PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

UNIVERSITY MOHAMED KHIDER OF BISKRA

Faculty of Exact Sciences and Sciences of Nature and Life

Thesis Submitted for the Degree of:

Doctorate in Applied Mathematics

Option: Probability

Edited by **Khouloud Makhlouf**

Title

Optimal control of stochastic systems with memory under noisy observations

In front of the Committee:

Professor	Nacer KHELIL	University of Biskra	President
Professor	Boulakhras GHERBAL	University of Biskra	Supervisor
Professor	Nacira AGRAM	University of Sweden	Co-Supervisor
Professor	Mokhtar HAFAYED	University of Biskra	Examiner
Professor	Khalil SAADI	Unversity of M'sila	Examiner

June 2023

Dedication

I dedicate this work to :

My parents

who support me and advise me in my academic career

To my brothers and my sisters, and my family who give love and liveliness.

To all those who have helped me - directly or indirectly - and those who shared with me the emotional moments during the accomplishment of this work and who warmly supported and encouraged throughout my journey.

Thanks a lot !

Khouloud Makhlouf

Acknowledgements

First and foremost, praises and thanks to **Allah**, for his help and blessings throughout the research work to complete my thesis successfully

I would like to thank my supervisor, Pr. **Boulakhras Gherbal** for his advice, support, and patience. Without his guidance and assistance, this thesis would not have been possible.

I would like to express my special thanks of gratitude to my co-supervisor, Pr. **Nacira Agram** from the Royal Institute of Technology (KTH), Sweden. who believed in me and gave me the golden opportunity to do this wonderful project.

Especially, I would like to thank Pr. **Bernt Øksendal** from the University of Oslo, Norway, and Pr. **Astrid Hilbert** from Linnaeus University (LNU), Växjö, Sweden. It was a great pleasure to work with these two nice people.

Special thanks to the members of the jury: Pr. **Nacer Khelil**, Pr.**Mokhtar Hafayed** from Biskra University and Pr.**Khalil Saadi** from M'sila University, to have accepted inspect the contents of my thesis, and also for there advice, remarks and orientations.

I would also like to thank all my colleagues in the Mathematics Department, Laboratory of Applied Mathematics, and Laboratory of Mathematical Analysis, Probability, and Optimization at the University of Mohamed Khider of Biskra.

Lastly, I would like to thank my family, for their love and their sacrifices that made it possible

KHOULOUD MAKHLOUF

Abstract

This thesis aims to study a new type of stochastic partial differential equations (SPDEs) with space interactions. By space interactions, we mean that the dynamics of the system at time t and position in space x also depend on the space-mean of values at neighbouring points.

In the first part, we introduce linear SPDEs. Then we prove the existence and uniqueness results (mild solution) for nonlinear SPDEs under linear growth and Lipschitz conditions on the coefficients. In the second part of this thesis, using results from Noisy Observation (nonlinear filtering), we transformed this noisy observation stochastic differential equation (SDE) control problem into full observation stochastic partial differential equations (SPDEs), and then we prove a sufficient and necessary maximum principle for the optimal control of SPDEs. In the third part of this thesis, we prove the existence and uniqueness of strong, smooth solutions of a class of stochastic partial differential equations with space interactions., and we show that, under some conditions, we use white noise theory to prove a positivity theorem for a class of SPDEs with space interactions. The solutions are positive for all times if the initial values are. Then we study the general optimization problem for such a system. Sufficient and necessary maximum principles for the optimal control of such systems are derived. Finally, we apply the results to study an example of optimal vaccination strategy for epidemics modelled as stochastic partial differential equations (SPDEs) with space interactions.

Contents

Symbols and Abbreviations

The main symbols and abbreviations used in this thesis:

$(\Omega, \mathscr{F}, \mathbb{P})$:	Probability space.
$\mathbb{P}$:	Probability measure
$\mathbb{P}_T$:	the product of the Lebesgue measure in $[0, T]$
$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$:	Filtered probability space
$B(t)$:	Brownian motion
$u(t,x)$:	control process
$\overline{Y}(t,x)$:	The average value of $Y(t, x+\cdot)$
A_x :	The second order partial differential operator acting on x
D :	An open set in $\mathbb{R}^n$
∂D:	The boundary of the set D
τ:	stopping time
$\diamond$:	Wick multiplication
$\nabla_x F$:	Fréchet derivative
i.e.:	Namely or that is.
a.s.:	almost surely.
a.e.:	almost everywhere.
SDE:	Stochastic Differential Equation
SPDEs:	Stochastic Partial Differential Equations
BSPDE:	Backward Stochastic Partial Differential Equations

General Introduction

0.1 Introduction

The theory of stochastic partial differential equations (SPDEs for short) finds applications in many scientific fields, such as physics, biology, chemistry, and finance. The main motivation for studying this type of equation is the filtering of partially observable processes. The first results on stochastic evolution equations started to appear in the early 1960s and were motivated by physics, filtering, and control theory. An important development, concerning the potential theory on infinite dimensional spaces, has been initiated by L. Gross [24] and Yu. Daleckij [14]. Basic results on the existence and uniqueness of solutions of SPDEs were obtained in the 1970s by A. Bensoussan, R. Temam [4],[5] E. Pardoux [19], M. Viot [36], and many others. SPDEs are a type of stochastic differential equation that is defined on an infinite dimensional space. A different method for studying stochastic partial differential equations, the so-called variational approach, was introduced by Pardoux [47], Krylov, Rozovskii [33] by Prévot and Röckner [50]. There is another method, the so-called semigroups generated by unbounded operators and mild solutions in [22] and [15]

Our objective in this thesis is to introduce a new type of generalised stochastic heat equation with *space interactions* as a model for population growth. This result is new (we refer to read[38]). By space interactions, we mean that the dynamics of the population density $Y(t,x)$ at a time t and a point x depends not only on its value and derivatives at x, but also on its values in a neighborhood of x. For example, define G to be a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y)\,dy; \quad \varphi \in L^2(\mathbb{R}^n), \tag{1}$$

where $V(\cdot)$ denotes Lebesgue volume and

$$K_r = \{y \in \mathbb{R}^n; |y| < r\}$$

is the ball of radius $r > 0$ in $\mathbb{R}^n$ centred at 0. Then

$$\overline{Y}_G(t,x) := G(x, Y(t,\cdot))$$

is the average value of $Y(t, x+\cdot)$ in the ball K_r.
More generally, if we are given a nonnegative measure (weight) $\rho(dy)$ of total mass 1, then the ρ weighted average of Y at x is defined by

$$\overline{Y}_\rho(t,x) := \int_D Y(t, x+y)\rho(dy).$$

We believe that by allowing interactions between populations at different locations, we get a better model for population growth, including the modelling of epidemics. For example, we know that COVID-19 is spreading by close contact in space.

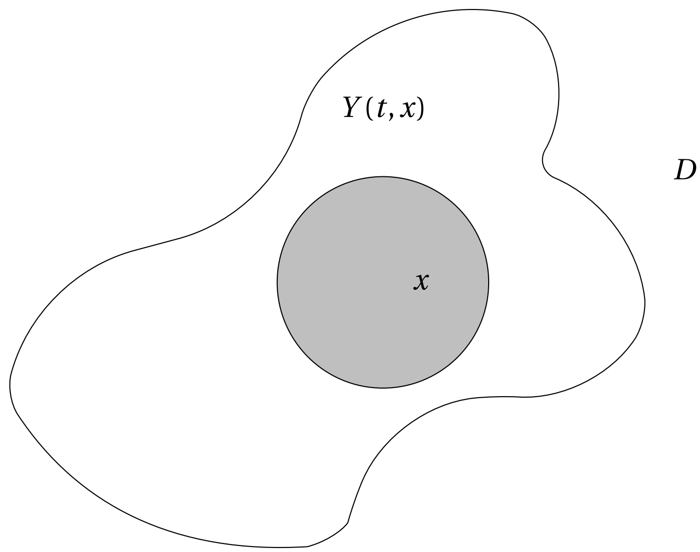
We illustrate the above by the following population growth model:

Example 1 *With G as in equation (1), suppose the density $Y(t,x)$ of a population at the time t and the point x satisfies the following space-interaction version of a reaction-diffusion equation:*

$$\begin{cases} dY(t,x) &= \left(\frac{1}{2}\Delta Y(t,x) + \alpha\overline{Y}(t,x) - u(t,x)Y(t,x)\right)dt + \beta Y(t,x)dB(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T)\times\partial D, \end{cases} \tag{2}$$

where α is a constant, ξ, η are given bounded functions, $\overline{Y}(t,x) = G(x, Y(t,\cdot))$ and $B(t) = B(t,\omega); (t,\omega) \in [0,T]\times\Omega$ is a Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$.

Here $u(t,x)$ is our control process, e.g. representing our harvesting or vaccine effort.



Then the equation (2) is a natural model for population growth with space interactions in an environment.

If $u(t,x)$ represents a vaccination effort rate at (t,x), we define the total expected utility $J_0(u)$ of the harvesting by an expression of the form

$$J_0(u) = \mathbb{E}\Big[\int_D \int_0^T U_1(u(t,x))\,dt\,dx + \int_D U_2(Y(T,x))\,dx\Big],$$

where U_1 and U_2 are given cost functions. The problem to find the optimal vaccination rate u^* is the following:

Problem 2 *Find $u^* \in \mathcal{U}$ such that*

$$J_0(u^*) = \inf_{u \in \mathcal{U}} J_0(u),$$

where $\mathcal{U}$ is a given family of admissible controls.

We will return to the example above after first discussing more general stochastic optimal control models with a system whose state $Y(t,x)$ at time t and at the point x satisfies an SPDE with a non-local space-interaction dynamics of the following type:

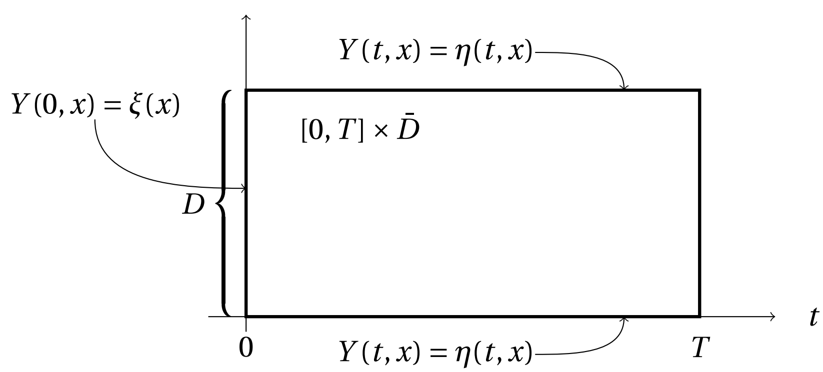
$$\begin{cases} dY(t,x) &= A_x Y(t,x)dt + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))dt \\ & +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))dB(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T)\times\partial D. \end{cases} \tag{3}$$

Here $dY(t,x)$ denotes the differential with respect to t while A_x is the second order partial differential operator acting on x of the form

$$A_x\phi(x) = \sum_{i,j=1}^{n} \alpha_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\beta_i(x)\frac{\partial\phi}{\partial x_i}; \quad \phi \in \mathscr{C}_0^2(\mathbb{R}^n). \tag{4}$$

The domain D is an open set in $\mathbb{R}^n$ with a Lipschitz boundary ∂D and closure $\overline{D}$. We extend $Y(t,x)$ to be a function on all of $[0,T]\times\mathbb{R}^n$ by setting

$$Y(t,x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \overline{D}.$$



Example 3 *In particular, the partial differential operator A_x could be the Laplacian Δ. or more generally an operator of the $div-grad$-form*

$$A_x(\varphi) = div(\alpha(x)\nabla\varphi)(x); \quad \varphi \in \mathscr{C}^2(D),$$

where div denotes the divergence operator, ∇ denotes the gradient and

$$\alpha(x) = [\alpha_{i,j}(x)]_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$$

is a nonnegative definite matrix for each x. Equations of this type are of interest because they represent important models in many situations, e.g. in physics (e.g. fluid flow in random media, see e.g Holden et al [25], in epidemiology and in biology, e.g. in population growth where $Y(t,x)$ represents the population density at t,x.

There are two well-known approaches to solve stochastic control problems: The Bellman dynamic programming [12] method and the Pontryagin maximum principle [30]. Because of the space-mean dependence in our model, the system is not Markovian, and it is not clear how to apply a dynamic programming approach. Instead, we will use a stochastic version of the Pontryagin maximum principle, which involves a coupled system of forward/backward SPDEs. Stochastic control of SPDEs has been studied widely in the literature, for example, we refer to Bensoussan [6], [7], [8], [9], Hu & Peng [27], Zhou [58], Øksendal [43], Fuhrman et al [20] and Øksendal *et al* [44], [45], [46] and the references therein. In the fundamental papers [6, 25] it is assumed that the diffusion coefficient of the system does not depend on the control, and in [6, 25], there is no space-mean dependence so they do not cover our situation.

In [35], a general maximum principle of optimal control of SPDEs is proved, with an adjoint equation (BSPDE) formulated in a weak setting. The general setting in [35] covers the situation we consider, except that in [35] only the case with the underlying space D being all of R^n is considered. Our approach deals with general D and is directly focused on the effect of the space interaction, with application to population modelling in mind.

Moreover, for our type of equation, we prove the smoothness and positivity of the solution. Specifically, in our case of a control problem for an SPDE with space-interaction in a subset D of R^n, we derive an explicit adjoint equation, which is a BSPDE, also with space-interaction dependence. We derive both sufficient and necessary maximum principles for this type of stochastic control problem. For related singular stochastic control with space interaction, we refer to Agram et al. [2].

This thesis is organized as follows:

Chapter 1: The aim of this chapter is to study stochastic partial differential equations (SPDEs). We introduce linear SPDEs, and we prove the existence and uniqueness of nonlinear SPDEs.

Chapter 2: In this chapter, using results from noisy observation (nonlinear filtering), we transform these noisy observation SDE control problems into full observation SPDEs and then we prove a sufficient and necessary maximum principle for the optimal control of SPDEs.

Chapter 3: (The results of this chapter were the subject of a paper published in international journal ESAIM: Control, Optimisation and Calculus of Variations, COCV 29,2023). In this chapter, we aim to prove the existence and uniqueness of strong, smooth solutions of a class of stochastic partial differential equations (SPDEs) with space interactions., and we show that, under some conditions, the solutions are positive for all times if the initial values are. Sufficient and necessary maximum principles for the optimal control of such systems are derived. Finally, we apply the results to study an optimal vaccine strategy problem for an epidemic by modelling the population density as a space mean stochastic reaction-diffusion equation.

Chapter 1

Preliminaries

Probability theory plays a vital role in the general study of stochastic calculus. Stochastic calculus is concerned with the study of stochastic processes, which involve randomness or noise. In this Chapter, we give preliminaries on stochastic calculus and semigroup theory which are required for this thesis. For more details, we refer for example to [42],[15],[22],[16].

1.1 Elements from Stochastic Calculus:

Definition 4 *A probability measure* $\mathbb{P}$ *on a measurable space* $(\Omega, \mathscr{F})$ *is a function* $\mathbb{P} : \mathscr{F} \to [0, 1]$ *such that*

(i) $P(\emptyset) = 0, \mathbb{P}(\Omega) = 1$

(ii) *if* $A_1, A_2, ... \in \mathscr{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *is disjoint* (*i.e.* $A_i \cap A_j = \emptyset$ *if* $i \neq j$) *then*

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 5 *A complete probability space is a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *if* $\mathscr{F}$ *contains all subsets* A *of* Ω *with* $\mathbb{P}$*-outer measure zero. That is,*

$$\mathbb{P}(A) = \inf\{\mathbb{P}(F), F \in \mathscr{F}, A \subset F\} = 0.$$

Definition 6 *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a given probability space. A function* $f : \Omega \to \mathbb{R}^n$ *is called* $\mathscr{F}$*-measurable* *if for all open sets $U \in \mathbb{R}^n$, we have,*

$$f^{-1}(U) = \{\omega \in \Omega, f(\omega) \in U\} \in \mathscr{F}.$$

Definition 7 *If Ω is a given set, then a σ-algebra $\mathscr{F}$ on Ω is a family of subsets of Ω with the following properties:*

(i) $\emptyset \in \mathscr{F}$,

(ii) $A \in \mathscr{F} \Longrightarrow A^c \in \mathscr{F}$,

(iii) $A_1, A_2, \ldots \in \mathscr{F} \Longrightarrow A := \cup_{i=1}^{\infty} A_i \in \mathscr{F}$.

Let H and U be two separable Hilbert spaces.

Definition 8 *A process $\Phi(t)$, $t \in [0,T]$ with values in $L(H,U)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal filtration $\mathscr{F}_t$, $t \in [0,T]$ is said to be elementary if there exists $0 = t_0 < t_1 < \ldots < t_k = T$, $k \in \mathbb{N}$ such that*

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m \mathbf{1}_{]t_m, t_{m+1}]}(t), 0 \le t \le T.$$

Where

(i) *Φ_m is $\mathscr{F}_{t_m}$-measurable with respect to the Borel σ-algebra on $L(H,U)$, $1 \le m \le k-1$*

(ii) *Φ_m takes only a finite number of values in $L(H,U)$, $1 \le m \le k-1$.*

Definition 9 *The stochastic integral for an elementary process $\Phi(t)$, $t \in [0,T]$ is defined by*

$$\int_0^t \Phi(s) dB(s) = \sum_{m=0}^{k-1} \Phi_m (B(t_{m+1} \wedge t) - B(t_m \wedge t)), 0 \le t \le T.$$

Definition 10 (Q-Wiener processes) *A U-valued stochastic process $B(t)$, $t \ge 0$, is called a Q-Wiener process if*

(a) $B(0) = 0$,

(b) *$B(t)$ has $\mathbb{P}$-a.s continuous trajectories,*

(c) *$B(t)$ has independent increments,*

(d)

$$\mathscr{L}(B(t)-B(s)) = \mathscr{L}(0,(t-s)Q), 0 \le s \le t \le T.$$

Note that there exists a complete orthonormal system $\{e_k\}$in U and a bounded sequence of nonnegative real numbers$\{\lambda_k\}$ such that

$$Qe_k = \lambda_k e_k, k \in \mathbb{N}.$$

Proposition 11 *Assume that $B(t)$is a Q-Wiener process. Then the following statements hold.*

- *B is a Gaussian process on U and*

$$\mathbb{E}(B(t)) = 0, Cov(B(t)) = tQ, t \ge 0.$$

- *For arbitrary $t \ge 0$, B has the expansion*

$$B(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j,$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}} \left\langle B(t), e_j \right\rangle, j \in \mathbb{N},$$

are real valued Brownian motions mutually independent on $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition 12 (Generalized Wiener processes) *Let $B(t), t \ge 0$, be a Wiener process on a Hilbert space U and let Q be its covariance operator. For each $a \in U$ define a real valued Wiener process $B_a(t)$, $t \ge 0$, by the formula*

$$B_a(t) = \langle a, B(t) \rangle, t \ge 0.$$

The transformation $a \to B_a$ is linear from U to the space of stochastic processes. Moreover it is *continuous in the following sense:*

$$\begin{aligned} \{a_n\} &\subset U, t \geq 0, \\ \lim_{n\to\infty} a_n &= a \Longrightarrow \lim_{n\to\infty} \mathbb{E}[B_a(t) - B_{a_n}(t)]^2 = 0. \end{aligned}$$

Remark 13 *The operator Q is self-adjoint and positive definite, we call it the covariance of the generalized Wiener process $a \to B_a$. If the covariance Q is the identity operator I then the generalized Wiener process is called a cylindrical Wiener process in U. Denote by U_0 the image $Q^{1/2}(U)$ with the induced norm. We call $Q^{1/2}(U)$ the reproducing kernel of the generalized Wiener process $a \to B_a$.*

Proposition 14 *Let U_1 be a Hilbert space such that $U_0 = Q^{1/2}(U)$ is embedded into U_1 with a Hillbert–Schmidt embedding J. Then the formula*

$$B(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j \beta_j(t), t \geq 0,$$

defines a U_1-valued Wiener process. Moreover, if Q_1 is the covariance of B then the spaces $Q_1^{1/2}(U_1)$ and $Q^{1/2}(U)$ are identical.

Definition 15 *Let H and U be two separable Hilbert spaces. A bounded linear operator $A : H \to U$ is called Hilbert-Schmidt if*

$$\sum_{k\in\mathbb{N}} \|Ae_k\|^2 < \infty,$$

where $e_k, k \in \mathbb{N}$ is an orthonormal basis of H. We denote the space of all Hilbert-Schmidt operators from H to U.

1.2 The Itô formula:

Assume that ϕ is an L_0^2-valued process stochastically integrable in$[0, T]$, φ a H-valued predictable process Bochner integrable on $[0, T]$, $\mathbb{P}$-a.s., and $Y(0)$ a $\mathscr{F}_0$-measurable H-valued random variable. Then the following process

$$Y(t) = Y(0) + \int_0^t \varphi(s)ds + \int_0^t \phi(s)dB(s), t \in [0, T],$$

is well-defined. Assume that a function $F : [0, T] \times H \to \mathbb{R}$ and its partial derivatives F_t, F_x, F_{xx}, are uniformly continuous on bounded subsets of $[0, T] \times H$.

Theorem 16 (The Itô formula) *Under the above conditions, $\mathbb{P}$-a.s., for all $t \in [0, T]$*

$$F(t, Y(t)) = F(0, Y(0)) + \int_0^t \langle F_x(s, Y(s)), \phi(s) dB(s) \rangle$$
$$+ \int_0^t \{F_t(s, Y(s)) + \langle F_x(s, Y(s)), \varphi(s) \rangle$$
$$+ \frac{1}{2} Tr[F_{xx}(s, Y(s))(\phi(s)Q^{1/2})(\phi(s)Q^{1/2})^*]\} ds.$$

(see [15]).

1.3 Cauchy problems:

Linear evolution equations, as parabolic, hyperbolic or delay equations, can often be formulated as an evolution equation in a Banach space E (see[15]):

$$\begin{cases} u'(t) & = A_1 u(t), t \geq 0, \\ u(0) & = x \in E, \end{cases} \tag{1.1}$$

with A_1 being a linear operator, in general unbounded, defined in a dense linear subspace $D(A_1)$ of E. In equation (1.1) $u'(t)$ stands for the strong derivative of $u(t)$

$$\lim_{h \to 0} \frac{u(t+h) - u(t)}{h} = u'(t).$$

The equation (1.1) is the Cauchy problem or the initial value problem.

Definition 17 *We say that the Cauchy problem (1.1) is well posed if:*

1. *for arbitrary $x \in D(A_1)$ there exists exactly one strongly differentiable function $u(t, x)$, $t \in [0, +\infty)$, satisfying (1.1) for all $t \in [0, +\infty)$,*

2. *if $\{x_n\} \in D(A_1)$ and $\lim_{n \to \infty} x_n = 0$, then for all $t \in [0, +\infty)$,*

$$\lim_{n \to \infty} u(t, x_n) = 0, \tag{1.2}$$

If the limit in (1.2) is uniform in t on compact subsets of $[0,+\infty)$ we say that the Cauchy problem (1.1) is uniformly well posed.

1.4 Elements of Semigroup Theory:

Let $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$ be Banach spaces. Denote by $\mathscr{L}(X,Y)$ the family of bounded linear operators from X to Y. $\mathscr{L}(X,Y)$ becomes a Banach space when equipped with the norm

$$\| T \|_{\mathscr{L}(X,Y)} = \sup_{x\in X,\|x\|_{X=1}} \| Tx \|_Y, T \in \mathscr{L}(X,Y).$$

For brevity,$\mathscr{L}(X)$ will denote the Banach space of bounded linear operators on X. Let X^* denote the dual space of all bounded linear functionals x^* on X. X^* is again a Banach space under the supremum norm

$$\| x^* \|_{X^*} = \sup_{x\in X,\|x\|_{X=1}} |\left\langle x, x^* \right\rangle|,$$

where $\langle .,. \rangle$ denotes the duality on $X \times X^*$. For $T \in \mathscr{L}(X,Y)$, the adjoint operator $T^* \in \mathscr{L}(X^*,Y^*)$ is defined by

$$\left\langle x, T^* y^* \right\rangle = \left\langle Tx, y^* \right\rangle, x \in X, y^* \in Y^*.$$

Let H be a real Hilbert space. A linear operator $T \in \mathscr{L}(H)$ is called symmetric if for all $h, g \in H$,

$$\left\langle Th, g \right\rangle_H = \left\langle h, Tg \right\rangle_H.$$

A symmetric operator T is called a nonnegative definite if for every $h \in H$,

$$\langle Th, h \rangle_H \geq 0.$$

Definition 18 *Let X be a Banach space. A semigroup $S(t) \in \mathscr{L}(X)$, $t \geq 0$, of bounded linear operators on a Banach space X such that*

(i) $S(0) = I$*, the identity operator on X.*

(ii) $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$, *Semigroup property.*

(iii) $\lim\limits_{t\to 0^+} S(t)x = x$ *for every* $x \in X$, *Strong continuity property.*

Let $S(t)$ *be a* C_0*-semigroup on a Banach space* X. *Then, there exist constants* $\alpha \geq 0$ *and* $M \geq 1$ *such that*

$$\| S(t) \|_{\mathscr{L}(X)} \leq Me^{\alpha t}, t \geq 0.$$

Remark 19 *If* $M = 0$, *then* $S(t)$ *is called a pseudo-contraction semigroup. If* $\alpha = 0$, *then* $S(t)$ *is called uniformly bounded, and if* $M = 1$ *and* $\alpha = 0$, *then* $S(t)$ *is called a semigroup of contractions.*

Definition 20 *The infinitesimal generator of a semigroup* $S(t)$ *is a linear operator* A *defined by*

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t}, exists\}$$

Where $D(A)$ *is the domain of* A, *and*

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}$$

If

$$\lim_{t\to 0^+} \| S(t) - I \|_{\mathscr{L}(X)} = 0.$$

A semigroup $S(t)$ *is called uniformly continuous.*

Theorem 21 (Hille-Yosida) *Let* $A : D(A) \subset X \to X$ *be a linear closed operator on* X. *Necessary and sufficient conditions for* A *to generate a* C_0*-semigroup* $S(t)$ *are*

(a) *A is the infinitesimal generator of a* C_0*-semigroup* $S(.)$ *such that*

$$\| S(t) \| \leq Me^{\alpha t}, \forall t \geq 0.$$

(b) $D(A)$ *is dense in* X, *the resolvent set* $\rho(A)$ *contains the interval* $(\alpha, +\infty)$ *and the following estimates hold*

$$\left\| R^k(\lambda, A) \right\| \leq \frac{M}{(\lambda - \alpha)^k}, \forall k \in \mathbb{N}.$$

Moreover if either (a) or (b) holds then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt, \forall x \in X, \lambda > \alpha.$$

Finally

$$S(t)x = \lim_{n\to\infty} e^{tA_n} x, \forall x \in X,$$

where

$$A_n = nAR(n, A),$$

and the following estimate holds

$$\left\| e^{tA_n} \right\| \le M e^{\frac{\alpha n t}{n-\alpha}}, \forall t > 0, n > \alpha.$$

The operators $A_n = AJ_n$ where $J_n = nR(n, A)$, $n > \alpha$, are called the Yosida approximations of A. The following properties of Yosida approximations will be frequently used.

1.5 Factorization formula:

Proposition 22 *Assume that $p > 1, r \ge 0, \alpha > \frac{1}{p} + r$ and that E_1, E_2 are Banach spaces such that*

$$|S(t)x|_{E_1} \le Mt^{-r} |x|_{E_2}, 0 \le t \le T, x \in E_2,$$

then G_a given by

$$G_a f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s) f(s) ds, t \in [0, T].$$

is a bounded linear operator from $L^p(0, T, E_2)$ into $C([0, T]; E_1)$.

Proof (see[15])

Assume now that U and H are Hilbert spaces and that B is a U-valued Wiener process. Denote

$$\begin{aligned} B_a(t) &= \int_0^t S(t-s)\Phi(s)dB(s), t \geq 0, \\ Y_\alpha(t) &= \int_0^t (t-s)^{\alpha-1}\Phi(s)dB(s), t \geq 0. \end{aligned}$$

The following result is a corollary of the stochastic Fubini theorem.

Theorem 23 *Assume that for some* $\alpha \in (0,1)$ *and all* $t \in [0,T]$,

$$\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-\sigma)^{-2\alpha}\mathbb{E}\left[S\,\|(t-\sigma)\Phi(\sigma)\|^2_{L_2^0}\right]d\sigma\right)^{1/2} ds < +\infty.$$

Then

$$B_a(t) = \frac{\sin\alpha\pi}{\pi}\int_0^t (t-s)^{\alpha-1}Y_\alpha(s)ds, t \in [0,T].$$

Thus

$$\begin{aligned} & \frac{\sin\alpha\pi}{\pi}\int_0^t (t-s)^{\alpha-1}Y_\alpha(s)ds \\ = & \frac{\sin\alpha\pi}{\pi}\int_0^t (t-s)^{\alpha-1}S(t-s)\left[\int_0^s (s-\sigma)^{-\alpha}S(s-\sigma)\Phi(\sigma)dB(\sigma)\right]ds \\ = & \frac{\sin\alpha\pi}{\pi}\int_0^t \left[\int_0^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}ds\right]S(s-\sigma)\Phi(\sigma)dB(\sigma). \end{aligned}$$

Since

$$\int_0^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}ds = \frac{\pi}{\sin\alpha\pi}, \sigma \in [0,T], \alpha \in (0,1).$$

Theorem 24 *If* $Tr\,Q < +\infty$ *then for arbitrary* $p > 2$,

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}|B_a(t) - B_{an}(t)|^p\right] = 0.$$

1.6 Useful results:

Theorem 25 (Burkholder-Davis-Gundy inequality) *Let $p > 0$, there are positive constants c_p and C_p such that, for any continuous local martingale $X = (X_t)_{t\geq 0}$, null in 0 :*

$$c_p\mathbb{E}\left[\langle X,X\rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[\sup_{t\geq 0}|X_t|^p\right] \leq C_p\mathbb{E}\left[\langle X,X\rangle_\infty^{p/2}\right].$$

Lemma 26 (Gronwall lemma) *Let $T < 0$, and let g be a measurable positive function bounded on $[0,T]$. Suppose there are two constants $a \geq 0, b \geq 0$, such that for all $t \in [0,T]$,*

$$g(t) \leq a + b\int_0^t g(s)ds,$$

then, we have for all $t \in [0,T]$,

$$g(t) \leq a\exp(bt).$$

Theorem 27 (Stochastic Fubini theorem) *Assume that $(E,\mathscr{E})$ is a measurable space and let*

$$\Phi : (t,\omega,x) \to \Phi(t,\omega,x)$$

be a measurable mapping from $(\Omega_T \times E, \mathscr{P}_T \times \mathscr{B}(E))$ into $(L_2^0, \mathscr{B}(L_2^0))$. Assume that

$$\int_E \|\Phi(.,.,x)\|_T\,\mu(dx) < +\infty,$$

then $\mathbb{P}$-a.s

$$\int_E \left[\int_0^T \Phi(t,x)dB(t)\right]\mu(dx) = \int_0^T \left[\int_E \Phi(t,x)\mu(dx)\right]dB(t).$$

proof (we refer to read [15])

Theorem 28 (Girsanov) *Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form*

$$\begin{cases} dY(t) &= \alpha(t,\omega)dt + dB(t); t \leq T, \\ Y(0) &= 0. \end{cases}$$

where $t \leq \infty$ is a given constant and $B(t)$ is n-dimensional Brownian motion. Put

$$M_t = \exp\left(-\int_0^t \alpha(s,\omega)dB_s - \frac{1}{2}\int_0^t \alpha^2(s,\omega)ds\right); t \leq T,$$

. Assume that $a(s,w)$ satisfies Novikov's condition

$$E[\exp(\frac{1}{2}\int_0^T a^2(s,w)ds] < \infty,$$

where $E = E_P$ is the expecation w.r.t.P. Define the measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}_T)$ by

$$dQ(\omega) = M_T(\omega)dP(\omega).$$

Then Q is a probability measure on $\mathscr{F}_T^{(n)}$ and $Y(t)$ is an n-dimensional Brownian motion w.r.t. Q, for $t \leq T$,and

$$\frac{dQ(\omega)}{dP(\omega)} = M_T(\omega) = \exp\left(-\int_0^T \alpha(s,\omega)dB_s - \frac{1}{2}\int_0^T \alpha^2(s,\omega)ds\right),$$

is called the Radon-Nikodym derivative.

Lemma 29 (Bayes'rule) *[42] Let be μ and ν be two probability measures on a measurable space $(\Omega; G)$ such that $d\nu(\omega) = f(\omega)d\mu(\omega)$ for some $f \in L^1(\mu)$. Let X be a random variable on $(\Omega; G)$ such that*

$$\mathbb{E}\nu[|X|] = \int_\Omega |X(\omega)| f(\omega)d\mu(\omega) < \infty.$$

Let $\mathscr{H}$ be a σ-algebra, $\mathscr{H} \subset \mathscr{G}$. Then

$$\mathbb{E}_\nu[X \mid \mathscr{H}].\mathbb{E}_\mu[f \mid \mathscr{H}] = \mathbb{E}\mu[fX \mid \mathscr{H}] \ a.s.$$

Proposition 30 (Holder's inequality) *Let $p, q \in [1, +\infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g are measurable applications, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proposition 31 (Young inequality) *Let $a, b \geq 0$ and $p, q \in]1, +\infty[$, with $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Definition 32 (Fréchet differentiable) *Let $f:\mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at x if there is a linear form $L:\mathbb{R}^n \to \mathbb{R}^m$ such that*

$$f(x+h) = f(x) + L(h) + \|h\|\,\varepsilon(h), \forall h \in \mathbb{R}^n,$$

$\varepsilon(h) \to 0$, when $h \to 0$. we can note $f'(x) = L$, we can replace $\|h\|\,\varepsilon(h)$ by $o(h)$. The gradient of f is the vector $\nabla f = \left(\frac{\partial f}{\partial x_i}\right)$.

Theorem 33 (The Riesz representation theorem) *Let H a Hilbet space and f a continuous linear form defined on H. Then there exists a unique vector $y \in H$ such that, for all $x \in H$,*

$$f(x) = \langle x, y \rangle.$$

And

$$\|f\|_{H^*} = \|y\|_H.$$

Definition 34 (Duncan-Mortensen-Zakai equation) *the Zakai equation is a linear stochastic partial differential equation (Linear SPDEs) for the un-normalized density of a hidden state. It was named after Moshe Zakai [57].*

In the state of the system evolves and observation equation have the form:

$$\begin{cases} dx(t) &= f(x,t)dt + dB(t), \\ dz(t) &= \beta(x,t)dt + d\widetilde{B}(t). \end{cases}$$

where are independent Wiener processes. Then the unnormalized conditional probability density $p(x,t)$ of the state at time t is given by the Zakai equation:

$$dp = L(p)dt + ph^\top dz,$$

where

$$L(p) = -\frac{\partial(f_i p)}{\partial x_i} + \frac{1}{2}\frac{\partial^2 p}{\partial x_i \partial x_j} \tag{1.3}$$

1.7 Sobolev Spaces

A Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norms of the function together with its derivatives up to a given order. Sobolev spaces are named after Sergei Sobolev, (We refer to read [1])

Definition 35 (Test Functions) *Let Ω be a domain in $\mathbb{R}^n$. A sequence $\{\Phi_j\}$ of functions belonging to $C_0^\infty(\Omega)$ is said to converge in the sense of the space $\mathscr{D}(\Omega)$ to the function $\Phi \in C_0^\infty(\Omega)$ provided the following conditions are satisfied:*

- *there exists $K \subset \Omega$ such that supp $(\Phi_j - \Phi) \subset K$ for every j, and*
- $\lim_{j\to\infty} D^\alpha \Phi_j(x) = D^\alpha \Phi(x)$ *uniformly on K for each multi-index α.*

Definition 36 (Schwartz Distributions) *The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of (Schwartz) distributions on Ω. $\mathscr{D}'(\Omega)$ is given the weak-star topology as the dual of $\mathscr{D}(\Omega)$, and is a locally convex TVS with that topology. We summarize the vector space and convergence operations in $\mathscr{D}'(\Omega)$ as follows: if S, T, T_j belong to $\mathscr{D}'(\Omega)$ and $c \in \mathbb{C}$, then*

$$\begin{aligned} (S+T)(\Phi) &= S(\Phi) + T(\Phi), \Phi \in \mathscr{D}(\Omega), \\ (cT)(\Phi) &= cT(\Phi), \Phi \in \mathscr{D}(\Omega), \end{aligned}$$

$T_j \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_j(\Phi) \to T(\Phi)$ in $\mathbb{C}$ for every $\Phi \in \mathscr{D}(\Omega)$.

Definition 37 (Derivatives of Distributions) *Let $u \in C^1(\Omega)$ and $\Phi \in \mathscr{D}(\Omega)$. Since Φ vanishes outside some compact subset of Ω, we obtain by integration by parts in the variable x_j*

$$\int_\Omega \left(\frac{\partial}{\partial x_j} u(x)\right) \Phi(x)\,dx = -\int_\Omega u(x) \left(\frac{\partial}{\partial x_j} \Phi(x)\right) dx.$$

Similarly, if $u \in C^{|\alpha|}(\Omega)$, then integration by parts $|\alpha|$ times leads to

$$\int_\Omega (D^\alpha u(x)) \Phi(x)\,dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \Phi(x)\,dx.$$

Definition 38 (The Sobolev Norms) *We define a functional* $\|.\|_{m,p}$*, where* m *is a positive integer and* $1 \leq p \leq \infty$*, as follows:*

$$\begin{aligned} \|u\|_{m,p} &= \left({}_{0\leq|\alpha|\leq m} \left\|D^{\alpha}u\right\|_{p}^{p} \right)^{1/p}, \text{if } 1 \leq p \leq \infty. \\ \|u\|_{m,\infty} &= \max_{0\leq|\alpha|\leq m} \left\|D^{\alpha}u\right\|_{\infty}. \end{aligned}$$

Definition 39 (Sobolev Spaces) *Assume that* $\Omega \in \mathbb{R}^n$*. For any positive integer* m *and* $1 \leq p \leq \infty$*. The Sobolev space* $W^{m,p}(\Omega)$ *Consists of functions* $u \in L^p(\Omega)$ *such that for every multi-index* α *with* $|\alpha| \leq m$*, the weak derivative* $D^{\alpha}u$ *exists and* $D^{\alpha}u \in L^p(\Omega)$*. Thus*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega), |\alpha| \leq m\}.$$

Definition 40 (White noise) *The white noise process is the measurable map*

$$w : \mathscr{S} \times \mathscr{S}' \to \mathbb{R},$$

given by

$$w(\phi,\omega) = w_{\phi}(\omega) = \langle \omega, \phi \rangle, \phi \in \mathscr{S}, \omega \in \mathscr{S}'.$$

From w_{ϕ} *we can construct a Wiener process* $W(t)$*,* $t \in \mathbb{R}$*, as follows:*

- *(Step1): The isometry* $\mathbb{E}\left[w_{\phi}^2\right] = \|\phi\|^2, \phi \in \mathscr{S}$*, holds true where, according to our notation, the left-hand side is*

$$\mathbb{E}\left[w_{\phi}^2\right] = \int_{\mathscr{S}'} \langle \omega, \phi \rangle^2 P(dw).$$

- *(Step2): Use Step 1 to define the value* $\langle \omega, \psi \rangle$ *for arbitrary* $\psi \in L^2(\mathbb{R})$*, as* $\langle \omega, \psi \rangle = \lim \langle \omega, \phi_n \rangle$*, where* $\phi_n \in \mathscr{S}, n = 1.2, ...,$ *and* $\phi_n \to \psi$ *in* $L^2(\mathbb{R})$*.*

- *(Step3): Use Step2 to define*

$$\widetilde{W}(t,\omega) = \langle \omega, \chi_{[0,t]} \rangle, t \in \mathbb{R},$$

  by choosing

$$\psi(s) = \chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0,t) (\text{or } s \in [t,0) \text{ if } t < 0), \\ 0 & \text{otherwise}, \end{cases}$$

which belongs to $L^2(\mathbb{R})$ *for all* $t \in \mathbb{R}$. *(see [16])*

Definition 41 (The Wick Product) *If* $Y = \sum_\alpha a_\alpha H_\alpha \in (\mathscr{S})^*$, $Z = \sum_\beta b_\beta H_\beta \in (\mathscr{S})^*$ *then the Wick product* $Y \diamond Z$ *of* Y *and* Z *is defined by*

$$Y \diamond Z = \sum_\alpha a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma.$$

Chapter 2

Stochastic Partial Differential Equations

A stochastic partial differential equation (SPDE) is an equation combining the features of equations with partial derivatives and stochastic differential equations. In the most general sense, an SPDE is a partial differential equation in which at least one of the following is random: coefficients, initial conditions, boundary conditions, the region in which the equation is considered, including the terminal time, and the driving force. There are two types of SPDEs:

(1) SPDEs as Stochastic Equations: A stochastic differential equation (SDE) describes an adapted stochastic process with values in a finite-dimensional Euclidean space and has a finite-dimensional initial condition (see Gikhman and Skorokhod [32], Khasminskii [51]).

(2) SPDEs as Partial Differential Equations: As partial differential equations, SPDEs can be classified according to the following features: the order of the equation, the type of the nonlinearity in the equation, the type of the initial and boundary conditions, elliptic/hyperbolic/parabolic,..., (see [40]).

This chapter consists of two parts. In the first part, we introduce linear SPDEs (see [15]) . In the second part, we prove the existence and uniqueness of the solutions nonlinear SPDEs under linear growth and Lipschitz conditions on the coefficients, we refer the reader to [15], using a method that involves semigroups generated by unbounded operators and results in constructing mild solutions.

2.1 Linear SPDEs:

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space together with a normal filtration $\mathscr{F}_t, t \geq 0$. We consider two Hilbert spaces H and U, and a Q-Wiener process $B(t)$ on $(\Omega, \mathscr{F}, \mathbb{P})$(see Definition 9). We assume that there exists a complete orthonormal system $\{e_k\}$ in U, a bounded sequence $\{\lambda_k\}$ of nonnegative real numbers such that

$$Qe_k = \lambda_k e_k, k \in \mathbb{N},$$

and a sequence $\{\beta_k\}$ of real independent Brownian motions such that

$$\langle B(t), u\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle u, e_k\rangle \beta_k(t), u \in U, t \geq 0.$$

We will consider the following linear affine equation

$$\begin{cases} dY(t) &= (AY(t) + f(t))dt + KdB(t), \\ Y(0) &= \xi. \end{cases} \tag{2.1}$$

Where

$$\begin{aligned} A &: D(A) \subset H \to H \\ K &: U \to H \end{aligned}$$

are linear operators and f is an H-valued stochastic process. We will assume that the deterministic Cauchy problem

$$\begin{cases} u'(t) &= Au(t), \\ u(0) &= x \in H, \end{cases}$$

is uniformly well posed (see Definition 17) and that K is bounded.

Hypothesis1.1 A generates a C_0-semigroup $S(.)$ in H and $K \in L(U, H)$. It is also natural to require the following.

Hypothesis1.2 **(i)** f is a predictable process with Bochner integrable trajectories on an arbitrary finite interval $[0,T]$.

(ii) $\xi$is $\mathscr{F}_0$-measurable.

An H-valued predictable process $Y(t)$, $t\in[0,T]$, is said to be a strong solution to (2.1) if $Y(t)$ takes values in $D(A)$,

$$\int_0^T |AY(s)|\,ds < +\infty, \mathbb{P}-a.s.$$

And for $t\in[0,T]$

$$Y(t) = \xi + \int_0^t [AY(s) + f(s)]ds + KB(t), \mathbb{P}-a.s.$$

This is a strong solution that should necessarily have continuous modification.

An H-valued predictable process $Y(t)$, $t\in[0,T]$, is said to be a weak solution to (2.1) if the trajectories of $Y(.)$ are $\mathbb{P}$-a.s. Bochner integrable and if for all $z\in D(A^*)$ and all $t\in[0,T]$ we have

$$\langle Y(t), z\rangle = \langle \xi, z\rangle + \int_0^t [\left\langle Y(s), A^* z\right\rangle + \left\langle f(s), z\right\rangle]ds + \langle KB(t), z\rangle, \mathbb{P}-a.s.$$

This definition is meaningful for a cylindrical Wiener process because the scalar processes $\langle KB(t), z\rangle$, $t\in[0,T]$

2.2 Nonlinear SPDEs:

The purpose of this section is to study mild solutions of nonlinear stochastic partial differential equations (SPDEs for short), we prove existence and uniqueness of the solutions of nonlinear SPDEs under linear growth and Lipschitz conditions on the coefficients.

2.2.1 Existence and uniqueness for nonlinear SPDEs:

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a compete probability space together with a normal filtration $\mathscr{F}_t, t\geq 0$. Let $\mathscr{P}$ and $\mathscr{P}_T$ will denote predictable σ-fields on $\Omega_\infty = [0,+\infty)\times\Omega$ and on $\Omega_T = [0,T]\times\Omega$ respectively. For any $T>0$ we define $\mathbb{P}_T$ to be the product of the Lebesgue measure in $[0,T]$ and the measure $\mathbb{P}$. We assume two Hilbert spaces H and U, and that B is a Q-Wiener process on $U\subset U_1$ and $U_0 = Q^{1/2}U$, (see proposition 14). Spaces U, H and $L_2^0 = L_2(U_0, H)$ are equipped with Borel σ-fields $\mathscr{B}(U)$, $\mathscr{B}(H)$ and $\mathscr{B}(L_2^0)$. Morever ξ is an H- valued random variable $\mathscr{F}_0$-measurable. We fix $T > 0$ and impose first the following conditions on coefficients A, F and K of the equation We proceed to study nonlinear equations

$$\begin{cases} dY(t) = & (AY(t) + F(t,Y))dt + K(t,Y)dB(t), \\ Y(0) = & \xi. \end{cases} \tag{2.2}$$

Where

$$\begin{aligned} A &: \quad D(A) \subset H \to H, \\ F &: \quad \Omega \times [0,T] \times H \to H, \\ K &: \quad \Omega \times [0,T] \times H \to L_2^0. \end{aligned}$$

Here, A is the generator of a C_0-semigroup of operators $S(t) = e^{tA}$, $t \geq 0$ in H (see Defintion 18). The initial condition ξ is an $\mathscr{F}_0$-measurable H-valued random variable.

Hypothesis:

(A0) F is measurable from $(\Omega_T \times H, \mathscr{P}_T \times \mathscr{B}(H))$ into $(H, \mathscr{B}(H))$.

(A1) K is measurable from $(\Omega_T \times H, \mathscr{P}_T \times \mathscr{B}(H))$ into $(L_2^0, \mathscr{B}(L_2^0))$.

(A2) A is the generator of a C_0-semigroup of operators $S(t) = e^{tA}$, $t \geq 0$ in H.

(A3) There exists a constant $C > 0$ such that for all $y, z \in H$, $t \in [0,T]$, $\omega \in \Omega$, we have

$$\left|F(t,\omega,y) - F(t,\omega,z)\right| + \left\|K(t,\omega,y) - K(t,\omega,z)\right\|_{L_2^0} \leq C\left|y - z\right| \tag{2.3}$$

and

$$\left|F(t,\omega,y)\right|^2 + \left\|K(t,\omega,y)\right\|_{L_2^0}^2 \leq C^2(1 + \left|y\right|^2). \tag{2.4}$$

Definition 42 *A stochastic process $Y(t)$ defined on a filtred probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\leq T}, \mathbb{P})$ and adapted to the filtration $\{\mathscr{F}_t\}_{t\leq T}$ is a* mild solution *of equation (2.2) if*

(1) *the following conditions hold:*

$$\mathbb{P}(\int_0^T \| \quad Y(t) \|_H \, dt < \infty) = 1,$$
$$\mathbb{P}(\int_0^T \| \quad F(t, Y) \|_H + \|K(t, Y)\|^2_{L_2^0} < \infty) = 1.$$

(2) *for every* $t \le T$, $\mathbb{P}$*-a.s.,*

$$Y(t) = S(t)\xi + \int_0^t S(t-s)F(s, Y)\,ds + \int_0^t S(t-s)K(s, Y)\,dB(s).$$

Proposition 43 *Assume Hypothesis 2.2.1 (A3) and that for arbitrary* $y, h \in H, u \in U$ *. The processes* $\langle F(.,.,y), h \rangle, \langle K(.,.,y)Q^{1/2}u, h \rangle$ *are predictable. Then Hypothesis 2.2.1 (A0)-(A1) are fulfilled. A predictable* H*-valued process* $Y(t), 0 \le t \le T$ *is said to be a mild solution of (2.2) if*

$$\mathbb{P}(\int_0^T |Y(s)|^2\, ds < +\infty) = 1, \tag{2.5}$$

and, for arbitrary $0 \le t \le T$, $\mathbb{P}$*-a.s.,*

$$Y(t) = S(t)\xi + \int_0^t S(t-s)F(s, Y(s))\,ds + \int_0^t S(t-s)K(s, Y(s))\,dB(s). \tag{2.6}$$

Theorem 44 *Assume that* ξ *is an* $\mathscr{F}_0$*- measurable* H*-valued random variable and Hypothesis 2.2.1 is satisfied.*

(1) *There exists a mild solution* Y *to (2.2) unique, up to equivalence, among the processes, satisfying*

$$\mathbb{P}(\int_0^T |Y(s)|^2\, ds < +\infty) = 1.$$

Moreover, Y *possesses a continuous modification.*

(2) *For any* $p \ge 2$ *there exists a constant* $C_{p,T} > 0$ *such that*

$$\sup_{0 \le t \le T} \mathbb{E}\left[|Y(t)|^p\right] \le C_{p,T}(1 + \mathbb{E}\left[|\xi|^p\right]). \tag{2.7}$$

(3) *For any* $p > 2$ *there exists a constant* $\widehat{C}_{p,T} > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y(t)|^p\right] \le \widehat{C}_{p,T}(1+\mathbb{E}\left[|\xi|^p\right]). \tag{2.8}$$

Proof. We first prove uniqueness. We show that if $Y_1(.)$ and $Y_2(.)$ are two processes satisfying (2.5) and (2.6) then, for arbitrary $0 \le t \le T$,

$$\mathbb{P}(Y_1(t) = Y_2(t)) = 1$$

For a fixed number $R > 0$ we define

$$\tau_i = \inf\{t \le T : \int_0^t |F(s, Y_i(s))|\, ds \ge R\}, i = 1,2$$

or

$$\tau_i = \inf\{t \le T : \int_0^t \|K(s, Y_i(s))\|^2_{L_2^0}\, ds \ge R\}, i = 1,2$$

and $\tau = \tau_1 \wedge \tau_2$. Let $\widehat{Y}_i(t) = I_{[0,\tau]}(t)Y_i(t), 0 \le t \le T, i = 1,2.$ then, for arbitrary $0 \le t \le T$, $\mathbb{P}$-a.s.

$$\begin{aligned}\widehat{Y}_i(t) &= I_{[0,\tau]}(t)S(t)\xi + I_{[0,\tau]}(t)\int_0^t I_{[0,\tau]}(s)S(t-s)F(s,\widehat{Y}_i(s))ds \\ &\quad + I_{[0,\tau]}(t)\int_0^t I_{[0,\tau]}(s)S(t-s)K(s,\widehat{Y}_i(s))dB(s).\end{aligned}$$

Consequently, for arbitrary $0 \le t \le T$, $\mathbb{P}$-a.s.

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{Y}_1(t) - \widehat{Y}_2(t)\right|^2\right] &\le 2\mathbb{E}\left\{\int_0^t \left|F(s,\widehat{Y}_1(s)) - F(s,\widehat{Y}_2(s))\right| ds\right\}^2 \\ &\quad + 2\mathbb{E}\left\{\int_0^t \|K(s,\widehat{Y}_1(s)) - K(s,\widehat{Y}_2(s))\|^2_{L_2^0}\, ds\right\}.\end{aligned} \tag{2.9}$$

By (2.3) and (2.4) we get

$$\mathbb{E}\left[\left|\widehat{Y}_1(t) - \widehat{Y}_2(t)\right|^2\right] \le 2C^2(T+1)\int_0^t \mathbb{E}\left[\left|\widehat{Y}_1(t) - \widehat{Y}_2(t)\right|^2\right] ds.$$

The boundedness of $\mathbb{E}\left[\left|\widehat{Y}_1(t) - \widehat{Y}_2(t)\right|^2\right]$,$0 \le t \le T$, and by the Gronwall lemma (sse Lemma 26) we have

$$\mathbb{E}\left[\left|\widehat{Y}_1(t)-\widehat{Y}_2(t)\right|^2\right]=0$$

. Therefore, for all $0\le t\le T$, one has

$$\mathbb{P}(Y_1(t)=Y_2(t))=1.$$

So the predictable processes $\widehat{Y}_1(.)$, $\widehat{Y}_2(.)$ are $\mathbb{P}_T$-a.s. identical. Since this is true for arbitrary$R>0$ therefore $Y_1(.)$ and $Y_2(.)$ are $\mathbb{P}_T$-a.s. identical. Taking into account that Y_1 and Y_2 are solutions of the equation (2.6) one easily deduces that for arbitrary $0\le t\le T, Y_1(t)=Y_2(t),\mathbb{P}$-a.s.

The proof of existence is based on the classical fixed point theorem for contractions. Denote by $\mathscr{H}_p$, $p\ge 2$, the Banach space of all the H-valued predictable processes Z defined on the time interval $[0,T]$ such that

$$\|Z\|_p=\left(\sup_{0\le t\le T}\mathbb{E}\left[|Z(t)|^p\right]\right)^{1/p}<+\infty.$$

If one identifies processes which are identical $\mathbb{P}_T$-a.s. then$\mathscr{H}_p$, with the norm $\|Z\|_p$, becomes a Banach space. Let $\mathscr{K}$ be the following transformation:

$$\begin{aligned}\mathscr{K}(Z)(t) &= S(t)\xi+\int_0^t S(t-s)F(s,Z(s))ds+\int_0^t S(t-s)K(s,Z(s))dB(s)\\ &= S(t)\xi+\mathscr{K}_1(Z)(t)+\mathscr{K}_2(Z)(t), 0\le t\le T, Z\in\mathscr{H}_p.\end{aligned}$$

We assume that $\mathbb{E}(|\xi|^p)<+\infty$ and show that $\mathscr{K}:\mathscr{H}_p\to\mathscr{H}_p$. As the composition of measurable mappings is measurable therefore, taking into account Hypothesis 2.2.1 ($A0$-$A3$), one obtains that the transformations $\mathscr{K}_1$ and $\mathscr{K}_2$ are well defined. Moreover

$$\begin{aligned}\|\mathscr{K}_1(Z)\|_p^p &\le M^p\mathbb{E}\left(\int_0^T|F(s,Z(s))|\,ds\right)^p\\ &\le T^{p-1}M^p\mathbb{E}\left[\int_0^T|F(s,Z(s))|^p\,ds\right]\\ &\le 2^{p/2-1}T^{p-1}M^pC^p\mathbb{E}\left[\int_0^T(1+|Z(s)|^p)ds\right]\\ &\le 2^{p/2-1}(TMC)^p(1+\|Z\|^p).\end{aligned}$$

where $M=\sup_{0\le t\le T}\|S(t)\|$. Consequently $\mathscr{K}_1:\mathscr{H}_p\to\mathscr{H}_p$. To show the same property for $\mathscr{K}_1$ we remark that, by Theorem [47] we find

$$
\begin{aligned}
\|\mathcal{K}_2(Z)\|_p^p &\leq \sup_{0\leq t\leq T}\mathbb{E}\left[\left|\int_0^t S(t-s)K(s,Z(s))dB(s)\right|^p\right]\\
&\leq M^p C_{p/2}\mathbb{E}\left[\int_0^T \|K(s,Z(s))\|_{L_2^0}^2\,ds\right]^{p/2}\\
&\leq M^p C_{p/2}C^p\mathbb{E}\left[\int_0^T (1+|Z(s)|^2)ds\right]^{p/2}\\
&\leq M^p C_{p/2}CT^{p/2-1}\mathbb{E}\left[\int_0^T (1+|Z(s)|^2)^{p/2}ds\right]\\
&\leq M^p C_{p/2}CT^{p/2-1}2^{p/2-1}\mathbb{E}\left[\int_0^T (1+|Z(s)|^p)ds\right]\\
&\leq M^p C_{p/2}C(2T)^{p/2-1}(T+\|Z\|_p)^p.
\end{aligned}
$$

Let Z_1 and Z_2 be arbitrary processes from $\mathcal{H}_p$ then

$$
\begin{aligned}
\|\mathcal{K}(Z_1)-\mathcal{K}(Z_2)\|_p &\leq \|\mathcal{K}_1(Z_1)-\mathcal{K}_1(Z_2)\|_p+\|\mathcal{K}_2(Z_1)-\mathcal{K}_2(Z_2)\|_p\\
&= I_1+I_2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_1^p &\leq \sup_{0\leq t\leq T}\mathbb{E}\left[\left|\int_0^t S(t-s)(F(s,Z_1(s))-F(s,Z_2(s)))ds\right|^p\right]\\
&\leq M^p\sup_{0\leq t\leq T}\mathbb{E}\left[\int_0^t [|F(s,Z_1(s))-F(s,Z_2(s))|]ds\right]^p\\
&\leq (MC)^p T^{p-1}\left[\int_0^T \mathbb{E}|Z_1(s)-Z_2(s)|^p\,ds\right]\\
&\quad (MC)^p T^p\sup_{0\leq t\leq T}\mathbb{E}\left[|Z_1(s)-Z_2(s)|^p\right]\\
&\leq (MC)^p T^p\|Z_1-Z_2\|_p^p.
\end{aligned}
$$

By Theorem [47] we have

$$
\begin{aligned}
I_2^p &\leq M^p C_{p/2}\mathbb{E}\left[\int_0^T \|K(s,Z_1(s))-K(s,Z_2(s))\|_{L_2^0}^2\,ds\right]^{p/2}\\
&\quad C_{p/2}(MC)^p T^{p/2-1}\mathbb{E}\int_0^T\left[\|K(s,Z_1(s))-K(s,Z_2(s))\|_{L_2^0}^2\right]ds\\
&\leq C_{p/2}(MC)^p T^{p/2-1}\|Z_1-Z_2\|_p^p.
\end{aligned}
$$

Summing up the obtained estimates we have:

$$\|\mathscr{K}(Z_1)-\mathscr{K}(Z_2)\|_p \leq CM(T^p + C_{p/2}T^{p/2})^{1/p}\,\|Z_1-Z_2\|_p\,, \tag{2.10}$$

for all $Z_1, Z_2 \in \mathscr{K}$. Consequently, if

$$MCT(1+C_{p/2}T^{1/2})^{1/p} < 1, \tag{2.11}$$

then the transformation $\mathscr{K}$ has unique fixed point Y in $\mathscr{H}_p$ which, as it is easy to see, is a solution of the equation (2.2).

To construct a solution when $E|\xi|^p = +\infty$, we show first that if ξ and η are two initial conditions satisfying $E|\xi|^p < +\infty, E\left|\eta\right|^p < +\infty$, and if $Y, Z \in \mathscr{H}_p$ are the corresponding solutions of equation (2.2), then

$$I_\Gamma Y(.) = I_\Gamma Z(.), \tag{2.12}$$

$\mathbb{P}$-a.s., where

$$\Gamma = \{\omega \in \Omega : \xi(\omega) = \eta(\omega)\}.$$

We define

$$Y^0 = S(.)\xi,\, Y^{k+1} = \mathscr{K}(Y^k), 0 \leq t \leq T, k \in \mathbb{N}.$$

Thus for $0 \leq t \leq T, \mathbb{P}$-a.s.

$$Y^{k+1}(t) = S(t)\xi + \int_0^t S(t-s)F(s, Y^k(s))ds + \int_0^t S(t-s)K(s, Y^k(s))dB(s).$$

Since I_Γ is an $\mathscr{F}_0$-measurable random variable, therefore $I_\Gamma K(., Y^k(.))$ is an L_2^0-predictable process and for $0 \leq t \leq T$,

$$\int_0^t S(t-s)I_\Gamma K(s, Y^k(s))dB(s) = I_\Gamma \int_0^t S(t-s)K(s, Y^k(s))dB(s).$$

Thus for $0 \le t \le T$,

$$I_\Gamma Y^{k+1}(t) = S(t)I_\Gamma\xi + \int_0^t S(t-s)I_\Gamma F(s,Y^k(s))ds + \int_0^t S(t-s)I_\Gamma K(s,Y^k(s))dB(s). \quad (2.13)$$

If for a similarly defined sequence

$$Y^0(t) = S(t)\eta, Y^{k+1}(t) = \mathscr{K}(Y^k), 0 \le t \le T, k \in \mathbb{N},$$

we have

$$I_\Gamma Y^k(.) = I_\Gamma Z^k(.), P_T - a.s.$$

Then

$$I_\Gamma F(., Y^k(.)) = I_\Gamma F(., Z^k(.)), I_\Gamma B(., Y^k(.)) = I_\Gamma K(., Z^k(.)), P_T - a.s.$$

Consequently

$$I_\Gamma Y^{k+1}(.) = I_\Gamma Z^{k+1}(.), \mathbb{P}_T - a.s.$$

Since the processes Y and Z are limits in the $\|.\|_p$ norm of the sequences $Y^k(.)$ and $Z^k(.)$ respectively, therefore (2.12) must be true. Moreover the process$I_\Gamma Y(.)$ satisfies the equation (2.2) with the initial condition $I_\Gamma\xi = I_\Gamma\eta$.

We now prove existence. Let us define, for $n \in \mathbb{N}$

$$\xi_n \begin{cases} \xi & if \quad |\xi| \le n, \\ 0 & if \quad |\xi| > n, \end{cases}$$

and denote by $Y_n(\cdot)$ the corresponding solution of (2.11). By the previous argument we have

$$Y_n(t) = Y_{n+1}(t), \{\omega \in \Omega : |\xi| \le n\}.$$

Then

$$Y(t) = \lim_{n\to\infty} Y_n(t), 0 \le t \le T,$$

is $\mathbb{P}$-a.s. well defined and satisfies the equation (2.2).

For proof of the existence of continuous modification of the mild solution assume first that $E|\xi|^{2r} < +\infty$ for some $r > 1$. From the first part of the theorem one knows that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\|Y(t)\|^{2r}\right] < +\infty,$$

Define

$$\Phi(t) = K(t, Y(t)), 0 \leq t \leq T,$$

and

$$I = \mathbb{E}\left[\int_0^T \|\Phi(t)\|_{L_2^0}^{2r} dt\right] = \mathbb{E}\left[\int_0^T \|K(t, Y(t))\|_{L_2^0}^{2r} dt\right].$$

By (2.3) and (2.4) we have

$$I \leq C^{2r}\mathbb{E}\left(\int_0^T (1 + |Y(t)|^2)^r dt\right) < +\infty.$$

Consequently Proposition (45) below implies that the process

$$\int_0^t S(t-s)K(s, Y(s))dB(s), 0 \leq t \leq T,$$

and therefore also $Y(t)$, $0 \leq t \leq T$, has a continuous modification. The case of initial conditions satisfying $\mathbb{E}\left[|\xi|^{2r}\right] = +\infty$ can be reduced to the case just considered by regarding initial conditions ξ_n

$$\xi_n \begin{cases} \xi & if \quad |\xi| \leq n, \\ 0 & if \quad |\xi| > n, \end{cases}$$

as in the proof of existence. Finally (2.8) follows again from Gronwall's lemma. The proof is complete. ■

We consider now the approximating problem

$$\begin{cases} dY_n = & (A_n Y + F(t, Y_n))dt + K(t, Y_t)dB(t), \\ Y_n(0) = & \xi. \end{cases} \tag{2.14}$$

where A_n are the Yosida approximations of A (see Theorem [21]), the problem (2.14) has a unique solution Y_n for any random variable ξ , $\mathscr{F}_0$-measurable. We will need the following result.

Proposition 45 *Let $p > 2, T > 0$ and let Φ be an L_2^0-valued predictable process such that*

$$\mathbb{E}\left[\int_0^T \|\Phi(s)\|_{L_2^0}^p ds\right] < +\infty.$$

There exists a constant $C_T > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |S(t-s)\Phi(s)dB(s)|^p\right] \le C_T \mathbb{E}\left[\int_0^T \|\Phi(s)\|_{L_2^0}^p ds\right]. \tag{2.15}$$

Moreover

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} \left|B_A^\Phi(t) - B_{A,n}(t)\right|^p\right] = 0, \tag{2.16}$$

where $B_{A,n}^\Phi$ is defined as

$$B_{A,n}^\Phi(t) = \int_0^t e^{(t-s)A_n}\Phi(s)dB(s), 0 \le t \le T,$$

and A_n are the Yosida approximations of A. where B_A^Φ has a continuous modification define by

$$W_A^\Phi = \int_0^t S(t-s)\Phi(s)dB(s)$$

Proof. *We will use the factorization method, Let $\alpha \in (\frac{1}{p}, \frac{1}{2})$, the stochastic Fubini theorem (Theorem27) implies that*

$$B_A^\Phi(t) = \frac{\sin \pi\alpha}{\pi} = \int_0^t (t-s)^{\alpha-1} S(t-s) Z(s) ds, 0 \le t \le T,$$

where

$$Z(s) = \int_0^s (s-\sigma)^{-\alpha} S(s-\sigma)\Phi(\sigma)dB(\sigma), 0 \le s \le T.$$

Since $\alpha > \frac{1}{p}$, applying Hölder's inequality (see Proposition [30]) one obtains that there exists a constant $C_{1,T} > 0$ such that

$$\sup_{0\le t\le T} \left|B_A^\Phi(t)\right|^p \le C_{1,T} \int_0^T |Z(s)|^p ds. \tag{2.17}$$

Moreover, by Theorem(47), there exists a constant $C_{2,T} > 0$*such that*

$$\mathbb{E}\left[|Z(s)|^p\right] \le C_{2,T} E\left(\int_0^s (s-\sigma)^{-2\alpha} \|\Phi(\sigma)\|^p_{L_2^0} d\sigma\right)^{p/2}. \tag{2.18}$$

Now, using the Young inequality (see Proposition [31]), we obtain that

$$\begin{aligned}\int_0^T \mathbb{E}\left[|Z(s)|^P ds\right] &\le C_{2,T}\mathbb{E}\left[\int_0^s \sigma^{-2\alpha} d\sigma\right]^{p/2} \int_0^s \|\Phi(\sigma)\|^{2r}_{L_2^0} d\sigma \\ &\le C_{3,T}\mathbb{E}\left[\int_0^T \|\Phi(\sigma)\|^p_{L_2^0} d\sigma\right].\end{aligned}$$

This finishes the proof of (2.15) with $C = C_{1,T}C_{2,T}$.

We now prove (2.16), we have

$$B^{\Phi}_{A,n}(t) = \frac{\sin \pi\alpha}{\pi}\int_0^t e^{(t-s)A_n}(t-s)^{\alpha-1} Z_n(s)ds,$$

where

$$Z_n(s) = \int_0^s e^{(s-\sigma)A_n}(s-\sigma)^{-\alpha}\Phi(\sigma)dB(\sigma).$$

Thus

$$\begin{aligned}B^{\Phi}_{A}(t) - B^{\Phi}_{A,n}(t) &= \frac{\sin \pi\alpha}{\pi}\int_0^t \left[S(t-s) - e^{(t-s)A_n}\right](t-s)^{\alpha-1} Z(s)ds \\ &= \frac{\sin \pi\alpha}{\pi}\int_0^t \left[S(t-s) - e^{(t-s)A_n}\right](t-s)^{\alpha-1} Z(s)ds \\ &\quad + \frac{\sin \pi\alpha}{\pi}\int_0^t e^{(t-s)A_n}(t-s)^{\alpha-1}\left[Z(s) - Z_n(s)\right]ds \\ &= I_n(t) + J_n(t).\end{aligned}$$

We proceed now in two steps.

Step 1 *Exactly as in Step 1 of the proof of theorem 24 we show that*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} |I_n(t)|^p\right] = 0. \tag{2.19}$$

Step 2

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|J_n(t)|^p\right]=0. \tag{2.20}$$

The following estimate is proved as (2.17)

$$\sup_{0\le t\le T}|J_n(t)|^p\le C_{2,T}\int_0^T|Z(s)-Z_n(s)|^p\,ds. \tag{2.21}$$

We now show

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T|Z(s)-Z_n(s)|^p\,ds\right]=0.$$

We define the operators

$$\mathcal{K}_n\Phi(s)=\int_0^s(s-\sigma)^{-\alpha}(S(s-\sigma)-e^{(s-\sigma)A_n})\Phi(s)dB(s).$$

Where $\mathcal{K}_n\Phi=Z-Z_n$. *We will show that if* $E\int_0^T\|\Phi(s)\|_{L_2^0}^p\,ds<\infty$, *then*

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T|\mathcal{K}_n\Phi(s)|^p\,ds\right]=0. \tag{2.22}$$

It follows from considerations following (2.18) that the operators

$$\mathcal{K}_n:L^p(\Omega\times[0,T];L_2^0)\to L^p(\Omega\times[0,T];H).$$

It is enough to prove (2.22) for a dense set of such that

$$\mathbb{E}\left[\int_0^T\left\|A^2\Phi(s)\right\|_{L_2^0}^p\,ds\right]<\infty.$$

In fact, the processes Z can by approximated as follows:

$$Z_m=(m(mI-A)^{-1})^2Z,\Phi_m=(m(mI-A)^{-1})^2\Phi.$$

By Theorem (47)

$$\mathbb{E}\left[|\mathcal{K}_n\Phi(s)|^p\right]\le c_pE\left[\int_0^s(s-\sigma)^{-2\alpha}\left\|(S(s-\sigma)-e^{(s-\sigma)A_n})\Phi(\sigma)\right\|_{L_2^0}^2d\sigma\right]^{p/2}.$$

However,

$$\left\|(S(s-\sigma)-e^{(s-\sigma)A_n})\Phi(\sigma)\right\|_{L_2^0}^2 \le \left(\frac{M}{n-\omega}\right)^2 \|A\Phi(s)\|_{L_2^0}^p,$$

and therefore

$$\mathbb{E}\left[|\mathcal{K}_n\Phi(s)|^p\right] \le c_p\left(\frac{M}{n-\omega}\right)^2 \mathbb{E}\left[\int_0^s (s-\sigma)^{-2\alpha}\|A\Phi(\sigma)\|_{L_2^0}^p\, d\sigma\right]^{p/2}.$$

By Young's inequality, it follows that

$$\mathbb{E}\left[\int_0^T |\mathcal{K}_n\Phi(s)|^p\right] \le c_p\left(\frac{M}{n-\omega}\right)^2\left(\int_0^T \sigma^{-2\alpha}\right)^{2/p}\mathbb{E}\left[\int_0^T \|A\Phi(\sigma)\|_{L_2^0}^p\, d\sigma\right].$$

and we get the required convergence. Finally, the existence of a continuous modification of W_A^Φ now follows easily from (2.16)

■

Proposition 46 *Under the hypotheses of Theorem (44), assume that $\xi \in L^p(\Omega,\mathcal{F},\mathbb{P})$ with $p \ge 2$, and let Y and Y_n be the solutions of problems (2.2) and (2.14) respectively. Then we have*

$$\lim_{n\to\infty}\sup_{0\le t\le T}\mathbb{E}\left[|Y(t)-Y_n(t)|^p\right]=0.$$

Moreover, if $p>2$

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|Y(t)-Y_n(t)|^p\right]=0.$$

Proof. The result follows from a straightforward application of the contraction principle depending on the parameter n, Theorem (47) and Proposition (45). ■

Theorem 47 *For every $p>0$ there exists $c_p>0$ such that for every $t\ge 0$,*

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s \Phi(\tau)dB(\tau)\right|^p\right] \le c_p\mathbb{E}\left[\left|\int_0^s \Phi(\tau)dB(\tau)\right|^p\right]^{p/2}.$$

proof (we refer to read [15])

Chapter 3

Partial (Noisy) Observation Optimal Control

In the theory of stochastic processes, filtering is the problem of estimating the state of a system as a set of observations. Filtering found applications in many fields from signal processing to finance. In 1960, R.E. Kalman published his famous paper [29] on recursive minimum variance estimation for linear Gaussian dynamical systems. Basically, the filter gives a procedure for estimating the state of a system which satisfies a noisy linear differential equation, based on a series of noisy observations. The problem of optimal non-linear filtering (even for the non-stationary case) was solved by Ruslan L. Stratonovich (1959, 1960)(we refer to read [54]and [55]), see also Harold J. Kushner's work [31] and Moshe Zakai's, who introduced a simplified dynamics for the unnormalized conditional law of the filter(see [57]) known as Zakai equation. In this chapter, using results from Noisy Observation (nonlinear filtering), we transform this noisy observations stochastic differential equation (SDE) control problem into a full observation stochastic partial differential equations (SPDEs for short) control problem, we refer the reader to [17], [46] and [59]. Then we obtain a sufficient and a necessary maximum principle for the optimal control of SPDEs.

3.1 Problem formulation:

We consider the signal process$X(t)$ with $X(t) = X^{(u)}(t)$ and the observation process $Z(t)$ are given respectively by the following system of stochastic differential equations (SDE) of the form:

- (Signal process)

$$\begin{cases} dX(t) = & \alpha(X(t), Z(t), u(t))dt + \sigma(X(t), Z(t), u(t))dB(t); t \in [0, T], \\ X(0) = & X_0 \end{cases} \tag{3.1}$$

- (Observation process)

$$\begin{cases} dZ(t) = & \beta(X(t))dt + d\widetilde{B}(t) \\ Z(0) = & 0. \end{cases} \tag{3.2}$$

$$\begin{aligned} \alpha &: \mathbb{R} \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}, \\ \sigma &: \mathbb{R} \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}, \\ \beta &: \mathbb{R} \to \mathbb{R}, \end{aligned}$$

are given deterministic functions. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space equipped with a natural filtration $\mathscr{F}_t = \sigma(B(s), \widetilde{B}(s), 0 \leq s \leq t\}$ where B and $\widetilde{B}$ be two independent standard Brownian motions.. The process $u(t)$ is our control process, assumed to have values in a given convex set $\mathbb{U} \subset \mathbb{R}$. Such that $u(t)$ be adapted to the filtration

$$\mathbb{G} := \{\mathscr{G}_t\}_{0 \leq t \leq T},$$

where $\mathscr{G}_t$ is the sigma-algebra generated by the observations process $Z(s), s \leq t$. A control $u(.)$ is called admissible if it takes values in U and is $\{\mathscr{G}_t\}_{t \geq 0}$ adapted. We call u(t) admissible if, in addition, (3.1) and (3.2) has a unique strong solution $(X(t), Z(t))$ such that

$$E[\int_0^T \left| f(X(t), u(t)) \right| dt + \left| g(X(T) \right|] < \infty.$$

Where $f : \mathbb{R} \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given functions. The set of all admissible controls is denoted by $\mathscr{A}_{\mathbb{G}}$ For $u \in \mathscr{A}_{\mathbb{G}}$ we define the performance functional

$$J(u) = E[\int_0^T f(X(t), u(t))dt + g(X(T))].$$

Problem 48 *We consider the noisy observation sochastic control poblem to find* $u^* \in \mathscr{A}_{\mathbb{G}}$ *such* *that*

$$\sup_{u \in \mathscr{A}_{\mathbb{G}}} J(u) = J(u^*) \tag{3.3}$$

We now proceed to show that this noisy observation SDE control problem can be transformed into a complete observation SPDE control problem (sse [17], [46]). To do this, first, we change the probability measure as followes:

Define the probability measure $\widetilde{P}$ by:

$$d\widetilde{P}(\omega) = M_t(\omega) dP(\omega).$$

where

$$M_t(\omega) = \exp(-\int_0^t \beta(X(s)) d\widetilde{B}(s) - \frac{1}{2}\int_0^t \beta^2(X(s)) ds).$$

By the Girsanov theorem (Theorem 28) that the observation process $Z(t)$ defined by (3.2) is a Brownian motion with respect to $\widetilde{P}$

Moreover, we have

$$dP(\omega) = K_t(\omega) d\widetilde{P}(\omega)$$

where

$$K_t = M_t^{-1}(\omega) = \exp(\int_0^t \beta(X(s)) d\widetilde{B}(s) + \frac{1}{2}\int_0^t \beta^2(X(s)) ds).$$

In (3.2), we have

$$d\widetilde{B}(t) = dZ(t) - \beta(X(t), Z(t), u(t)) dt$$

then

$$K_t = M_t^{-1}(\omega) = \exp(\int_0^t \beta(X(s)) dZ(s) - \frac{1}{2}\int_0^t \beta^2(X(s)) ds).$$

For $\varphi \in C_0^2(\mathbb{R})$ and fixed $g \in \mathbb{R}$, $c \in \mathbb{U}$ define the integro-differential operator $A = A_{g,c}$ by

$$A_{g,c}\varphi(x) = \alpha(x, g, c)\frac{\partial \varphi}{\partial x}(x) + \frac{1}{2}\sigma^2(x, g, c)\frac{\partial^2 \varphi}{\partial x^2}, \tag{3.4}$$

and let A^* be the adjoint of A, in the sense that

$$(A\varphi, \psi)_{L^2(\mathbb{R})} = (\varphi, A^*\psi)_{L^2(\mathbb{R})} \tag{3.5}$$

for all $\varphi, \psi C_0^2(\mathbb{R})$.

Suppose that there exists a stochastic process $y(t,x)$ such that

$$E_{\widetilde{P}}[\varphi(X(t))K_t \mid \mathcal{G}_t] = \int_{\mathbb{R}} \varphi(x) y(t,x) dx \tag{3.6}$$

for all bounded measurable functions φ. Then $y(t,x)$ is called the unnormalized conditional density of $X(t)$ given the observation filtration $\mathcal{G}_t$. Note that by the Bayes rule (see Lemma 29) we have

$$E[\varphi(X(t)) \mid \mathcal{G}_t] = \frac{E_{\widetilde{P}}[\varphi(X(t))K_t \mid \mathcal{G}_t]}{E_{\widetilde{P}}[K_t \mid \mathcal{G}_t]} \tag{3.7}$$

Then $y(t,x)$ exists under certain conditions and satisfies the following integro-SPDE, called the Duncan–Mortensen–Zakai equation (see Definition 34):

$$\begin{cases} dy(t,x) = & A^*_{Z(t),u(t)} y(t,x)dt + \beta(x) y(t,x) dZ(t), \\ y(0,x) = & F(x). \end{cases} \tag{3.8}$$

If (3.6) holds, we get

$$\begin{aligned}
J(u) &= \mathbb{E}\left[\int_0^T f(X(t),u(t))dt + g(X(T))\right] \\
&= \mathbb{E}_{\widetilde{P}}\left[\int_0^T f(X(t),u(t))K_t dt + g(X(T))K_T\right] \\
&= \mathbb{E}_{\widetilde{P}}\left[\int_0^T \mathbb{E}_{\widetilde{P}}\left[f(X(t),u(t))K_t \mid \mathcal{G}_t\right] dt + \mathbb{E}_{\widetilde{P}}\left[g(X(T))K_t \mid \mathcal{G}_t\right]\right] \\
&= \mathbb{E}_{\widetilde{P}}\left[\int_0^T \mathbb{E}_{\widetilde{P}}\left[f(X(t),v)K_t \mid \mathcal{G}_t\right]_{v=u(t)} dt + \mathbb{E}_{\widetilde{P}}\left[g(X(T))K_t \mid \mathcal{G}_t\right]\right] \\
&= \mathbb{E}_{\widetilde{P}}\left[\int_0^T \int_{\mathbb{R}} [f(x,u(t))y(t,x)dxdt + \int_{\mathbb{R}} g(x)y(T,x)dx\right] \\
&= J_{\widetilde{P}}(u).
\end{aligned}$$

This transforms the noisy observation stochastic control problem (3.3) into a full observation SPDE control problem of the type we have discussed in the previous sections.

Theorem 49 *[17] we transform the noisy observation SDE control to full into SPDE control assuming that (3.6) and (3.7) hold. Then the solution $u^*(t)$ of the noisy observation SDE Control Problem (3.3) coincides with the solution u^* of the following stochastic partial differential equa-*

tions (SPDEs for short) control problem:

Problem 50 *Find $u^* \in \mathscr{A}_{\mathbb{G}}$ such that*

$$\sup_{u \in \mathscr{A}_{\mathbb{G}}} J_{\widetilde{P}}(u) = J_{\widetilde{P}}(u^*),$$

where

$$J_{\widetilde{P}}(u) = \mathbb{E}_{\widetilde{P}}\left[\int_0^T \int_{\mathbb{R}} f(x, u(t)) y(t, x) dx dt + \int_{\mathbb{R}} g(x) y(T, x) dx\right],$$

where $y(t, x)$ solves the stochastic partial differential equations (SPDEs) (3.8).

3.2 Stochastic maximum principle for SPDEs:

Stochastic control of the stochastic partial differential equations (SPDEs) arising from partial observation control has been studied by Mortensen [39], using a dynamic programming approach, and subsequently by Bensoussan, using a maximum principle method. See [9] and the references therein. In this section, We prove a sufficient and necessary maximum principle for the optimal control of SPDEs (see [46]).

Let $T > 0$ and let D be an open set in $\mathbb{R}^n$ with C^1 boundary ∂D. Suppose that the state $Y(t, x) \in \mathbb{R}$ of a system at time $t \in [0, T]$ and at the point $x \in \overline{D} = D \cup \partial D$ is given by stochastic partial differential equation of the form

$$dY(t, x) = \Big[AY(t, x) + b(t, x, Y(t, x), u(t, x))\Big] dt \qquad (3.9)$$
$$+ \sigma(t, x, Y(t, x), u(t, x)) dB(t), \; (t, x) \in (0, T) \times D$$

with boundary conditions

$$Y(0, x) = \xi(x); \; x \in \overline{D} \qquad (3.10)$$

$$Y(t, x) = \eta(t, x); \; (t, x) \in [0, T] \times \partial D \qquad (3.11)$$

The operator A is a linear integro-differential operator acting on x.

The equation (3.9) for Y is interpreted in the weak (variational) sense, i.e., $Y(t, .)$ satisfies the equation

$$
\begin{aligned}
(Y(t,.),\phi)_{L^2(D)} &= (a,\phi)_{L^2(D)} + \int_0^t (Y(s,.),A^*\phi)_{L^2(D)}ds \\
&\quad + \int_0^t (b(s,Y(s,.),\phi)_{L^2(D)}ds + \int_0^t (\sigma(s,Y(s,.),\phi)_{L^2(D)}dB_s
\end{aligned}
$$

for all smooth functions ϕ with compact support in D. Here

$$
(\psi,\phi)_{L^2(D)} = \int_D \psi(x)\phi(x)dx
$$

is the L^2 inner product on D and A^* is the adjoint of the operator A, in the sense that

$$
(A\psi,\phi)_{L^2(D)} = (\psi, A^*\phi)_{L^2(D)} \tag{3.12}
$$

for all smooth L^2 functions ψ,ϕ with compact support in D.

The process $B(t) = B(t,\omega); t \geq 0, \omega \in \Omega$ is a Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$, while $u(t,x) = u(t,x,\omega)$ is our control process. We assume that $u(t,x)$ has values in a given convex set $U \subset \mathbb{R}^k$ and that $u(t,x,.)$ is $\mathscr{F}_t$-mesurable for all $(t,x) \in [0,T] \times D$, i.e., that $u(t,x)$ is adapted for all $x \in D$. The functions $b : [0,T] \times D \times \mathbb{R} \times U \to \mathbb{R}$ and $\sigma : [0,T] \times D \times \mathbb{R} \times U \to \mathbb{R}$ are given C^1 functions. The boundary value functions $\xi : \overline{D} \to \mathbb{R}$ and $\eta : (0,T) \times \partial D \to \mathbb{R}$ are assumed to be deterministic and C^1.

We call the control process $u(t,x)$ admissible if the corresponding stochastic partial differential equation in (3.9), (3.10) and (3.11) has a unique, strong solution $Y(.) \in L^2(\lambda \times P)$, where λ is Lebesgue measure on $[0,T] \times \overline{D}$, and with values in a given set $S \subset \mathbb{R}$. The set of admissible controls is denoted by $\mathscr{A}$.

we define the performance $J(u)$ obtained by $u \in \mathscr{A}$ of the form

$$
J(u) = E\Big[\int_0^T \Big(\int_D f(t,x,Y(t,x),u(t,x))\mathrm{d}x\Big)\mathrm{d}t + \int_D g(x,Y(T,x))\mathrm{d}x\Big]. \tag{3.13}
$$

Where $f : [0,T] \times D \times \mathbb{R} \times U \to \mathbb{R}$ and $g : D \times \mathbb{R} \to \mathbb{R}$ are given lower bounded C^1 functions and E denotes the expectation with respect to P.

Problem 51 *Find $u^* \in \mathscr{A}$ such that*

$$\sup_{u \in \mathscr{A}} J(u) = J(u^*) \tag{3.14}$$

This is an optimal control problem for the stochastic partial differential equation (SPDE).

3.2.1 A sufficient Maximum Principle:

We now formulate a sufficient maximum principle for optimal control.

Define the Hamiltonian $H : [0,T] \times D \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

by

$$H(t,x,y,u,,p,q) = f(t,x,y,u) + b(t,x,y,u)p + \sigma(t,x,y,u)q. \tag{3.15}$$

We define the adjoint process $p(t,x), q(t,x)$ as the solution of BSPDE

$$\begin{cases} dp(t,x) = -\{A^* p(t,x) + (\frac{\partial H}{\partial y})(t,x,Y(t,x),u(t,x),p(t,x),q(t,x))\}dt + q(t,x)dB(t); (t,x) \in (0,T) \times D \\ p(T,x) = \frac{\partial g}{\partial y}(x,Y(T,x)); \quad x \in \overline{D} \\ p(t,x) = 0; \quad (t,x) \in [0,T] \times \partial D, \end{cases} \tag{3.16}$$

Theorem 52 (A Sufficient Maximum Principle) *[46], [17] Let $\hat{u} \in \mathcal{A}$ with corresponding solutions $\hat{Y}(t,x), \hat{p}(t,x), \hat{q}(t,x)$ of (3.9) and (3.16) respectively. Suppose that*

(1) *$y \to g(x,y)$ is concave for all x,*

(2) *$(y,u) \to H(y,u) := H(t,x,y,u,\widehat{p}(t,x),\widehat{q}(t,x)); y \in \mathbb{R}, u \in \mathbb{U}$, is concave for all t,x,*

(3) *$\sup_{u \in \mathbb{U}} H[t,x,\widehat{Y}(t,x),u,\widehat{p}(t,x),\widehat{q}(t,x)] = H[t,x,\widehat{Y}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)]$ for all t,x.*

Then $\widehat{u}(t,x)$ is an optimal control for the stochastic control Problem (51).

Proof. *Let $u \in \mathcal{A}$ be an arbitrary admissible control with corresponding solution of (3.9) and (3.16) be $Y(t,x)$ and $p(t,x)$, $q(t,x)$, respectively. For simplicity of notation, we write*

$$f = f(t,x,Y(t,x),u(t,x)), \widehat{f} = f(t,x,\widehat{Y}(t,x),\widehat{u}(t,x))$$

and similarly with $b, \widehat{b}, \sigma, \widehat{\sigma}$. and so on. Moreover put

$$\begin{aligned} H(t,x) &= H(t,x,Y(t,x),u(t,x),\widehat{p}(t,x),\widehat{q}(t,x)), \\ \widehat{H}(t,x) &= H(t,x,\widehat{Y}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)). \end{aligned}$$

In the following we write $\widetilde{f} = \widehat{f} - f, \widetilde{b} = \widehat{b} - b, \widetilde{\sigma} = \widehat{\sigma} - \sigma$.

Consider

$$J(u) - J(\widehat{u}) = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= E\left[\int_0^T \left(\int_D \{\widehat{f}(t,x) - f(t,x)\} dx\right) dt\right], \\ I_2 &= E\left[\int_D \{\widehat{g}(x) - g(x)\} dx\right]. \end{aligned}$$

By the definition of H we have

$$I_1 = E\left[\int_0^T \int_D \{\widetilde{H}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\} dx dt\right], \tag{3.17}$$

Since g is concave with respect to y we have

$$g(x, Y(T,x)) - \widehat{g}(x, Y(T,x)) \leq \frac{\partial g}{\partial y}(x, \widehat{Y}(T,x)).\widetilde{Y}(T,x). \tag{3.18}$$

where

$$\widetilde{Y}(T,x) := Y(T,x) - \widehat{Y}(T,x),$$

and

$$\widetilde{H}(t,x) := H(t,x) - \widehat{H}(t,x).$$

We get

$$\begin{aligned} I_2 &\leq \mathbb{E}[\int_D \frac{\partial g}{\partial y}(x, \widehat{Y}(T,x))\widetilde{Y}(T,x)dx] \\ &= \mathbb{E}[\int_D \widehat{p}(T,x)\widetilde{Y}(T,x)dx] \\ &= \mathbb{E}\left[\int_D \left(\int_0^T \widehat{p}(t,x)d\widetilde{Y}(t,x) + \int_0^T \widetilde{Y}(t,x)d\widehat{p}(t,x) + \int_0^T d[\widehat{p},\widetilde{Y}]_t\right)dx\right] \\ &= \mathbb{E}\left[\int_D \int_0^T \{\widehat{p}(t,x)[A\widetilde{Y}(t,x) + \widetilde{b}(t,x) - \widetilde{Y}(t,x)\{A^*\widehat{p}(t,x) + \frac{\partial \widehat{H}(t,x)}{\partial y}\} + \widetilde{\sigma}(t,x)\widehat{q}(t,x)\}dtdx\right]. \end{aligned} \tag{3.19}$$

Where

$$\frac{\partial \widehat{H}(t,x)}{\partial y} = \frac{\partial H}{\partial y}(t,x,y,\widehat{Y}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)).$$

By a small extension of (3.12) we get

$$\int_D \widetilde{Y}(t,x)A^*\widehat{p}(t,x)dx = \int_D \widehat{p}(t,x)A\widetilde{Y}(t,x)dx \tag{3.20}$$

Therefore, adding (3.17)-(3.19) and using (3.20) we get

$$\begin{aligned} J(u) - J(\widehat{u}) \quad &\leq \\ & E[\int_D (\int_0^T \{H(t,x) - \widehat{H}(t,x) - [\widehat{p}(t,x)A\widetilde{Y}(t,x) + \widetilde{Y}(t,x)\frac{\partial \widehat{H}(t,x)}{\partial y}]\}dt)dx]. \end{aligned} \tag{3.21}$$

Hence

$$J(u) - J(\widehat{u}) \leq E[\int_D (\int_0^T H(t,x) - \widehat{H}(t,x) - \nabla_{\widehat{Y}}\widehat{H}(\widetilde{Y})(t,x)\}dt)dx]. \tag{3.22}$$

where

$$\nabla_{\widehat{Y}}.\widehat{H}(\widetilde{Y}) = \nabla_y \widehat{H}(\widetilde{Y})$$

By the concavity assumption of H *in* (y,u) *we have*

$$H(t,x) - \widehat{H}(t,x) \leq \nabla_{\widehat{Y}} \widehat{H}(Y - \widehat{Y})(t,x) + \frac{\partial \widehat{H}}{\partial u}(t,x)(u(t,x) - \widehat{u}(t,x))$$

and the maximum condition implies that

$$\frac{\partial \widehat{H}}{\partial u}(t,x)(\widehat{u}(t,x) - u(t,x)) \leq 0.$$

Hence by (3.22) we get

$$J(u) - J(\widehat{u}) \leq 0.$$

Since $u \in \mathscr{A}$ *was arbitrary, this shows that* $\widehat{u}$ *is optimal.* ■

3.2.2 A Necessary Maximum Principle:

We proceed to prove a corresponding necessary maximum principle, we need the following assumptions about the set of admissible control processes:

- For all $t_0 \in [0,T]$ and all bounded $\mathscr{F}_{t_0}$-measurable random variables $\alpha(x,\omega)$, the control $\theta(t,x,\omega)$ defined by

  $$\theta(t,x,\omega) := \mathbf{1}_{[t_0,T]}\alpha(x,\omega)$$

  belong to $\mathscr{A}$.

- For all $u, \beta_0 \in \mathscr{A}$ with $\beta_0(t,x) \leq K \leq \infty$ for all t,x define

  $$\delta(t,x) = \frac{1}{2K} dist(u(t,x), \partial V) \wedge 1 > 0$$

  and put

  $$\beta(t,x) = \delta(t,x)\beta_0(t,x) \tag{3.23}$$

then the control

$$\widetilde{u}(t,x) = u(t,x) + a\beta(t,x) \in \mathscr{A}, t \in [0,T]$$

for all $a \in (-1,1)$.

- For all β as in (3.23) the derivative process

$$\eta(t,x) = \frac{d}{da} Y^{u+a\beta}(t,x)|_{a=0}. \tag{3.24}$$

exists, and belongs to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

$$\begin{cases} d\eta(t,x) = [A\eta(t,x) + \frac{\partial b}{\partial y}(t,x)\eta(t,x) + \frac{\partial b}{\partial u}(t,x)\beta(t,x)]dt \\ +[\frac{\partial \sigma}{\partial y}(t,x)\eta(t,x) + \frac{\partial \sigma}{\partial u}(t,x)\beta(t,x)]dB(t) \\ \quad (t,x) \in [0,T] \times D, \\ \eta(0,x) = \frac{d}{da} Y^{u+a\beta}(0,x)|_{a=0} = 0, \\ \eta(t,x) = 0; (t,x) \in [0,T] \times \partial D. \end{cases} \tag{3.25}$$

Theorem 53 (Necessary maximum principle) *[46], [17] Let $\hat{u} \in \mathscr{A}$ Then the following are equivalent:*

1.

$$\frac{d}{da} J(\widehat{u} + a\beta)|_{a=0} = 0,$$

for all bounded $\beta \in \mathscr{A}$ of the form (3.23).

2.

$$\frac{\partial H}{\partial u}(t,x)_{u=\widehat{u}} = 0, \textit{for all } (t,x) \in [0,T] \times D$$

Proof. *We can Write*

$$\frac{d}{da} J(u + a\beta)|_{a=0} = I_1 + I_2$$

where

$$I_1 = \frac{d}{da} E[\int_D \int_0^T f(t,x,Y^{u+a\beta}(t,x), u(t,x) + a\beta(t,x)) dt dx] \parallel_{a=0}$$

and

$$I_2 = \frac{d}{da}E[\int_D g(x, Y^{u+a\beta}(T,x))dx]\,|_{a=0}\,.$$

By our assumptions on f and g and by (3.24) we have

$$I_1 = E[\int_D \int_0^T \{\frac{\partial f}{\partial y}(t,x)\eta(t,x) + \frac{\partial f}{\partial u}(t,x)\beta(t,x)\}dtdx], \tag{3.26}$$

$$I_2 = E[\int_D \frac{\partial g}{\partial y}(x, Y(T,x))\eta(T,x)dx] = E[\int_D p(T,x)\eta(T,x)dx]. \tag{3.27}$$

By Itô formula

$$\begin{aligned}
I_2 &= E[\int_D p(T,x)\eta(T,x)dx] = E[\int_D \int_0^T p(t,x)d\eta(t,x)dx \\
&\quad + \int_D \int_0^T \eta(t,x)dp(t,x)dx + \int_D \int_0^T d[p,\eta](t,x)dx] \\
&= E[\int_D \int_0^T p(t,x)\{A\eta(t,x) \\
&\quad + \frac{\partial b}{\partial y}(t,x)\eta(t,x) + \frac{\partial b}{\partial u}(t,x)\beta(t,x)\}dtdx \\
&\quad + \int_D \int_0^T p(t,x)\{\frac{\partial \sigma}{\partial y}(t,x)\eta(t,x) + \frac{\partial \sigma}{\partial u}(t,x)\beta(t,x)\}dB_t \\
&\quad - \int_D \int_0^T \eta(t,x)[A^* p(t,x) + \frac{\partial H}{\partial y}(t,x)]dtdx \\
&\quad + \int_D \int_0^T q(t,x)\{\frac{\partial \sigma}{\partial y}(t,x)\eta(t,x) + \frac{\partial \sigma}{\partial u}(t,x)\beta(t,x)\}dtdx \\
&\quad + \int_D \int_0^T \eta(t,x)q(t,x)dB_t dx \\
&= E[\int_D (\int_0^T \{p(t,x)A\eta(t,x))\}dt \\
&\quad + \int_0^T \eta(t,x)\{p(t,x)\frac{\partial b}{\partial y}(t,x) + q(t,x)\frac{\partial \sigma}{\partial y}(t,x) - A^* p(t,x) - \frac{\partial H}{\partial y}(t,x)\}dt \\
&\quad + \int_0^T \beta(t,x)\{p(t,x)\frac{\partial b}{\partial u}(t,x) + q(t,x)\frac{\partial \sigma}{\partial u}(t,x)\}dt)dx] \\
&= E[-\int_D \int_0^T \eta(t,x)\frac{\partial f}{\partial y}\}dt + \int_0^T \{\frac{\partial H}{\partial u}(t,x) - \frac{\partial f}{\partial u}(t,x)\}\beta(t,x)\}dtdx] \\
&= -I_1 + E[\int_D \int_0^T \frac{\partial H}{\partial u}(t,x)\beta(t,x)dtdx].
\end{aligned} \tag{3.28}$$

Adding (3.26) and (3.28) we get

$$\frac{d}{da}J(u+a\beta)|_{a=0} = I_1 + I_2 = E[\int_D \int_0^T \frac{\partial H}{\partial u}(t,x)\beta(t,x)dtdx].$$

We conclude that

$$\frac{d}{da}J(u+a\beta)|_{a=0}$$

if and only if

$$E[\int_D \int_0^T \frac{\partial H}{\partial u}(t,x)\beta(t,x)dtdx] = 0,$$

for all bounded $\beta \in \mathscr{A}$. of the form (3.23)

Now apply this to $\beta(t,x) = \theta(t,x)$, we get that this is again equivalent to

$$\frac{\partial H}{\partial u}(t,x) = 0 \textit{ for all } (t,x) \in [0,T] \times D.$$

■

3.3 Controls which are independent of x

In many situations, for example in connection with partial (noisy) observation control, it is of interest to study the case when the controls $u(t) = u(t,x)$ are not allowed to depend on the space variable x. Thus we let the set $\mathscr{A}_1$ of admissible controls defined by:

$$\mathscr{A}_1 = \{u \in \mathscr{A}, u(t,x) = u(t)\}$$

where control u des not depend on x. With the performance functional $J(u)$ as in Problem (3.14), the problem is now the following:

Problem 54 *For each find $u_1^* \in \mathscr{A}_1$ such that*

$$\sup_{u \in \mathscr{A}_1} J(u) = J(u_1^*). \tag{3.29}$$

Theorem 55 (Sufficient SPDE maximum principle for controls which are independent of x) *[17][46]* *Let $\hat{u} \in \mathscr{A}_1$ with corresponding solutions $\hat{Y}(t,x)$ of (3.9) and $\hat{p}(t,x)$, $\hat{q}(t,x)$ of (3.16) respectively. Suppose that*

(1) $y \to g(x,y)$ *is concave for all* x,

(2) $(y,u) \to H(y,u) := H(t,x,y,u,\widehat{p}(t,x),\widehat{q}(t,x)); y \in \mathbb{R}, u \in \mathbb{U}$, *is concave for all* t,x,

(3) $\sup_{u \in \mathbb{U}} H[t,x,\widehat{Y}(t,x),u,\widehat{p}(t,x),\widehat{q}(t,x)] = H[t,x,\widehat{Y}(t,x),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x)]$ *for all* t,x.

Then $\widehat{u}(t)$ *is an optimal control for the stochastic control problem in (Problem 3.29).*

Proof. Let $u \in \mathscr{A}_1$ be an arbitrary admissible control with corresponding solution of (3.9) and (3.16) be $Y(t,x)$ and $p(t,x)$, $q(t,x)$, respectively. For simplicity of notation, we write

$$f = f(t,x,Y(t,x),u(t)), \widehat{f} = f(t,x,\widehat{Y}(t,x),\widehat{u}(t))$$

and similarly with $b, \widehat{b}, \sigma, \widehat{\sigma}$. and so on. Moreover put

$$\begin{aligned} H(t,x) &= H(t,x,Y(t,x),u(t),\widehat{p}(t,x),\widehat{q}(t,x)), \\ \widehat{H}(t,x) &= H(t,x,\widehat{Y}(t,x),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x)). \end{aligned}$$

In the following we write $\widetilde{f} = \widehat{f} - f, \widetilde{b} = \widehat{b} - b, \widetilde{\sigma} = \widehat{\sigma} - \sigma$.

Consider

$$J(u) - J(\widehat{u}) = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= E\left[\int_0^T \left(\int_D \{\widehat{f}(t,x) - f(t,x)\} dx\right) dt\right], \\ I_2 &= E\left[\int_D \{\widehat{g}(x) - g(x)\} dx\right]. \end{aligned}$$

By the definition of H we have

$$I_1 = E\left[\int_0^T \int_D \{\widetilde{H}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\} dx dt\right], \tag{3.30}$$

Since g is concave with respect to y we have

$$g(x,Y(T,x)) - \widehat{g}(x,Y(T,x)) \leq \frac{\partial g}{\partial y}(x,\widehat{Y}(T,x)).\widetilde{Y}(T,x). \tag{3.31}$$

where

$$\widetilde{Y}(T,x) := Y(T,x) - \widehat{Y}(T,x),$$

and

$$\widetilde{H}(t,x) := H(t,x) - \widehat{H}(t,x).$$

We get

$$\begin{aligned} I_2 &\leq \mathbb{E}[\int_D \frac{\partial g}{\partial y}(x,\widehat{Y}(T,x))\widetilde{Y}(T,x)dx] \\ &= \mathbb{E}[\int_D \widehat{p}(T,x)\widetilde{Y}(T,x)dx] \\ &= \mathbb{E}\left[\int_D \left(\int_0^T \widehat{p}(t,x)d\widetilde{Y}(t,x) + \int_0^T \widetilde{Y}(t,x)d\widehat{p}(t,x) + \int_0^T d[\widehat{p},\widetilde{Y}]_t\right)dx\right] \\ &= \mathbb{E}\left[\int_D \int_0^T \{\widehat{p}(t,x)[A\widetilde{Y}(t,x) + \widetilde{b}(t,x) - \widetilde{Y}(t,x)\{A^*\widehat{p}(t,x) + \frac{\partial \widehat{H}(t,x)}{\partial y}\} + \widetilde{\sigma}(t,x)\widehat{q}(t,x)\}dtdx\right]. \end{aligned} \tag{3.32}$$

Where

$$\frac{\partial \widehat{H}(t,x)}{\partial y} = \frac{\partial H}{\partial y}(t,x,y,\widehat{Y}(t,x),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x)). \tag{3.33}$$

By a slight extension of (3.33) we get

$$\int_D \widetilde{Y}(t,x)A^*\widehat{p}(t,x)dx = \int_D \widehat{p}(t,x)A\widetilde{Y}(t,x)dx \tag{3.34}$$

Therefore, adding (3.30)-(3.32) and using (3.34) we get

$$\begin{aligned} J(u) - J(\widehat{u}) \quad &\leq \\ & E[\int_D (\int_0^T \{H(t,x) - \widehat{H}(t,x) - [\widehat{p}(t,x)A\widetilde{Y}(t,x) + \widetilde{Y}(t,x)\frac{\partial \widehat{H}(t,x)}{\partial y}]\}dt)dx]. \end{aligned} \tag{3.35}$$

Hence

$$J(u) - J(\widehat{u}) \leq E[\int_D (\int_0^T H(t,x) - \widehat{H}(t,x) - \nabla_{\widehat{Y}}\widehat{H}(\widetilde{Y})(t,x)\}dt)dx]. \tag{3.36}$$

where

$$\nabla_{\widehat{Y}}.\widehat{H}(\widetilde{Y}) = \nabla_y \widehat{H}(\widetilde{Y})$$

By the concavity assumption of H in (y, u) we have

$$H(t,x) - \widehat{H}(t,x) \leq \nabla_{\widehat{Y}} \widehat{H}(Y - \widehat{Y})(t,x) + \frac{\partial \widehat{H}}{\partial u}(t,x)(u(t) - \widehat{u}(t))$$

and the maximum condition implies that

$$\frac{\partial \widehat{H}}{\partial u}(t,x)(\widehat{u}(t) - u(t)) \leq 0.$$

Hence by (3.36) we get

$$J(u) - J(\widehat{u}) \leq 0.$$

Since $\mathscr{A}_1$ was arbitrary, this shows that $\widehat{u}$ is optimal. ■

We proceed as in Theorem 53 to establish a corresponding necessary maximum principle for controls which do not depend on x. We assume the follwing:

- For all $t_0 \in [0, T]$ and all bounded $\mathscr{H}_{t_0}$-measurable random variables $\alpha(x, \omega)$, the control $\theta(t, \omega)$ defined by

  $$\theta(t, \omega) := \mathbf{1}_{[t_0, T]} \alpha(\omega)$$

  belong to $\mathscr{A}_1$.

- For all $u, \beta_0 \in \mathscr{A}_1$ with $\beta_0 \leq K \leq \infty$ for all t define

  $$\delta(t) = \frac{1}{2K} dist(u(t), \partial V) \wedge 1 > 0$$

  and put

  $$\beta(t) = \delta(t)\beta_0(t) \tag{3.37}$$

  then the control

  $$\widetilde{u}(t) = u(t) + a\beta(t) \in \mathscr{A}, t \in [0, T]$$

  for all $a \in (-1, 1)$.

- For all β as in (3.37) the derivative process

$$\eta(t,x) = \frac{d}{da} Y^{u+a\beta}(t,x)|_{a=0}. \tag{3.38}$$

exists, and belongs to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

$$\begin{cases} d\eta(t,x) = [A\eta(t,x) + \frac{\partial b}{\partial y}(t,x)\eta(t,x) + \frac{\partial b}{\partial u}(t,x)\beta(t,x)]dt \\ +[\frac{\partial \sigma}{\partial y}(t,x)\eta(t,x) + \frac{\partial \sigma}{\partial u}(t,x)\beta(t,x)]dB(t) \\ \quad (t,x) \in [0,T] \times D, \\ \eta(0,x) = \frac{d}{da} Y^{u+a\beta}(0,x)|_{a=0} = 0, \\ \eta(t,x) = 0; (t,x) \in [0,T] \times \partial D. \end{cases} \tag{3.39}$$

Theorem 56 (Necessary SPDE maximum principle for controls which are independent of x) *[17, 46] Let $\hat{u} \in \mathscr{A}$ Then the following are equivalent:*

1.

$$\frac{d}{da} J(\widehat{u} + a\beta)|_{a=0} = 0,$$

for all bounded $\beta \in \mathscr{A}$ of the form (3.37).

2.

$$\frac{\partial H}{\partial u}(t,x)_{u=\widehat{u}} = 0, \textit{for all } (t,x) \in [0,T] \times D$$

Proof. The proof is analogous to the proof of Theorem 53. ■

Chapter 4

Stochastic Partial Differential Equations with space interactions and application to population modelling

The purpose of this chapter is to introduce a new type of generalised stochastic partial differential equations (SPDEs) with space interactions as a model for population growth this result is new [38]. The SPDEs have *space interactions*, where the dynamics of the system at time t and position in space x also depend on the space-mean of values at neighbouring points. Our goal in this chapter is to prove the existence and uniqueness of a strong, smooth solution of a class of space interaction SPDEs, including the application studied in Section 4. In this application, we have an example in which the density $Y(t,x)$ of infected individuals in a population in a random/noisy environment changes over time t and space point x according to the following space interaction reaction-diffusion. And we give an iterative procedure for finding the solution (Theorem 57). Then we use white noise theory to prove a positivity theorem for a class of SPDEs with space interactions (Theorem 58). White noise $W(t)$ is formally defined as a derivative of the Brownian motion:

$$W_t = \frac{d}{dt}B(t),$$

(we refer to read [34],[16]) and we prove that the solution is positive if the initial values are (Theorem 59). Subsequently, in Section 3 of this chapter, we study the general optimization problem for such a system. We derive both suffcient and necessary maximum principles for optimal control. See (Theorem 66 and Theorem 67). Finally, as an illustration of our results, in Section 4, we study an example about optimal vaccination strategy for epidemics modelled as an SPDE with space interactions.

4.1 Solutions of SPDEs with space interactions, and positivity

In this section, we prove the existence and uniqueness of a strong, smooth solution of stochastic partial differential equations (SPDEs) with space interactions. We are not aiming to prove this for the most general SPDE of this type, but we settle for a class of SPDEs which includes the application in Section 4. Thus, for simplicity, we consider only the case when $A_x = L$ given by

$$L = \frac{1}{2}\Delta := \frac{1}{2}\sum_{k=1}^{k=n}\frac{\partial^2}{\partial x_k^2}, \text{ and } D = \mathbb{R}^n,$$

but it is clear that our method can also be applied to more general situations.
Fix $t > 0$, and let $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots, \ldots\}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m; m = 1, 2, \ldots.$
For functions $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ (the family of functions in $\mathscr{C}(\mathbb{R}^n)$ with compact support), we define the Sobolev norm (see Definition 38)

$$\left|f\right|_k = \sum_{|\alpha|\le k}\Big(\int_{\mathbb{R}^n}|\partial^\alpha f(x)|^2 dx\Big)^{\frac{1}{2}}; \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n,$$

and we define the Sobolev space $\mathbb{H}^k$ (see Definition 39) to be the closure of $C_0^\infty(\mathbb{R}^n)$ in this norm. Note that $\mathbb{H}^k$ is a Hilbert space for all k. Also, note that if $f \in \mathbb{H}^{k+2}$ then $Lf \in \mathbb{H}^k$, because

$$\left|Lf\right|_k = \sum_{|\alpha|\le k}\Big(\int_{\mathbb{R}^n}\left|\partial^\alpha Lf(x)\right|^2 dx\Big)^{\frac{1}{2}} \le \frac{1}{2}\sum_{|\alpha|\le k+2}\Big(\int_{\mathbb{R}^n}\left|\partial^\alpha Lf(x)\right|^2 dx\Big)^{\frac{1}{2}} = \frac{1}{2}\left|f\right|_{k+2}. \tag{4.1}$$

Let $\mathscr{Y}_k^{(t)}$ denote the family of adapted random fields $Y(s,x) = Y(s,x,\omega)$, such that

$$\|Y\|_{t,k} < \infty,$$

where

$$||Y||_{t,k} = \mathbb{E}\left[\sup_{s\le t}\{|Y(s,.)|_k^2\}\right]^{\frac{1}{2}}, \tag{4.2}$$

and let $\mathscr{Y}^{(t)}$ be the intersection of all the spaces $\mathscr{Y}_k^{(t)}; k \in \mathbb{N}_0$, with the norm

$$||Y||_t^2 := \sum_{k=1}^{\infty} 2^{-k} ||Y||_{t,k}^2. \tag{4.3}$$

In the following we let

$$\varphi \mapsto \overline{\varphi}(x)$$

be any averaging operator such that there exists a constant C_1 such that

$$|\overline{\varphi}|_k \le C_1 |\varphi|_k \text{ for all } \varphi, k. \tag{4.4}$$

This holds, for example, if $\overline{\varphi}(x) = \int \varphi(x+y)\rho(dy)$ for some measure ρ of total mass 1.

We can now prove the following:

Theorem 57 *Let $\xi \in \mathscr{Y}^{(T)}$ be deterministic and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic.*

(i) Then there exists a unique solution $Y(t,x) \in \mathscr{Y}^{(T)}$ of the following SPDE with space interactions:

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)ds + \int_0^t \overline{Y}(s,x)ds + \int_0^t h(s)Y(s,x)dB(s); \quad t \in [0,T].$$

(ii) Moreover, the solution $Y(t,x)$ can be found by iteration, as follows:
Choose $Y_0 \in \mathscr{Y}^{(T)}$ arbitrary deterministic and define inductively Y_m to be the solution of

$$Y_m(t,x) = \xi(x) + \int_0^t LY_m(s,x)ds + \int_0^t \overline{Y}_{m-1}(s,x)ds + \int_0^t h(s)Y_m(s,x)dB(s); \quad t \in [0,T]; m = 1,2,.... \tag{4.5}$$

Then

$$Y_m \to Y \text{ in } \mathscr{Y}^{(T)} \text{ when } m \to \infty.$$

Proof. Part(i): In the first part of the proof, we are concerned on proving the existence and the uniqueness of the solution of $Y(t,x) \in \mathscr{Y}^{(T)}$. Define the operator $F : \mathscr{Y}^{(T)} \to \mathscr{Y}^{(T)}$ by $F(Z) = Y^Z$ where Y^Z is the solution of the equation

$$Y^Z(t,x) = \xi(x) + \int_0^t LY^Z(s,x)ds + \int_0^t \overline{Z}(s,x)ds + \int_0^t Y^Z(s,.)h(s)dB(s).$$

For $i = 1,2$ choose $Z_i \in \mathscr{Y}^{(T)}$ and define $Y_i = Y^{Z_i} =: F(Z_i)$ to be the solution of the SPDE

$$Y_i(t,x) = \xi(x) + \int_0^t LY_i(s,x)\,ds + \int_0^t \overline{Z_i}(s,x)\,ds + \int_0^t Y_i(s,\cdot)\,h(s)\,dB(s).$$

Note that here Z_i (and hence $\overline{Z_i}$, is given for each i. Therefore the existence and uniqueness of the solution Y_i follows by the general existence and uniqueness theorems for solutions of SPDEs. e.g. as given in Theorem 3.3 in [22]. Define

$$\widetilde{Y} = Y_1 - Y_2,$$
$$\widetilde{Z}(t,x) = Z_1(t,x) - Z_2(t,x)$$
$$\widetilde{\overline{Z}} = \overline{Z}_1 - \overline{Z}_2.$$

Then

$$\widetilde{Y}(t,x) = \int_0^t L\widetilde{Y}(s,x)\,ds + \int_0^t \widetilde{\overline{Z}}(s,x)\,ds + \int_0^t \widetilde{Y}(s,x)\,h(s)\,dB(s).$$

Hence

$$\left|\widetilde{Y}(s,.)\right|_k \le \int_0^s \left|L\widetilde{Y}(r,.)\right|_k dr + \int_0^s \left|\widetilde{\overline{Z}}(r,.)\right|_k dr + \left|\int_0^s \widetilde{Y}(r,\cdot)h(r)dB(r)\right|_k \tag{4.6}$$

By (4.1) we have

$$\left|L\widetilde{Y}(r,.)\right|_k \le \left|\widetilde{Y}(r,.)\right|_{k+2}, \tag{4.7}$$

and from (4.4) we get

$$\left|\overline{Z}\right|_k \leq C_1 |Z|_k \text{ for all } k. \tag{4.8}$$

Then by (4.6), (4.7) and (4.8), we get

$$\begin{aligned} & \mathbb{E}\left[\sup_{s\leq t}\left|\widetilde{Y}(s,.)\right|_k^2\right] \qquad (4.9)\\ \leq\ & 3\mathbb{E}\left[\sup_{s\leq t}\left(\int_0^s \left|\widetilde{Y}(r,.)\right|_{k+2} dr\right)^2\right] + 3C_1\mathbb{E}\left[\sup_{s\leq t}\left(\int_0^s \left|\widetilde{Z}(r,.)\right| dr\right)^2\right] \\ & +3\mathbb{E}\left[\sup_{s\leq t}\left|\int_0^s \widetilde{Y}(r,\cdot)\, h(r)\, dB(r)\right|_k^2\right]. \end{aligned}$$

By the Burkholder-Davis-Gundy inequality (see Theorem 25) for Hilbert spaces (see e.g. [41]), there exists a constant C_2 such that

$$\begin{aligned} \mathbb{E}\left[\sup_{s\leq t}\left|\int_0^s \widetilde{Y}(r,\cdot)\, h(r)\, dB(r)\right|_k^2\right] &\leq C_2\mathbb{E}\left[\int_0^t \left|\widetilde{Y}(r,.)\right|_k^2 h^2(r)\, dr\right] \\ &\leq C_2 h_0^2 t\mathbb{E}\left[\sup_{s\leq t}\left|\widetilde{Y}(s,.)\right|_k^2\right];\text{ where } h_0^2 = \sup_{s\in[0,T]} |h(s)|^2. \end{aligned}$$

Combining the above we get, if $0 \leq t \leq 1$,

$$\mathbb{E}\left[\sup_{s\leq t}\left|\widetilde{Y}(s,.)\right|_k^2\right] \tag{4.10}$$

$$\leq 3t^2\mathbb{E}\left[\sup_{r\leq t}\left|\widetilde{Y}(r,.)\right|_{k+2}^2\right] + 3C_1 t^2\mathbb{E}\left[\sup_{r\leq t}\left|\widetilde{Z}(r,.)\right|_k^2\right] \tag{4.11}$$

$$+3C_2 h_0^2 t\mathbb{E}\left[\sup_{r\leq t}\left|\widetilde{Y}(r,.)\right|_k^2\right]. \tag{4.12}$$

In other words,

$$||\widetilde{Y}||_{t,k}^2 \leq 3t^2||\widetilde{Y}||_{t,k+2}^2 + 3C_1 t^2||\widetilde{Z}||_{t,k}^2 + 3C_2 h_0^2 t||\widetilde{Y}||_{t,k}^2. \tag{4.13}$$

Note that

$$\sum_{k=1}^{\infty} 2^{-k}||\widetilde{Y}||_{t,k+2}^2 = \sum_{j=3}^{\infty} 2^{-(j-2)}||\widetilde{Y}||_{t,j} \leq 4\sum_{j=3}^{\infty} 2^{-j}||\widetilde{Y}||_{t,j} \leq 4\sum_{k=1}^{\infty} 2^{-k}||\widetilde{Y}||_{t,k} = 4||\widetilde{Y}||_t.$$

Therefore, by multiplying the terms in (4.13) by 2^{-k} and summing over k, we get

$$||\widetilde{Y}||_t^2 = \sum_{k=1}^{\infty} 2^{-k} ||\widetilde{Y}||_{t,k}^2 \leq 12 t^2 ||\widetilde{Y}||_t^2 + 3C_1 t^2 ||\widetilde{Z}||_t^2 + 3C_2 h_0^2 t ||\widetilde{Y}||_t^2,$$

or

$$(1 - 12t^2 - 3C_2 h_0^2 t) ||\widetilde{Y}||_t^2 \leq 3C_1 t^2 ||\widetilde{Z}||_t^2.$$

Hence, if $t_0 > 0$ is chosen so small that

$$\frac{3C_1 t_0^2}{1 - 12t_0^2 - 3C_2 h_0^2 t_0} < 1,$$

we obtain that the map

$$Z \to Y^Z = F(Z)$$

is a contraction on $\mathscr{Y}^{(t_0)}$. Therefore, by the Banach fixed point theorem there exists a fixed point $\widehat{Y}$ of this map. Then $\widehat{Y}$ solves the SPDE

$$\begin{cases} d\widehat{Y}(t,x) = L\widehat{Y}(t,x)\,dt + \overline{\widehat{Y}}(t,x)\,dt + \widehat{Y}(t,x)\,h(t)\,dB(t); & t \in [0, t_0], \\ \widehat{Y}(0,x) = \xi(x); & x \in \mathbb{R}^n. \end{cases}$$

Uniqueness follows by a similar argument.

Since the constants do not depend on t_0, we can repeat the argument starting from t_0 and hence by induction obtain a solution $Y(t,x) \in \mathscr{Y}^{(2t_0)}$. Repeating this argument we thus obtain a solution $Y \in \mathscr{Y}^{(T)}$. This proves part (i).

Part (ii): The second part of the theorem follows by the Banach fixed point theorem on the Banach space $\mathscr{Y}^{(T)}$. ∎

4.2 The non-homogeneous stochastic heat equation and positivity

In this section we will prove positivity of the solutions $Y(t,x)$ of SPDEs of the form

$$\begin{cases} dY(t,x) & = LY dt + K(t,x)\,dt + h(t)\,Y(t)\,dB(t), \\ Y(0,x) & = \xi(x); \quad x \in \mathbb{R}^n, \end{cases}$$

where the function $\xi \in \mathscr{Y}^{(T)}$ is deterministic and positive, $h : [0,T] \mapsto \mathbb{R}$ is bounded and deterministic and $K(t,x) = K(t,x,\omega) : [0,T] \times \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$ is a given positive random field.

To motivate our method, we first recall the following basic results about the classical heat equation:

Let $L = \frac{1}{2}\triangle$ and consider the equation

$$\begin{cases} dY(t,x) & = LY dt + K(t,x)\,dt, \\ Y(0,x) & = \xi(x); \quad x \in \mathbb{R}^n, \end{cases} \tag{4.14}$$

where $\xi \in \mathscr{Y}^{(T)}$ and $K \in L^2([0,T] \times \mathbb{R}^n)$ are given deterministic functions. Define the operator $P_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$P_t f(x) = \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} f(y) \exp\left(-\frac{|x-y|^2}{2t}\right) dy, \tag{4.15}$$

then

$$\frac{d}{dt} P_t f = L(P_t f),$$

and if we define

$$Y(t,x) = P_t \xi(x) + \int_0^t P_{t-s}(K(s,.))(x)\,ds,$$

we get

$$\begin{aligned} \frac{d}{dt} Y(t,x) &= L(P_t \xi)(x) + P_0(K(t,.))(x) + \int_0^t L(P_{t-s}(K(s,.)))(x)\,ds \\ &= LY(t,x) + K(t,x). \end{aligned}$$

Hence

$$Y(t,x) \text{ solves the heat equation (4.14).}$$

Next, consider the case

$$dY(t,x) = LY\,dt + K(t,x)\,dt + \theta(t)\,Y(t,x)\,dt.$$

Multiply the equation by

$$Z(t) = \exp\left(-\int_0^t \theta(s)\,ds\right).$$

Then the equation becomes

$$d(Z(t)\,Y(t,x)) = L(Z(t)\,Y(t,x))\,dt + Z(t)\,K(t,x)\,dt.$$

Hence, if we put

$$\widehat{Y} = Z(t)\,Y(t,x),$$

then $\widehat{Y}$ solves the equation

$$\begin{cases} d\widehat{Y}(t,x) & = L\widehat{Y}\,dt + Z(t)\,K(t,x)\,dt, \\ \widehat{Y}(0,x) & = \xi(x), \end{cases}$$

and we are back to the previous case.

Finally, consider the SPDE

$$dY(t,x) = LY\,dt + K(t,x)\,dt + h(t)\,Y(t)\,dB(t), \tag{4.16}$$

where h is a given bounded deterministic function and $K(t,x)$ is stochastic and adapted, and $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x)\,dt\,dx] < \infty$. We handle this case by using white noise calculus on the Hida space $(\mathscr{S})^*$ of stochastic distributions: We introduce *white noise* $W_t \in (\mathscr{S})^*$ (see Definition 40) defined by

$$W_t = \frac{d}{dt}B(t),$$

and then we see that equation (4.16) can be written

$$\frac{d}{dt}Y(t,x) = LY + K(t,x) + Y(t)\,h(t) \diamond W_t,$$

where $\diamond$ denotes Wick multiplication or (wick product) (see Definition 41). We refer to e.g. [16] for more information about white noise calculus. If we Wick-multiply this equation by

$$Z_t := \exp^{\diamond}\left(-\int_0^t h(s)\,dB(s)\right),$$

where in general $\exp^{\diamond}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}\phi^{\diamond n}; \phi \in (\mathscr{S})^*$ is the Wick exponential, we get

$$Z_t \diamond \frac{d}{dt}Y(t,x) = L(Y \diamond Z_t) + K \diamond Z_t + Y(t)\,h(t) \diamond W_t \diamond Z_t. \tag{4.17}$$

Now

$$\frac{d}{dt}(Z_t \diamond Y) = Z_t \diamond \frac{d}{dt}Y(t) - Y(t) \diamond Z_t \diamond h(t)\,W_t, \tag{4.18}$$

and hence (4.17) can be written as

$$\frac{d}{dt}\underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} = L\underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} + K(t,x) \diamond Z_t.$$

This has the same form as (4.14). Hence the solution $\widehat{Y}$ is

$$\widehat{Y}(t,x) = P_t\xi(x) + \int_0^t P_{t-s}(K(s,.))(x) \diamond Z_s ds.$$

Now we go back from $\widehat{Y}$ to Y and get the solution

$$\begin{aligned} Y(t,x) &= \widehat{Y}(t,x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right) \\ &= P_t\xi(x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right) \\ &+ \int_0^t P_{t-s}(K(s,.))(x) \diamond \exp^{\diamond}\left(\int_s^t h(r)\,dB(r)\right) ds. \end{aligned} \tag{4.19}$$

Note that

$$\exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right) = \exp\left(\int_0^t h(s)\,dB(s) - \frac{1}{2}\int_0^t h^2(s)\,ds\right) > 0.$$

Recall the Gjessing-Benth lemma (see [11], [23] or Theorem 2.10.6 in [25] or Proposition 13 in [10]), which states that

$$\phi \diamond \exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right) = \left(\tau_{-h}\phi\right)\exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right),$$

where, for $\phi : \Omega \mapsto \mathbb{R}$, we define $\tau_{-h}\phi(\omega) = \phi(\omega - h); \omega \in \Omega$ to be the shift operator on Ω.

Using this in (4.19) we conclude that if

$$\xi \geq 0 \text{ and } K \geq 0 \text{ then } Y \geq 0.$$

We summarize what we have proved as follows:

Theorem 58 *Assume that* $\xi \in \mathscr{Y}^{(T)}$ *is deterministic,* $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x)\,dt\,dx] < \infty$ *and let* $h : [0,T] \mapsto [0,T]$ *be bounded deterministic.*

1. *Then the unique solution* $Y(t,x) \in \mathscr{Y}^{(T)}$ *of the non-homogeneous SPDE*

$$\begin{aligned} dY(t,x) &= LY\,dt + K(t,x)\,dt + h(t)\,Y(t)\,dB(t), \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n \end{aligned}$$

is given by

$$\begin{aligned} Y(t,x) = (\tau_{-h}P_t\xi)(x)\exp^{\diamond}\left(\int_0^t h(s)\,dB(s)\right) \\ + \int_0^t (\tau_{-h}P_{t-s}(K(s,.))(x)\exp^{\diamond}(\int_s^t h(r)dB(r))ds, \end{aligned}$$

where $\exp^{\diamond}(\int_s^t h(r)dB(r)) = \exp(\int_s^t h(r)dB(r) - \frac{1}{2}\int_s^t h^2(r)dr); \quad 0 \leq s \leq t \leq T.$

2. *In particular, if* $\xi(x) \geq 0$ *and* $K(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*, then* $Y(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$.

Combining this with Theorem 57 we get

Theorem 59 *(Positivity) Assume that $\xi \in \mathscr{Y}^{(T)}$ is deterministic and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic. Let $Y(t,x) \in \mathscr{Y}^{(T)}$ be the unique solution of the following SPDE with space interactions:*

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)ds + \int_0^t \overline{Y}(s,x)ds + \int_0^t h(s)Y(s,x)dB(s); \quad t \in [0,T], \tag{4.20}$$

given by Theorem 58.

Then if $\xi(x) \geq 0$ for all $x \in \mathbb{R}^n$, we have $Y(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Proof.

By Theorem 57 we know that the solution of (4.20) can be obtained as the limit when $m \to \infty$ of the sequence $Y_m(t,x)$ defined recursively by the equation (4.5). Then by Theorem 58, part 2, we know that $Y_m(t,x) \geq 0$ for all t,x,m. We conclude that $Y(t,x) \geq 0$ for all (t,x). ■

Remark 60 *The results from this and the previous section can be extended to equations of the form*

$$dY(t,x) = \left[LY(t,x) + \gamma(t,x)Y(t,x)\right]dt + \overline{Y}(t,x)dt + h(t)dB(t); \quad t \in [0,T], \tag{4.21}$$

for a given adapted process $\gamma \in \mathscr{Y}^{(T)}$.To see this we apply the arguments above with the operator L replaced by the operator $\widehat{L}$ defined by $\widehat{L}\varphi = L\varphi + \gamma\varphi; \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$. We omit the details.

4.3 The optimization problem

In general, if $\mathscr{X}, \mathscr{Y}$ are two Banach spaces and $F : \mathscr{X} \to \mathscr{Y}$ if Fréchet differentiable at $x \in \mathscr{X}$, then we let $\nabla_x F$ denote the Fréchet derivative of F at x (see Definition 32). It is a linear operator from $\mathscr{X}$ to $\mathscr{Y}$ and the action of $\nabla_x F$ to $h \in \mathscr{X}$ is denoted by $\nabla_x F(h) = \langle \nabla_x F, h \rangle \in \mathscr{Y}$. Recall that if F is Fréchet differentiable at x with Fréchet derivative $\nabla_x F$, then F has a directional derivative

$$D_x F(h) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(x + \epsilon h) - F(x)) \tag{4.22}$$

in all directions $h \in \mathscr{X}$ and

$$D_x F(h) = \nabla_x F(h) = \langle \nabla_x F, h \rangle. \tag{4.23}$$

In particular, note that if F is a linear operator, then $\nabla_x F = F$ for all x.

4.3.1 The Hamiltonian and the adjoint BSPDE

We now give a general formulation of the problem we consider.

Let A_x be a linear second order partial differential operator given by

$$A_x \phi(x) = \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(x) \frac{\partial \phi}{\partial x_i}; \quad \phi \in \mathscr{C}_0^2(\mathbb{R}^n). \tag{4.24}$$

Let $T > 0$ and assume that the state $Y(t,x)$ at time $t \in [0,T]$ and at the point $x \in \overline{D} := D \cup \partial D$ satisfies the following non-local quasilinear stochastic heat equation:

$$\begin{cases} dY(t,x) &= A_x Y(t,x)dt + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))dt \\ &\quad +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))dB(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{4.25}$$

We make the following assumptions on *$(\alpha, \beta, b, \sigma, \xi, \eta)$*

(a) $(\alpha_{ij}(x))_{1 \le i,j \le n}$ is a given symmetric nonnegative definite $n \times n$ matrix with eigenvalues bounded away from 0 and with entries $\alpha_{ij}(x) \in \mathscr{C}^4(D) \cap \mathscr{C}(\overline{D})$ for all $i,j = 1,2,\ldots,n$.

(b) $\beta_i(x) \in \mathscr{C}^3(D) \cap \mathscr{C}(\overline{D})$ for all $i = 1,2,\ldots,n$.

(c) The functions b and σ are $\mathbb{F}$-adapted, $\mathscr{C}^2$ with respect to y and u and admit uniformly bounded derivatives.

(d) $\xi \in L^2(D)$, and $\eta \in L^2([0,T] \times D \times \Omega)$ in $\mathbb{F}$-adapted.

We call the equation (4.25) a *stochastic partial differential equation with space-interactions.*

In general, the formal adjoint A^* of an operator A is defined by the identity

$$(A\phi, \psi) = (\phi, A^*\psi), \quad \text{for all } \phi, \psi \in \mathscr{C}_0^2(D),$$

where $(\phi_1, \phi_2) := \langle \phi_1, \phi_2 \rangle_{L^2(D)} = \int_D \phi_1(x)\phi_2(x)dx$ is the inner product in $L^2(D)$ and $\mathscr{C}_0^2(D)$ is the set of twice differentiable functions with compact support in D. In our case we have

$$A_x^*\phi(x) = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(\alpha_{ij}(x)\phi(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\beta_i(x)\phi(x)); \quad \phi \in \mathscr{C}^2(D).$$

We interpret Y as a weak (variational) solution to (4.25), in the sense that

$$\begin{aligned} \langle Y(t), \phi \rangle_{L^2(D)} &= \langle \xi(x), \phi \rangle_{L^2(D)} + \int_0^t \langle Y(s), A_x^*\phi \rangle_{L^2(D)} ds \\ &+ \int_0^t \langle b(s, Y(s)), \phi \rangle_{L^2(D)} ds + \int_0^t \langle \sigma(s, Y(s)), \phi \rangle_{L^2(D)} dB(s); \phi \in \mathscr{C}_0^2(D). \end{aligned}$$

For simplicity, in the above equation, we have not written all the arguments of b, σ.
In the following, we will assume that there is a unique strong solution of (4.25). It is not known to us under what conditions this is the case for general D. In the case when $D = \mathbb{R}^n$ it follows by Proposition 12.1 in [35] that there exists a unique weak solution $Y(t;x)$ of (4.25) for all given initial values $\xi \in L^2(D)$. In Section 2 of this chapter, we have proved that there is a unique smooth, strong positive solution of equation (4.21) if $\xi > 0$ and $D = \mathbb{R}^n$. The process $u(t,x) = u(t,x,\omega)$ is our control process, assumed to have values in a given convex set $U \subset \mathbb{R}^k$.

Definition 61 *We call the control process $u(t,x)$ admissible if $u(t;x)$ is $\mathbb{F}$-predictable for all $(t,x) \in [0,T] \times D$ and $u(t,x) \in U$ for all t,x. The set of admissible controls is denoted by $\mathscr{U}$.*
The performance functional (cost) associated to the control u is assumed to have the form

$$J(u) = \mathbb{E}\left[\int_0^T \int_D f(t,x,Y(t,x),Y(t,.),u(t,x))dxdt + \int_D g(x,Y(T,x),Y(T,.))dx\right]; u \in \mathscr{U}. \quad (4.26)$$

We make the following assumptions on (f,g):

(e) The function $f(t,x,y,\varphi,u)$ is $\mathbb{F}$-adapted and the function $g(x,y,\varphi)$ is $\mathscr{F}_T$-measurable. They are assumed to be bounded, $\mathscr{C}^2$ with respect to y, φ, u, with uniformly bounded derivatives.

We consider the following problem of optimal control of a solution of an SPDE:

Problem 62 *Find $\widehat{u} \in \mathcal{U}$ such that*

$$J(\widehat{u}) = \inf_{u \in \mathcal{U}} J(u). \tag{4.27}$$

As mentioned in the Introduction (**General Introduction**), this type of problem has been studied by many authors, and it may in some sense be considered as a special case of the general problem discussed in [35], except that we are considering strong solutions on $[0, T] \times D$, where D a given open subset of $\mathbb{R}^n$, with given boundary values on ∂D. Moreover, our approach is specifically focused on the stochastic reaction-diffusion equation with space interaction presented in Section 1 of this chapter, and therefore gives more explicit results.
To study this problem we define the associated *Hamiltonian* $H : [0, T] \times D \times \mathbb{R} \times L(\mathbb{R}^n) \times U \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} H(t, x, y, \varphi, u, p, q) := H(t, x, y, \varphi, u, p, q, \omega) = f(t, x, y, \varphi, u) + b(t, x, y, \varphi, u)p \\ + \sigma(t, x, y, \varphi, u)q. \end{aligned} \tag{4.28}$$

In general, if $h : L^2(D) \mapsto L^2(D)$ is Fréchet differentiable map, then its Fréchet derivative (gradient) at $\varphi \in L^2(D)$ denoted by $\nabla_\varphi h = \nabla h$ is a bounded linear map on the Hilbert space $L^2(D)$, and by the Riesz representation theorem (see Theorem 33) we can represent it by a function $\nabla h(x, y) \in L^2(D \times D)$. We denote the action of ∇h on a function $\psi \in L^2(D)$ by $\langle \nabla h, \psi \rangle$.

Hence

$$\langle \nabla h, \psi \rangle (x) := \int_D \nabla h(x, y)\psi(y)dy; \quad \text{for all } \psi \in L^2(D). \tag{4.29}$$

Remark 63 • *Note in particular that if $h : L^2(D) \mapsto L^2(D)$ is linear, then*

$$\nabla h(x, y) = h(x, y)$$

.

- *Also note that from (4.29) it follows by the Fubini theorem that*

$$\int_D \langle \nabla h, \psi \rangle (x) dx = \int_D \int_D \nabla h(x,y)\psi(y) dy dx = \int_D \int_D \nabla h(y,x)\psi(x) dx dy$$
$$= \int_D \left(\int_D \nabla h(y,x) dy \right) \psi(x) dx = \int_D \overline{\nabla} h(x)\psi(x) dx,$$

where

$$\overline{\nabla} h(x) := \int_D \nabla h(y,x) dy. \tag{4.30}$$

Example 64 **a)** *Assume that $h : L^2(D) \mapsto L^2(D)$ is given by*

$$h(\varphi) = \langle h, \varphi \rangle (x) = G(x, \varphi(\cdot)) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y) dy. \tag{4.31}$$

Then

$$\langle \nabla_\varphi h, \psi \rangle (x) = \langle h, \psi \rangle (x) = \frac{1}{V(K_r)} \int_{K_r} \psi(x+y) dy.$$

Therefore $\nabla h(x,y)$ is given by the identity

$$\int_D \nabla h(x,y)\psi(y) dy = \frac{1}{V(K_r)} \int_{K_r} \psi(x+y) dy; \quad \psi \in L^2(D).$$

Substituting $z = x + y$ this can be written

$$\int_D \nabla_\varphi^* h(x,y)\psi(y) dy = \frac{1}{V(K_r)} \int_{x+K_r} \psi(z) dz = \int_D \frac{\mathbf{1}_{x+K_r}(y)}{V(K_r)} \psi(y) dy.$$

Since this is required to hold for all ψ, we conclude the following:

b) *Suppose that h is given by (4.31). Then*

$$\nabla h(x,y) = \frac{\mathbf{1}_{x+K_r}(y)}{V(K_r)},$$

and

$$\overline{\nabla}_\varphi h(x) = \int_D \nabla_\varphi h(y,x) dy = \frac{1}{V(K_r)} \int_D \mathbf{1}_{y+K_r}(x) dy = \frac{1}{V(K_r)} \int_D \mathbf{1}_{x-K_r}(y) dy$$
$$= \frac{V((x-K_r)\cap D)}{V(K_r)} = \frac{V((x+K_r)\cap D)}{V(K_r)},$$

since $K_r = -K_r$.

We associate with the Hamiltonian the following backward stochastic partial differential equations (BSPDE for short) given by

$$dp(t,x) = -\Big[A_x^* p(t,x) + \tfrac{\partial H}{\partial y}(t,x) + \overline{\nabla} H(t,x)\Big]dt + q(t,x)dB(t), \tag{4.32}$$

with boundary/terminal values

$$\begin{cases} p(T,x) &= \frac{\partial g}{\partial y}(x) + \overline{\nabla} g(x); \quad x \in D, \\ p(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases} \tag{4.33}$$

where we have used the simplified notation

$$H(t,x) = H(t,x,y,\varphi,u,p,q)|_{y=Y(t,x),\varphi=Y(t,\cdot),u=u(t,x),p=p(t,x),q=q(t,x)},$$

and similarly we have used the notation $g(x)$ for $g(x,Y(T,x),Y(T,\cdot))$. Here A_x^* denotes the adjoint of the operator A_x.
Note that in differential of p in (4.32) can be written explicity as follows:

$$\begin{aligned} dp(t,x) = -\Big[&\sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} p(t,x) \\ &+ \sum_{i=1}^{n} \Big(-\beta_i(x) + 2\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \alpha_{ij}(x) \Big) \frac{\partial}{\partial x_i} p(t,x) \\ &\Big(-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \beta_i(x) + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \alpha_{ij}(x) \Big) p(t,x) + \Big((\frac{\partial}{\partial y} + \overline{\nabla}) b(t,x) \Big) p(t,x) \\ &+ \Big((\frac{\partial}{\partial y} + \overline{\nabla}) \sigma(t,x) \Big) q(t,x) + \Big(\frac{\partial}{\partial y} + \overline{\nabla} \Big) f(t,x) \Big] dt + q(t,x) dB(t). \end{aligned}$$

To the best of our knowledge, the existence and uniqueness of a solution of (4.32)-(4.33) is not known in general. However, note that (for given u) the equation (4.32), regarded as a BSPDE in the unknown $\mathscr{Y}^{(T)} \times \mathscr{Y}^{(T)}$-valued processes (p,q), is linear. Therefore, in view of our general assumptions (a)-(d) above, the existence and uniqueness of solution follows from e.g.Theorem 2.1 in [18], provided that the terms $\overline{\nabla} b(t,x), \overline{\nabla}\sigma(t,x)$ and $\overline{\nabla} f(t,x)$ satisfy condition (F_m) in [18]. To this end, it suffices that b, σ and f depend linearly on φ and in a space-averaging manner, as in the example with h in (4.31) above. In particular, this holds in the application studied in Section 4 of this chapter.

Remark 65 *Here, as in Sections 1 and 2, of this chapter we are primarily interested in strong solutions* $(p,q) \in \mathcal{Y}^{(T)} \times \mathcal{Y}^{(T)}$*, but weak solutions are also of interest. A pair* (p,q) *of random fields is said to be a weak solution to the backward SPDE (4.32)-(4.33) if, for all* $\phi \in \mathcal{C}_0^2(D)$*,*

$$\langle p(t,.),\phi(.)\rangle - \langle p(T,.),\phi(.)\rangle = \int_t^T \langle A_x^* p(s,.),\phi(.)\rangle ds + \int_t^T \langle \tfrac{\partial H}{\partial y}(t,.) + \overline{\nabla} H(t,.),\phi(.)\rangle ds$$
$$- \int_t^T \langle q(s,.),\phi(.)\rangle dB(s); \quad \text{a.s. for each } t \in [0,T].$$

Hence, we observe that p admits the following mild representation

$$p(t,x) = P_{T-t}\Big(p(T,x)\Big) + \int_t^T P_{s-t}\Big(\tfrac{\partial H}{\partial y}(t,x) + \overline{\nabla} H(t,x)\Big) ds - \int_t^T P_{s-t}\Big(q(s,x)\Big) dB(s); \quad 0 \le t \le T,$$

where P_t *denotes the semigroup of the operator* A^**.*

4.3.2 A sufficient maximum principle approach (I)

We now formulate a sufficient version (a verification theorem) of the maximum principle for the optimal control of the problem (4.25)-(4.27).

In the special case when $D = \mathbb{R}^n$ the result follows from Theorem 12.21 in [25]. We give direct proof for our situation, with general D.

Theorem 66 (Sufficient Maximum Principle (I)) *Suppose* $\hat{u} \in \mathcal{U}$*, with corresponding* $\widehat{Y}(t,x), \widehat{p}(t,x), \widehat{q}(t,x)$*. Suppose the functions* $(y,\varphi) \mapsto g(x,y,\varphi)$ *and* $(y,\varphi,u) \mapsto H(t,x,y,\varphi,u,\widehat{p}(t,x),\widehat{q}(t,x))$ *are convex for each* $(t,x) \in [0,T] \times D$*. Moreover, suppose that, for all* $(t,x) \in [0,T] \times D$*,*

$$\min_{v \in U} H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),v,\widehat{p}(t,x),\widehat{q}(t,x))$$
$$= H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\hat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)).$$

Then $\hat{u}$ *is an optimal control.*

Proof. Consider

$$J(u) - J(\widehat{u}) = I_1 + I_2,$$

where

$$I_1 = \mathbb{E}\left[\int_0^T \int_D \{f(t,x,Y(t,x),Y(t,\cdot),u(t,x)) - f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x))\} dx dt\right],$$

and

$$I_2 = \int_D \mathbb{E}\left[g(x,Y(T,x),Y(T,\cdot)) - g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\right] dx.$$

By convexity on g together with the identities (4.29)-(4.30) (by putting $\nabla h(x,y) = \nabla_\varphi \widehat{g}(T,x)$ and $\psi = (Y(T,.) - \widehat{Y}(T,.)))$, we get

$$\begin{aligned}
I_2 &\geq \int_D \mathbb{E}\left[\tfrac{\partial \widehat{g}}{\partial y}(T,x)(Y(T,x) - \widehat{Y}(T,x)) + \left\langle \nabla_\varphi \widehat{g}(T,x), (Y(T,\cdot) - \widehat{Y}(T,\cdot))\right\rangle\right] dx \\
&= \int_D \mathbb{E}\left[\tfrac{\partial \widehat{g}}{\partial y}(T,x)(Y(T,x) - \widehat{Y}(T,x)) + \overline{\nabla} \widehat{g}(T,x)(Y(T,x) - \widehat{Y}(T,x))\right] dx \\
&= \int_D \mathbb{E}\left[\widehat{p}(T,x)(Y(T,x) - \widehat{Y}(T,x))\right] dx \\
&= \int_D \mathbb{E}\left[\widehat{p}(T,x)\widetilde{Y}(T,x)\right] dx,
\end{aligned}$$

where we put

$$\widetilde{Y}(t,x) = Y(t,x) - \widehat{Y}(t,x); (t,x) \in [0,T] \times D. \tag{4.34}$$

Applying the Itô formula to $\widehat{p}(t,x)\widetilde{Y}(t,x)$, we have

$$\begin{aligned}
I_2 \geq \int_0^T \int_D \mathbb{E}\Big[&\widehat{p}(t,x)\{A_x \widetilde{Y}(t,x) + \widetilde{b}(t,x)\} - \widetilde{Y}(t,x)\{A_x^* \widehat{p}(t,x) \\
&+ \frac{\partial \widehat{H}}{\partial y}(t,x) + \overline{\nabla}_\varphi^* \widehat{H}(t,x)\} + \widehat{q}(t,x)\widetilde{\sigma}(t,x)\Big] dx dt,
\end{aligned} \tag{4.35}$$

where

$$\widetilde{b}(t) = b(t) - \widehat{b}(t),\ \widetilde{\sigma}(t) = \sigma(t) - \widehat{\sigma}(t). \tag{4.36}$$

Since $\widetilde{Y}(t,x) = \widehat{p}(t,x) \equiv 0$, for all $(t,x) \in (0,T) \times \partial D$, we get

$$\int_D \widehat{p}(t,x) A_x \widetilde{Y}(t,x) dx = \int_D \widetilde{Y}(t,x) A_x^* \widehat{p}(t,x) dx. \tag{4.37}$$

Substituting (4.37) in (4.35), yields

$$I_2 \geq \int_0^T \int_D \mathbb{E}\left[\widehat{p}(t,x)\widetilde{b}(t,x) - \widetilde{Y}(t,x)\left\{\tfrac{\partial \widehat{H}}{\partial y}(t,x) + \overline{\nabla}\widehat{H}(t,x)\right\} + \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right] dxdt. \tag{4.38}$$

Using the definition of the Hamiltonian H in (4.28), and putting

$$\begin{aligned}\widetilde{H}(t,x) &= H(t,x,Y(t,x),Y(t,\cdot),u(t,x),\widehat{p}(t,x),\widehat{q}(t,x)) \\ &\quad - H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)),\end{aligned} \tag{4.39}$$

we get

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_0^T \int_D \{\widetilde{H}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\} dxdt\right] \\ &\geq \mathbb{E}\left[\int_0^T \int_D \left\{\frac{\partial \widehat{H}}{\partial y}(t,x)\widetilde{Y}(t,x) + \langle \nabla \widehat{H}(t,x), \widetilde{Y}(t,\cdot)\rangle \right.\right. \\ &\quad \left.\left. + \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right\} dxdt\right], \end{aligned} \tag{4.40}$$

where the last inequality holds because of the concavity assumption of H.

Summing (4.38) and (4.40), and using (4.29), (4.30), we end up with

$$I_1 + I_2 \geq \mathbb{E}\left[\int_0^T \int_D \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x) dxdt\right].$$

By the maximum condition of H we have

$$J(u) - J(\widehat{u}) \geq \mathbb{E}\left[\int_0^T \int_D \tfrac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x) dxdt\right] \geq 0.$$

■

4.3.3 A necessary maximum principle approach (I)

We now go to the other version of the necessary maximum principle which can be seen as an extension of Pontryagin's maximum principle to SPDE with space-mean dynamics. In the case when $D = \mathbb{R}^n$a version of the necessary maximum principle is proved in [35]. Here concavity assumptions are not required . We consider the following:
Given arbitrary controls $u, \widehat{u} \in \mathscr{U}$ with u bounded, we define the following convex perturbation

$$u^\theta := \widehat{u} + \theta u; \quad \theta \in [0,1].$$

Note that, thanks to the convexity of U, we also have $u^\theta \in \mathscr{U}$. We denote by $Y^\theta := Y^{u^\theta}$ and by $\widehat{Y} := Y^{\widehat{u}}$ the solution processes of (4.25) corresponding to u^θ and $\widehat{u}$, respectively.

Define the derivative process $Z(t,x)$ by

$$Z(t,x) = \lim_{\theta \to 0} \frac{1}{\theta}(Y^\theta(t,x) - \widehat{Y}(t,x)) \text{ (limit in } \mathscr{Y}^{(T)}). \tag{4.41}$$

Then, by our assumptions on f, g, b and σ it is easy to see that $Z(t,x)$ exists and satisfies the following equation:

$$\begin{cases} dZ(t,x) &= \left\{ A_x Z(t,x) + \frac{\partial b}{\partial y}(t,x) Z(t,x) + \langle \nabla b(t,x), Z(t,\cdot)\rangle + \frac{\partial b}{\partial u}(t,x) u(t,x) \right\} dt \\ &\quad + \left\{ \frac{\partial \sigma}{\partial y}(t,x) Z(t,x) + \langle \nabla \sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial \sigma}{\partial u}(t,x) u(t,x) \right\} dB(t), \\ Z(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D, \\ Z(0,x) &= 0; \quad x \in D. \end{cases} \tag{4.42}$$

Note that (4.42), regarded as an SPDE in the unknown $\mathscr{Y}^{(T)}$-valued process Z, is linear and hence the existence and uniqueness of solution follows from e.g. Theorem 3.3 in [22].

Theorem 67 (Necessary Maximum Principle (I)) *Let $\widehat{u}(t,x)$ be an optimal control and $\widehat{Y}(t,x)$ the corresponding trajectory and adjoint processes $(\widehat{p}(t,x), \widehat{q}(t,x))$. Then we have*

$$\left.\frac{\partial \widehat{H}}{\partial u}\right|_{u=\widehat{u}}(t,x) = 0; \qquad a.s.$$

Proof. Since $\widehat{u}$ is optimal we get, by the definition (20) of J, dominated convergence and the chain rule,

$$
\begin{aligned}
0 &\le \underline{\lim}_{\theta\to 0}\frac{J(u^\theta)-J(\widehat{u})}{\theta}\\
&= \underline{\lim}_{\theta\to 0}\frac{1}{\theta}\mathbb{E}\Big[\int_D \{g(x,Y^\theta(T),Y^\theta(T,\cdot))-g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\}dx\\
&+\int_D\int_0^T \{f(t,x,Y^\theta(t,x),Y^\theta(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}dtdx\Big]\\
&= \mathbb{E}\Big[\int_D \underline{\lim}_{\theta\to 0}\frac{1}{\theta}\{g(x,Y^\theta(T),Y^\theta(T,\cdot))-g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\}dx\\
&+\int_D\int_0^T \underline{\lim}_{\theta\to 0}\frac{1}{\theta}\{f(t,x,Y^\theta(t,x),Y^\theta(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}dtdx\Big]\\
&= \mathbb{E}\Big[\int_D \frac{\partial g}{\partial y}(x,\widehat{Y}^\theta(T,x),\widehat{Y}^\theta(T,\cdot))\underline{\lim}_{\theta\to 0}\frac{1}{\theta}(Y^\theta(t,x)-\widehat{Y}(t,x))\\
&+\Big\langle \nabla g(x,\widehat{Y}^\theta(T,x),\widehat{Y}^\theta(T,\cdot)),\underline{\lim}_{\theta\to 0}\frac{1}{\theta}(Y^\theta(t,\cdot)-\widehat{Y}(t,\cdot))\Big\rangle dx\\
&+\int_D\int_0^T \underline{\lim}_{\theta\to 0}\frac{1}{\theta}\{f(t,x,Y^\theta(t,x),Y^\theta(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}dtdx\Big]
\end{aligned}
$$

Therefore, writing $\frac{\partial\widehat{g}}{\partial y}(T,x)=\frac{\partial g}{\partial y}(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))$ and $\frac{\partial\widehat{f}}{\partial y}(t,x)=\frac{\partial f}{\partial y}(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x))$ and similarly with $\nabla\widehat{g}(T,x),\nabla\widehat{f}(t,x)$ we obtain

$$
\begin{aligned}
0 &\le \underline{\lim}_{\theta\to 0}\tfrac{J(u^\theta)-J(\widehat{u})}{\theta}\\
&= \mathbb{E}\left[\int_D\Big\{\frac{\partial\widehat{g}}{\partial y}(T,x)Z(T,x)+\big\langle\nabla\widehat{g}(T,x),Z(T,\cdot)\big\rangle\Big\}dx\right]\\
&+\mathbb{E}\left[\int_0^T\int_D\left\{\frac{\partial\widehat{f}}{\partial y}(t,x)Z(t,x)+\big\langle\nabla\widehat{f}(t,x),Z(t,\cdot)\big\rangle+\frac{\partial\widehat{f}}{\partial u}(t,x)u(t,x)\right\}dxdt\right].
\end{aligned}
\tag{4.43}
$$

By (4.29) and the BSPDE for $\widehat{p}(t,x)$, we have

$$
\mathbb{E}\left[\int_D\Big\{\frac{\partial\widehat{g}}{\partial y}(T,x)Z(T,x)+\big\langle\nabla\widehat{g}(T,x),Z(T,\cdot)\big\rangle\Big\}dx\right]=\mathbb{E}\left[\int_D\widehat{p}(T,x)Z(T,x)dx\right],
$$

The Itô formula applied to the product $\widehat{p}(t,x)\cdot Z(t,x)$, where $\widehat{p}$ and Z are the associated equations (4.42), (4.32)-(4.33), respectively, to the optimal control $\widehat{u}$, combined with the definition of $\widehat{H}$ in (4.28), leads to

$$\mathbb{E}\Big[\int_D \widehat{p}(T,x)Z(T,x)dx\Big] = \mathbb{E}\Big[\int_0^T \int_D \Big(\widehat{p}(t,x)dZ(t,x) + Z(t,x)d\widehat{p}(t,x)$$
$$+ \int_0^T \int_D \Big\{\widehat{q}(t,x)\Big(\frac{\partial\sigma}{\partial y}(t,x)Z(t,x) + \langle\nabla\sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial\sigma}{\partial u}(t,x)u(t,x)\Big)\Big\}dtdx\Big]$$
$$= \mathbb{E}\Big[\int_0^T \int_D \Big\{\widehat{p}(t,x)\Big(A_x Z(t,x) + \frac{\partial b}{\partial y}(t,x)Z(t,x) + \langle\nabla b(t,x), Z(t,\cdot)\rangle + \frac{\partial b}{\partial u}(t,x)u(t,x)\Big)$$
$$+ Z(t,x)\Big(-A_x^*\widehat{p}(t,x) - \frac{\partial\widehat{H}}{\partial y}(t,x) - \overline{\nabla}\widehat{H}(t,x)\Big)$$
$$+ \Big\{\widehat{q}(t,x)\Big(\frac{\partial\sigma}{\partial y}(t,x)Z(t,x) + \langle\nabla\sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial\sigma}{\partial u}(t,x)u(t,x)\Big)\Big\}dtdx\Big].$$

Substituting this in (4.43), we get

$$0 \le \mathbb{E}\Big[\int_0^T \int_D \frac{\partial\widehat{H}}{\partial u}(t,x)u(t,x)dxdt\Big].$$

In particular, if we apply this to

$$u(t,x) = \mathbf{1}_{[s,T]}(t)\alpha(x),$$

where $\alpha(x)$ is bounded and $\mathscr{F}_s$-measurable we get

$$0 \ge \mathbb{E}\Big[\int_s^T \int_D \frac{\partial\widehat{H}}{\partial u}(t,x)\alpha(x)dxdt\Big].$$

Since this holds for all such α (positive or negative) and all $s \in [0,T]$, we conclude that

$$0 = \frac{\partial\widehat{H}}{\partial u}(t,x); \quad \text{for a.a. } t,x.$$

■

4.3.4 Controls which are independent of x

In many situations, for example in connection with partial observation control, it is of interest to study the case when the controls $u(t) = u(t,\omega)$ are not allowed to depend on the space variable x. Let us denote the set of such controls $u \in \mathscr{U}$ by $\overline{\mathscr{U}}$. Then the corresponding control problem is to find $\widehat{u} \in \overline{\mathscr{U}}$ such that

$$J(\widehat{u}) = \inf_{u \in \overline{\mathscr{U}}} J(u).$$

Theorem 68 (Sufficient Maximum Principle (II)) *Suppose $\widehat{u} \in \overline{\mathcal{U}}$, with corresponding $\widehat{Y}(t,x), \widehat{p}(t,x), \widehat{q}(t,x)$. Suppose the functions $(y,\varphi) \mapsto g(x,y,\varphi)$ and $(y,\varphi,u) \mapsto H(t,x,y,\varphi,u,\widehat{p}(t,x),\widehat{q}(t,x)$ are convex for each $(t,x) \in [0,T] \times D$. Moreover, suppose the following average minimum condition,*

$$\min_{v\in U}\left\{\int_D H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),v,\widehat{p}(t,x),\widehat{q}(t,x))dx\right\}$$
$$=\int_D H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x))dx.$$

Proof. *Consider*

$$J(u)-J(\widehat{u})=I_1+I_2,$$

where

$$I_1=\mathbb{E}\left[\int_0^T\int_D\{f(t,x,Y(t,x),Y(t,\cdot),u(t))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t))\}dxdt\right],$$

and

$$I_2=\int_D\mathbb{E}\left[g(x,Y(T,x),Y(T,\cdot))-g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\right]dx.$$

By convexity on g together with the identities (4.29)-(4.30) (by putting $\nabla h(x,y)=\nabla_\varphi\widehat{g}(T,x)$ and $\psi=(Y(T,.)-\widehat{Y}(T,.))$), we get

$$\begin{aligned}I_2&\geq\int_D\mathbb{E}\left[\tfrac{\partial\widehat{g}}{\partial y}(T,x)(Y(T,x)-\widehat{Y}(T,x))+\left\langle\nabla_\varphi\widehat{g}(T,x),(Y(T,\cdot)-\widehat{Y}(T,\cdot))\right\rangle\right]dx\\&=\int_D\mathbb{E}\left[\tfrac{\partial\widehat{g}}{\partial y}(T,x)(Y(T,x)-\widehat{Y}(T,x))+\overline{\nabla}\widehat{g}(T,x)(Y(T,x)-\widehat{Y}(T,x))\right]dx\\&=\int_D\mathbb{E}\left[\widehat{p}(T,x)(Y(T,x)-\widehat{Y}(T,x))\right]dx\\&=\int_D\mathbb{E}\left[\widehat{p}(T,x)\widetilde{Y}(T,x)\right]dx,\end{aligned}$$

where we put

$$\widetilde{Y}(t,x)=Y(t,x)-\widehat{Y}(t,x);(t,x)\in[0,T]\times D. \tag{4.44}$$

Applying the Itô formula to $\widehat{p}(t,x)\widetilde{Y}(t,x)$, we have

$$\begin{aligned}I_2\geq\int_0^T\int_D\mathbb{E}\Big[&\widehat{p}(t,x)\{A_x\widetilde{Y}(t,x)+\widetilde{b}(t,x)\}-\widetilde{Y}(t,x)\{A_x^*\widehat{p}(t,x)\\&+\frac{\partial\widehat{H}}{\partial y}(t,x)+\overline{\nabla}_\varphi^*\widehat{H}(t,x)\}+\widehat{q}(t,x)\widetilde{\sigma}(t,x)\Big]dxdt,\end{aligned} \tag{4.45}$$

where

$$\widetilde{b}(t) = b(t) - \widehat{b}(t),\ \widetilde{\sigma}(t) = \sigma(t) - \widehat{\sigma}(t). \tag{4.46}$$

Since $\widetilde{Y}(t,x) = \widehat{p}(t,x) \equiv 0$, *for all* $(t,x) \in (0,T) \times \partial D$, *we get*

$$\int_D \widehat{p}(t,x) A_x \widetilde{Y}(t,x) dx = \int_D \widetilde{Y}(t,x) A_x^* \widehat{p}(t,x) dx. \tag{4.47}$$

Substituting (4.47) *in* (4.45), *yields*

$$I_2 \geq \int_0^T \int_D \mathbb{E}\left[\widehat{p}(t,x)\widetilde{b}(t,x) - \widetilde{Y}(t,x)\left\{\tfrac{\partial \widehat{H}}{\partial y}(t,x) + \overline{\nabla}\widehat{H}(t,x)\right\} + \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right] dxdt. \tag{4.48}$$

Using the definition of the Hamiltonian H *in* (4.28), *and putting*

$$\widetilde{H}(t,x) = H(t,x,Y(t,x),Y(t,\cdot),u(t),\widehat{p}(t,x),\widehat{q}(t,x)) - H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x)),$$

we get

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_0^T \int_D \{\widetilde{H}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\} dxdt\right] \\ &\geq \mathbb{E}\left[\int_0^T \int_D \left\{\frac{\partial \widehat{H}}{\partial y}(t,x)\widetilde{Y}(t,x) + \left\langle \nabla \widehat{H}(t,x), \widetilde{Y}(t,\cdot)\right\rangle\right.\right. \\ &\quad \left.\left. + \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right\} dxdt\right], \end{aligned} \tag{4.49}$$

where the last inequality holds because of the concavity assumption of H.

Summing (4.48) *and* (4.49), *and using* (4.29), (4.30), *we end up with*

$$I_1 + I_2 \quad \geq \mathbb{E}\left[\int_0^T \int_D \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t) dxdt\right].$$

By the maximum condition of H *we have*

$$J(u) - J(\widehat{u}) \geq \mathbb{E}\left[\int_0^T \int_D \tfrac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t) dxdt\right] \geq 0.$$

■ *Then* $\widehat{u}$ *is an optimal control.*

Theorem 69 (Necessary Maximum Principle (II)) *Let* $\widehat{u}(t)$ *be an optimal control and* $\widehat{Y}(t,x)$ *the corre-*

sponding trajectory and adjoint processes $(\widehat{p}(t,x), \widehat{q}(t,x))$. *Then we have*

$$\int_D \frac{\partial \widehat{H}}{\partial u}\Big|_{u=\widehat{u}}(t,x)dx = 0; \quad a.s.\ dt \times d\mathbb{P}.$$

Proof. The proof is analogous to the proof of Theorem 67 ■

4.4 Application to vaccine optimisation

In this section we study an example, assuming that the *density* $Y(t,x)$ *of infected individuals in a population* in a random/noisy environment changes over time t and space point x according to the following space-interaction reaction-diffusion equation

$$\begin{cases} dY(t,x) &= \frac{1}{2}\Delta Y(t,x)dt + \Big(\alpha \overline{Y}(t,x) - u(t,x)Y(t,x)\Big)dt + \beta Y(t,x)dB(t), \\ Y(0,x) &= \xi(x) \geq 0; \quad x \in D, \\ Y(t,x) &= \eta(t,x) \geq 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases}$$

where α, β are given constants modelling the effect on the growth $dY(t,x)$ of the term $\overline{Y}$ and of the noise, respectively, and $\overline{Y}(t,x) = G(x, Y(t,\cdot))$, where, as before, G is a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y)dy; \quad \varphi \in L^2(D),$$

with $V(\cdot)$ denoting Lebesgue volume and

$$K_r = \{y \in \mathbb{R}^n; |y| < r\}$$

is the ball of radius $r > 0$ in $\mathbb{R}^n$ centered at 0.

By a slight extension of Theorem 59 (see Remark 60), we know that $Y(t,x) \geq 0$ for all t, x.

If $u(t,x)$ represents our vaccine effort rate at (t,x), we define the total expected cost $J(u)$ of the effort by

$$J(u) = \mathbb{E}\Big[\frac{\rho}{2}\int_D \int_0^T u(t,x)^2 Y(t,x)dtdx + \int_D h_0(x)Y(T,x)dx\Big],$$

where $\rho > 0$ is a constant, and $h_0(x) > 0$ is a bounded function. Here we may regard the first quadratic term as the cost of the vaccination effort, with unit price ρ, and the second term as the cost of having remaining infection at time T. In this case the Hamiltonian is

$$H(t,x,y,\overline{y},p,q) = (\alpha\overline{y} - uy)p + \beta yq + \frac{\rho}{2}u^2 y,$$

and the adjoint equation satisfies

$$\begin{cases} dp(t,x) = -\left[\frac{1}{2}\Delta p(t,x) - u(t,x)p(t,x) + \overline{\nabla}_{\overline{y}}H(t,x) + \beta q(t,x) + \frac{\rho}{2}u^2(t,x)\right]dt + q(t,x)dB(t), \\ p(T,x) = h_0(x); \quad x \in D \\ p(t,x) = 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases} \tag{4.50}$$

where, by Example 64, $\overline{\nabla}_{\overline{y}}H(t,x) = \upsilon_D(x)\alpha p(t,x)$, with $\upsilon_D(x) := \frac{V((x+K_r)\cap D)}{V(K_r)}$.
The first order condition for an optimal $u = \widehat{u}$ for H together with the requirement that $Y(t,x) > 0$, lead to

$$\widehat{u}(t,x) = \frac{p(t,x)}{\rho}.$$

Hence the pair of random fields $(\widehat{p},\widehat{q})$ becomes

$$\begin{cases} d\widehat{p}(t,x) &= -\left[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + \upsilon_D(x)\alpha\widehat{p}(t,x) + \beta\widehat{q}(t,x)\right]dt + \widehat{q}(t,x)dB(t), \\ \widehat{p}(T,x) &= h_0(x); \quad x \in D, \\ \widehat{p}(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{4.51}$$

Since h_0 and all the coefficients of this equation are deterministic, we can conclude that $\widehat{q} = 0$ and (4.51) reduces to the deterministic partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}\widehat{p}(t,x) &= -\left[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + \upsilon_D(x)\alpha\widehat{p}(t,x)\right], \\ \widehat{p}(T,x) &= h_0(x); \quad x \in D, \\ \widehat{p}(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases}$$

This is a (deterministic) Fujita type backward quadratic reaction diffusion equation. We could also from the beginning have allowed $h_0(x)$ to be random and satisfy $\mathbb{E}\left[\int_D h_0^2(x)dx\right] < \infty$. Then the equation (4.51) would have become a nonlinear backward *stochastic* reaction-diffusion equation. We will not discuss this further here, but refer to Bandle, & Levine [3], Dalang et al [13] and Fujita [21] and the references therein for more information.

conclusion

In this thesis, we have used the stochastic partial differential equations theory. In the first part, We introduced linear SPDEs. We proved the existence and uniqueness of nonlinear SPDEs Then in the second part, We have used results from noisy Observation (nonlinear filtering), and we transformed these noisy observations stochastic differential equation (SDE) control problem into a complete observation stochastic partial differential equations (SPDEs for short). We proved a sufficient and necessary maximum principle for the optimal control of SPDEs. Finally, in the third part, we have used a new type of non-local stochastic partial differential equations (SPDEs). The SPDEs have space interactions, in the sense that the dynamics of the system at time t and position in space x also depend on the space-mean of values at neighboring points. We have proved the existence and uniqueness of solutions of a class of SPDEs with space interactions, and we have shown that, under some conditions. In case we have the solutions positive for all times if the initial values are. Then we have proved sufficient and necessary maximum principles for optimal control. Finally, we have applied the results to study an optimal vaccine strategy problem for an epidemic by modeling the population density as a space-mean stochastic reaction-diffusion equation. The results of the third part were the subject of a paper published in international journal ESAIM: Control, Optimisation, and Calculus of Variations, COCV 29,2023.

Bibliography

[1] Adams,R.A., & Fournier, J.J. (2003). sobolev Spaces.

[2] Agram, N., Hilbert, A., & Øksendal, B. (2020). Singular control of SPDEs with space-mean dynamics. Mathematical Control & Related Fields, 10(2), 425.

[3] Bandle, & Levine, H. (1994). Fujita type phenomena for reaction-diffusion equations with convection like terms. Differential and Integral Equations, 7(5-6), 1169-1193.

[4] A. Bensoussan and R. Temam, ´equations aux dérivées partielles stochastiques non linéaires, I. Israel J. Math. 11, 95-129, 1972

[5] Bensoussan, A., & Temam, R. (1973). Equations stochastiques du type Navier-Stokes. Journal of Functional Analysis, 13(2), 195-222.

[6] Bensoussan, A. (1983). Stochastic maximum principle for distributed parameter systems. J. Franklin Inst. 315, 387-406.

[7] Bensoussan, A. (1983). Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. Stochastics: An International Journal of Probability and Stochastic Processes, 9(3), 169-222.

[8] Bensoussan, A. (1991). Stochastic maximum principle for systems with partial information and application to the separation principle. Applied Stochastic Analysis. Gordon and Breach, 157-172.

[9] Bensoussan, A. (2004). Stochastic Control of Partially Observable Systems. Cambridge University Press.

[10] Benth, F. E. (1997). On the positivity of the stochastic heat equation. Potential Analysis, 6(2), 127-148.

[11] Benth, F. E., & Gjessing, H. K. (1994). A non-linear parabolic equation with noise: a reduction method.

[12] Bellman. Dynamic Programming. Princeton Univ. Press, Princeton, New Jersey, 1957.

[13] Dalang, R. C., Khoshnevisan, D., & Zhang, T. (2019). Global solutions to stochastic reaction-diffusion equations with super-linear drift and multiplicative noise. Annals of Probability, 47(1), 519-559.

[14] DALECKII Yu. L. (1966) Differential equations with functional derivatives and stochastic equations for generalized random processes, Dokl. Akad. Nauk SSSR, 166, 1035–1038.

[15] Da Prato, G., & Zabczyk, J. (2014). Stochastic equations in infinite dimensions. Cambridge University Press.

[16] Di Nunno, G., Øksendal, B., & Proske, F. (2009). Malliavin Calculus for Lévy Processes with Applications to Finance. Springer.

[17] Draouil, O., & Øksendal, B. (2018). Optimal insider control of stochastic partial differential equations. Stochastics and Dynamics, 18(01), 1850014.

[18] Du, K., Tang, S. & Zhang, Q. (2013). Wm, p-solution ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space. Journal of Differential Equations, 254(7), 2877-2904.

[19] E. Pardoux, Équations aux dérivés Partielles Stochastiques Non Linéaires Montones. Thése, Université Paris XI.

[20] Fuhrman, M., Ying Hu, Y. & Tessitore, G. (2013). Stochastic maximum principle for optimal control of SPDEs. Applied Mathematics & Optimization 68.2, 181-217.

[21] Fujita, H. (1966). On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 13, 109-124.

[22] Gawarecki, L., & Mandrekar, V. (2010). Stochastic Differential Equations in Infinite Dimensions: with Applications to Stochastic Partial Differential Equations. Springer Science & Business Media.

[23] Gjessing, H. K. (1995). Wick calculus with applications to anticipating stochastic differential equations. Institut Mittag-Leffler.

[24] GROSS L. (1967) Potential theory in Hilbert spaces, J. Funct. Anal., 1, 123–181.

[25] Holden, H., Øksendal, B., Ubøe, J. & Zhang, T. (2010). Stochastic Partial Differential Equations. A Modelling, White Noise Functional Approach. Second Edition. Springer.

[26] Hu, Y., Ma, J., & Yong, J. (2002). On semi-linear degenerate backward stochastic partial differential equations. Probability Theory and Related Fields, 123(3), 381-411.

[27] Hu, Y., & Peng, S. (1990). Maximum principle for semilinear stochastic evolution control systems. Stochastics and Stochastic Reports, 33(3-4), 159-180.

[28] Hairer, M. (2009). An introduction to stochastic PDEs. arXiv preprint arXiv:0907.4178.

[29] Kalman R.E. On the General Theory of Control System. Proc. IFAC Congr 1st. 1, pp. 481-491, 1961

[30] Kopp, R. E. (1962). Pontryagin maximum principle. In Mathematics in Science and Engineering (Vol. 5, pp. 255-279). Elsevier.

[31] Kushner, Harold. (1967). Nonlinear filtering: The exact dynamical equations satisfied by the conditional mode. Automatic Control, IEEE Transactions on Volume 12, Issue 3, Jun 1967 Page(s): 262 - 267

[32] I. Gikhman and A.V. Skorokhod. The Theory of Stochastic Processes, Springer, Berlin (1974).

[33] Krylov N. V. & Rozovskii B. L. (1981) Stochastic evolution equations, translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki, 14 (1979), 71–146, Plenum Publishing Corp.

[34] Kuo, H. H. (1996). White noise distribution theory (Vol. 5). CRC press.

[35] Q. Lü and X. Zhang, Mathematical Control Theory for Stochastic Partial Differential Equations. Springer, Berlin (2021).

[36] M. Viot, Solutions faibles d'équations aux dérivé Partielles non Linéaires . Thése, Université Paris (1976).

[37] Ma, J., & Yong, J. (1999). On linear, degenerate backward stochastic partial differential equations. Probability Theory and Related Fields, 113(2), 135-170.

[38] Makhlouf, K., Agram, N., Hilbert, A., & Øksendal, B. (2023). SPDEs with space interactions and application to population modelling. ESAIM. COCV, 29.

[39] Mortensen, R.E. 1966. Stochastic optimal control with noisy observations. Int. J. Control 4:455–464.

[40] Lototsky, S. V., & Rozovsky, B. L. (2017). Stochastic partial differential equations (pp. 6-22). Cham: Springer.

[41] Marinelli, C., & Röckner, M. (2016). On the maximal inequalities of Burkholder, Davis and Gundy. Expositiones Mathematicae 34 (1), 1-26.

[42] Øksendal,B.Stochastic Differential Equations An Introduction With Applications,6th Edition, Springer(2013).

[43] Øksendal, B. (2005). Optimal control of stochastic partial differential equations. Stochastic Analysis and Applications, 23(1), 165-179.

[44] Øksendal, B., Proske, F., & Zhang, T. (2005). Backward stochastic partial differential equations with jumps and application to optimal control of random jump fields. Stochastics 77(5), 381-399.

[45] Øksendal, B., Sulem, A., & Zhang, T. (2014). Singular control and optimal stopping of SPDEs, and backward SPDEs with reflection. Mathematics of Operations Research, 39(2), 464-486.

[46] Øksendal, B., & Sulem, A.(2019). Applied Stochastic Control of Jump Diffusions. Third Edition. Springer.

[47] Pardoux E. (1975) Equations aux derivées partielles stochastiques nonlinéaire monotones, Thése, Université Paris XI.

[48] Pardoux, É. (1980). Stochastic partial differential equations and filtering of diffusion processes. Stochastics, 3(1-4), 127-167.

[49] Pardoux, É. (1991). Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. In Ecole d'Eté de Probabilités de Saint-Flour XIX—1989 (pp. 68-163). Springer.

[50] Prévôt, C., & Röckner, M. (2007). A Concise Course on Stochastic Partial Differential Equations. Springer.

[51] R. Khasminskii. Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Alphen aan den Rijn (1980).

[52] Röckner, M., & Zhang, T. (2007). Stochastic evolution equations of jump type: existence, uniqueness and large deviation principles. Potential Analysis, 26(3), 255-279.

[53] Smoller, J. (2012). Shock waves and reaction-diffusion equations (Vol. 258). Springer Science & Business Media.

[54] Stratonovich, R. L. (1959). Optimum nonlinear systems which bring about a separation of a signal with constant parameters from noise. Radiofizika, 2:6, pp. 892-901.

[55] Stratonovich, R.L. (1960). Application of the Markov processes theory to optimal filtering. Radio Engineering and Electronic Physics, 5:11, pp.1-19.

[56] Wloka, J. (1987) Partial Differential Equations. Cambridge University Press.

[57] Zakai, Moshe (1969), On the optimal filtering of diffusion processes. Zeit. Wahrsch. 11 230–243

[58] Zhou, X. Y. (1993). On the necessary conditions of optimal controls for stochastic partial differential equations. SIAM Journal on Control and Optimization, 31(6), 1462-1478.

[59] Yong, J., & Zhou, X. Y. (1999). Stochastic controls: Hamiltonian systems and HJB equations (Vol. 43). Springer Science & Business Media.