonclassical diffusion equations*. Boundary Value Problems, no. 84 (2017) 1-11.

[143] X. Wang, Y. Zhang, and L. Yang : *Attractor for the nonautonomous classical reaction diffusion equation with fading memory*. Chinese Annals of Mathematics, Series A, vol. 40, no. 1 (2019) 79-92.

[144] G.F. Webb : *An abstract semilinear Volterra integrodifferential equation*. Proc. Amer. Math. Soc. 69 , no. 2 (1978) 255-260.

[145] Y. Yamada : *On a certain class of semilinear Volterra diffusion equations* . J. Math. Anal. Appl., 88 (1982) 433-457.

[146] Y. Yamada : *Asymptotic stability for some systems of semilinear Volterra diffusion equations.* J. Differ. Equations, 52 (1984) 295-326.

[147] Q.S. Zhang : *A blow-up result for a nonlinear wave equation with damping, the critical case.* C. R. Acad. Sci. Paris Série I Math. 333 (2) (2001) 109-114.

[148] X. Zhang, E. Zuazua : *On the optimality of the observability inequalities for Kirchhoff plate systems with potentials in unbounded domains.* Hyperbolic Problems, Theory, Numerics and Applications, Springer, (2008) 233-243.

[149] H. Zhao, J. Nuen : *Decay estimate of solutions for a semi-linear wave equation* . Springer (2015).