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Stochastic Maximum Principle under Sublinear Expectation

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Dedication

To my beloved family.

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Biskra, February 02, 2023

Meriyam Dassa

Abstract

This thesis extends the famous Pontryagin's stochastic maximum principle to the case of volatility uncertainty and ambiguity which is modelled by G-Brownian motion (G-SMP) where we present two research topics, the first one is divided into four parts. In the first part, we introduce an optimal control problem where the state equation is driven by G-Brownian motion and the cost functional is given of risk-neutral type. We prove the stability of controlled stochastic differential equations driven by G-Brownian motion (G-SDEs in short) with respect to the control variable by using the convex perturbation method, in which the set of admissible controls is convex. In the second part, we introduce the G-adjoint process and the G-adjoint equation by using the resolvent method and the G-martingale representation theorem. In the third part, we establish necessary and sufficient optimality conditions for this model. Lastly, we illustrate our main result by giving an example of a linear-quadratic problem where we solve the Riccati-type equation.

The second topic is characterising the problem of optimal control under a risk-sensitive control model. Both the admissible control and the system dynamics are defined in the same way as those of the first topic. The only difference is the way of defining the performance criterion. Instead of minimizing the direct cost, we aim to minimize a convex disutility function of the cost. As a preliminary step, we clarify the relationship between risk-neutral and risk-sensitive loss functional. Secondly, we are doing a simple reformulation of risk-sensitive problem as a standard risk-neutral problem under G-expectation. Thus, An intermediate G-SMP is obtained by a standard application of risk-neutral result. Thirdly, we prove the equivalence relation between G-expected exponential utility and G-quadratic backward stochastic differential equation. Finally, we deal with the example of Merton's problem with power utility.

Key words: G-expectation, G-SDE, G-SMP, Risk-sensitive control.

Résumé

Cette thèse étend le principe du maximum stochastique de Pontryagin dans le cas de l'incertitude de la volatilité et l'ambiguïté qui est modélisée par G-mouvement Brownien (G-PMS) où nous présentons deux sujets de recherche, le premier est divisé en quatre parties. Dans la première partie, nous introduisons un problème de contrôle optimal où l'équation d'état est gouvernée par G-mouvement Brownian et le fonctionnel de coût est donné de type risque-neutre. Nous démontrons la stabilité des équations différentielles stochastiques contrôlées gouvernée par G-mouvement Brownien par rapport à la variable de contrôle en utilisant la méthode de perturbation convexe dans laquelle l'ensemble des contrôles admissibles est convexe. Dans la deuxième partie, nous introduisons le G-processus adjoint et la G-équation adjoint en utilisant la méthode de la resolvant et le théorème de la representation des G-martingales. Dans la troisième partie, nous établissons des conditions necessaires et suffisantes d'optimalité pour ce modèle. Enfin, nous illustrons notre résultat principal en donnant un exemple de problème linéaire-quadratique où nous résolvons l'équation de type Riccati.

Le deuxième sujet consiste à caractériser le problème du contrôle optimal avec une fonction de performance de risque-sensible modèle de contrôle. Le contrôle admissible et la dynamique du système sont définis dans de la même manière que ceux du premier sujet. La seule différence est la façon de définir le critère de performance. Au lieu de minimiser le coût direct, nous visons à minimiser un fonction de désutilité convexe du coût. Dans un premier temps, nous clarifions la relation entre risque-neutre et risque-sensible. Deuxièmement, nous faisons une reformulation simple au problème du risque-sensible comme un problème de risque-neutre sous G-espérance. Ainsi, un intermédiaire G-PMS est obtenu par une application standard du résultat de risque-neutre. Troisièmement, nous démontrons la relation d'équivalence entre G-espérance d'utilité exponentielle et G-équation différentielle stochastique rétrograde quadratique.

Enfin, nous traitons l'exemple du problème de Merton avec l'utilité de puissance.

Mots Clés: G-espérance, G-EDS, G-PMS, Contrôle de risque-sensible.

Contents

Symbols and Abbreviations

For readers' convenience, we list the different symbols and abbreviations used in this thesis as follows.

Symbols

$\mathbb{E}$	:	Sublinear expectation.
$\mathbb{E}_G$	:	G-expectation.
B	:	G-Brownian motion.
$\langle B\rangle$	:	Quadratic variation process of B.
$(\Omega, \mathcal{H}, \mathbb{E})$	:	Sublinear expectation space.
$(\mathcal{F}_t)_{t\in[0,T]}$	:	Filtration generated by the canonical process B.
$\mathbb{R}^n$	:	n-dimensional Euclidean space.
$\mathbb{R}^{n\times d}$	:	The space of $n \times d$ matrices with real values.
U_{ad}	:	The set of all admissible controls.
$\lvert\cdot\rvert$	:	Euclidean norm on $\mathbb{R}^n$.
$\mathcal{T}$	:	Appearing in the superscripts denotes the transpose of a matrix or a vector.
$\widehat{u}$	:	Optimal control.
u^θ	:	Perturbed control.
θ	:	Perturbation index.
ε	:	Risk sensitivity index.
$\nabla_x f$	:	The gradient vector of the function f in the vector x.
$D_x\varphi$	:	The gradient matrix of the vectorial function φ in the vector x.
1_A	:	The indicator function of A.

Abbreviations

a.s.	:	Almost-surely.
q.s.	:	Quasi-surely.
q.c.	:	Quasi-continuous.
G-SDE	:	G-stochastic differential equation.
G-SMP	:	G-stochastic maximum principle.
BDG	:	Burkholder-Davis-Gundy.
G-BSDE	:	G-backward stochastic differential equation.
G-QBSDE	:	G-quadratic backward stochastic differential equation.

Introduction

Traditionally, we quantify the risk in our statement on the uncertain events in our daily life with a unique probability measure. However, sometimes in our probability assessment we should not ignore the variety of uncertainties in the financial markets and we are faced with the so-called Knightian uncertainty, model uncertainty or ambiguity (see Knight Frank [20]), which indicates that the decision-makers have a sceptical attitude on the model they used and they are unable to obtain objectively an accurate form of the model and this is due to incomplete information or vague concepts and principles. In early 1953, when Allais paradox was introduced [1], the economists discovered that the theory of "expected utility" based on linear mathematical expectation posed many questions. A question then arises: can we find a new theory that can be a natural generalisation of a linear expectation? In particular, preserving, as much as possible, the properties of the classical linear expectation. As an answer to this question, Peng proposed in [25] a new notion of nonlinear expectation more dynamic, called G-expectation. Considerably, the notion of nonlinear mathematical expectation then has been developed and a new concept of sublinear expectation has been given by Peng [30] by using a functional analysis theory. The G-expectation framework is a generalisation of the classical probability system based on the sublinear expectation and generated by a nonlinear heat equation with a given infinitesimal generator G to deal with the phenomena that cannot be described by a single probabilistic model. These phenomena are closely related to the long-existing concern about model uncertainty in many fields. Similar to the concepts of the classical framework, Peng [29] established the notions of distribution and independence in this new context. However, the distributions and independence in the G-framework are quite different from the classical setup. These distinctions bring difficulty when applying the idea of this framework to general probabilistic practice. Therefore, a fundamental and unavoidable problem is better understanding G-version concepts from a probabilistic perspective.

In this thesis, we work in a G-Brownian motion setting that turns out to be a good framework to develop stochastic calculus of Itô's type. We can also use the related stochastic calculus, including the Itô's formula, G-SDEs, martingale representation and G-BSDEs, as developed in [14, 29, 33, 36, 37, 38, 39]. Many economic and financial problems involve volatility uncertainty, refer to Epstein and Ji [11], which is characterized by a family of nondominated probability measures. Volatility uncertainty has been investigated in the literature by following two approaches, by introducing an abstract sublinear expectation space (see [33]) or by capacity theory. These two methods are strongly related and have been proved by Denis et al [8]. The link between these two approaches is the representation of the sublinear expectation $\mathbb{E}$ associated with the G-Brownian motion as a supremum of ordinary expectations over a tight family of probability measures $\mathcal{P}$, whose elements are mutually singular[1]. To solve the super-replication problem in an uncertainty volatility model, Denis and Martini [9] independently introduced a notion of upper expectation and the related capacity theory. Recently, Peng [26, 28] established a nonlinear expectation theory. A random variable X with "G-normal distribution" is defined via the heat equation. With this single nonlinear distribution and as a special and typical case, Peng [33] studied a fully nonlinear expectation, called G-expectation $\mathbb{E}_G[.]$ under which the canonical process is a G-Brownian motion and the corresponding time-conditional G-expectation $\mathbb{E}_G[.|\mathcal{F}_t]$ on a space of random vari-

[1] Two measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ on Ω are singular if there exists $A \subset \Omega$ such that $\mathbb{P}(A) = 1$ and $\widetilde{\mathbb{P}}(A) = 0$. They are equivalent, if for every $A, \mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$. Thus $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ singular implies that they are not equivalent, but the converse is false.

ables completed under the norm $\left(\mathbb{E}_G\left[|.|^p\right]\right)^{\frac{1}{p}}$, denoted by $\mathbb{L}^p_G(\Omega_T)$, $p \geq 1$. Under this G-framework, a new type of Brownian motion $B = (B_1, ..., B_d)$ has independent, stationary and G-normally distributed increments called G-Brownian motion with $B_0 = 0$, for more details see [31]. We call an increasing process $(\langle B\rangle_t)_{t\geq 0}$ the quadratic variation process of G-Brownian motion B and for all $1 \leq \mathtt{i}, \mathtt{j} \leq d$, $\left\langle B^{\mathtt{i}}, B^{\mathtt{j}}\right\rangle = \langle B\rangle^{\mathtt{ij}}$ is its cross-variation process, it characterises the part of statistic uncertainty of G-Brownian motion. It is important for understanding the nature of G-Brownian motion to keep in mind that its quadratic variation $\langle B\rangle$ is not a deterministic process, unlike the classical case, but it is absolutely continuous with the density tak value in a fixed set (for example, $[\underline{\sigma}^2, \overline{\sigma}^2]$ for $d = 1$). Each $\mathbb{P} \in \mathcal{P}$ can then be seen as a model with a different scenario for the quadratic variation. That justifies why G-Brownian motion is a good framework for investigating model uncertainty. G-Brownian motion and its quadratic variation plays a central role in the related nonlinear stochastic analysis. Indeed, the stochastic integrals with respect to G-Brownian motion and its quadratic variation have been first introduced by Peng in his pioneer work [30], which are initially defined on the simple process space and later extended as a linear operator on Banach completions. Thereafter, the G-stochastic calculus is further developped, for example, in [12, 21, 23]. In papers [25, 29, 30], as in the classical case, Peng has introduced the definition of the Itô's integral by Riemann-Stieltjes's sums the process of the form

$$\eta_t = \sum_{k=0}^{N-1} \xi_k 1_{[t_k, t_{k+1}[}(t)\,. \tag{1}$$

In the following, the $\mathbb{M}^{0,2}_G(0,T)$-space of the processes is completed for the standard norm $\left(\frac{1}{T}\mathbb{E}_G\left[\int_0^T |.|^2\, dt\right]\right)^{1/2}$ (see Peng [30]). This completed space is noted by $\mathbb{M}^2_G(0,T)$. After getting a G-Itô's inequality instead of the Itô's isometry in classical case, Peng demonstrated that Itô's integral of the form $\mathcal{I}(\eta) = \int \eta_t dB_t$ can be seen as a continuous and linear functional on $\mathbb{M}^{0,2}_G(0,T)$, and that it can be extended uniquely to the completed space $\mathbb{M}_G^2(0,T)$. Subsequently, Li and Peng [21] defined Itô's integral for a process of the form (1), where the ξ_k are replaced by bounded random variables. Similarly, they have shown that the definition of this new integral can be extended to the completed $\mathbb{M}_G^2(0,T)$. Moreover, the notion of quasi-sure with respect to the associated Choquet capacity is introduced by Denis et al. to the G-framework. After having defined the stochastic integrals within the framework of G-expectation and similar to their classical counterparts, stochastic differential equations driven by G-Brownian motion (G-SDEs) are well defined in the quasi-sure sense and their solvability can be established by the contracting mapping theory under Lipschitz's assumptions. The first work on G-SDE has been carried out by Peng [30] by using the fixed point theorem. Then in [2, 41] with the following form for all $0 \leq t \leq T, 1 \leq \mathfrak{i}, \mathfrak{j} \leq d$

$$\begin{cases} dx_t = b(t,x_t)\,dt + h_{\mathfrak{ij}}(t,x_t)\,d\langle B\rangle_t^{\mathfrak{ij}} + \sigma_{\mathfrak{i}}(t,x_t)\,dB_t^{\mathfrak{i}}, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{2}$$

where the coefficients $b, h_{\mathfrak{ij}}$ and $\sigma_{\mathfrak{i}}$ are uniformly Lipschitz and the G-SDE (2) has a unique strong solution. In addition, other researchers have been interested in G-SDE, for example, the work of Lin [23] and Bai and Lin [24]. Overall these papers were carried out under a common assumption which ensures that the coefficients are linearly growing. Note that in these papers on G-SDEs, some tools for stochastic analysis in the framework of G-expectation are developed. For example, Itô's Formula was introduced in Peng's paper [29] and was generalised by Gao [12], Li and Peng [21] and Zhang et al. [47]. An inequality of Burkholder-Davis-Gundy type (BDG in short) has been also proved in this new framework. In contrast to the classical martingale representation, the G-martingale is decomposed into two parts: the G-Itô's type integral part $X_t = \int_0^t Z_s dB_s$, which is called symmetric G-martingale, in the sense that $-X_t$ is still a G-martingale and the decreasing G-martingale part K, which vanishes in the classical theory, but it plays a significant role in this new context (see [33, 36]). However, the challenging problem of wellposedness for optimal control problem remained open until now. The stochastic optimal control problem is important in control theory. The maximum principle, necessary and sufficient conditions for optimality, is one of the central results. A lot of work has been done on this topic in the classical expectation. However, in the G-expectation, few papers we are aware of and which deal with the G-stochastic maximum principle are [3, 13, 40, 46], where the controlled G-SDE is given in risk-neutral type control.

The main objective of this thesis is to study two research topics about G-stochastic maximum principle. For the first topic, we study G-stochastic maximum principle for risk-neutral control problem where the system is governed by the nonlinear n-dimensional controlled G-stochastic differential equation for all $0 \leq t \leq T, 1 \leq \mathfrak{i}, \mathfrak{j} \leq d$

$$\begin{cases} dx_t = b\left(t, x_t, u_t\right) dt + h_{\mathfrak{ij}}\left(t, x_t, u_t\right) d\left\langle B\right\rangle_t^{\mathfrak{ij}} + \sigma_{\mathfrak{i}}\left(t, x_t, u_t\right) dB_t^{\mathfrak{i}}, \\ x\left(0\right) = x \in \mathbb{R}^n. \end{cases} \tag{3}$$

The cost functional that is minimized over the class of admissible controls has the following form

$$\mathcal{J}\left(u\right) = \mathbb{E}_G\left[g\left(x_T^u\right) + \int_0^T l\left(t, x_t^u, u_t\right) dt + \int_0^T m_{\mathfrak{ij}}\left(t, x_t, u_t\right) d\left\langle B\right\rangle_t^{\mathfrak{ij}}\right].$$

Xu [46] studied this problem. Based on the subadditivity of $\mathbb{E}[\cdot]$, he obtained the variational inequality by the classical variational method. Moreover, Hu and Ji [13] thought that the classical variational method which was used in [46] can not be applied for (3) and they establish a stochastic maximum principle for stochastic recursive optimal control problems in the G-setting by the linearization and weak convergence but still using the worst-case approach where they use the minimax theorem to obtain the variational inequality under a reference probability $\mathbb{P}^*$ and the stochastic maximum principle holds then under such a $\mathbb{P}^*$-a.s, which is the main difference with respect to our approach.

In recent papers, Sun et al. [40] proved a stochastic maximum principle for controlled processes driven by G-Brownian motion. Then they obtained the maximum condition in terms of the $\mathcal{H}$-function, plus some convexity conditions constitute sufficient conditions for optimality. Biagini et al. [3] studied also a stochastic maximum principle of controlled G-SDE under the assumption of the existence of the strongly robust optimal control but their control problem is different from the one in [13] and they considered delayed information and adapted the stochastic maximum principle to the G-framework to find necessary and sufficient conditions for the existence of a strongly robust optimal control by using the G-adjoint equation directly without proof. However, our contribution proposes different methods to prove this problem. In order to derive the G-stochastic maximum principle, we must first verify the stability of controlled G-SDE. Moreover, we introduce the variational equation and variational inequality. Furthermore, we prove that the obtained stochastic maximum principle is also a sufficient condition under some convex assumptions.

In the G-framework, the only result we are aware of about the existence of an optimal control for this problem has been given by Redjil et al. [35], which is the existence of an optimal relaxed control where the stochastic differential equation is considered with jump diffusion, the celebrate Chattering lemma was generalised in the G-framework and the existence of relaxed optimal control was proved using the approximtion of trajectories such that each G-relaxed control is considered as a limit of a sequence of strict controls.

For the second topic, our system is governed by controlled G-stochastic differential equation and it is given for all $0 \leq t \leq T$ by

$$\begin{cases} dx_t^u = & b\left(t, x_t^u, u_t\right) dt + h\left(t, x_t^u, u_t\right) d\left\langle B\right\rangle_t + \sigma\left(t, x_t^u, u_t\right) dB_t \\ x\left(0\right) = & x_0 \in \mathbb{R}. \end{cases}$$

The objective of the risk-sensitive optimal control problem is to minimize, over the set

U_{ad} of all admissible controls, the functional cost $\mathcal{J}^{\varepsilon}(.)$ of the form

$$\mathcal{J}^{\varepsilon}(u) = \mathbb{E}_G \left[\exp \varepsilon \left(g\left(x_T^u\right) + \int_0^T l\left(t, x_t^u, u_t\right) dt + \int_0^T m\left(t, x_t^u, u_t\right) d\left\langle B\right\rangle_t \right) \right]. \quad (4)$$

A stochastic control $\widehat{u}$ is called optimal if it solves $\mathcal{J}^{\varepsilon}(\widehat{u}) = \inf_{u \in U_{ad}} \mathcal{J}^{\varepsilon}(u)$ by using the fact that the set of admissible controls U_{ad} is convex.

The risk-sensitive maximum principle for optimal stochastic control derived from the classical one, that is an immediate generalisation of the G-stochastic Pontryagin maximum principle.

In the past decades, much research has attracted attention to control problems with risk-sensitive performance functional in the classical expectation. We note that necessary optimality conditions for risk-sensitive cost functional, where the systems are governed by a stochastic differential equation, have been studied by [22]. We also realize that necessary optimality conditions for stochastic controls, where the systems are governed by a nonlinear forward stochastic differential equation with jumps, have been studied by [18] in the case where the set of admissible controls is convex and [19] in the general case with application to finance. Furthermore, the case of systems governed by a mean-field stochastic differential equation has been studied by [10]. Chala [4] developed Pontryagin's risk-sensitive stochastic maximum principle for backward stochastic differential equations with application. In addition, in [6] the authors proved the use of Girsanov's theorem to describe the risk-sensitive problem and application to optimal control. Moreover, Chala [5] studied a risk-sensitive stochastic control problem of a nonlinear system in which the variable control has two components, the first being continuous and the second being singular.

Most formulations of the G-stochastic maximum principle have been studied in a risk-neutral type where the performance functionals are G-expected values of stage-additive payoff functions. However, not all behaviours can be captured by risk-neutral type controls. One way of capturing risk-averse and risk-seeking behaviours is by exponentiating the performance functional before G-expectation. More information on the risk-sensitive control can be found in [42, 43, 44].

There has been renewed interest in the cost criterion (4) during the past decade. The primary reason is the original one: when $\varepsilon > 0$ the use of the exponential reduces the possibility of rare, but devastating large excursions of the state process. This control problem has attracted more recent attention because of the interesting connections between risk-sensitive control, game theory and mathematical finance.

Finally, when the process incorporates three sources of uncertainty: drift, diffusion and volatility uncertainty are control-dependent in the case of G-expectation, the risk-sensitive stochastic maximum principle is not very different from the classical one, where the G-maximum condition has additional terms involving the second G-adjoint variable and the risk-sensitive parameter. That is, the optimal control depends explicitly on these quantities.

Our work brings together two important subjects of actual intensive research, Pontryagin's stochastic maximum principle for risk-neutral control problem under Peng's sublinear G-expectation on one side and for risk-sensitive control problem popularised by P. Whittle [42] on the other side.

Let us briefly describe the contents of this thesis:

In the first chapter, we present the most important results and the basic definitions of the G-framework, the sublinear expectation, G-normal distribution, G-Brownian motion and G-stochastic calculus which are used throughout this thesis.

In the second chapter, we study an optimal control problem where the state equation is driven by G-Brownian motion. First, introducing the problem and the various assumptions used throughout this chapter. Second, proving the stability between the perturbed solution and the optimal solution of controlled stochastic differential equations driven by G-Brownian motion, where we introduce three estimation's lemmas about the solution of controlled G-SDE by using the convex perturbation method, in which the set of admissible controls is convex. Third, we introduce in detail the G-adjoint process and G-adjoint equation by using the resolvent method and the G-martingale representation theorem. Moreover, we give our first and second main results in this chapter, the necessary as well as sufficient optimality conditions for G-SDE. In conclusion, We give an example of a linear-quadratic problem.

In the third chapter, we discuss a risk-sensitive control problem while the state is described by G-SDE with an exponential of integral cost functional**.** First, we formulate the problem of risk-sensitive control problem and give the various assumptions and proofs of some results about the stability between the perturbed solution and the optimal solution of the controlled G-stochastic differential equations. Second, we establish mean-variance uncertainty of loss functional. Third, this part is devoted to applying and proving in detail risk-neutral control problem to solve risk-sensitive control problem. Then, we give and prove the relationship between the G-expected exponential utility and the G-quadratic backward stochastic differential equation. Lastly, applying the obtained results to solve a Merton-type problem with power utility.

Relevant Papers

The content of this thesis was the subject of the following papers:

1. M. Dassa and A. Chala, Stochastic maximum principle for optimal control problem under G-expectation utility, Random Operators and Stochastic Equations (ROSE), 2(30), 121-135 (2022).

2. M. Dassa and A. Chala, G-Stochastic Maximum Principle for Risk-Sensitive Control Problem and its Applications, (accepted).

3. M. Dassa and A. Chala, Applying G-Stochastic Maximum Principle for Risk-Neutral Control Problem to Solve Risk-Sensitive Control Problem, (Under re-

view).

International Communications

Several communications in control theory were done:

1. M. Dassa and A. Chala, G-Adjoint Equation for Stochastic Optimal Control, International Conference "Dynamic Control and Optimization" (DCO 2021), 3-5 of February, 2021, Aveiro, Portugal.

2. M. Dassa and A. Chala, Stability of Controlled Stochastic Differential Equations Driven by G-Brownian Motion, The First International Conference on Pure and Applied Mathematics (IC-PAM'21), May 26-27, 2021, Ouargla, Algeria.

3. M. Dassa and A. Chala, Pontryagin's Stochastic Maximum Principle for Stochastic Differential Equations driven by G-Brownian Motion, The 1st International Conference on Innovative Academic Studies, 10-13 September, 2022, Konya, Turkey.

National Communications

1. M. Dassa and A. Chala, Basic Theory of Sublinear Expectations, Study days in mathematics (LAM and LAMPO-2022), 12-14 december, Biskra, Algeria.

2. M. Dassa and A. Chala, Stochastic Control Systems Driven by G-Brownian Motion for the Multi-Dimensional Situation, The First National Applied Mathematics Seminar (1st-NAMS'23), 14-15 May, 2023, Biskra, Algeria.

Chapter 1

Preliminaries in G-Framework

In this introductory chapter, we introduce a few notations and basic results in the framework of sublinear expectation, G-expectation, G-Brownian motion and related G-stochastic calculus, which are required in the following chapters. Furthermore, the readers interested in more details on this topic are referred to [27, 28, 29, 30, 31, 32, 34], the book of Peng [33] and the references therein.

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω, such that $\mathcal{H}$ satisfies the following conditions:

1) $c \in \mathcal{H}$ for each constant c.

2) $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. We will treat elements of $\mathcal{H}$ as random variables.

1.1 Sublinear expectation

Definition 1.1 ***(Sublinear expectation)*** *A sublinear expectation* $\mathbb{E}$ *is a functional* $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ *satisfying the following properties for all* $X, Y \in \mathcal{H}$*:*

1) Monotonicity: If $X \leq Y$*, then* $\mathbb{E}(X) \leq \mathbb{E}(Y)$*.*

2) Constant preserving: $\mathbb{E}(c) = c$*, for each constant* c*.*

3) Sublinearity: $\mathbb{E}(X+Y) \leq \mathbb{E}(X) + \mathbb{E}(Y)$*, or* $\mathbb{E}(X) - \mathbb{E}(Y) \leq \mathbb{E}(X-Y)$*.*

4) Positive homogeneity: $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$ *for all* $\lambda \in \mathbb{R}_+$*.*

Remark 1.1 *The triple* $(\Omega, \mathcal{H}, \mathbb{E})$ *is called sublinear expectation space.*

Remark 1.2 *If it further satisfies* $\mathbb{E}(-X) = -\mathbb{E}(X)$ *for all* $X \in \mathcal{H}$*, then* $\mathbb{E}$ *is called a linear expectation.*

We will consider the space $\mathcal{H}$ of random variables having the following property: if $X_i \in \mathcal{H}$, $i = 1, ..., n$ then

$$\varphi(X_1, ..., X_n) \in \mathcal{H}, \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of all continuous real-valued functions φ defined on $\mathbb{R}^n$ satisfying the following local Lipschitz condition

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)\,|x - y| \quad \forall x, y \in \mathbb{R}^n,$$

where the constant $C > 0$ and the integer $m \in \mathbb{N}$ depend on φ.

In this thesis, we mainly use $C_{l,Lip}(\mathbb{R}^n)$ for convenience of techniques. In practice, $C_{l,Lip}(\mathbb{R}^n)$ can be replaced by $C_{b,Lip}(\mathbb{R}^n)$ the space of bounded and Lipschitz continuous functions.

Theorem 1.1 *(**Representation of a sublinear expectation**) Let* $\mathbb{E}$ *be a functional defined on a linear space* $\mathcal{H}$ *satisfying subadditivity and positive homogeneity. Then there exists a family of linear functionals* $\mathbb{E}^\theta : \mathcal{H} \to \mathbb{R}$*, indexed by* $\theta \in \Theta$*, such that,*

$$\mathbb{E}(X) = \sup_{\theta \in \Theta} \mathbb{E}^\theta(X), \text{ for } X \in \mathcal{H}, \tag{1.1}$$

where

$$\mathbb{E}^\theta(X) = \int_\Omega X(w)\, d\mathbb{P}_\theta(w).$$

Moreover, for each $X \in \mathcal{H}$*, there exists* $\theta_X \in \Theta$ *such that* $\mathbb{E}(X) = \mathbb{E}_{\theta_X}(X)$*.*

Furthermore, if $\mathbb{E}$ is a sublinear expectation, then the corresponding $\mathbb{E}_\theta$ is a linear expectation.

Proof. The proof of this theorem can be found in Peng [33]. ■

Theorem 1.2 ***(Robust Daniell-Stone Theorem [33])*** *Assume that* $(\Omega, \mathcal{H}, \mathbb{E})$ *is a sublinear expectation space satisfying*

$$\mathbb{E}[X_i] \to 0, \ \ as\ i \to \infty, \tag{1.2}$$

for each sequence $\{X_i\}_{i=1}^{\infty}$ *of random variables in* $\mathcal{H}$ *such that* $X_i(\omega) \downarrow 0$ *for each* $\omega \in \Omega$*. Then there exists a family of probability measures* $\{\mathbb{P}_\theta\}_{\theta\in\Theta}$ *defined on the measurable space* $(\Omega, \sigma(\mathcal{H}))$ *such that*

$$\mathbb{E}[X] = \max_{\theta\in\Theta} \int_\Omega X(\omega)d\mathbb{P}_\theta, \ \ for\ each\ X \in \mathcal{H}. \tag{1.3}$$

Here $\sigma(\mathcal{H})$ *is the smallest* σ*-algebra generated by* $\mathcal{H}$*.*

We will express the notions of distribution and independence of random vectors using test functions in $C_{l,Lip}(\mathbb{R}^n)$.

Definition 1.2 ***(Distribution)*** *For an* n*-dimensional random vector* $X = (X_1, ..., X_n)$ *for* $X_i \in \mathcal{H}$*,* $i = 1, 2, \cdots, n$*, set*

$$F_X(\varphi) := \mathbb{E}(\varphi(X)) : \varphi \in C_{l,Lip}(\mathbb{R}^n) \longmapsto \mathbb{R}.$$

The triplet $(\mathbb{R}^n, C_{l,Lip}(\mathbb{R}^n), F_X)$ *forms a nonlinear expectation space.* F_X *is called the distribution of random vector* X *on* $(\Omega, \mathcal{H}, \mathbb{E})$*.*

In this case F_X is also a sublinear expectation. Furthermore, there exists a family of probability measures $\{F_X(\theta,.)\}_{\theta\in\Theta}$ defined on $(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$ such that

$$F_X[\varphi]=\sup_{\theta\in\Theta}\int_{\mathbb{R}^n}\varphi(x)F_X(\theta,dx),\ \text{for each}\ \varphi\in C_{l,Lip}(\mathbb{R}^n).$$

Thus $F_X[.]$ characterizes the uncertainty of the distributions of X.

Definition 1.3 *Let X_1 and X_2 be two n-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1,\mathcal{H}_1,\mathbb{E}_1)$ and $(\Omega_2,\mathcal{H}_2,\mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X_1\overset{d}{=}X_2$, if*

$$\mathbb{E}_1[\varphi(X_1)]=\mathbb{E}_2\left[\varphi(X_2)\right]\ \textit{for all}\ \varphi\in C_{b,Lip}(\mathbb{R}^n).$$

We say that the distribution of X_1 is stronger than that of X_2 if

$$\mathbb{E}_1[\varphi(X_1)]\geq\mathbb{E}_2\left[\varphi(X_2)\right],\ \textit{for each}\ \varphi\in C_{b,Lip}(\mathbb{R}^n).$$

The distribution of $X\in\mathcal{H}$ has the following typical parameters:

$$\overline{\mu}:=\mathbb{E}[X],\underline{\mu}:=-\mathbb{E}[-X],\overline{\sigma}^2=\mathbb{E}[X^2],\underline{\sigma}^2=-\mathbb{E}[-X^2].$$

The subsets $\left[\underline{\mu},\overline{\mu}\right]$ and $[\underline{\sigma}^2,\overline{\sigma}^2]$ characterizes the mean-uncertainty and the variance-uncertainty of X.

Proposition 1.1 *(See **[33]**) Let $(\Omega,\mathcal{H},\mathbb{E})$ be a sublinear expectation space and X,Y be two random variables such that $\mathbb{E}[Y]=-\mathbb{E}[-Y]$, i.e., Y has no mean-uncertainty. Then we have*

$$\mathbb{E}[X+\alpha Y]=\mathbb{E}[X]+\alpha\mathbb{E}[Y]\ \textit{for}\ \alpha\in\mathbb{R}.$$

In particular, if $\mathbb{E}[Y] = \mathbb{E}[-Y] = 0$*, then*

$$\mathbb{E}[X + \alpha Y] = \mathbb{E}[X].$$

Definition 1.4 ***(Independence)*** *For two random vectors* $Y = (Y_1, , ..., Y_m)$ *for* $Y_j \in \mathcal{H}$ *and* $X = (X_1, ..., X_n)$ *for* $X_i \in \mathcal{H}$*, if for all* $\varphi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^m)$

$$\mathbb{E}\left[\varphi(X, Y)\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi(x, Y)\right]_{x=X}\right].$$

We say that Y *is independent from* X*.*

Remark 1.3 *The situation "*Y *is independent of* X*" often appears when* Y *occurs after* X*, thus a robust expectation should take the information of* X *into account.*

1.1.1 G-Normal distribution

In what follows, we introduce a new definition of a special type of distribution, which plays the same role as the classical normal distribution in probability and statistics theory.

Definition 1.5 *Let* $(\Omega, \mathcal{H}, \mathbb{E})$ *be a sublinear expectation space, and* $X = (X_1, ..., X_n)$ *be* n*-dimensional random vector, is called G-normally distributed with zero mean if for each* $a, b \geq 0$*, we have*

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\overline{X}$ *is an independent copy of* X*, i.e.,* $\overline{X} \overset{d}{=} X$ *and* $\overline{X}$ *independent of* X*.*

Definition 1.6 *The distribution of* X *is characterized by sublinear function* $G : \mathbb{S}_n \to \mathbb{R}$ *defined by*

$$G\left(A\right) = G_X\left(A\right) := \frac{1}{2}\mathbb{E}\left(\langle AX, X\rangle\right), \text{ for all } A \in \mathbb{S}_n, \tag{1.4}$$

where $\mathbb{S}_n$ denotes the collection of $n \times n$ symmetric matrices.

The G-normal distribution is characterized by a nonlinear heat equation as follows.

Proposition 1.2 *An n-dimensional random vector $X = (X_1, ..., X_n)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is G-normally distributed if and only if for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, the function u defined by*

$$u(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], (t,x) \in [0,\infty) \times \mathbb{R}.$$

is the unique viscosity solution of the following G-heat equation:

$$\begin{cases} \partial_t u - G(D_x^2 u) = 0; \\ u(0,x) = \varphi(x), \end{cases}$$

where $D_x^2 u = \left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right)_{i,j=1}^n$ is the Hessian matrix of u and the function $G : \mathbb{S}_n \to \mathbb{R}$ is a monotonic, sublinear mapping on $\mathbb{S}_n$, which implies that there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_n^+$ such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} tr\,[AB]\,,$$

$\mathbb{S}_n^+$ denotes the collection of nonnegative elements in $\mathbb{S}_n$.

Definition 1.7 ***(Stochastic process)*** *Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. $(X_t)_{t\geq 0}$ is called an n-dimensional stochastic process if for each $t \geq 0$, X_t is an n-dimensional random vector in $\mathcal{H}$.*

1.1.2 G-Brownian motion

Now we introduce the notion of G-Brownian motion, which is Brownian motion related to G-normal distribution in a space of a sublinear expectation, for more details see Peng

[31].

Definition 1.8 ***(G-Brownian motion)*** *A process $(B_t(w))_{t\geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called Brownian motion under $\mathbb{E}$ (or G-Brownian motion), if for each $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_n < t < \infty$, $B_{t_1}, B_{t_2}, ..., B_{t_n} \in \mathcal{H}$ and the following properties are satisfied:*
1) $B_0(w) = 0$.
2) For each $t, s \geq 0$, the increments satisfy $B_{t+s} - B_t \overset{d}{=} B_s$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$.
3) $|B_t|^3 \in \mathcal{H}$ and $\dfrac{\mathbb{E}\left(|B_t|^3\right)}{t} \to 0$ as $t \downarrow 0$.

Moreover, if $\mathbb{E}(B_t) = \mathbb{E}(-B_t) = 0$, then $(B_t)_{t\geq 0}$ is called a symmetric G-Brownian motion.

Remark 1.4 *Here the letter G indicates that the B_t is G-normal distributed with*

$$\begin{aligned} G(\alpha) &:= \frac{1}{2}\mathbb{E}(\langle \alpha X, X\rangle) = \frac{1}{2}\mathbb{E}\left(\alpha X^2\right) \\ &= \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right), \end{aligned}$$

where $\overline{\sigma}^2 := \mathbb{E}(X^2) \geq -\mathbb{E}(-X^2) := \underline{\sigma}^2$, for every $\alpha \in \mathbb{R}$. Notice that $X^+ = X \vee 0$ and $X^- = (-X)^+$ and

$$a \wedge b = \min\{a, b\} = \frac{1}{2}(a + b - |a - b|),\ a \vee b = -[(-a) \wedge (-b)].$$

1.1.3 G-expectation

Definition 1.9 *Let $\Omega = C_0(\mathbb{R}^+, \mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in[0,T]}$ vanishing at the origin, equipped with the distance (the uniform convergence on compact* *intervals topology)*

$$\rho(\omega^{(1)},\omega^{(2)}) = \sum_{i=1}^{\infty} 2^{-i} \left(\left(\max_{t\in[0,i]} \left|\omega^{(1)} - \omega^{(2)}\right| \right) \wedge 1 \right), \quad \omega^{(1)},\omega^{(2)} \in \Omega.$$

The canonical process $(B_t)_{t\geq 0}$ *is defined by* $B_t(\omega) := \omega_t$, *for* $(t,\omega) \in [0,\infty) \times \Omega$ *and denote by* $\mathcal{B}(\Omega)$ *the Borel* σ*-algebra of* Ω. *Let*

$$\mathcal{H} = Lip(\Omega) := \left\{ \varphi(\omega_{t_1},...,\omega_{t_n}) : t_1,...,t_n \in [0,\infty) \text{ and } \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n}) \text{ for all } n \in \mathbb{N} \right\}.$$

A G-expectation $\mathbb{E}_G$ *is a sublinear expectation on* $(\mathcal{H},\Omega)$ *defined as follows: for* $X \in Lip(\Omega)$ *of the form*

$$X = \varphi(B_{t_1} - B_{t_0},...,B_{t_n} - B_{t_{n-1}}), \quad \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n}) \text{ and } 0 = t_0 < t_1 < ... < t_n < \infty,$$

we set

$$\mathbb{E}_G[X] := \mathbb{E}\left[\varphi\left(\xi_1\sqrt{t_1 - t_0},...,\xi_n\sqrt{t_n - t_{n-1}}\right)\right],$$

where $\xi_1,...,\xi_n$ *are d-dimensional random variables on the sublinear expectation space* $(\widetilde{\Omega},\widetilde{\mathcal{H}},\mathbb{E})$ *such that for each* $i = 1,...,n$, ξ_i *is G-normally distributed and independent of* $(\xi_1,...,\xi_{i-1})$. *We denote by* $\mathbb{L}_G^p(\Omega)$ *for* $p \geq 1$, *the completion of Lip*(Ω) *under the norm* $\|X\|_{\mathbb{L}_G^p} := (\mathbb{E}_G[|X|^p])^{1/p}$. *Then it is easy to check that* $\mathbb{E}_G$ *is also a sublinear expectation on the space* $(\Omega,\mathbb{L}_G^2(\Omega))$, *where* $\mathbb{L}_G^2(\Omega)$ *is a Banach space and the canonical process* $B_t(\omega) := \omega_t$ *is a G-Brownian motion.*

Following [33] and [8], we introduce the notations: for each $t \in [0,\infty)$

1) $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$, $\mathcal{F}_t := \mathcal{B}(\Omega_t)$,

2) $L^0(\Omega)$ is the space of all $\mathcal{B}(\Omega)$-measurable real functions,

3) $L^0(\Omega_t)$ is the space of all $\mathcal{B}(\Omega_t)$-measurable real functions,

4) $Lip(\Omega_t) := Lip(\Omega) \cap L^0(\Omega_t)$ and $\mathbb{L}_G^p(\Omega_t) := \mathbb{L}_G^p(\Omega) \cap L^0(\Omega_t)$,

5) Let $\mathbb{M}_G^{0,p}(0,T)$ be the collection of processes in the following form: for a given partition of the set $\{t_0 < ... < t_N\}$ of $[0,T]$

$$\eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) 1_{[t_i,t_{i+1}[}(t),$$

where $\xi_i \in \mathbb{L}_G^p(\Omega_{t_i})$, we denote by $\mathbb{M}_G^p(0,T), \mathbb{H}_G^p(0,T), \mathbb{S}_G^p(0,T)$ the completion of $\mathbb{M}_G^{0,p}(0,T)$ under the norm

$$\|\eta\|_{\mathbb{M}_G^p} = \left(\mathbb{E}_G\left[\int_0^T |\eta(t)|^p\, dt\right]\right)^{1/p},$$

$$\|\eta\|_{\mathbb{H}_G^p} = \left(\mathbb{E}_G\left[\int_0^T |\eta(t)|^2\, dt\right]^{\frac{p}{2}}\right)^{1/p},$$

$$\|Z\|_{\mathbb{S}_G^p} := \left(\mathbb{E}_G\left[\sup_{t\in[0,T]} |Z_t|^p\right]\right)^{1/p}$$

respectively.

7) Set

$$\mathcal{P} = \left\{\mathbb{P} : \mathbb{P} \text{ is a probability on } (\Omega, \mathcal{B}(\Omega)), \mathbb{E}^{\mathbb{P}}(X) \leq \mathbb{E}(X) \text{ for } X \in \mathbb{L}_G^1(\Omega)\right\}.$$

The set $\mathcal{P}$ is relatively weakly compact and thus its completion $\overline{\mathcal{P}}$ is weakly compact. Therefore, we can naturally define the Choquet capacity $\mathcal{C}(.)$ by $\mathcal{C}(A) := \sup_{P\in\overline{\mathcal{P}}} P(A)$, $A \in \mathcal{B}(\Omega)$ and introduce the notions of Choquet capacity and quasi-sure.

Definition 1.10 ***(Upper probability)*** *The set function $\mathcal{C}$ is called an upper probability associated with $\mathcal{P}$.*

One can easily verify the following theorem.

Theorem 1.3 ***(Choquet capacity)*** *The upper probability $\mathcal{C}(.)$ is a Choquet capacity, i.e. (see [33]):*

1) $0 \leq \mathcal{C}(A) \leq 1, \forall A \subset \Omega$.

2) If $A \subset B$, *then* $\mathcal{C}(A) \leq \mathcal{C}(B)$.

3) If $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\Omega)$, *then* $\mathcal{C}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathcal{C}(A_n)$.

4) If $(A_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{B}(\Omega) : A_n \uparrow A = \cup_{n=1}^{\infty} An$, *then* $\mathcal{C}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathcal{C}(A_n)$.

Definition 1.11 *(**Quasi-sure**) A set* $A \in \mathcal{B}(\Omega)$ *is a* $\mathcal{C}$*-polar, if* $\mathcal{C}(A) = 0$. *A property is said to hold "quasi-surely" (q.s.) with respect to* $\mathcal{C}$, *if it holds true outside a* $\mathcal{C}$*-polar set.*

1) We say that a random variable Y *is a version of* X *if* $X = Y$ *q.s.*

2) A random variable X *is said to be quasi-continuous (q.c. in short), if for every* $\epsilon > 0$ *there exists an open set* O *such that* $\mathcal{C}(O) < \epsilon$ *and* $X|_{O^c}$ *is continuous.*

Theorem 1.4 *(Theorem 18 and 25 in [8]) For each* $p \geq 1$ *one has*

$$\mathbb{L}_G^p(\Omega) = \left\{ X \in L^0(\Omega) : X \textit{ has a q.c. version and } \lim_{n\to\infty} \mathbb{E}_G\left[|X|^p 1_{\{|X|>n\}}\right] = 0 \right\}.$$

Definition 1.12 ***(Conditional G-expectation)*** *For each random variable* $X \in Lip(\Omega_T)$ *of the following form:*

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) : t_1, ..., t_n \in [0, T],$$

the conditional G-expectation $\mathbb{E}_G\left[. \,|\Omega_{t_i}\right], i = 1, ..., n$, *is defined as follows*

$$\begin{aligned}\mathbb{E}_G\left[X \,|\Omega_{t_i}\right] &= \mathbb{E}_G\left[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \,|\Omega_{t_i}\right] \\ &= \widetilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_i} - B_{t_{i-1}}),\end{aligned}$$

where

$$\widetilde{\varphi}(x_1, ..., x_i) = \mathbb{E}\left[\varphi(x_1, ..., x_i, \xi_{i+1}\sqrt{t_{i+1} - t_i}, ..., \xi_n\sqrt{t_n - t_{n-1}})\right].$$

If $t \in (t_i, t_{i+1})$, then the conditional G-expectation $\mathbb{E}_{t_i}[X]$ could be defined by reformulating X as

$$X = \widehat{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_t - B_{t_i}, B_{t_{i+1}} - B_t, ..., B_{t_n} - B_{t_{n-1}}), \quad \widehat{\varphi} \in C_{l,Lip}(\mathbb{R}^{n+1}).$$

For each given $t \in [0,T]$, the conditional G-expectation $\mathbb{E}_G[.\,|\Omega_t] : Lip(\Omega_T) \to Lip(\Omega_t)$ can be also extended as a mapping $\mathbb{E}_G[\cdot|\Omega_t] : \mathbb{L}_G^1(\Omega_T) \to \mathbb{L}_G^1(\Omega_t)$ and satisfies the following properties:

1) If $\xi, \eta \in \mathbb{L}_G^1(\Omega_t)\,; \xi \leq \eta$, then $\mathbb{E}_G[\xi\,|\Omega_s] \leq \mathbb{E}_G[\eta\,|\Omega_s]$ for all $s \leq t$;

2) If $\xi \in \mathbb{L}_G^1(\Omega_t)$ and $\eta \in \mathbb{L}_G^1(\Omega)$ then $\mathbb{E}_G[\xi + \eta\,|\Omega_t] = \xi + \mathbb{E}_G[\eta\,|\Omega_t]$;

3) $\mathbb{E}_G[\xi + \eta\,|\Omega_t] \leq \mathbb{E}_G[\xi\,|\Omega_t] + \mathbb{E}_G[\eta\,|\Omega_t]$;

4) If $\xi \in L^0(\Omega_t)$ is bounded, $\eta \in \mathbb{L}_G^1(\Omega)$, then $\mathbb{E}_G[\xi\eta\,|\Omega_t] = \xi^+\mathbb{E}_G[\eta\,|\Omega_t] + \xi^-\mathbb{E}_G[-\eta\,|\Omega_t]$;

5) If $\xi \in \mathbb{L}_G^1(\Omega)$ then $\mathbb{E}_G[\mathbb{E}_G[\xi\,|\Omega_t]\,|\Omega_s] = \mathbb{E}_G[\xi\,|\Omega_{s\wedge t}]$.

Definition 1.13 *(**G-martingale**) A process $(X_t)_{t\in[0,T]}$ with $X_t \in \mathbb{L}_G^1(\Omega_t)\,, 0 \leq t \leq T$, is called a G-martingale (respectively, G-supermartingale, G-submartingale) if for all $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}_G[X_t\,|\Omega_s] = X_s \quad (\textit{respectively}, \ \leq Xs, \ \geq Xs).$$

The process $(X_t)_{t\in[0;T]}$ is called symmetric G-martingale if $-X$ is also a G-martingale.

1.2 G-stochastic calculus

1.2.1 Itô's integral with respect to G-Brownian motion

In this part, we discuss the stochastic integrals with respect to the G-Brownian motion and its quadratic variation. In [33] Chapter 03, Peng introduces Itô's type stochastic integral with respect to the G-Brownian motion.

Definition 1.14 *For $\eta \in \mathbb{M}_G^{0,2}(0,T)$, we can define*

$$\mathcal{I}(\eta) = \int_0^T \eta_s dB_s := \sum_{i=1}^{N-1} \xi_i(w)\left(B_{t_{i+1}} - B_{t_i}\right).$$

Remark 1.5 *The mapping $\mathcal{I} : \mathbb{M}_G^{0,2}(0,T) \to \mathbb{L}_G^2(\Omega_T)$ can be continuously extended to $\mathcal{I} : \mathbb{M}_G^2(0,T) \to \mathbb{L}_G^2(\Omega_T)$, then for every $\eta \in \mathbb{M}_G^2(0,T)$.*
Moreover, this extension of $\mathcal{I}$ satisfies, for each $\eta \in \mathbb{M}_G^2(0,T)$

$$\mathbb{E}_G(\mathcal{I}(\eta)) = 0, \text{ and } \mathbb{E}_G\left(\mathcal{I}^2(\eta)\right) \leq \overline{\sigma}^2 \mathbb{E}_G\left(\int_0^T \eta^2(t)\, dt\right). \tag{1.5}$$

Proposition 1.3 *Let $\eta, \theta \in \mathbb{M}_G^2(0,T)$ and let $0 \leq s \leq r \leq t \leq T$. Then we have*
1) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.
2) $\int_s^t (\alpha\eta_u + \theta_u)\, dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, if α is bounded and in $\mathbb{L}_G^1(\Omega_s)$.
3) $\mathbb{E}_G\left[X + \int_r^T \eta_u dB_u \,|\Omega_s\right] = \mathbb{E}_G[X\,|\Omega_s]$ for all $X \in \mathbb{L}_G^1(\Omega)$.

1.2.2 Quadratic variation process of G-Brownian motion

Dissimilar to the classical theory, the quadratic variation of G-Brownian motion B is not always a deterministic process, and can be formulated in $\mathbb{L}_G^2(\Omega_{t_i})$ by

$$\begin{aligned}
\langle B \rangle_t &:= \lim_{N\to\infty} \sum_{i=1}^{N-1} \left(B_{t_{i+1}} - B_{t_i}\right)^2 \\
&= \lim_{N\to\infty} \sum_{i=1}^{N-1} \left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \lim_{N\to\infty} \sum_{i=1}^{N-1} 2B_{t_i}^2\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) \\
&= B_t^2 - 2\int_0^t B_s dB_s.
\end{aligned}$$

$(\langle B \rangle_t)_{t\geq 0}$ is an increasing process with $B_0 = 0$. We call it the quadratic variation process of the G-Brownian motion B. It characterizes the part of statistic uncertainty of G-Brownian motion. We have the following isometry.

Proposition 1.4 *Let $\eta \in \mathbb{M}_G^2(0,T)$. Then*

$$\mathbb{E}_G\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \mathbb{E}_G\left[\int_0^T \eta_t^2 d\langle B\rangle_t\right].$$

Proof. We refer the readers to the Proposition 3.4.5 conducted by Peng [33]. ■

All distributional uncertainty of the G-Brownian motion B is concentrated in $\langle B\rangle$. Moreover, $\langle B\rangle$ itself is a typical process with mean uncertainty and variance-uncertainty. We have the upper bound $\mathbb{E}_G\left[\langle B\rangle_t^2\right] \leq 10\overline{\sigma}^4 t^2$.

1.2.3 G-Itô's formula

We are going now to give the general form of G-Itô's formula. We start with

$$X_t^\nu = X_0^\nu + \int_0^t b^\nu(t)dt + \int_0^t h_{\mathfrak{i}\mathfrak{j}}^\nu(t)\, d\langle B\rangle_t^{\mathfrak{i}\mathfrak{j}} + \int_0^t \sigma_{\mathfrak{i}}^\nu(t)\, dB_t^{\mathfrak{i}}, \quad \nu = 1,\ldots,n, \mathfrak{i},\mathfrak{j} = 1,\ldots,d.$$

Theorem 1.5 *(See [33]) Let X be a C^2-function on $\mathbb{R}^n$ such that $\partial^2_{x^\mu x^\nu}\varphi$ satisfies polynomial growth condition for $\mu,\nu = 1,\ldots,n$. Let b^ν, $\sigma_{\mathfrak{i}}^\nu$ and $h_{\mathfrak{i}\mathfrak{j}}^\nu, \nu = 1,\ldots,n, \mathfrak{i},\mathfrak{j} = 1,\ldots,d$ be bounded processes in $\mathbb{M}_G^2(0,T)$. Then for each $t \geq 0$ we have in $\mathbb{L}_G^2(\Omega_t)$*

$$\begin{aligned}\varphi(X_t) - \varphi(X_s) &= \int_s^t \partial_{x^\nu}\varphi(X_u)\,\sigma_{\mathfrak{i}}^\nu dB_u^{\mathfrak{i}} + \int_s^t \partial_{x^\nu}\varphi(X_u)\, b^\nu(u)du \\ &\quad + \int_s^t \left[\partial_{x^\nu}\varphi(X_u)\, h_{\mathfrak{i}\mathfrak{j}}^\nu(u) + \frac{1}{2}\partial^2_{x^\mu x^\nu}\varphi(X_u)\,\sigma_{\mathfrak{i}}^\mu(u)\,\sigma_{\mathfrak{j}}^\nu(u)\right] d\langle B\rangle_u^{\mathfrak{i}\mathfrak{j}}.\end{aligned}$$

1.2.4 G-martingales representation theorem

In contrast to the classical martingale representation, the G-martingale is decomposed into two parts: the G-Itô's type integral part $X_t = \int_0^t Z_s dB_s$, which is called symmetric G-martingale, in the sense that $-X_t$ is still a G-martingale; the decreasing G-martingale part K, which vanishes in the classical theory. However, it plays a significant role in this new context (see [36, 33]).

Theorem 1.6 ***(G-Martingales representation theorem)*** *Let ξ be in $\mathbb{L}_G^2(\Omega_T)$. Then the martingale $\mathbb{E}_G[\xi\,|\mathcal{F}_t]$ has a continuous quasi-modification $Y \in \mathbb{S}_G^2(0,T)$ given by*

$$Y_t = \mathbb{E}_G[\xi] + \int_0^t Z_s dB_s + K_t,$$

where $Z \in \mathbb{H}_G^2(0,T)$ and K is a non-increasing continuous G-martingale with $K_0 = 0$ and $K_T \in \mathbb{L}_G^2(\Omega_T)$. Moreover, the above decomposition is unique.

Proof. The proof might be found in [36]. ■

1.2.5 G-backward stochastic differential equation

In this subsection, we give a short introduction to G-backward stochastic differential equations (G-BSDEs in short) and their solutions which are a key tool to consider the maximum principle. We consider the following G-BSDE

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)\,ds + \int_t^T g(s, Y_s, Z_s)\,d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t). \quad (1.6)$$

For simplicity, we denote by $\mathcal{G}_G^2(0,T)$ the collection of process (Y, Z, K) such that $Y \in \mathbb{S}_G^2(0,T), Z \in \mathbb{H}_G^2(0,T), K$ is a non-increasing $G-$martingale with $K_0 = 0$ and $K_T \in \mathbb{L}_G^2(\Omega_T)$.

Theorem 1.7 *Under the condition of Lipschitz on the coefficients f and g with respect to (Y, Z), then the G-BSDE (1.6) has unique strong solution $(Y, Z, K) \in \mathcal{G}_G^2(0,T)$.*

Proof. See the paper of Hu et al [14]. ■

Lemma 1.1 ***(Integration by parts)*** *Suppose*

$\mathbb{E}_G\left[(X_T^i)^2\right] < \infty$ *for* $i = 1, 2$ *where*

$$X_t^i = X_0^i + \int_0^t b^i(s, X_s)\,ds + \int_0^t h_{\mathrm{ij}}^i(s, X_s)\,d\langle B\rangle_s^{\mathrm{ij}} + \int_0^t \sigma_{\mathrm{i}}^i(s, X_s)\,dB_s^{\mathrm{i}}.$$

Then

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_s^1 dX_s^2 + \int_0^t X_s^2 dX_s^1 + \left\langle X^1, X^2\right\rangle_t.$$

In this case the quadratic covariation is

$$\left\langle X^1, X^2\right\rangle_t = \int_0^t \sigma_{\mathrm{i}}^1(s, X_s)\,\sigma_{\mathrm{j}}^2(s, X_s)^{\mathcal{T}}\,d\langle B\rangle_s^{\mathrm{ij}}.$$

The following lemmas of Burkholder-Davis-Gundy (BDG in short) are essential in the field of stochastic control, see Gao [12].

Lemma 1.2 ***(BDG1)*** *For each* $p \geq 1$ *and* $\eta \in \mathbb{M}_G^2(0, T)$, *we have the following inequality*

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\int_0^t \eta_s d\langle B\rangle_s\right|^p\right] \leq \overline{\sigma}^{2p} T^{p-1} \int_0^T \mathbb{E}_G(|\eta_s|^p)\,ds. \tag{1.7}$$

Lemma 1.3 ***(BDG2)*** *For each* $p \geq 2$ *and* $\eta \in \mathbb{M}_G^2(0, T)$, *Then there exist some constant* C_p *depending only on* p *and* T, *such that*

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\int_0^t \eta_s dB_s\right|^p\right] \leq C_p T^{\frac{p}{2}-1} \int_0^T \mathbb{E}_G(|\eta_s|^p)\,ds. \tag{1.8}$$

1.2.6 Girsanov's type transformation for G-expectation

In the next, we present a result, which is called the G-Girsanov's Theorem, it plays an important role in the application, especially in economics and optimal control to change a G-Brownian motion with a drift to a G-Brownian motion under the transformation of

G-expectation. In G-Girsanov's theorem application, we can visit the papers [45, 15]. In the application of G-Itô's calculus, G-Girsanov's theorem is used frequently since it transforms a class of processes to Brownian motion with an equivalent probability measure transformation.

Assumption 1.1 *There exists an* $\varepsilon_0 > 0$ *such that*

$$\mathbb{E}_G\left[\exp\left\{\left(\frac{1}{2}+\varepsilon_0\right)\int_0^T Z^2(s,\omega)\,d\langle B\rangle_s\right\}\right]<\infty. \tag{1.9}$$

Here, $Z(s,\omega)$ *is such that the integral exists under the G-framework.*

Define

$$\xi(B_t) := \exp\left\{\int_0^t Z(s,\omega)\,dB_s - \frac{1}{2}\int_0^t Z^2(s,\omega)\,d\langle B\rangle_s\right\}.$$

For any $X \in Lip(\Omega_T)$, introduce

$$\widetilde{\mathbb{E}}_G[X] = \mathbb{E}_G[\xi(B_T)X].$$

$$\widetilde{\mathbb{E}}_G[X|\mathcal{F}_t] = [\xi(B_t)]^{-1}\,\mathbb{E}_G[\xi(B_T)X|\mathcal{F}_t].$$

Theorem 1.8 *If Assumption* 1.1 *holds, then* $B_t - \int_0^t Z(s,\omega)\,d\langle B\rangle_s$ *is a G-Brownian motion under* $\widetilde{\mathbb{E}}_G$*.*

Proof. The proof might be found in [45]. ■

Chapter 2

G-Stochastic Maximum Principle for Risk-Neutral Control Problem

In this chapter, the system is governed by the nonlinear n-dimensional controlled G-stochastic differential equation and the cost functional has a volatility term, that is a generalization of our results in [7]. For all $0 \leq t \leq T, 1 \leq \mathfrak{i}, \mathfrak{j} \leq d$

$$\begin{cases} dx_t = b\left(t, x_t, u_t\right) dt + h_{\mathfrak{ij}}\left(t, x_t, u_t\right) d\left\langle B\right\rangle_t^{\mathfrak{ij}} + \sigma_{\mathfrak{i}}\left(t, x_t, u_t\right) dB_t^{\mathfrak{i}}, \\ x\left(0\right) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $b, h_{\mathfrak{ij}}$ and $\sigma_{\mathfrak{i}}$ are uniformly Lipschitz, x_0 is the initial state, B_t is a G-Brownian motion that satisfies

$$\left\langle B\right\rangle_t = B_t^2 - 2\int_0^t B_s dB_s, \qquad 0 \leq t \leq T.$$

The cost functional to be minimized over the class of admissible controls has the form

$$\mathcal{J}\left(u\right) = \mathbb{E}_G\left[g\left(x_T^u\right) + \int_0^T l\left(t, x_t^u, u_t\right) dt + \int_0^T m_{\mathfrak{ij}}\left(t, x_t, u_t\right) d\left\langle B\right\rangle_t^{\mathfrak{ij}}\right].$$

A stochastic control $\widehat{u}$ is called optimal if it solves $\mathcal{J}(\widehat{u}) = \inf_{u \in U_{ad}} \mathcal{J}(u)$.

Our objective in this chapter is to establish necessary and sufficient optimality conditions for this model. The idea is to use the fact that the set of admissible controls U_{ad} is convex and all the terms of (2.1) are controlled. Then, we establish necessary optimality conditions by using the convex perturbation method. More precisely, if we denote by $\widehat{u}$ an optimal control and u is an arbitrary element of U_{ad}, then with a sufficiently small $\theta > 0$ and for each $t \in [0, T]$, we can define a perturbed control as follows $u_t^\theta = \widehat{u}_t + \theta (u_t - \widehat{u}_t)$. By using the fact that the coefficients b, h and σ are uniformly Lipschitz with respect to (x, u), then the G-stochastic maximum principle is obtained directly in the global form.

This chapter is organized as follows. In section 01, we formulate the problem and give the various assumptions used throughout this chapter. In section 02, we study some estimations of the solution of the G-SDE. In section 03, we introduce in detail the adjoint process and adjoint equation. In section 04, we give our first and second main results, the necessary and sufficient optimality conditions of optimality for the G-SDE. In the last section, we apply the necessary and sufficient G-stochastic maximum principle to the linear-quadratic problem.

2.1 Control problem under G-expectation

Let T be a strictly positive real number and let U be a nonempty convex subset of $\mathbb{R}^n$.

Definition 2.1 *An admissible control u is $\mathcal{F}_t-$adapted process with valued in U, such that $u \in \mathbb{M}_G^2(0, T)$. We denote by U_{ad} the set of all admissible controls.*

For any admissible control $u \in U_{ad}$ and initial state $x_0 \in \mathbb{R}^n$, we consider the following n-dimensional progressive SDE driven by the d-dimensional G-Brownian motion $B =$

$(B_1, ..., B_d)$ for each given $0 \leq t \leq T < \infty$

$$\begin{cases} dx_t & = & b(t, x_t, u_t)\, dt + h_{\mathrm{ij}}(t, x_t, u_t)\, d\langle B\rangle_t^{\mathrm{ij}} + \sigma_{\mathrm{i}}(t, x_t, u_t)\, dB_t^{\mathrm{i}}, \\ x(0) & = & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.2}$$

where $\langle B\rangle^{\mathrm{ij}} = \left\langle B^{\mathrm{i}}, B^{\mathrm{j}}\right\rangle$ is the cross-variation process of B, for $1 \leq \mathrm{i}, \mathrm{j} \leq d$. We recall the following assumptions.

Assumption 2.1 *We will work under the following standard assumptions:*

(H1) $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $h_{\mathrm{ij}} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ *and* $\sigma_{\mathrm{i}} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ *are given functions satisfying* $b(., x, u)$, $h_{\mathrm{ij}}(., x, u)$, *and* $\sigma_{\mathrm{i}}(., x, u) \in \mathbb{M}_G^2(0,T)$ *for each* $x \in \mathbb{R}^n$ *and* $u \in U$.

(H2) There exists constant κ *such that* $|\varphi(t, x, u) - \varphi(t, y, v)| \leq \kappa(|x-y| + |u-v|)$ *for each* $x, y \in \mathbb{R}^n$ *and* $u, v \in U$ *where* $\varphi := b, h_{\mathrm{ij}}, \sigma_{\mathrm{i}}$.

Theorem 2.1 *(See Peng [30]) Under the above assumptions, for every* $u \in U_{ad}$ *the equation* (2.2) *has unique strong solution* $x \in \mathbb{S}_G^2(0,T)$.

We define the criterion to be minimized, with terminal cost, under G-expectation type, as follows

$$\begin{aligned} \mathcal{J}(u) &= \mathbb{E}_G\left[g(x_T^u) + \int_0^T l(t, x_t^u, u_t)\, dt + \int_0^T m_{\mathrm{ij}}(t, x_t, u_t)\, d\langle B\rangle_t^{\mathrm{ij}}\right] \\ &= \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[g(x_T^u) + \int_0^T l(t, x_t^u, u_t)\, dt + \int_0^T m_{\mathrm{ij}}(t, x_t, u_t)\, d\langle B\rangle_t^{\mathrm{ij}}\right] \\ &= \sup_{\mathbb{P}\in\mathcal{P}} J^{\mathbb{P}}(u), \end{aligned} \tag{2.3}$$

where $l : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ and $m_{\mathrm{ij}} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$. Our objective is to minimize the functional $\mathcal{J}$ over U_{ad}. If $\widehat{u} \in U_{ad}$ is an optimal control, that is

$$\mathcal{J}(\widehat{u}) = \inf_{u \in U_{ad}} \mathcal{J}(u). \tag{2.4}$$

Remark 2.1 *The set U_{ad} of all admissible controls is convex.*

A control that solves the problem $\{(2.2),(2.3),(2.4)\}$ is called optimal. If an optimal control minimises the cost $\mathcal{J}$ over U_{ad} exists, we seek necessary as well as sufficient conditions for optimality checked by this control in the form of G-stochastic maximum principle.

We will use a method which consists in perturbing the optimal control $\widehat{u}$ as follows

$$u_t^{\theta}=\widehat{u}_t+\theta\left(u_t-\widehat{u}_t\right). \tag{2.5}$$

We denote by x_t^{θ} the trajectory of the system corresponding to u^{θ} as follows

$$\begin{cases} dx_t^{\theta}=b\left(t,x_t^{\theta},u_t^{\theta}\right)dt+h_{\mathrm{ij}}\left(t,x_t^{\theta},u_t^{\theta}\right)d\left\langle B\right\rangle_t^{\mathrm{ij}}+\sigma_{\mathrm{i}}\left(t,x_t^{\theta},u_t^{\theta}\right)dB_t^{\mathrm{i}}, \\ x^{\theta}\left(0\right)=x_0; \qquad\qquad 0\leq t\leq T. \end{cases} \tag{2.6}$$

Assumption 2.2 *We will work under the following assumptions:*

(H1) b, h_{ij}, σ_{i}, l and m_{ij} are continuously differentiable with respect to (x,u).

(H2) The derivatives of b, h_{ij}, σ_{i}, l and m_{ij} are uniformly bounded by $C\left(1+|x|+|u|\right)$.

(H3) g is continuously differentiable with respect to x and its derivative is bounded uniformly by $C\left(1+|x|\right)$.

Since U_{ad} is convex, we have the perturbed control $u^{\theta}\in U_{ad}$, hence u^{θ} is an admissible control and from the optimality of $\widehat{u}$, we have

$$\mathcal{J}\left(u^{\theta}\right)\geq\mathcal{J}\left(\widehat{u}\right).$$

Thus

$$\begin{aligned} &\mathbb{E}_G\left[g\left(x_T^{\theta}\right)+\int_0^T l\left(t,x_t^{\theta},u_t^{\theta}\right)dt+\int_0^T m_{\mathrm{ij}}\left(t,x_t^{\theta},u_t^{\theta}\right)d\left\langle B\right\rangle_t^{\mathrm{ij}}\right] \\ &-\mathbb{E}_G\left[g\left(\widehat{x}_T\right)+\int_0^T l\left(t,\widehat{x}_t,\widehat{u}_t\right)dt+\int_0^T m_{\mathrm{ij}}\left(t,\widehat{x}_t,\widehat{u}_t\right)d\left\langle B\right\rangle_t^{\mathrm{ij}}\right]\geq 0. \end{aligned}$$

In this case, using the property 03 of definition 1.1, we get the following first form of variational inequality

$$\begin{aligned}\mathbb{E}_G\Bigg[&\left(g\left(x_T^\theta\right)-g\left(\widehat{x}_T\right)\right)+\int_0^T\left(l\left(t,x_t^\theta,u_t^\theta\right)-l\left(t,\widehat{x}_t,\widehat{u}_t\right)\right)dt\\&+\int_0^T\left(m_{\mathrm{ij}}\left(t,x_t^\theta,u_t^\theta\right)-m_{\mathrm{ij}}\left(t,\widehat{x}_t,\widehat{u}_t\right)\right)d\left\langle B\right\rangle_t^{\mathrm{ij}}\Bigg]\geq 0.\end{aligned} \tag{2.7}$$

It is easy to show that (2.2) and (2.6) have a unique strong solution and in the next section we are going to prove some basic estimates about it.

2.2 Stability of controlled G-SDE

In this section, we prove the stability between the perturbed solution and the optimal solution of the controlled G-stochastic differential equation with respect to the control variable. We introduce the short-hand notations for the sake of simplicity: $\varrho\left(t,x_t,u_t\right)=\varrho\left(t\right),\varrho\left(t,x_t^\theta,u_t^\theta\right)=\varrho^\theta\left(t\right),\varrho\left(t,\widehat{x}_t,\widehat{u}_t\right)=\widehat{\varrho}\left(t\right)$, for all $\varrho=b,h_{\mathrm{ij}},\sigma_{\mathrm{i}},l$ and m_{ij}.

Lemma 2.1 *Let $\widehat{u}\in U_{ad}$ be an optimal control and $(\widehat{x})$ the corresponding trajectory. Then under the assumptions* 2.2*, we have*

$$\lim_{\theta\to 0}\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|x_t^\theta-\widehat{x}_t\right|^2\right]=0. \tag{2.8}$$

Proof. Let $\widehat{x}$, x^θ denote the solution of equations (2.2) and (2.6) respectively. By the property of the absolute value and as $\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{n} a_i^2$, we have

$$
\begin{aligned}
&\left|x_t^\theta - \widehat{x}_t\right|^2 \\
&\leq 6 \left[\left| \int_0^t \left(b\left(s, x_s^\theta, u_s^\theta\right) - b\left(s, \widehat{x}_s, u_s^\theta\right) \right) ds \right|^2 + \left| \int_0^t \left(b\left(s, \widehat{x}_s, u_s^\theta\right) - b\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) ds \right|^2 \right. \\
&+ \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) d\left\langle B \right\rangle_s^{\mathrm{ij}} \right|^2 + \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) d\left\langle B \right\rangle_s^{\mathrm{ij}} \right|^2 \\
&\left. + \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) dB_s^{\mathrm{i}} \right|^2 + \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) dB_s^{\mathrm{i}} \right|^2 \right].
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\sup_{t\in[0,T]} \left|x_t^\theta - \widehat{x}_t\right|^2 \leq 6 &\left[\sup_{t\in[0,T]} \left| \int_0^t \left(b\left(s, x_s^\theta, u_s^\theta\right) - b\left(s, \widehat{x}_s, u_s^\theta\right) \right) ds \right|^2 \right. \\
&+ \sup_{t\in[0,T]} \left| \int_0^t \left(b\left(s, \widehat{x}_s, u_s^\theta\right) - b\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) ds \right|^2 \\
&+ \sup_{t\in[0,T]} \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) d\left\langle B \right\rangle_s^{\mathrm{ij}} \right|^2 \\
&+ \sup_{t\in[0,T]} \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) d\left\langle B \right\rangle_s^{\mathrm{ij}} \right|^2 \\
&+ \sup_{t\in[0,T]} \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) dB_s^{\mathrm{i}} \right|^2 \\
&\left. + \sup_{t\in[0,T]} \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) dB_s^{\mathrm{i}} \right|^2 \right].
\end{aligned}
$$

By the third property of the sublinear expectation, we have

$$
\mathbb{E}_G \left[\sup_{t\in[0,T]} \left|x_t^\theta - \widehat{x}_t\right|^2 \right] \leq 6 \left\{ I_1 + I_2 + I_3 \right\},
$$

where

$$\begin{aligned} I_1 = {} & \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(b\left(s, x_s^\theta, u_s^\theta\right) - b\left(s, \widehat{x}_s, u_s^\theta\right) \right) ds \right|^2 \right] \\ & + \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(b\left(s, \widehat{x}_s, u_s^\theta\right) - b\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) ds \right|^2 \right]. \end{aligned}$$

$$\begin{aligned} I_2 = {} & \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) d\left\langle B\right\rangle_s^{\mathrm{ij}} \right|^2 \right] \\ & + \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) d\left\langle B\right\rangle_s^{\mathrm{ij}} \right|^2 \right]. \end{aligned}$$

$$\begin{aligned} I_3 = {} & \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) \right) dB_s^{\mathrm{i}} \right|^2 \right] \\ & + \mathbb{E}_G \left[\sup_{t\in[0,T]} \left| \int_0^t \left(\sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right) dB_s^{\mathrm{i}} \right|^2 \right]. \end{aligned}$$

By using G-Hölder inequality and G-BDG inequalities $(1.7), (1.8)$, we obtain

$$\begin{aligned} I_1 \leq {} & T\int_0^T \mathbb{E}_G \left[\left| b\left(s, x_s^\theta, u_s^\theta\right) - b\left(s, \widehat{x}_s, u_s^\theta\right) \right|^2 \right] ds \\ & + T\int_0^T \mathbb{E}_G \left[\left| b\left(s, \widehat{x}_s, u_s^\theta\right) - b\left(s, \widehat{x}_s, \widehat{u}_s\right) \right|^2 \right] ds. \\ I_2 \leq {} & \overline{\sigma}^4 T\int_0^T \mathbb{E}_G \left[\left| h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) \right|^2 \right] ds \\ & + \overline{\sigma}^4 T\int_0^T \mathbb{E}_G \left[\left| h_{\mathrm{ij}}\left(s, \widehat{x}_s, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right|^2 \right] ds. \\ I_3 \leq {} & C_2\int_0^T \mathbb{E}_G \left[\left| \sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) \right|^2 \right] ds \\ & + C_2\int_0^T \mathbb{E}_G \left[\left| \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, \widehat{u}_s\right) \right|^2 \right] ds. \end{aligned}$$

Since the coefficients b, h_{ij} and σ_{i} are Lipschitz with respect to (x, u), we easily obtain the following estimate

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|x_t^\theta-\widehat{x}_t\right|^2\right]\leq 6T\kappa\int_0^T\mathbb{E}_G\left|x_s^\theta-\widehat{x}_s\right|^2ds+6T\kappa\int_0^T\mathbb{E}_G\left|u_s^\theta-\widehat{u}_s\right|^2ds$$
$$+6\overline{\sigma}^4T\kappa\int_0^T\mathbb{E}_G\left|x_s^\theta-\widehat{x}_s\right|^2ds+6\overline{\sigma}^4T\kappa\int_0^T\mathbb{E}_G\left|u_s^\theta-\widehat{u}_s\right|^2ds$$
$$+6C_2\kappa\int_0^T\mathbb{E}_G\left|x_s^\theta-\widehat{x}_s\right|^2ds+6C_2\kappa\int_0^T\mathbb{E}_G\left|u_s^\theta-\widehat{u}_s\right|^2ds.$$

Then, we have

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|x_t^\theta-\widehat{x}_t\right|^2\right]\leq\varsigma\int_0^T\mathbb{E}_G\left|x_s^\theta-\widehat{x}_s\right|^2ds+\varsigma\theta^2\int_0^T\mathbb{E}_G\left|u_s-\widehat{u}_s\right|^2ds.$$

where $\varsigma=6\kappa(T+\overline{\sigma}^4T+C_2)$ and $\overline{\sigma}^4, C_2$ are the constants appeared in BDG inequalities. Using Gronwall's inequality and sending $\theta\to 0$, we arrive at the desired result (2.8). ■

Lemma 2.2 *We assume that the assumptions* 2.2 *are satisfied. Then* $Y(.)$ *is the solution of the following variational equation*

$$\begin{cases} dY_t &= \left(D_x\widehat{b}\,(t)\,Y_t+D_u\widehat{b}\,(t)\,(u_t-\widehat{u}_t)\right)dt \\ &\quad+\left(D_x\widehat{h}_{\mathrm{ij}}\,(t)\,Y_t+D_u\widehat{h}_{\mathrm{ij}}\,(t)\,(u_t-\widehat{u}_t)\right)d\left\langle B\right\rangle_t^{\mathrm{ij}} \\ &\quad+\left(D_x\widehat{\sigma}_{\mathrm{i}}\,(t)\,Y_t+D_u\widehat{\sigma}_{\mathrm{i}}\,(t)\,(u_t-\widehat{u}_t)\right)dB_t^{\mathrm{i}}, \\ Y_0 &= 0, \end{cases}\tag{2.9}$$

where $Y_t=\lim_{\theta\to0}\frac{1}{\theta}\left(x_t^\theta-\widehat{x}_t\right)$*, and*

$$D_x\widehat{k}:=\begin{pmatrix}\partial_{x_1}\widehat{k}_1&\cdots&\partial_{x_n}\widehat{k}_1\\ \vdots&\ddots&\vdots\\ \partial_{x_1}\widehat{k}_n&\cdots&\partial_{x_n}\widehat{k}_n\end{pmatrix},\ D_u\widehat{k}:=\begin{pmatrix}\partial_{u_1}\widehat{k}_1&\cdots&\partial_{u_n}\widehat{k}_1\\ \vdots&\ddots&\vdots\\ \partial_{u_1}\widehat{k}_n&\cdots&\partial_{u_n}\widehat{k}_n\end{pmatrix}\in\mathbb{R}^{n\times n},$$

for all $k = b, h_{\mathrm{ij}}$ *and* σ_{i}.

$$\left(D_x\widehat{k}\,(t)\,Y_t + D_u\widehat{k}\,(t)\,(u_t - \widehat{u}_t)\right)$$
$$= \begin{pmatrix} \partial_{x_1}\widehat{k}_1 & \cdots & \partial_{x_n}\widehat{k}_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1}\widehat{k}_n & \cdots & \partial_{x_n}\widehat{k}_n \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} + \begin{pmatrix} \partial_{u_1}\widehat{k}_1 & \cdots & \partial_{u_n}\widehat{k}_1 \\ \vdots & \ddots & \vdots \\ \partial_{u_1}\widehat{k}_n & \cdots & \partial_{u_n}\widehat{k}_n \end{pmatrix} \begin{pmatrix} (u-\widehat{u})_1 \\ \vdots \\ (u-\widehat{u})_n \end{pmatrix}.$$

Proof. We have

$$x_t^\theta - \widehat{x}_t = \int_0^t \left(b\left(s, x_s^\theta, u_s^\theta\right) - b\left(s, \widehat{x}_s, \widehat{u}_s\right)\right) ds + \int_0^t \left(h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right) - h_{\mathrm{ij}}\left(s, \widehat{x}_s, \widehat{u}_s\right)\right) d\left\langle B\right\rangle_s^{\mathrm{ij}}$$
$$+ \int_0^t \left(\sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right) - \sigma_{\mathrm{i}}\left(s, \widehat{x}_s, \widehat{u}_s\right)\right) dB_s^{\mathrm{i}}.$$

For all $k = b, h_{\mathrm{ij}}$ and σ_{i}, we have

$$k\left(s, x_s^\theta, u_s^\theta\right) = k\left(s, \widehat{x}_s + \left(x_s^\theta - \widehat{x}_s\right), \widehat{u}_s + \left(u_s^\theta - \widehat{u}_s\right)\right)$$

Using Taylor's expansion with an integral remain on the functions $b\left(s, x_s^\theta, u_s^\theta\right), h_{\mathrm{ij}}\left(s, x_s^\theta, u_s^\theta\right)$ and $\sigma_{\mathrm{i}}\left(s, x_s^\theta, u_s^\theta\right)$ at the point $\left(\widehat{x}_s, \widehat{u}_s\right)$,

$$k\left(s, x_s^\theta, u_s^\theta\right) = k\left(s, \widehat{x}_s, \widehat{u}_s\right)$$
$$+ \int_0^1 D_x k\left(s, \widehat{x}_s + \lambda\left(x_s^\theta - \widehat{x}_s\right), \widehat{u}_s + \lambda\left(u_s^\theta - \widehat{u}_s\right)\right)\left(x_s^\theta - \widehat{x}_s\right) d\lambda$$
$$+ \int_0^1 D_u k\left(s, \widehat{x}_s + \lambda\left(x_s^\theta - \widehat{x}_s\right), \widehat{u}_s + \lambda\left(u_s^\theta - \widehat{u}_s\right)\right)\left(u_s^\theta - \widehat{u}_s\right) d\lambda.$$

By (2.5), we get

$$\begin{aligned}&\frac{1}{\theta}\left(k\left(s,x_s^\theta,u_s^\theta\right)-k\left(s,\widehat{x}_s,\widehat{u}_s\right)\right) \\ &=\int_0^1\frac{1}{\theta}D_xk\left(s,\widehat{x}_s+\lambda\left(x_s^\theta-\widehat{x}_s\right),\widehat{u}_s+\lambda\theta\left(u_s-\widehat{u}_s\right)\right)\left(x_s^\theta-\widehat{x}_s\right)d\lambda \\ &+\int_0^1D_uk\left(s,\widehat{x}_s+\lambda\left(x_s^\theta-\widehat{x}_s\right),\widehat{u}_s+\lambda\theta\left(u_s-\widehat{u}_s\right)\right)\left(u_s-\widehat{u}_s\right)d\lambda.\end{aligned} \tag{2.10}$$

Since all the derivatives of $b, h_{\mathrm{ij}}, \sigma_{\mathrm{i}}$ are continuous and bounded with respect to (x,u), using the Lebesgue's bounded convergence theorem and the result (2.8), then the limit when θ goes to zero for every member in the side of (2.10) gives us the desired result.

■

Lemma 2.3 *Under the assumptions* 2.2, *we have*

$$\lim_{\theta\to 0}\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\frac{1}{\theta}\left(x_t^\theta-\widehat{x}_t\right)-Y_t\right|^2\right]=0, \tag{2.11}$$

where Y *is given by* (2.9) *.*

Proof. By replacing $\widehat{x}_t, x_t^\theta$ and Y_t by their values in equations $(2.2),(2.6)$ and (2.9) respectively, if we put for simplification $\widetilde{x}_t^\theta=\frac{1}{\theta}\left(x_t^\theta-\widehat{x}_t\right)-Y_t$, where $\widetilde{x}_t^\theta:[0,T]\times\Omega\to\mathbb{R}^n$

and $\widetilde{\Lambda}_s^\theta := \left(s, \widehat{x}_s + \lambda\theta\left(\widetilde{x}_s^\theta + Y_s\right), \widehat{u}_s + \lambda\theta\left(u_s - \widehat{u}_s\right)\right)$, we get

$$
\begin{aligned}
\widetilde{x}_t^\theta &= \int_0^1 \int_0^t D_x b\left(\widetilde{\Lambda}_s^\theta\right)\left(\widetilde{x}_s^\theta + Y_s\right) ds d\lambda + \int_0^1 \int_0^t D_u b\left(\widetilde{\Lambda}_s^\theta\right)\left(u_s - \widehat{u}_s\right) ds d\lambda \\
&+ \int_0^1 \int_0^t D_x h_{\mathsf{ij}}\left(\widetilde{\Lambda}_s^\theta\right)\left(\widetilde{x}_s^\theta + Y_s\right) d\left\langle B\right\rangle_s^{\mathsf{ij}} d\lambda + \int_0^1 \int_0^t D_u h_{\mathsf{ij}}\left(\widetilde{\Lambda}_s^\theta\right)\left(u_s - \widehat{u}_s\right) d\left\langle B\right\rangle_s^{\mathsf{ij}} d\lambda \\
&+ \int_0^1 \int_0^t D_x \sigma_{\mathsf{i}}\left(\widetilde{\Lambda}_s^\theta\right)\left(\widetilde{x}_s^\theta + Y_s\right) dB_s^{\mathsf{i}} d\lambda + \int_0^1 \int_0^t D_u \sigma_{\mathsf{i}}\left(\widetilde{\Lambda}_s^\theta\right)\left(u_s - \widehat{u}_s\right) dB_s^{\mathsf{i}} d\lambda \\
&- \int_0^t \left(D_x \widehat{b}\left(s\right) Y_s + D_u \widehat{b}\left(s\right)\left(u_s - \widehat{u}_s\right)\right) ds \\
&- \int_0^t \left(D_x \widehat{h}_{\mathsf{ij}}\left(s\right) Y_s + D_u \widehat{h}_{\mathsf{ij}}\left(s\right)\left(u_s - \widehat{u}_s\right)\right) d\left\langle B\right\rangle_s^{\mathsf{ij}} \\
&- \int_0^t \left(D_x \widehat{\sigma}_{\mathsf{i}}\left(s\right) Y_s + D_u \widehat{\sigma}_{\mathsf{i}}\left(s\right)\left(u_s - \widehat{u}_s\right)\right) dB_s^{\mathsf{i}}.
\end{aligned}
$$

According to the sublinearity of the G-expectation $\mathbb{E}_G$, G-Hölder inequality and G-BDG inequalities $(1.7), (1.8)$, we have

$$
\begin{aligned}
&\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\widetilde{x}_t^\theta\right|^2\right] \\
&\leq C_1 \int_0^1 \int_0^t \mathbb{E}_G\left|D_x b\left(\widetilde{\Lambda}_s^\theta\right)\widetilde{x}_s^\theta\right|^2 ds d\lambda + C_2 \int_0^1 \int_0^t \mathbb{E}_G\left|D_x h_{\mathsf{ij}}\left(\widetilde{\Lambda}_s^\theta\right)\widetilde{x}_s^\theta\right|^2 ds d\lambda \\
&+ C_3 \int_0^1 \int_0^t \mathbb{E}_G\left|D_x \sigma_{\mathsf{i}}\left(\widetilde{\Lambda}_s^\theta\right)\widetilde{x}_s^\theta\right|^2 ds d\lambda + \mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^\theta\right|^2\right],
\end{aligned}
$$

where

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^\theta\right|^2\right] \le C_4\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_x b\left(\widetilde{\Lambda}_s^\theta\right) - D_x\widehat{b}\left(s\right)\right\} Y_s d\lambda\right|^2 ds$$
$$+ C_5\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_x h_{\mathrm{ij}}\left(\widetilde{\Lambda}_s^\theta\right) - D_x\widehat{h}_{\mathrm{ij}}\left(s\right)\right\} Y_s d\lambda\right|^2 ds$$
$$+ C_6\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_x \sigma_{\mathrm{i}}\left(\widetilde{\Lambda}_s^\theta\right) - D_x\widehat{\sigma}_{\mathrm{i}}\left(s\right)\right\} Y_s d\lambda\right|^2 ds$$
$$+ C_7\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_u b\left(\widetilde{\Lambda}_s^\theta\right) - D_u\widehat{b}\left(s\right)\right\} \left(u_s - \widehat{u}_s\right) d\lambda\right|^2 ds$$
$$+ C_8\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_u h_{\mathrm{ij}}\left(\widetilde{\Lambda}_s^\theta\right) - D_u\widehat{h}_{\mathrm{ij}}\left(s\right)\right\} \left(u_s - \widehat{u}_s\right) d\lambda\right|^2 ds$$
$$+ C_9\int_0^t \mathbb{E}_G\left|\int_0^1 \left\{D_u \sigma_{\mathrm{i}}\left(\widetilde{\Lambda}_s^\theta\right) - D_u\widehat{\sigma}_{\mathrm{i}}\left(s\right)\right\} \left(u_s - \widehat{u}_s\right) d\lambda\right|^2 ds.$$

Since all the derivatives of b, h_{ij} and σ_{i} are continuous and bounded and by using Lebesgue's bounded convergence theorem, we obtain the limit of the term $\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^\theta\right|^2\right]$ equals 0 when θ goes to 0. So that, we can use G-Gronwall's inequality to find the limit (2.11). ■

The next lemma gives us the second and the third forms of the variational inequality, which are the principal tools to establish the G-stochastic maximum principle in the following sections.

Lemma 2.4 *Assume that hypotheses* 2.2 *hold, then we have*

$$0 \le \mathbb{E}_G\left[\int_0^T \left(\left(\nabla_x\widehat{l}\left(t\right)\right)^{\mathcal{T}} Y_t + \left(\nabla_u\widehat{l}\left(t\right)\right)^{\mathcal{T}}\left(u_t - \widehat{u}_t\right)\right) dt\right.$$
$$\left. + \int_0^T \left(\left(\nabla_x\widehat{m}_{\mathrm{ij}}\left(t\right)\right)^{\mathcal{T}} Y_t + \left(\nabla_u\widehat{m}_{\mathrm{ij}}\left(t\right)\right)^{\mathcal{T}}\left(u_t - \widehat{u}_t\right)\right) d\left\langle B\right\rangle_t^{\mathrm{ij}} + \left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} Y_T\right], \tag{2.12}$$

and

$$0 \leq \mathbb{E}_G \left[\int_0^T \left(\left(\nabla_x \widehat{l}\,(t) \right)^{\mathcal{T}} Y_t + \left(\nabla_u \widehat{l}\,(t) \right)^{\mathcal{T}} (u_t - \widehat{u}_t) \right) dt \right] + \mathbb{E}_G \left[\int_0^T \left((\nabla_x \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} Y_t + (\nabla_u \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} (u_t - \widehat{u}_t) \right) d \langle B \rangle_t^{\mathrm{ij}} \right] + \mathbb{E}_G \left[(\nabla_x g(\widehat{x}_T))^{\mathcal{T}} Y_T \right], \tag{2.13}$$

where

$$\nabla_x g(\widehat{x}_T) := \begin{pmatrix} \partial_{x_1} g(\widehat{x}_T) \\ \vdots \\ \partial_{x_n} g(\widehat{x}_T) \end{pmatrix}, \nabla_x \widehat{l}\,(t) := \begin{pmatrix} \partial_{x_1} \widehat{l}\,(t) \\ \vdots \\ \partial_{x_n} \widehat{l}\,(t) \end{pmatrix}, \nabla_u \widehat{l}\,(t) := \begin{pmatrix} \partial_{u_1} \widehat{l}\,(t) \\ \vdots \\ \partial_{u_n} \widehat{l}\,(t) \end{pmatrix} \in \mathbb{R}^n,$$

$$\nabla_x \widehat{m}_{\mathrm{ij}}(t) = \begin{pmatrix} \partial_{x_1} \widehat{m}_{\mathrm{ij}}(t) \\ \vdots \\ \partial_{x_n} \widehat{m}_{\mathrm{ij}}(t) \end{pmatrix}, \nabla_u \widehat{m}_{\mathrm{ij}}(t) := \begin{pmatrix} \partial_{u_1} \widehat{m}_{\mathrm{ij}}(t) \\ \vdots \\ \partial_{u_n} \widehat{m}_{\mathrm{ij}}(t) \end{pmatrix} \in \mathbb{R}^n.$$

Proof. We use the same techniques as in lemma 2.3, we consider the perturbed control u^θdefined in (2.5). On one hand, from the variational inequality (2.7) and if we put $\Lambda_t^\theta := \left(t, \widehat{x}_t + \lambda \left(x_t^\theta - \widehat{x}_t\right), \widehat{u}_t + \lambda\theta \left(u_t - \widehat{u}_t\right)\right)$, we get

$$\begin{aligned} 0 \leq \mathbb{E}_G \Bigg[& \int_0^1 \frac{1}{\theta} \left(\nabla_x g \left(\widehat{x}_T + \lambda \left(x_T^\theta - \widehat{x}_T \right) \right) \right)^{\mathcal{T}} \left(x_T^\theta - \widehat{x}_T \right) d\lambda \\ & + \int_0^T \int_0^1 \left(\frac{1}{\theta} \left(\nabla_x l \left(\Lambda_t^\theta \right) \right)^{\mathcal{T}} \left(x_t^\theta - \widehat{x}_t \right) + \left(\nabla_u l \left(\Lambda_t^\theta \right) \right)^{\mathcal{T}} (u_t - \widehat{u}_t) \right) d\lambda dt \\ & + \int_0^T \int_0^1 \left(\frac{1}{\theta} \left(\nabla_x m_{\mathrm{ij}} \left(\Lambda_t^\theta \right) \right)^{\mathcal{T}} \left(x_t^\theta - \widehat{x}_t \right) + \left(\nabla_u m_{\mathrm{ij}} \left(\Lambda_t^\theta \right) \right)^{\mathcal{T}} (u_t - \widehat{u}_t) \right) d\lambda d \langle B \rangle_t^{\mathrm{ij}} \Bigg]. \end{aligned}$$

On the other hand, from the variational inequality (2.7) and by using the subadditivity property of G-expectation Definition 1.1 we get

$$\begin{aligned} 0 \leq \mathbb{E}_G \left[g\left(x_T^\theta\right) - g(\widehat{x}_T) \right] + \mathbb{E}_G \left[\int_0^T \left(l\left(t, x_t^\theta, u_t^\theta\right) - l(t, \widehat{x}_t, \widehat{u}_t) \right) dt \right] \\ + \mathbb{E}_G \left[\int_0^T \left(m_{\mathrm{ij}}\left(t, x_t^\theta, u_t^\theta\right) - m_{\mathrm{ij}}(t, \widehat{x}_t, \widehat{u}_t) \right) d \langle B \rangle_t^{\mathrm{ij}} \right]. \end{aligned}$$

For the second time, we put $\Lambda_t^\theta := \left(t, \widehat{x}_t + \lambda\left(x_t^\theta - \widehat{x}_t\right), \widehat{u}_t + \lambda\theta\left(u_t - \widehat{u}_t\right)\right)$, we get

$$
\begin{aligned}
0 \leq \Bigg\{ & \mathbb{E}_G\left[\int_0^1 \frac{1}{\theta}\left(\nabla_x g\left(\widehat{x}_T + \lambda\left(x_T^\theta - \widehat{x}_T\right)\right)\right)^{\mathcal{T}}\left(x_T^\theta - \widehat{x}_T\right) d\lambda\right] \\
& + \mathbb{E}_G\left[\int_0^T \int_0^1 \left(\frac{1}{\theta}\left(\nabla_x l\left(\Lambda_t^\theta\right)\right)^{\mathcal{T}}\left(x_t^\theta - \widehat{x}_t\right) + \left(\nabla_u l\left(\Lambda_t^\theta\right)\right)^{\mathcal{T}}\left(u_t - \widehat{u}_t\right)\right) d\lambda dt\right] \\
& + \mathbb{E}_G\left[\int_0^T \int_0^1 \left(\frac{1}{\theta}\left(\nabla_x m_{\mathrm{ij}}\left(\Lambda_t^\theta\right)\right)^{\mathcal{T}}\left(x_t^\theta - \widehat{x}_t\right) + \left(\nabla_u m_{\mathrm{ij}}\left(\Lambda_t^\theta\right)\right)^{\mathcal{T}}\left(u_t - \widehat{u}_t\right)\right) d\lambda d\langle B\rangle_t^{\mathrm{ij}}\right].
\end{aligned}
$$

Since the derivative of g, l and m_{ij} are continuous and bounded, then from (2.8),(2.11) and letting θ going to 0, we get (2.12) and (2.13). ■

2.3 G-adjoint process and G-adjoint equation

We consider the linear form of the equation (2.9)

$$
\begin{cases}
d\Phi(t) & = \Phi(t) D_x\widehat{b}(t) dt + \Phi(t) D_x\widehat{h}_{\mathrm{ij}}(t) d\langle B\rangle_t^{\mathrm{ij}} + \Phi(t) D_x\widehat{\sigma}_{\mathrm{i}}(t) dB_t^{\mathrm{i}}; 0 \leq t \leq T, \\
\Phi(0) & = I_n,
\end{cases}
\tag{2.14}
$$

where $\Phi : [0,T] \times \Omega \to \mathbb{R}^{n\times n}$ is a symmetric matrix. This equation is linear and has bounded coefficients, then it admits a unique strong solution. Moreover, this solution is invertible and its inverse Ψ is given by

$$
\begin{cases}
d\Psi(t) = & -\Psi(t) D_x\widehat{b}(t) dt + \Psi(t)\left(D_x\widehat{\sigma}_{\mathrm{i}}(t)\left(D_x\widehat{\sigma}_{\mathrm{j}}(t)\right)^{\mathcal{T}} - D_x\widehat{h}_{\mathrm{ij}}(t)\right) d\langle B\rangle_t^{\mathrm{ij}} \\
& -\Psi(t) D_x\widehat{\sigma}_{\mathrm{i}}(t) dB_t^{\mathrm{i}}; \qquad 0 \leq t \leq T, \\
\Psi(0) = & I_n.
\end{cases}
\tag{2.15}
$$

In fact, by applying G-Itô's formula on the C^2-function $f\left(\Phi\left(t\right)\right)=\left(\Phi\left(t\right)\right)^{-1}$ we have

$$\begin{aligned} f\left(\Phi\left(t\right)\right) &= f\left(\Phi\left(0\right)\right)+\int_0^t f'\left(\Phi\left(s\right)\right)\Phi\left(s\right)D_x\widehat{b}\left(s\right)ds \\ &\quad+\int_0^t f'\left(\Phi\left(s\right)\right)\Phi\left(s\right)D_x\widehat{\sigma}_{\mathsf{i}}\left(s\right)dB_s^{\mathsf{i}}+\int_0^t f'\left(\Phi\left(s\right)\right)\Phi\left(s\right)D_x\widehat{h}_{\mathsf{ij}}\left(s\right)d\left\langle B\right\rangle_s^{\mathsf{ij}} \\ &\quad+\frac{1}{2}\int_0^t \Phi\left(s\right)D_x\widehat{\sigma}_{\mathsf{i}}\left(s\right)f''\left(\Phi\left(s\right)\right)\Phi\left(s\right)D_x\widehat{\sigma}_{\mathsf{j}}\left(s\right)d\left\langle B\right\rangle_s^{\mathsf{ij}}. \end{aligned}$$

Then

$$\begin{aligned} \left(\Phi\left(t\right)\right)^{-1} &= I_n-\int_0^t\left(\Phi\left(s\right)\right)^{-1}D_x\widehat{b}\left(s\right)ds-\int_0^t\left(\Phi\left(s\right)\right)^{-1}D_xD_x\widehat{h}_{\mathsf{ij}}\left(s\right)d\left\langle B\right\rangle_s^{\mathsf{ij}} \\ &\quad-\int_0^t\left(\Phi\left(s\right)\right)^{-1}D_x\widehat{\sigma}_{\mathsf{i}}\left(t\right)dB_t^{\mathsf{i}}+\int_0^t\left(\Phi\left(s\right)\right)^{-1}D_x\widehat{\sigma}_{\mathsf{i}}\left(t\right)\left(D_x\widehat{\sigma}_{\mathsf{j}}\left(t\right)\right)^{\mathcal{T}}d\left\langle B\right\rangle_s^{\mathsf{ij}}. \end{aligned}$$

By naming $\left(\Phi\left(t\right)\right)^{-1}:=\Psi\left(t\right)$, we get

$$\begin{aligned} \Psi\left(t\right) &= I_n-\int_0^t\Psi\left(s\right)D_x\widehat{b}\left(s\right)ds-\int_0^t\Psi\left(s\right)D_x\widehat{h}_{\mathsf{ij}}\left(s\right)d\left\langle B\right\rangle_s^{\mathsf{ij}} \\ &\quad-\int_0^t\Psi\left(s\right)D_x\widehat{\sigma}_{\mathsf{i}}\left(t\right)dB_t^{\mathsf{i}}+\int_0^t\Psi\left(s\right)D_x\widehat{\sigma}_{\mathsf{i}}\left(t\right)\left(D_x\widehat{\sigma}_{\mathsf{j}}\left(t\right)\right)^{\mathcal{T}}d\left\langle B\right\rangle_s^{\mathsf{ij}}. \end{aligned}$$

In addition, the processes Φ and Ψ are continuous and also

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\Phi\right|^2\right]+\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\Psi\right|^2\right]<\infty. \tag{2.16}$$

By using the resolvent method, we put

$$R\left(t\right)=\Psi\left(t\right)Y\left(t\right). \tag{2.17}$$

Hence, we apply integration by parts for G-Itô's processes to $R(t)$, we get

$$\begin{aligned} dR(t) &= d[\Psi(t) Y(t)] \\ &= \Psi(t) dY(t) + d\Psi(t) Y(t) + d\langle \Psi, Y \rangle_t \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = \Psi(t) \Big[\Big(D_x \widehat{b}(t) Y_t + D_u \widehat{b}(t)(u_t - \widehat{u}_t) \Big) dt + \Big(D_x \widehat{h}_{ij}(t) Y_t + D_u \widehat{h}_{ij}(t)(u_t - \widehat{u}_t) \Big) d\langle B \rangle_t^{ij} + \left(D_x \widehat{\sigma}_i(t) Y_t + D_u \widehat{\sigma}_i(t)(u_t - \widehat{u}_t) \right) dB_t^i \Big].$$

$$I_2 = \Big[-\Psi(t) D_x \widehat{b}(t) dt + \Psi(t) \Big(D_x \widehat{\sigma}_i(t) \left(D_x \widehat{\sigma}_j(t) \right)^{\mathcal{T}} - D_x \widehat{h}_{ij}(t) \Big) d\langle B \rangle_t^{ij} - \Psi(t) D_x \widehat{\sigma}_i(t) dB_t^i \Big] Y(t).$$

$$I_3 = -\Psi(t) D_x \widehat{\sigma}_i(t) \left(D_x \widehat{\sigma}_j(t) Y_t + D_u \widehat{\sigma}_j(t)(u_t - \widehat{u}_t) \right) d\langle B \rangle_t^{ij}.$$

Then

$$\begin{aligned} dR(t) &= \Psi(t) D_u \widehat{b}(t)(u_t - \widehat{u}_t) dt + \Psi(t) D_u \widehat{\sigma}_i(t)(u_t - \widehat{u}_t) dB_t^i \\ &\quad + \Psi(t) \Big(D_u \widehat{h}_{ij}(t)(u_t - \widehat{u}_t) - D_x \widehat{\sigma}_i(t) \left(D_u \widehat{\sigma}_j(t) \right)^{\mathcal{T}} (u_t - \widehat{u}_t) \Big) d\langle B \rangle_t^{ij}. \end{aligned}$$

We suppose that

$$\Gamma = \Phi(T) \nabla_x g(\widehat{x}_T) + \int_0^T \Phi(s) \nabla_x \widehat{l}(s) ds + \int_0^T \Phi(s) \nabla_x \widehat{m}_{ij}(s) d\langle B \rangle_s^{ij}. \tag{2.18}$$

$$\xi(t) = \mathbb{E}_G[\Gamma | \mathcal{F}_t] - \int_0^t \Phi(s) \nabla_x \widehat{l}(s) ds - \int_0^t \Phi(s) \nabla_x \widehat{m}_{ij}(s) d\langle B \rangle_s^{ij}. \tag{2.19}$$

We observe from (2.17) , (2.18) and (2.19) that

$$\mathbb{E}_G\left[\left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} Y_T\right] = \mathbb{E}_G\left[\left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} \Phi\left(T\right) R\left(T\right)\right] = \mathbb{E}_G\left[\left(\xi\left(T\right)\right)^{\mathcal{T}} R\left(T\right)\right].$$

In fact, (2.17) equivalent to $Y_T = \Phi\left(T\right) R\left(T\right)$, such that

$$\mathbb{E}_G\left[\left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} Y_T\right] = \mathbb{E}_G\left[\left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} \Phi\left(T\right) R\left(T\right)\right].$$

On the other hand, using properties of conditional sublinear expectation, we get

$$\begin{aligned}
&\mathbb{E}_G\left[\left(\xi\left(T\right)\right)^{\mathcal{T}} R\left(T\right)\right] \\
&= \mathbb{E}_G\Bigg[\Bigg(\mathbb{E}_G\Bigg[\Bigg(\Phi\left(T\right)\nabla_x g\left(\widehat{x}_T\right) + \int_0^T \Phi\left(s\right)\nabla_x \widehat{l}\left(s\right) ds + \int_0^T \Phi\left(s\right)\nabla_x \widehat{m}_{\mathrm{ij}}\left(s\right) d\left\langle B\right\rangle_t^{\mathrm{ij}}\Bigg) \mid \mathcal{F}_T\Bigg] \\
&- \int_0^T \Phi\left(s\right)\nabla_x \widehat{l}\left(s\right) ds - \int_0^T \Phi\left(s\right)\nabla_x \widehat{m}_{\mathrm{ij}}\left(s\right) d\left\langle B\right\rangle_t^{\mathrm{ij}}\Bigg)^{\mathcal{T}} R\left(T\right)\Bigg] \\
&= \mathbb{E}_G\Bigg[\Bigg(\Phi\left(T\right)\nabla_x g\left(\widehat{x}_T\right) + \int_0^T \Phi\left(s\right)\nabla_x \widehat{l}\left(s\right) ds + \int_0^T \Phi\left(s\right)\nabla_x \widehat{m}_{\mathrm{ij}}\left(s\right) d\left\langle B\right\rangle_t^{\mathrm{ij}} \\
&- \int_0^T \Phi\left(s\right)\nabla_x \widehat{l}\left(s\right) ds - \int_0^T \Phi\left(s\right)\nabla_x \widehat{m}_{\mathrm{ij}}\left(s\right) d\left\langle B\right\rangle_t^{\mathrm{ij}}\Bigg)^{\mathcal{T}} R\left(T\right)\Bigg] \\
&= \mathbb{E}_G\left[\left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}} \Phi\left(T\right) R\left(T\right)\right].
\end{aligned}$$

According to the result obtained in [33]. Under the assumption $\mathbb{E}_G\left[\left|\Gamma\right|^2 \left|\mathcal{F}_t\right.\right] < \infty$ and by the G-martingales representation theorem there exists a couple $\left(Z, K\right)$, where $Z \in \mathbb{H}_G^2(0,T)$ and K is a non-increasing continuous G-martingale with $K_0 = 0$ and $K_T \in \mathbb{L}_G^2(\Omega_T)$, such that

$$\mathbb{E}_G\left[\Gamma \left|\mathcal{F}_t\right.\right] = \mathbb{E}_G\left[\Gamma\right] + \int_0^t Z_{\mathrm{i}}(s) dB_s^{\mathrm{i}} + K_t.$$

The equation (2.19) can be rewritten as

$$d\xi(t) = -\Phi(t)\nabla_x \widehat{l}(t)\,dt - \Phi(t)\nabla_x \widehat{m}_{\mathsf{ij}}(t)\,d\langle B\rangle_t^{\mathsf{ij}} + Z_{\mathsf{i}}(t)dB_t^{\mathsf{i}} + dK_t.$$

Using integration by parts for G-Itô's processes to $(\xi(t))^{\mathcal{T}} R(t)$, we get

$$\begin{aligned}
&d\left((\xi(t))^{\mathcal{T}} R(t)\right)\\
&= (\xi(t))^{\mathcal{T}}\Big[\Psi(t) D_u\widehat{b}(t)(u_t - \widehat{u}_t)\,dt + \Psi(t) D_u\widehat{\sigma}_{\mathsf{i}}(t)(u_t-\widehat{u}_t)\,dB_t^{\mathsf{i}}\\
&+\Psi(t)\Big(D_u\widehat{h}_{\mathsf{ij}}(t)(u_t-\widehat{u}_t) - D_x\widehat{\sigma}_{\mathsf{i}}(t)(D_u\widehat{\sigma}_{\mathsf{j}}(t))^{\mathcal{T}}(u_t-\widehat{u}_t)\Big)\,d\langle B\rangle_t^{\mathsf{ij}}\Big]\\
&+\Big[-\Phi(t)\nabla_x\widehat{l}(t)\,dt - \Phi(t)\nabla_x\widehat{m}_{\mathsf{ij}}(t)\,d\langle B\rangle_t^{\mathsf{ij}} + Z_{\mathsf{i}}(t)dB_t^{\mathsf{i}} + dK_t\Big]^{\mathcal{T}} R(t)\\
&+\Psi(t) D_u\widehat{\sigma}_{\mathsf{i}}(t)(u_t-\widehat{u}_t) Z_{\mathsf{i}}(t)d\langle B\rangle_t^{\mathsf{ij}}.
\end{aligned}$$

Then

$$\begin{aligned}
&d\left((\xi(t))^{\mathcal{T}} R(t)\right)\\
&= (\xi(t))^{\mathcal{T}}\Psi(t) D_u\widehat{b}(t)(u_t-\widehat{u}_t)\,dt + (\xi(t))^{\mathcal{T}}\Psi(t) D_u\widehat{\sigma}_{\mathsf{i}}(t)(u_t-\widehat{u}_t)\,dB_t^{\mathsf{i}}\\
&+(\xi(t))^{\mathcal{T}}\Psi(t)\Big(D_u\widehat{h}_{\mathsf{ij}}(t)(u_t-\widehat{u}_t) - D_x\widehat{\sigma}_{\mathsf{i}}(t)(D_u\widehat{\sigma}_{\mathsf{j}}(t))^{\mathcal{T}}(u_t-\widehat{u}_t)\Big)\,d\langle B\rangle_t^{\mathsf{ij}}\\
&+(Z_{\mathsf{i}}(t))^{\mathcal{T}}(\Psi(t) D_u\widehat{\sigma}_{\mathsf{j}}(t)(u_t-\widehat{u}_t))\,d\langle B\rangle_t^{\mathsf{ij}} - \left(\nabla_x\widehat{l}(t)\right)^{\mathcal{T}}\Phi(t) R(t)\,dt\\
&-(\nabla_x\widehat{m}_{\mathsf{ij}}(t))^{\mathcal{T}}\Phi(t) R(t)\,d\langle B\rangle_t^{\mathsf{ij}} + (Z_{\mathsf{i}}(t))^{\mathcal{T}} R(t)\,dB_t^{\mathsf{i}} + (dK_t)^{\mathcal{T}} R(t).
\end{aligned}$$

Then, if we put

$$\widehat{p}^G(t) = \Psi(t)\xi(t), \tag{2.20}$$

$$\widehat{Q}_{\mathsf{i}}^G(t) = \Psi(t) Z_{\mathsf{i}}(t) - D_x\widehat{\sigma}_{\mathsf{i}}(t)\widehat{p}^G(t), \tag{2.21}$$

where $\widehat{p}^G(t)$, $\widehat{Q}_{\mathrm{i}}^G(t) \in \mathbb{R}^n$. Then we get

$$\begin{aligned}
&\mathbb{E}_G\left[(\xi(T))^{\mathcal{T}} R(T)\right] \\
&= \mathbb{E}_G\left[\int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{b}(t)(u_t - \widehat{u}_t)\,dt - \int_0^T \left(\nabla_x \widehat{l}(t)\right)^{\mathcal{T}} Y_t dt\right. \\
&- \int_0^T (\nabla_x \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} Y_t d\langle B\rangle_t^{\mathrm{ij}} + \int_0^T (dK_t)^{\mathcal{T}} R(t) \\
&\left. + \int_0^T \left(\left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{h}_{\mathrm{ij}}(t) + \widehat{Q}_{\mathrm{i}}^G(t) D_u\widehat{\sigma}_{\mathrm{j}}(t)\right)(u_t - \widehat{u}_t)\, d\langle B\rangle_t^{\mathrm{ij}}\right].
\end{aligned} \tag{2.22}$$

By replacing (2.22) into (2.13), we obtain

$$\begin{aligned}
0 &\leq \mathbb{E}_G\left[\int_0^T \left(\left(\nabla_x \widehat{l}(t)\right)^{\mathcal{T}} Y_t + \left(\nabla_u \widehat{l}(t)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\right) dt\right] \\
&+ \mathbb{E}_G\left[\int_0^T \left((\nabla_x \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} Y_t + (\nabla_u \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} (u_t - \widehat{u}_t)\right) d\langle B\rangle_t^{\mathrm{ij}}\right] \\
&+ \mathbb{E}_G\left[\int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{b}(t)(u_t - \widehat{u}_t)\,dt - \int_0^T \left(\nabla_x \widehat{l}(t)\right)^{\mathcal{T}} Y_t dt\right. \\
&- \int_0^T (\nabla_x \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} Y_t d\langle B\rangle_t^{\mathrm{ij}} + \int_0^T \left(\left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{h}_{\mathrm{ij}}(t)\right. \\
&\left.\left. + \left(\widehat{Q}_i^G(t)\right)^{\mathcal{T}} D_u\widehat{\sigma}_{\mathrm{j}}(t)\right)(u_t - \widehat{u}_t)\, d\langle B\rangle_t^{\mathrm{ij}} + \int_0^T (dK_t)^{\mathcal{T}} R(t)\right].
\end{aligned}$$

And by (2.12), we get

$$\begin{aligned}
0 \leq \mathbb{E}_G&\left[\int_0^T \left(\nabla_u \widehat{l}(t)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\,dt + \int_0^T (\nabla_u \widehat{m}_{\mathrm{ij}}(t))^{\mathcal{T}} (u_t - \widehat{u}_t)\, d\langle B\rangle_t^{\mathrm{ij}}\right. \\
&+ \int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{b}(t)(u_t - \widehat{u}_t)\,dt + \int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}} D_u\widehat{h}_{\mathrm{ij}}(t)(u_t - \widehat{u}_t)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}dt \\
&\left. + \int_0^T \left(\widehat{Q}_{\mathrm{j}}^G(t)\right)^{\mathcal{T}} D_u\widehat{\sigma}_{\mathrm{i}}(t)(u_t - \widehat{u}_t)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}dt + \int_0^T (dK_t)^{\mathcal{T}} R(t)\right].
\end{aligned} \tag{2.23}$$

We define the Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{d\times n} \times U \to \mathbb{R}$ by

$$\begin{aligned}
H(t) &= H\left(t, x_t, p_t^G, Q_{\mathrm{i}}^G(t), u_t\right) = l(t) + m_{\mathrm{ij}}(t)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt} \\
&+ \left((b(t))^{\mathcal{T}} + (h_{\mathrm{ij}}(t))^{\mathcal{T}}\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}\right) p^G(t) + (\sigma_{\mathrm{i}}(t))^{\mathcal{T}} Q_{\mathrm{j}}^G(t)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}.
\end{aligned} \tag{2.24}$$

Hence, (2.23) becomes

$$0 \leq \mathbb{E}_G \left[\int_0^T \left(\nabla_u H \left(t, \widehat{x}_t, \widehat{p}^G(t), \widehat{Q}_{\mathsf{i}}^G(t), \widehat{u}_t \right) \right)^{\mathcal{T}} (u_t - \widehat{u}_t)\, dt + \int_0^T (dK_t)^{\mathcal{T}} R(t) \right].$$

First, by replacing $\xi(t)$ with its value in (2.20), we get the G-adjoint process as

$$\begin{aligned} \widehat{p}^G(t) &= \Psi(t)\,\xi(t) \\ &= \mathbb{E}_G \Bigg[\Psi(t)\,\Phi(T)\,\nabla_x g(\widehat{x}_T) + \Psi(t) \int_t^T \Phi(s)\,\nabla_x \widehat{l}(s)\, ds \\ &\quad + \Psi(t) \int_t^T \Phi(s)\,\nabla_x \widehat{m}_{\mathsf{ij}}(s)\, d\langle B \rangle_t^{\mathsf{ij}} \,|\mathcal{F}_t \Bigg] \\ &= \mathbb{E}_G \left[\Psi(t)\,\Phi(T)\,\nabla_x g(x_T)\,|\mathcal{F}_t \right] + \Psi(t) \int_t^T \Phi(s)\,\nabla_x \widehat{l}(s)\, ds \\ &\quad + \Psi(t) \int_t^T \Phi(s)\,\nabla_x \widehat{m}_{\mathsf{ij}}(s)\, d\langle B \rangle_t^{\mathsf{ij}}. \end{aligned} \tag{2.25}$$

Second, by applying integration by parts for G-Itô's processes to (2.20), we find the G-adjoint equation satisfied by the above G-adjoint process

$$\begin{aligned} d\widehat{p}^G(t) = {} & \Psi(t) \left[-\Phi(t)\,\nabla_x \widehat{l}(t)\, dt - \Phi(t)\,\nabla_x \widehat{m}_{\mathsf{ij}}(t)\, d\langle B \rangle_t^{\mathsf{ij}} + Z_{\mathsf{i}}(t) dB_t^{\mathsf{i}} + dK_t \right] \\ & + \xi(t) \Big[-\Psi(t)\, D_x \widehat{b}(t)\, dt + \Psi(t) \left(D_x \widehat{\sigma}_{\mathsf{i}}(t) \left(D_x \widehat{\sigma}_{\mathsf{j}}(t) \right)^{\mathcal{T}} - D_x \widehat{h}_{\mathsf{ij}}(t) \right) d\langle B \rangle_t^{\mathsf{ij}} \\ & - \Psi(t)\, D_x \widehat{\sigma}_{\mathsf{i}}(t)\, dB_t^{\mathsf{i}} \Big] - Z_{\mathsf{i}}(t) \Psi(t)\, D_x \widehat{\sigma}_{\mathsf{i}}(t)\, d\langle B \rangle_t^{\mathsf{ij}} \\ = {} & -\nabla_x \widehat{l}(t)\, dt - \nabla_x \widehat{m}_{\mathsf{ij}}(t)\, d\langle B \rangle_t^{\mathsf{ij}} - \left(D_x \widehat{b}(t) \right)^{\mathcal{T}} \widehat{p}^G(t)\, dt \\ & - \left(D_x \widehat{h}_{\mathsf{ij}}(t) \right)^{\mathcal{T}} \widehat{p}^G(t) \frac{d\langle B \rangle_t^{\mathsf{ij}}}{dt} dt - \left(D_x \widehat{\sigma}_{\mathsf{i}}(t) \right)^{\mathcal{T}} \big(\Psi(t)\, Z_{\mathsf{j}}(t) \\ & - D_x \widehat{\sigma}_{\mathsf{j}}(t)\, \widehat{p}^G(t) \big) \frac{d\langle B \rangle_t^{\mathsf{ij}}}{dt} dt + \left(\Psi(t)\, Z_{\mathsf{i}}(t) - D_x \widehat{\sigma}_{\mathsf{i}}(t)\, \widehat{p}^G(t) \right) dB_t^{\mathsf{i}} + \Psi(t)\, dK_t. \end{aligned}$$

By (2.21), we obtain

$$\begin{aligned} d\widehat{p}^{G}(t) = &-\left(\nabla_x \widehat{l}(t)\right)^{\mathcal{T}} dt - \nabla_x \widehat{m}_{\mathsf{ij}}(t)\, d\langle B\rangle_t^{\mathsf{ij}} - \left(D_x \widehat{b}(t)\right)^{\mathcal{T}} \widehat{p}^{G}(t)\, dt \\ &- \left(D_x \widehat{h}_{\mathsf{ij}}(t)\right)^{\mathcal{T}} \widehat{p}^{G}(t) \frac{d\langle B\rangle_t^{\mathsf{ij}}}{dt} dt + \widehat{Q}_{\mathsf{i}}^{G}(t) dB_t^{\mathsf{i}} \\ &- \left(D_x \widehat{\sigma}_{\mathsf{i}}(t)\right)^{\mathcal{T}} \widehat{Q}_{\mathsf{j}}^{G}(t) \frac{d\langle B\rangle_t^{\mathsf{ij}}}{dt} dt + \Psi(t)\, dK_t. \end{aligned}$$

By (2.24) , we have

$$\begin{aligned} \nabla_x \widehat{H}(t) &= \nabla_x H\left(t, \widehat{x}_t, \widehat{p}^{G}(t), \widehat{Q}_{\mathsf{i}}^{G}(t), \widehat{u}_t\right) \\ &= \nabla_x \widehat{l}(t) + \nabla_x \widehat{m}_{\mathsf{ij}}(t)\, d\langle B\rangle_t^{\mathsf{ij}} + \left(D_x \widehat{b}(t)\right)^{\mathcal{T}} \widehat{p}^{G}(t) \\ &+ \left(\left(D_x \widehat{h}_{\mathsf{ij}}(t)\right)^{\mathcal{T}} \widehat{p}^{G}(t) + \left(D_x \widehat{\sigma}_{\mathsf{i}}(t)\right)^{\mathcal{T}} \widehat{Q}_{\mathsf{j}}^{G}(t)\right) \frac{d\langle B\rangle_t^{\mathsf{ij}}}{dt}. \end{aligned}$$

Then, we get

$$\begin{aligned} d\widehat{p}^{G}(t) &= -\nabla_x \widehat{H}(t)\, dt + \widehat{Q}_{\mathsf{i}}^{G}(t)\, dB_t^{\mathsf{i}} + \Psi(t)\, dK_t, \\ \widehat{p}^{G}(T) &= \nabla_x g(\widehat{x}_T). \end{aligned} \tag{2.26}$$

Remark 2.2 *We prefer to call* (2.26) *a backward stochastic differential equation under G-expectation (see Peng [28]), or a fully nonlinear BSDE, instead of 2BSDE.*

2.4 A stochastic maximum principle for G-SDE

2.4.1 A necessary maximum principle

We can now state necessary optimality conditions for stochastic control problem (2.2) , (2.3) , and (2.4) for G-SDE in the global form.

Theorem 2.2 ***(Necessary optimality conditions for G-SDE)*** *Let $\widehat{u}$ be an optimal control minimizing the functional $\mathcal{J}$ over U_{ad} and $\widehat{x}$ be the solution of* (2.2) *associated with $\widehat{u}$. Then, there exists unique process $\left(\widehat{p}^{G}, \widehat{Q}_{\mathsf{i}}^{G}, K\right)$ of the G-BSDE system* *(called G-adjoint equation), and by using the notation given in the last section, we have*

$$\begin{cases} d\widehat{p}^{G}(t) = -\nabla_x \widehat{H}(t)\,dt + \widehat{Q}_{\mathrm{i}}^{G}(t)\,dB_t^{\mathrm{i}} + \Psi(t)\,dK_t; \quad 0 \le t \le T, \\ \widehat{p}^{G}(T) = \nabla_x g(\widehat{x}_T). \end{cases}$$

where $\left(\widehat{p}^{G}, \widehat{Q}_{\mathrm{i}}^{G}, K\right) \in \mathcal{G}_G^2(0,T)$, *such that, for every* $u \in U_{ad}$, *we have*

$$\mathbb{E}_G\left[\int_0^T \left(\nabla_u H\left(t, \widehat{x}_t, \widehat{p}^{G}(t), \widehat{Q}_{\mathrm{i}}^{G}(t), \widehat{u}_t\right)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\,dt + \int_0^T (dK_t)^{\mathcal{T}} R(t)\right] \ge 0. \tag{2.27}$$

Proof. By using the above procedure, we obtain (2.27) where the Hamiltonian given by (2.24) and the terms of the adjoint process $\left(\widehat{p}^{G}, \widehat{Q}_{\mathrm{i}}^{G}, K\right)$ given by $(2.20), (2.21)$ respectively. Hence, since $\widehat{u}$ is an optimal control, the variational inequality (2.13) translates into (2.27), for all $u \in U_{ad}$, almost every $0 \le t \le T$ and this completed the proof of Theorem 2.2. ■

2.4.2 A sufficient maximum principle

In this section, we study when the necessary optimality condition (2.27) becomes sufficient. For any $u \in U_{ad}$, we denote x, $\widehat{x}$ the solution of equation (2.2) controlled by u and $\widehat{u}$ respectively.

Assumption 2.3 *The function g is convex with respect to the state variable.*

Assumption 2.4 *The Hamiltonian H is convex with respect to (x, u).*

Theorem 2.3 ***(Sufficient optimality conditions for G-SDE)*** *Assume the assumptions* $(2.1) - (2.4)$ *are satisfied. Then,* $\widehat{u}$ *is an optimal solution of the stochastic control problem* $\{(2.2), (2.3), (2.4)\}$ *if it satisfies* (2.27).

Proof. Let $\widehat{u}$ be an arbitrary element of U_{ad} (candidate to be optimal). For any $u \in U_{ad}$, and using the definition of the sublinear expectation $\mathbb{E}_G$ (**??**), we have for all $\mathbb{P} \in \mathcal{P}$ q.s.

$$\begin{aligned}
&\frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \\
&= \frac{1}{\theta}\mathbb{E}^{\mathbb{P}}\left[g\left(x_T^{\theta}\right) - g\left(\widehat{x}_T\right) + \int_0^T \left(l^{\theta}(t) - \widehat{l}(t)\right) dt + \int_0^T \left(m_{\mathrm{ij}}^{\theta}(t) - \widehat{m}_{\mathrm{ij}}(t)\right) d\langle B\rangle_t^{\mathrm{ij}}\right].
\end{aligned}$$

By the convexity of the function g, we get

$$g\left(x_T^{\theta}\right) - g\left(\widehat{x}_T\right) \geq \left(\nabla_x g\left(\widehat{x}_T\right)\right)^{\mathcal{T}}\left(x_T^{\theta} - \widehat{x}_T\right) = \left(\widehat{p}^G(T)\right)^{\mathcal{T}}\left(x_T^{\theta} - \widehat{x}_T\right).$$

Using G-Itô's formula to $\left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(x_t^{\theta} - \widehat{x}_t\right)$, we have

$$\begin{aligned}
&\left(\widehat{p}^G(T)\right)^{\mathcal{T}}\left(x_T^{\theta} - \widehat{x}_T\right) = -\int_0^T \left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}}\left(x_t^{\theta} - \widehat{x}_t\right) dt \\
&+ \int_0^T \left(\widehat{Q}_{\mathrm{i}}^G(t)\right)^{\mathcal{T}}\left(x_t^{\theta} - \widehat{x}_t\right) dB_t^{\mathrm{i}} + \int_0^T \left(dK_t\right)^{\mathcal{T}}\Psi(t)\left(x_t^{\theta} - \widehat{x}_t\right) \\
&+ \int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(b^{\theta}(t) - \widehat{b}(t)\right) dt + \int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(h_{\mathrm{ij}}^{\theta}(t) - \widehat{h}_{\mathrm{ij}}(t)\right)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}dt \\
&+ \int_0^T \left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(\sigma_{\mathrm{i}}^{\theta}(t) - \widehat{\sigma}_{\mathrm{i}}(t)\right) dB_t^{\mathrm{i}} + \int_0^T \left(\widehat{Q}_{\mathrm{i}}^G(t)\right)^{\mathcal{T}}\left(\sigma_{\mathrm{i}}^{\theta}(t) - \widehat{\sigma}_{\mathrm{i}}(t)\right)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}dt. \quad (2.28)
\end{aligned}$$

By the definition of H (2.24), we can write

$$\begin{aligned}
&\int_0^T \left(l^{\theta}(t) - \widehat{l}(t)\right) dt + \int_0^T \left(m_{\mathrm{ij}}^{\theta}(t) - \widehat{m}_{\mathrm{ij}}(t)\right)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}dt \\
&= \int_0^T \left[H^{\theta}(t) - \widehat{H}(t) - \left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(b^{\theta}(t) - \widehat{b}(t)\right)\right. \\
&\left. - \left(\left(\widehat{p}^G(t)\right)^{\mathcal{T}}\left(h_{\mathrm{ij}}^{\theta}(t) - \widehat{h}_{\mathrm{ij}}(t)\right) + \left(\widehat{Q}_{\mathrm{i}}^G(t)\right)^{\mathcal{T}}\left(\sigma_{\mathrm{i}}^{\theta}(t) - \widehat{\sigma}_{\mathrm{i}}(t)\right)\right)\frac{d\langle B\rangle_t^{\mathrm{ij}}}{dt}\right] dt. \quad (2.29)
\end{aligned}$$

By the convexity of the Hamiltonian H with respect to (x, u), we have

$$H^{\theta}(t) - \widehat{H}(t) \geq \left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}} \left(x_t^{\theta} - \widehat{x}_t\right) + \left(\nabla_u \widehat{H}(t)\right)^{\mathcal{T}} \left(u_t^{\theta} - \widehat{u}_t\right).$$

Adding (2.28) and (2.29), hence, As K is non-increasing G-martingale and B is a symmetric G-martingale, we get for all $\mathbb{P} \in \mathcal{P}$ q.s.

$$\frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \geq \frac{1}{\theta}\mathbb{E}^{\mathbb{P}}\left[\left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}} \left(u_t^{\theta} - \widehat{u}_t\right) dt + \int_0^T (dK_t)^{\mathcal{T}} \Psi(t) \left(x_t^{\theta} - \widehat{x}_t\right)\right].$$

Then

$$\frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \geq \mathbb{E}^{\mathbb{P}}\left[\left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\, dt + \frac{1}{\theta}\int_0^T (dK_t)^{\mathcal{T}} \Psi(t) \left(x_t^{\theta} - \widehat{x}_t\right)\right].$$

Then, by sending $\theta \to 0$, thanks to lemma 2.2, thus

$$\lim_{\theta \to 0} \frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \geq \mathbb{E}^{\mathbb{P}}\left[\left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\, dt + \int_0^T (dK_t)^{\mathcal{T}} \Psi(t)\, Y_t\right].$$

Hence

$$\lim_{\theta \to 0} \frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \geq \mathbb{E}^{\mathbb{P}}\left[\left(\nabla_x \widehat{H}(t)\right)^{\mathcal{T}} (u_t - \widehat{u}_t)\, dt + \int_0^T (dK_t)^{\mathcal{T}} R(t)\right].$$

By (2.27), the last inequality implies that for all $\mathbb{P} \in \mathcal{P}$ q.s.

$$\lim_{\theta \to 0} \frac{1}{\theta}\{J^{\mathbb{P}}(u^{\theta}) - J^{\mathbb{P}}(\widehat{u})\} \geq 0.$$

Thus $\mathcal{J}(u) - \mathcal{J}(\widehat{u}) \geq 0$. This proves that the control $\widehat{u}$ is optimal for the problem $\{(2.2), (2.3), (2.4)\}$. ■

2.5 LQ problem

As an application of our result, in this section, we consider a one-dimensional linear quadratic (LQ) control problem for G-Stochastic Maximum principle in which the state equation dynamics is

$$\begin{cases} dx_t^u &= (a_t x_t + b_t u_t + c_t)\, dt + \sigma_t dB_t, \\ x_0^u &= \xi \in \mathbb{R}. \end{cases} \tag{2.30}$$

Our linear quadratic functional cost is given by

$$\mathcal{J}(u) = \mathbb{E}_G \left[\int_0^T l\left(t, x_t^u, u_t\right) dt + g\left(x_T^u\right) \right], \tag{2.31}$$

where

$$l\left(t, x_t^u, u_t\right) = \frac{1}{2}\left(m_t u_t^2 + n_t x_t^2\right), \text{ and } g\left(x_T^u\right) = \frac{1}{2}\left(x_T^u\right)^2. \tag{2.32}$$

The Hamiltonian H defined by

$$H\left(t, x_t^u, u_t, \widehat{p}^G(t), \widehat{Q}^G(t)\right) = \frac{1}{2}\left(m_t u_t^2 + n_t x_t^2\right) + \widehat{p}^G(t)\left(a_t x_t + b_t u_t + c_t\right) + \widehat{Q}^G(t)\, \sigma_t \frac{d\langle B\rangle_t}{dt}. \tag{2.33}$$

We want to minimize (2.31) subject to (2.30) by choosing u over U_{ad}, we can check that all the assumptions in section 05, are satisfied. Hence, we may apply Theorem 2.2 to solve our G-Linear Quadratic stochastic optimal control problem $\{(2.30), (2.31)\}$.

Let $(\widehat{x}_t, \widehat{u})$ be an optimal solution, and the G-adjoint equation (2.26) can be written by

$$\begin{cases} d\widehat{p}^G(t) &= -\left(\widehat{p}^G(t)\, a_t + n_t \widehat{x}_t\right) dt + \widehat{Q}^G(t)\, dB_t + d\widehat{K}_t, \\ \widehat{p}^G(T) &= \widehat{x}_T. \end{cases} \tag{2.34}$$

Minimizing the Hamiltonian (2.33) , we obtain

$$\widehat{u}_t = -\frac{b_t}{m_t}\widehat{p}^G(t). \tag{2.35}$$

Substituting (2.35) into the G-SDE (2.30), we get

$$\begin{cases} d\widehat{x}_t &= \left(a_t\widehat{x}_t - \dfrac{b_t^2}{m_t}\widehat{p}^G(t) + c_t\right)dt + \sigma_t dB_t, \\ \widehat{x}_0 &= \xi \in \mathbb{R}. \end{cases} . \tag{2.36}$$

Similarly, substituting (2.35) into the G-BSDE (2.34) , we get

$$\begin{cases} d\widehat{p}^G(t) &= -\left(\widehat{p}^G(t)\, a_t + n_t\widehat{x}_t\right)dt + \widehat{Q}^G(t)\, dB_t + dK_t, \\ \widehat{p}^G(T) &= \widehat{x}_T. \end{cases} \tag{2.37}$$

Unfortunately, such a system cannot be solved explicitly. for this raison let us guess that (2.37) admits the solution of the following form

$$\widehat{p}^G(t) = \varphi(t)\,\widehat{x}_t + \chi(t), \tag{2.38}$$

for some deterministic differentiable functions $\varphi(t)$ and $\chi(t)$. Application of Itô's formula to (2.38), gives

$$d\widehat{p}^G(t) = \left[\varphi'(t)\,\widehat{x}_t + \varphi(t)\left(a_t\widehat{x}_t - \frac{b_t^2}{m_t}\widehat{p}^G(t) + c_t\right) + \chi'(t)\right]dt \tag{2.39}$$
$$+ \varphi(t)\,\sigma_t dB_t.$$

Replacing (2.38) into (2.39), we get

$$d\widehat{p}^G(t) = \left[\left(\varphi'(t) - \frac{b_t^2}{m_t}\varphi^2(t) + \varphi(t)\, a_t\right)\widehat{x}_t - \frac{b_t^2}{m_t}\varphi(t)\,\chi(t) + \chi'(t) + c_t\right]dt \tag{2.40}$$
$$+ \varphi(t)\,\sigma_t dB_t.$$

On the other hand, after substituting (2.38) into (2.37), we arrive at

$$\begin{cases} d\widehat{p}^{G}(t) &= -(\varphi(t)\, a_t + n_t)\, \widehat{x}_t dt + \widehat{Q}^{G}(t)\, dB_t + dK_t, \\ \widehat{p}^{G}(T) &= \widehat{x}_T. \end{cases} \tag{2.41}$$

By equating the coefficients of (2.40) and (2.41), it gives

$$\left(\widehat{p}^{G}(t), \widehat{Q}^{G}(t)\right) = \left(\varphi(t)\, \widehat{x}_t + \chi(t), \varphi(t)\, \sigma_t\right), \tag{2.42}$$

where $\varphi(t)$ is the solution of the following Riccati type equation

$$\begin{cases} \varphi'(t) - \dfrac{b_t^2}{m_t}\varphi^2(t) + 2a_t\varphi(t) + n_t = 0, \\ \varphi(T) = 1, \end{cases} \tag{2.43}$$

and $\chi(t)$ is a solution of the following ordinary differential equation

$$\begin{cases} \chi'(t) - \dfrac{b_t^2}{m_t}\varphi(t)\,\chi(t) + c_t = 0, \\ \chi(T) = 0. \end{cases} \tag{2.44}$$

Replacing the value of functions φ and χ in the optimal control from state (2.35), we get this feedback form of the optimal control

$$\widehat{u}_t = -\frac{b_t}{m_t}\left(\varphi(t)\, \widehat{x}_t + \chi(t)\right). \tag{2.45}$$

Assuming that the discriminant $\Delta = 4a^2 + 4n\dfrac{b^2}{m} > 0$, we obtain

$$dt = \frac{m}{b^2}\frac{d\varphi(t)}{\varphi^2(t) - 2a_t\dfrac{m}{b^2}\varphi(t) + \dfrac{m}{b^2}n_t}.$$

If we denote

$$L(t)=\frac{\varphi(T)-\delta_2}{\varphi(T)-\delta_1}\exp\left(2a(T-t)\right),\ \delta_1=-\frac{\frac{2am}{b^2}-\sqrt{\Delta}}{2},\ \text{and } \delta_2=-\frac{\frac{2am}{b^2}+\sqrt{\Delta}}{2}.$$

Then, we get

$$2a(T-t)=\int_t^T\left(\frac{1}{\varphi(s)-\delta_2}-\frac{1}{\varphi(s)-\delta_1}\right)d\varphi(s).$$

$$\varphi(t)=\frac{\delta_1-\delta_2 L(t)}{1-L(t)}. \tag{2.46}$$

The explicit solution of the equation (2.44) is

$$\chi(t)=\left[\exp\left(\int_t^T\frac{b^2}{m}\varphi(s)\,ds\right)\right]\left[\int_t^T-c\exp\left(\int_t^T\frac{b^2}{m}\varphi(r)\,dr\right)ds\right], \tag{2.47}$$

where $\varphi(s)$ is determined by (2.46).

To summarize, we have the following result.

Corollary 2.1 *The Riccati equation* (2.43) *admits an explicit solution* (2.46)*. Moreover equation* (2.44) *admits an explicit solution* (2.47)*.*

Moreover, by Theorem 2.3, we have the following verification result.

Corollary 2.2 *If equations* (2.43) *and* (2.44) *admit the solutions* $\varphi(.)$ *and* $\chi(.)$ *respectively, then the feedback control* (2.45) *of our G-linear quadratic stochastic optimal control problem* $\{(2.30),(2.31)\}$ *is optimal.*

Theorem 2.4 *If the equations* (2.43) *admits the solutions* $\varphi(.)$ *given by* (2.46)*, then the optimal control of our G-Linear Quadratic stochastic optimal control problem* $\{(2.30),(2.31)\}$ *has the state feedback form* (2.45)*.*

Chapter 3

Solving the Risk-Sensitive Control Problem Using G-Stochastic Maximum Principle for Risk-Neutral Control Problem

In this chapter, we discuss the risk-sensitive control problem where the state is described by G-SDE with an exponential of integral cost functional which is fundamentally different from the existing results. We have proved in detail four results. The first is the stability between the perturbed solution and the optimal solution of the controlled G-stochastic differential equations, while the second is the mean-variance uncertainty of loss functional. Third, we apply the results of risk-neutral control problem type to solve risk-sensitive control problem where the last result is the relation between G-expected exponential utility and G-quadratic backward stochastic differential equation. In addition, an important contribution of this chapter is the method of proof, which consists of two steps outlined as follows. The first is a simple reformulation of the risk-sensitive problem as a standard risk-neutral problem by augmenting the state with an auxiliary process (this new state process represents the evolution of the cost over time). An intermediate G-maximum principle is then obtained by a standard application of risk-neutral result to the augmented problem where the second adjoint equation is a nonlinear G-backward stochastic differential equation (G-BSDE see [14]) and the first adjoint equation is linear. We study a class of stochastic control problems of the type (for simplicity in one dimension)

$$\begin{cases} dx_t &= b\left(t, x_t, u_t\right) dt + h\left(t, x_t, u_t\right) d\left\langle B\right\rangle_t + \sigma\left(t, x_t, u_t\right) dB_t, \\ x\left(0\right) &= x_0 \in \mathbb{R}^n, \end{cases}$$

where b, h and σ are uniformly Lipschitz, x_0 is the initial state. The cost functional to be minimized over the class of admissible controls has the form

$$\mathbb{E}_G\left[\exp \varepsilon\left(g\left(x_T^u\right) + \int_0^T l\left(t, x_t^u, u_t\right) dt + \int_0^T m(t, x_t^u, u_t) d\left\langle B\right\rangle_t\right)\right].$$

This chapter is organized as follows. In section 01, we formulate the problem and give the various assumptions used throughout this part and proof of some results about the stability between the perturbed solution and the optimal solution of the controlled G-stochastic differential equations. In section 02, we introduce the mean-variance uncertainty of loss functional that has been studied by [6] with details in the classical case of linear expectation. Section 03, is devoted to applying and proving in detail risk-neutral control problem to solve our problem of risk-sensitive control problem. In section 04, we give and prove the relationship between the G-expected exponential utility and the G-quadratic backward stochastic differential equations. In the last section, we give an example of Merton's problem with power utility and here we basically use G-Girsanov's theorem.

3.1 Statement of risk-sensitive control problem under G-expectation

Let T be a strictly positive real number and U is a nonempty convex subset of $\mathbb{R}$. A controller then intervenes on the system via an $\mathcal{F}_t$-adapted stochastic process u. The set of those controls is called admissible and denoted by U_{ad}. When the controller acts with u under almost every $\mathbb{P} \in \mathcal{P}$,

$$U_{ad} = \left\{ u | \ u \text{ is } \mathcal{F}_t\text{-adapted process with valued in } U, \ u \in \mathbb{M}_G^2 (0,T) \right\}.$$

Let us now consider some systems, whose evolution is described (for simplicity) by the canonical G-Brownian motion B, $u \in U_{ad}$ and initial state $x_0 \in \mathbb{R}$. Then for all $0 \leq t \leq T$ we set

$$\begin{cases} dx^u(t) &= b(t, x_t^u, u_t) dt + h(t, x_t^u, u_t) d\langle B \rangle_t + \sigma(t, x_t^u, u_t) dB_t \\ x^u(0) &= x_0 \in \mathbb{R}. \end{cases} \tag{3.1}$$

We define the risk-sensitive cost functional under G-expectation type associated with (3.1) with terminal cost as follows

$$\begin{aligned} \mathcal{J}^\varepsilon(u) &= \mathbb{E}_G \left[\exp \varepsilon \left(g(x_T^u) + \int_0^T l(t, x_t^u, u_t) dt + \int_0^T m(t, x_t^u, u_t) d\langle B \rangle_t \right) \right], \qquad (3.2) \\ &= \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\exp \varepsilon \left(g(x_T^u) + \int_0^T l(t, x_t^u, u_t) dt + \int_0^T m(t, x_t^u, u_t) d\langle B \rangle_t \right) \right], \\ &:= \sup_{\mathbb{P} \in \mathcal{P}} J^{\mathbb{P}\varepsilon}(u), \end{aligned}$$

where $u \in U_{ad}$ and ε is the risk sensitivity index.

Assumption 3.1 *We recall the following assumptions*

(A1) The functions $b, h, \sigma, l, m : [0,T] \times \mathbb{R} \times U \to \mathbb{R}$ *are given and satisfying* $b(., x, u)$,

$h(.,x,u)$, $\sigma(.,x,u)$, $l(.,x,u)$, $m(.,x,u) \in \mathbb{M}_G^2(0,T)$ and $g:\mathbb{R}\to\mathbb{R}$ for each $x\in\mathbb{R}$ and $u\in U$.

(A2) There exists constant κ such that

$|\varphi(t,x,u)-\varphi(t,y,w)| \leq \kappa(|x-y|+|u-w|)$ for each $x,y\in\mathbb{R}$ and $u,w\in U$ where $\varphi := b,h,\sigma,l,m$.

Theorem 3.1 *(See [30]) Under the above assumption, for every $u\in U_{ad}$ the equation* (3.1) *has unique strong solution $x\in\mathbb{S}_G^2(0,T)$ and the standard estimates show that $\mathbb{E}_G\left[\sup_{t\in[0,T]}|x_t|^2\right]<\infty$.*

Our objective is to minimize the functional $\mathcal{J}^\varepsilon$ over U_{ad}. If $\widehat{u}\in U_{ad}$ is an optimal control, that is

$$\mathcal{J}^\varepsilon(\widehat{u}) = \inf_{u\in U_{ad}} \mathcal{J}^\varepsilon(u). \tag{3.3}$$

Remark 3.1 *The set U_{ad} of all admissible controls is convex.*

A control that solves the problem $\{(3.1),(3.2),(3.3)\}$ is called optimal. If an optimal control minimises the cost $\mathcal{J}^\varepsilon$ over U_{ad} exists, we seek necessary optimality conditions checked by this control in the form of G-stochastic maximum principle.

Assumption 3.2 *We will work under the following standard assumptions*

(A1) Assume b, h, σ, l and m are continuously differentiable with respect to (x,u).

(A2) All the derivatives of b, h, σ, l and m are uniformly bounded by $C(1+|x|+|u|)$.

(A3) g is continuously differentiable with respect to x and its derivative is uniformly bounded by $C(1+|x|\ )$.

Remark 3.2 *It is not easy to solve the problem $\{(3.1),(3.2),(3.3)\}$ and seek the necessary condition by the classical method. For this reason, we may follow a new method.*

We may introduce an auxiliary state process y_t^u which is solution of the following G-SDE

$$dy_t^u = l(t,x_t^u,u_t)\,dt + m(t,x_t^u,u_t)d\langle B\rangle_t, \qquad y_0^u = 0.$$

From the above auxiliary process, our control problem of $\{(3.1),(3.2),(3.3)\}$ is equivalent to

$$\begin{cases} \inf_{u\in U_{ad}} \mathbb{E}_G\left[f^{\varepsilon}\left(x_T^u, y_T^u\right)\right]; \\ \text{subject to} \\ dy_t^u = l\left(t, x_t^u, u_t\right)dt + m(t, x_t^u, u_t)d\left\langle B\right\rangle_t; \\ dx_t^u = b\left(t, x_t^u, u_t\right)dt + h\left(t, x_t^u, u_t\right)d\left\langle B\right\rangle_t + \sigma\left(t, x_t^u, u_t\right)dB_t; \\ y^u\left(0\right) = 0; \quad x^u\left(0\right) = x_0; \end{cases} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{J}^{\varepsilon}\left(u\right) &= \mathbb{E}_G\left[\exp\varepsilon\left(g\left(x_T^u\right) + \int_0^T l\left(t, x_t^u, u_t\right)dt + \int_0^T m(t, x_t^u, u_t)d\left\langle B\right\rangle_t\right)\right]; \\ &= \mathbb{E}_G\left[\exp\varepsilon\left(g\left(x_T^u\right) + y_T^u\right)\right]; \qquad u\in U_{ad}; \\ &= \mathbb{E}_G\left[f^{\varepsilon}\left(x_T^u, y_T^u\right)\right]. \end{aligned}$$

We will use a method which consists in perturbing the optimal control $\widehat{u}$ as follows

$$u_t^{\theta} = \widehat{u}_t + \theta\left(u_t - \widehat{u}_t\right). \tag{3.5}$$

We denote by $x_t^{\theta}, y_t^{\theta}$ the trajectories of the system corresponding to u_t^{θ} as follow

$$\begin{cases} dy_t^{\theta} = l\left(t, x_t^{\theta}, u_t^{\theta}\right)dt + m(t, x_t^{\theta}, u_t^{\theta})d\left\langle B\right\rangle_t; \qquad 0\leq t\leq T. \\ dx_t^{\theta} = b\left(t, x_t^{\theta}, u_t^{\theta}\right)dt + h\left(t, x_t^{\theta}, u_t^{\theta}\right)d\left\langle B\right\rangle_t + \sigma\left(t, x_t^{\theta}, u_t^{\theta}\right)dB_t; \\ y^{\theta}\left(0\right) = 0; \quad x^{\theta}\left(0\right) = x. \end{cases} \tag{3.6}$$

Since U_{ad} is convex, we have the perturbed control $u^{\theta}\in U_{ad}$, hence u^{θ} is an admissible control and from the optimality of $\widehat{u}$, we have

$$\mathcal{J}^{\varepsilon}\left(u^{\theta}\right) \geq \mathcal{J}^{\varepsilon}\left(\widehat{u}\right).$$

Then

$$\mathbb{E}_G\left[f^{\varepsilon}\left(x_T^{\theta}, y_T^{\theta}\right)\right] - \mathbb{E}_G\left[f^{\varepsilon}\left(\widehat{x}_T, \widehat{y}_T\right)\right] \geq 0.$$

In this case, using the property 03 of definition 1.1, we get

$$\mathbb{E}_G\left[f^{\varepsilon}\left(x_T^{\theta}, y_T^{\theta}\right) - f^{\varepsilon}\left(\widehat{x}_T, \widehat{y}_T\right)\right] \geq 0. \tag{3.7}$$

The next Lemma gives the stability between the perturbed solution and the optimal solution of the controlled G-stochastic differential equations.

Lemma 3.1 *Let $\widehat{u} \in U_{ad}$ be an optimal control and $(\widehat{x}, \widehat{y})$ the corresponding trajectories. Then under the assumptions* 3.1 *and* 3.2*, we have*

$$\lim_{\theta \to 0} \mathbb{E}_G\left[\sup_{t \in [0,T]} \left|x_t^{\theta} - \widehat{x}_t\right|^2\right] = 0. \tag{3.8}$$

$$\lim_{\theta \to 0} \mathbb{E}_G\left[\sup_{t \in [0,T]} \left|y_t^{\theta} - \widehat{y}_t\right|^2\right] = 0. \tag{3.9}$$

Proof. Let $\widehat{x}, \widehat{y}, x^{\theta}, y^{\theta}$ denote the solution of the systems (3.4) and (3.6) respectively. The proof of (3.8) is the same as lemma 3.9 [7]. For the second limit (3.9), Applying G-Hölder's inequality and G-BDG inequality (1.7) yields that

$$\begin{aligned}
\mathbb{E}_G\left[\sup_{t \in [0,T]} \left|y_t^{\theta} - \widehat{y}_t\right|^2\right] &\leq 4T \int_0^T \left\{\mathbb{E}_G\left[\left|l\left(s, x_s^{\theta}, u_s^{\theta}\right) - l\left(s, \widehat{x}_s, u_s^{\theta}\right)\right|^2\right]\right. \\
&+ \mathbb{E}_G\left[\left|l\left(s, \widehat{x}_s, u_s^{\theta}\right) - l\left(s, \widehat{x}_s, \widehat{u}_s\right)\right|^2\right]\Big\} ds \\
&+ 4\overline{\sigma}^4 T \int_0^T \left\{\mathbb{E}_G\left[\left|m\left(s, x_s^{\theta}, u_s^{\theta}\right) - m\left(s, \widehat{x}_s, u_s^{\theta}\right)\right|^2\right]\right. \\
&+ \mathbb{E}_G\left[\left|m\left(s, \widehat{x}_s, u_s^{\theta}\right) - m\left(s, \widehat{x}_s, \widehat{u}_s\right)\right|^2\right]\Big\} ds.
\end{aligned}$$

Since the coefficients l and m are Lipschitz with respect to (x, u), we easily obtain the following estimate

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|y_t^\theta-\widehat{y}_t\right|^2\right]\leq 4T\kappa(1+\overline{\sigma}^4)\int_0^T\mathbb{E}_G\left|x_s^\theta-\widehat{x}_s\right|^2ds$$
$$+4T\kappa(1+\overline{\sigma}^4)\int_0^T\mathbb{E}_G\left|u_s^\theta-\widehat{u}_s\right|^2ds.$$

By sending $\theta\to 0$ and using the first limit we arrive at the desired result (3.9). ■

We introduce the short hand notation for the sake of simplicity: $\varrho(t,x_t^u,u_t)=\varrho^u(t)$, $\varrho(t,\widehat{x}_t,\widehat{u}_t)=\widehat{\varrho}(t)$, $\varrho\left(t,x_t^\theta,u_t^\theta\right)=\varrho^\theta(t)$ and $\varrho_x(t)=\frac{d\varrho}{dx}(t,x_t,u_t)$ for all $\varrho=b,h,\sigma,l$ and m.

Lemma 3.2 *We assume that the assumptions* 3.2 *are satisfied, then we have*

$$\begin{cases}dY_t=\left(\widehat{l}_x(t)X_t+\widehat{l}_u(t)(u_t-\widehat{u}_t)\right)dt\\ \qquad+\left(\widehat{m}_x(t)X_t+\widehat{m}_u(t)(u_t-\widehat{u}_t)\right)d\langle B\rangle_t,\\ dX_t=\left(\widehat{b}_x(t)X_t+\widehat{b}_u(t)(u_t-\widehat{u}_t)\right)dt\\ \qquad+\left(\widehat{h}_x(t)X_t+\widehat{h}_u(t)(u_t-\widehat{u}_t)\right)d\langle B\rangle_t\\ \qquad+\left(\widehat{\sigma}_x(t)X_t+\widehat{\sigma}_u(t)(u_t-\widehat{u}_t)\right)dB_t,\\ Y_0=0,\qquad X_0=0,\end{cases}\tag{3.10}$$

where $Y_t=\lim_{\theta\to0}\frac{1}{\theta}\left(y_t^\theta-\widehat{y}_t\right)$ *and* $X_t=\lim_{\theta\to0}\frac{1}{\theta}\left(x_t^\theta-\widehat{x}_t\right)$ *are the variational equations along the optimal control.*

Proof. Using Taylor's expansion with an integral remain on the function $b\left(t,x_t^\theta,u_t^\theta\right)$, $h\left(t,x_t^\theta,u_t^\theta\right)$, $\sigma\left(t,x_t^\theta,u_t^\theta\right)$, $l\left(t,x_t^\theta,u_t^\theta\right)$ and $m\left(t,x_t^\theta,u_t^\theta\right)$ at the point $(\widehat{x},\widehat{u})$ and using (3.5), we get for all $\varphi = b, h, \sigma, l$ and m

$$\begin{aligned}&\frac{1}{\theta}\left(\varphi\left(t, x_t^\theta, u_t^\theta\right) - \varphi\left(t, \widehat{x}_t, \widehat{u}_t\right)\right) \qquad (3.11)\\ &= \int_0^1 \frac{1}{\theta}\varphi_x\left(t, \widehat{x}_t + \lambda\left(x_t^\theta - \widehat{x}_t\right), \widehat{u}_t + \lambda\theta\left(u_t - \widehat{u}_t\right)\right)\left(x_t^\theta - \widehat{x}_t\right) d\lambda \\ &+ \int_0^1 \varphi_u\left(t, \widehat{x}_t + \lambda\left(x_t^\theta - \widehat{x}_t\right), \widehat{u}_t + \lambda\theta\left(u_t - \widehat{u}_t\right)\right)\left(u_t - \widehat{u}_t\right) d\lambda.\end{aligned}$$

Since all the derivatives of b, h, σ, l and m are continuously with respect to (x, u) and by using the limit with respect to θ goes to zero for every member in the side of (3.11) we obtain the desired result. ■

Lemma 3.3 *Under the assumptions* 3.2*, we have*

$$\lim_{\theta \to 0} \mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\frac{1}{\theta}\left(x_t^\theta - \widehat{x}_t\right) - X_t\right|^2\right] = 0, \qquad (3.12)$$

$$\lim_{\theta \to 0} \mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\frac{1}{\theta}\left(y_t^\theta - \widehat{y}_t\right) - Y_t\right|^2\right] = 0, \qquad (3.13)$$

where (X, Y) *is the solution of the system given by* (3.10)*.*

Proof. For the first limit, we follow the same as lemma 3.11 [7]. For the second limit, by replacing $\widehat{y}_t, y_t^\theta$ and Y_t with their values in equations (3.4), (3.6) and (3.10) respectively, if we put for simplification $\widetilde{y}_t^\theta = \frac{1}{\theta}\left(y_t^\theta - \widehat{y}_t\right) - Y_t$, where $\widetilde{y}_t^\theta : [0,T] \times \mathbb{R} \times U \to \mathbb{R}$ and $\widetilde{\Lambda}_s^\theta := \left(s, \widehat{x}_s + \lambda\theta\left(\widetilde{x}_s^\theta + X_s\right), \widehat{u}_s + \lambda\theta\left(u_s - \widehat{u}_s\right)\right)$, we get

$$\begin{aligned}\widetilde{y}_t^\theta &= \int_0^1\int_0^t l_x\left(\widetilde{\Lambda}_s^\theta\right)\left(\widetilde{x}_s^\theta + X_s\right) ds d\lambda + \int_0^1\int_0^t l_u\left(\widetilde{\Lambda}_s^\theta\right)\left(u_s - \widehat{u}_s\right) ds d\lambda \\ &+ \int_0^1\int_0^t m_x\left(\widetilde{\Lambda}_s^\theta\right)\left(\widetilde{x}_s^\theta + X_s\right) d\langle B\rangle_s d\lambda + \int_0^1\int_0^t m_u\left(\widetilde{\Lambda}_s^\theta\right)\left(u_s - \widehat{u}_s\right) d\langle B\rangle_s d\lambda \\ &- \int_0^t \left(l_x(s) X_s + \widehat{l}_u(s)\left(u_s - \widehat{u}_s\right)\right) ds - \int_0^t \left(\widehat{m}_x(s) X_s + \widehat{m}_u(s)\left(u_s - \widehat{u}_s\right)\right) d\langle B\rangle_s.\end{aligned}$$

According to the sublinearity of the G-expectation $\mathbb{E}_G$, G-Hölder inequality and the limit (3.12), we have

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\widetilde{y}_t^{\theta}\right|^2\right]\leq C_1\int_0^1\int_0^t\mathbb{E}_G\left|l_x\left(\widetilde{\Lambda}_s^{\theta}\right)\widetilde{x}_s^{\theta}\right|^2 dsd\lambda$$
$$+C_2\int_0^1\int_0^t\mathbb{E}_G\left|m_x\left(\widetilde{\Lambda}_s^{\theta}\right)\widetilde{x}_s^{\theta}\right|^2 dsd\lambda+\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^{\theta}\right|^2\right],$$

where

$$\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^{\theta}\right|^2\right]\leq C_3\int_0^t\mathbb{E}_G\left|\int_0^1\left\{l_x\left(\widetilde{\Lambda}_s^{\theta}\right)-\widehat{l}_x(s)\right\}X_s d\lambda\right|^2 ds$$
$$+C_4\int_0^t\mathbb{E}_G\left|\int_0^1\left\{m_x\left(\widetilde{\Lambda}_s^{\theta}\right)-\widehat{m}_x(s)\right\}X_s d\lambda\right|^2 ds$$
$$+C_5\int_0^t\mathbb{E}_G\left|\int_0^1\left\{l_u\left(\widetilde{\Lambda}_s^{\theta}\right)-\widehat{l}_u(s)\right\}(u_s-\widehat{u}_s)\,d\lambda\right|^2 ds$$
$$+C_6\int_0^t\mathbb{E}_G\left|\int_0^1\left\{m_u\left(\widetilde{\Lambda}_s^{\theta}\right)-\widehat{m}_u(s)\right\}(u_s-\widehat{u}_s)\,d\lambda\right|^2 ds.$$

Since all the derivatives of l and m are continuous and bounded and by using the Lebesgue's bounded convergence theorem, we obtain the limit of the term $\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\rho_t^{\theta}\right|^2\right]$ equals 0 when θ goes to 0. So that, we can use (3.12) to find the limit (3.13) . ■

In the following part, we explore the relationship between risk-neutral and risk-sensitive performance cost functional.

3.2 Mean-variance uncertainty of loss functional

We denote by $A^{\varepsilon}(T)=\exp\varepsilon\left(g\left(x_T^u\right)+y_T^u\right)$, and we can put also

$$\Theta(T)=g(x_T^u)+y_T^u. \tag{3.14}$$

The risk-sensitive loss functional under G-expectation is given by

$$\Psi(\varepsilon) := \frac{1}{\varepsilon} \log\left[\mathbb{E}_G\left[\exp\left(\varepsilon\Theta(T)\right)\right]\right]. \tag{3.15}$$

Lemma 3.4 *Let $\Psi(\varepsilon)$ be the loss functional written as* (3.15)*, where $\Theta(T)$ is given by* (3.14)*. Then, if the risk-sensitive index ε is small, the loss functional $\Psi(\varepsilon)$ can be verified the following*

$$\Psi(\varepsilon) \leq \overline{\mu} + \frac{\varepsilon}{2}(\overline{\sigma}^2 - \overline{\mu}^2) + o(\varepsilon), \tag{3.16}$$

where $\overline{\mu} := \mathbb{E}_G\left[\Theta(T)\right]; \overline{\sigma}^2 := \mathbb{E}_G\left[\Theta^2(T)\right]$ and $\overline{\mu}^2 := \mathbb{E}_G\left[\Theta(T)\right]^2$.

Proof. Applying the limited expansion on the function $f(x) = \exp(\varepsilon x)$ with rang two around zero, yields that

$$f(x) = \exp(\varepsilon x) = \sum_{n=0}^{2} \frac{(\varepsilon x)^n}{n!} = 1 + \varepsilon x + \frac{(\varepsilon x)^2}{2} + o(\varepsilon).$$

Replacing x by $\Theta(T)$, we get

$$\exp\left(\varepsilon\Theta(T)\right) = 1 + \varepsilon\Theta(T) + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon).$$

By taking G-expectation and using its properties, we have

$$\begin{aligned}\mathbb{E}_G\left[\exp\left(\varepsilon\Theta(T)\right)\right] &= \mathbb{E}_G\left[1 + \varepsilon\Theta(T) + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon)\right] \\ &= 1 + \mathbb{E}_G\left[\varepsilon\Theta(T) + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon)\right].\end{aligned}$$

Then

$$\log\left[\mathbb{E}_G\left[\exp\left(\varepsilon\Theta(T)\right)\right]\right] = \log\left[1 + \mathbb{E}_G\left[\varepsilon\Theta(T) + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon)\right]\right].$$

If we take $y = \mathbb{E}_G\left[\varepsilon\Theta(T) + \frac{(\varepsilon\Theta(T))^2}{2} + o(\varepsilon)\right]$, and by using the limited expansion of the function $g(y) = \ln(1+y)$ with rang two in neighborhood of zero, we get

$$g(y) = \ln(1+y) = \sum_{n=1}^{2} \frac{(-1)^{n-1}}{n} y^n.$$

Hence, replacing y by its value, we obtain

$$\begin{aligned}
\log\left[\mathbb{E}_G\left[\exp\left(\varepsilon\Theta(T)\right)\right]\right] &= \mathbb{E}_G\left[\varepsilon\Theta(T) \ + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon)\ \right] \\
&- \frac{1}{2}\mathbb{E}_G\left[\varepsilon\Theta(T) \ + \frac{\left(\varepsilon\Theta(T)\right)^2}{2} + o(\varepsilon)\ \right]^2 + o(\varepsilon) \\
&\leq \varepsilon\mathbb{E}_G\left[\Theta(T)\right] \ + \frac{\varepsilon^2}{2}\mathbb{E}_G\left[\Theta^2(T)\right] - \frac{\theta^2}{2}\mathbb{E}_G\left[\Theta(T)\right]^2 \\
&- \frac{\varepsilon^4}{8}\mathbb{E}_G\left[\Theta^2(T)\right]^2 + \ldots + o(\varepsilon) \\
&\leq \varepsilon\mathbb{E}_G\left[\Theta(T)\right] + \frac{\varepsilon^2}{2}\left(\mathbb{E}_G\left[\Theta^2(T)\right] - \mathbb{E}_G\left[\Theta(T)\right]^2\right) + o(\varepsilon).
\end{aligned}$$

By replacing $\overline{\mu} := \mathbb{E}_G\left[\Theta(T)\right]; \overline{\sigma}^2 := \mathbb{E}_G\left[\Theta^2(T)\right]$ and $\overline{\mu}^2 := \mathbb{E}_G\left[\Theta(T)\right]^2$ we get the desired result. ■

Remark 3.3 *The right-hand side of* (3.16) *tends to* $\overline{\mu}$ *since* ε *goes to* 0.

3.3 Applying risk-neutral G-stochastic maximum principle

Let us introduce the G-adjoint equations of controlled G-SDEs with risk-sensitive performance cost. We suppose that the assumptions 3.1 − 3.2 hold. Then, for all $0 \leq t \leq T$ there exists a unique $\mathcal{F}_t$-adapted triplet $(\overrightarrow{p}^G(t), \overrightarrow{q}^G(t), \overrightarrow{k}^G(t))$ where

$$\overrightarrow{p}^G(t) := \begin{pmatrix} p_1^G(t) \\ p_2^G(t) \end{pmatrix}; \overrightarrow{q}^G(t) := \begin{pmatrix} q_1^G(t) \\ q_2^G(t) \end{pmatrix}; \overrightarrow{k}^G(t) := \begin{pmatrix} k_1^G(t) \\ k_2^G(t) \end{pmatrix};$$

that solves the following system matrix of G-BSDE

$$\begin{cases} \begin{pmatrix} dp_1^G(t) \\ dp_2^G(t) \end{pmatrix} = -\left[\begin{pmatrix} 0 & 0 \\ \widehat{l}_x(t) + \widehat{m}_x(t)\frac{d\langle B\rangle_t}{dt} & \widehat{b}_x(t) + \widehat{h}_x(t)\frac{d\langle B\rangle_t}{dt} \end{pmatrix} \begin{pmatrix} p_1^G(t) \\ p_2^G(t) \end{pmatrix} \right. \\ \qquad \left. + \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\sigma}_x(t)\frac{d\langle B\rangle_t}{dt} \end{pmatrix} \begin{pmatrix} q_1^G(t) \\ q_2^G(t) \end{pmatrix} \right] dt + \begin{pmatrix} q_1^G(t) \\ q_2^G(t) \end{pmatrix} dB_t \\ \qquad + \begin{pmatrix} dk_1^G(t) \\ dk_2^G(t) \end{pmatrix}; \\ \begin{pmatrix} p_1^G(T) \\ p_2^G(T) \end{pmatrix} = \begin{pmatrix} f_y^\varepsilon(x_T, y_T) \\ f_x^\varepsilon(x_T, y_T) \end{pmatrix}; \end{cases} \tag{3.17}$$

where, for $i = 1, 2$, $\left(p_i^G, q_i^G, k_i^G\right) \in \mathcal{G}_G^2(0, T)$. To this end, we may define the Hamiltonian H^ε associated with the optimal state dynamics $(\widehat{x}_t, \widehat{y}_t)_{t\in[0,T]}$ and the couple of adjoint processes $(\overrightarrow{p}^G(t), \overrightarrow{q}^G(t), \overrightarrow{k}^G(t))$ given by

$$\begin{aligned} H^\varepsilon(t) &= H^\varepsilon(t, \widehat{x}_t, \widehat{y}_t, \widehat{u}_t, \overrightarrow{p}^G(t), \overrightarrow{q}^G(t)) \\ &= \left[\widehat{l}(t) + \widehat{m}(t)\frac{d\langle B\rangle_t}{dt}\right] p_1^G(t) + \left[\widehat{b}(t) + \widehat{h}(t)\frac{d\langle B\rangle_t}{dt}\right] p_2^G(t) + \widehat{\sigma}(t)\frac{d\langle B\rangle_t}{dt} q_2^G(t). \end{aligned} \tag{3.18}$$

In the rest of this section, we give a proof of Theorem 3.2. The main ingredients are Taylor's expansion with an integral remainder of the state trajectory and the cost functional with respect to the perturbation of the control variable. Let us make it more precise below.

Theorem 3.2 *Suppose that the assumptions* 3.1 *and* 3.2 *hold. If* $(\widehat{x}_t, \widehat{y}_t)_{t\in[0,T]}$ *is an optimal solution of the risk-neutral control problem* (3.4)*. Then, there exist* $\mathcal{F}_t$*-adapted* *processes* (p_1^G, q_1^G, k_1^G) *and* (p_2^G, q_2^G, k_2^G) *that satisfy* (3.17), *such that*

$$\begin{aligned}&\mathbb{E}_G\left[\int_0^T H_u^{\varepsilon}\left(t,\widehat{x}_t,\widehat{y}_t,\widehat{u}_t,\overrightarrow{p}_t^G,\overrightarrow{q}_t^G\right)(u_t-\widehat{u}_t)\,dt\right.\\&\left.-\int_0^T X_t^- dk_2^G(t)-\int_0^T Y_t^- dk_1^G(t)\right]\geq 0,\end{aligned}\tag{3.19}$$

for all $u\in U_{ad}$, *almost every* $t\in[0,T]$.

Proof. By applying Taylor's expansion on the function $f^{\varepsilon}\left(x_T^{\theta},u_T^{\theta}\right)$ at the point $(\widehat{x}_T,\widehat{u}_T)$, we get

$$f^{\varepsilon}\left(x_T^{\theta},y_T^{\theta}\right)-f^{\varepsilon}\left(\widehat{x}_T,\widehat{y}_T\right)=f_x^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(x_T^{\theta}-\widehat{x}_T\right)+f_y^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(y_T^{\theta}-\widehat{y}_T\right).$$

We have $\widetilde{x}_t^{\theta}=\frac{1}{\theta}\left(x_t^{\theta}-\widehat{x}_t\right)-X_t$ and $\widetilde{y}_t^{\theta}=\frac{1}{\theta}\left(y_t^{\theta}-\widehat{y}_t\right)-Y_t$, then

$$\begin{aligned}\widetilde{x}_t^{\theta}+X_t&=\frac{1}{\theta}\left(x_t^{\theta}-\widehat{x}_t\right).\\\widetilde{y}_t^{\theta}+Y_t&=\frac{1}{\theta}\left(y_t^{\theta}-\widehat{y}_t\right).\end{aligned}$$

By using (3.7), we get

$$\begin{aligned}0&\leq\frac{1}{\theta}\mathbb{E}_G\left[f_x^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(x_T^{\theta}-\widehat{x}_T\right)+f_y^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(y_T^{\theta}-\widehat{y}_T\right)\right]\\&=\mathbb{E}_G\left[f_x^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(\widetilde{x}_T^{\theta}+X_T\right)+f_y^{\varepsilon}(\widehat{x}_T,\widehat{y}_T)\left(\widetilde{y}_T^{\theta}+Y_T\right)\right]\\&=\mathbb{E}_G\left[p_2^G(T)\left(\widetilde{x}_T^{\theta}+X_T\right)+p_1^G(T)\left(\widetilde{y}_T^{\theta}+Y_T\right)\right].\end{aligned}$$

By sending $\theta\to 0$, we get

$$0\leq\mathbb{E}_G\left[p_2^G(T)X_T+p_1^G(T)Y_T\right].\tag{3.20}$$

Using integration by parts for G-Itô's processes to $p_2^G(t)X_t$ and $p_1^G(t)Y_t$, we obtain

$$\begin{aligned}
&d\left(p_2^G(t)X_t\right)\\
&= p_2^G(t)dX_t + X_t dp_2^G(t) + \left\langle p_2^G, X\right\rangle_t.\\
&= p_2^G(t)\left[\left(\widehat{b}_x(t)X_t + \widehat{b}_u(t)(u_t - \widehat{u}_t)\right)dt\right.\\
&+ \left(\widehat{h}_x(t)X_t + \widehat{h}_u(t)(u_t - \widehat{u}_t)\right)d\langle B\rangle_t + \left(\widehat{\sigma}_x(t)X_t + \widehat{\sigma}_u(t)(u_t - \widehat{u}_t)\right)dB_t\\
&+ X_t\left[-\left(\left(\widehat{l}_x(t) + \widehat{m}_x(t)\frac{d\langle B\rangle_t}{dt}\right)p_1^G(t) + \left(\widehat{b}_x(t) + \widehat{h}_x(t)\frac{d\langle B\rangle_t}{dt}\right)p_2^G(t)\right.\right.\\
&\left.\left.+\widehat{\sigma}_x(t)\frac{d\langle B\rangle_t}{dt}q_2^G(t)\right)dt + q_2^G(t)dB_t + dk_2^G(t)\right]\\
&+ \left(\widehat{\sigma}_x(t)X_t + \widehat{\sigma}_u(t)(u_t - \widehat{u}_t)\right)q_2^G(t)d\langle B\rangle_t,
\end{aligned}$$

and

$$\begin{aligned}
&d\left(p_1^G(t)Y_t\right)\\
&= p_1^G(t)dY_t + Y_t dp_1^G(t) + \left\langle dp_1^G, dY\right\rangle_t.\\
&= p_1^G(t)\left[\left(\widehat{l}_x(t)X_t + \widehat{l}_u(t)(u_t - \widehat{u}_t)\right)dt\right.\\
&\left.+ \left(\widehat{m}_x(t)X_t + \widehat{m}_u(t)(u_t - \widehat{u}_t)\right)d\langle B\rangle_t\right] + Y_t\left(q_1^G(t)dB_t + dk_1^G(t)\right).
\end{aligned}$$

As G-Itô's integral is a symmetric G-martingale and k_1^G, k_2^G are non-increasing continuous G-martingales, then by Proposition 1.3.7 in [33], we obtain

$$\begin{aligned}
&\mathbb{E}_G\left[p_2^G(T)X_T + p_1^G(T)Y_T\right]\\
&= \mathbb{E}_G\left[\int_0^T p_2^G(t)\widehat{b}_u(t)(u_t - \widehat{u}_t)dt + \int_0^T p_2^G(t)\widehat{h}_u(t)(u_t - \widehat{u}_t)d\langle B\rangle_t\right.\\
&+\int_0^T \widehat{\sigma}_u(t)(u_t - \widehat{u}_t)q_2^G(t)d\langle B\rangle_t + \int_0^T p_1^G(t)\widehat{l}_u(t)(u_t - \widehat{u}_t)dt\\
&\left.+\int_0^T p_1^G(t)\widehat{m}_u(t)(u_t - \widehat{u}_t)d\langle B\rangle_t - \int_0^T X_t^- dk_2^G(t) - \int_0^T Y_t^- dk_1^G(t)\right].
\end{aligned}$$

Then by (3.20) and the definition of H^{ε} we get

$$\mathbb{E}_G\left[\int_0^T H_u^{\varepsilon}\left(t, \widehat{x}_t, \widehat{y}_t, \widehat{u}_t, \overrightarrow{p}_t^G, \overrightarrow{q}_t^G\right)(u_t - \widehat{u}_t)\, dt - \int_0^T X_t^- dk_2^G(t) - \int_0^T Y_t^- dk_1^G(t) \geq 0\right].$$

■

3.4 G-expected exponential utility and G-QBSDE

In this part, we tend to prove the relationship between the G-expected exponential utility and the G-quadratic backward stochastic differential equation (G-QBSDE). First of all, it is very important to write the G-expected exponential utility under the following form

$$\begin{aligned}\exp(\varepsilon \Upsilon_t^{\varepsilon}) &= \mathbb{E}_G\left[A^{\varepsilon}(t,T)\,|\mathcal{F}_t\right] \\ &= \mathbb{E}_G\left[\exp \varepsilon\left(g(x_T^u) + \int_t^T m^u(s) d\left\langle B\right\rangle_s + \int_t^T l^u(s) ds\right) |\mathcal{F}_t\right].\end{aligned} \tag{3.21}$$

The process Υ^{ε} is the first component of the $\mathcal{F}_t$-adapted triplet of processes $(\Upsilon^{\varepsilon}, N, K)$, which is the unique solution of the following G-QBSDE

$$\begin{cases} d\Upsilon_t^{\varepsilon} &= -l^u(t)dt - \left(m^u(t) + \frac{\varepsilon}{2}\left|N(t)\right|^2\right) d\left\langle B\right\rangle_t + N(t)dB_t + dK_t, \\ \Upsilon_T^{\varepsilon} &= g(x_T^u), \end{cases} \tag{3.22}$$

where $\Upsilon^{\varepsilon} \in \mathbb{S}_G^2(0,T)$, $N \in \mathbb{H}_G^2(0,T)$, K is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in \mathbb{L}_G^2(\Omega_T)$.

For more details about the expected exponential utility optimisation and about the G-QBSDE, the reader can visit the papers [5] and [16, 17].

Lemma 3.5 *The assertions* (3.21) *and* (3.22) *are equivalent.*

Proof. Assume that

$$\exp\left(\varepsilon\Upsilon_t^\varepsilon\right)=\mathbb{E}_G\left[\exp\left(\varepsilon\left(\int_t^T l^u(s)ds+\int_t^T m^u(s)d\left\langle B\right\rangle_s+g(x_T^u)\right)\right)|\mathcal{F}_t\right].$$

Multiplying the two sides of the above expression by $\exp\left(\varepsilon\left(\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)$. Fortunately, we have $\exp\left(\varepsilon\left(\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)$ is a continuous function. Thus, it is a measurable function w.r.t. $\mathcal{F}_t$ and $\exp\left(\varepsilon\left(\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)>0$. Then by the positive homogeneity property of the sublinear expectation, we obtain

$$\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)=\mathbb{E}_G\left[A^\varepsilon(T)\,|\mathcal{F}_t\right],\tag{3.23}$$

where $\mathcal{F}_0=\{\varnothing,\Omega\}$.

According to the G-martingale representation theorem (see Theorem 1.6) and under the assumption $A^\theta(T)\in\mathbb{L}_G^2(\Omega_T)$, there exist $Z\in\mathbb{H}_G^2(0,T)$ and a non-increasing continuous G-martingale $\overline{K}$ with $\overline{K}_0=0$ and $\overline{K}_T\in\mathbb{L}_G^2(\Omega_T)$, such that for every $t\in[0,T]$ we get

$$\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)=\mathbb{E}_G\left[A^\varepsilon(T)\right]+\int_0^t Z(s)ds+\overline{K}_t\qquad\text{q.s..}$$

For $t=0$, we have $\exp\left(\varepsilon\Upsilon_0^\varepsilon\right)=\mathbb{E}_G\left[A^\varepsilon(T)\right]$ q.s.. Then

$$\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)-\exp\left(\varepsilon\Upsilon_0^\varepsilon\right)=\int_0^t Z(s)dB_s+\overline{K}_t.\tag{3.24}$$

Applying G-Itô's formula on (3.24), we obtain

$$\begin{aligned}&\varepsilon\left(l^u(t)dt+m^u(t)d\left\langle B\right\rangle_t\right)+\varepsilon d\Upsilon_t^\varepsilon+\frac{\varepsilon^2}{2}\left\langle d\Upsilon_t^\varepsilon,d\Upsilon^\varepsilon\right\rangle_t\\&=\exp\left(-\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\left\langle B\right\rangle_s\right)\right)\left(Z(t)dB_t+d\overline{K}_t\right)\qquad\text{q.s..}\end{aligned}\tag{3.25}$$

Hence the right side of (3.25) is a G-martingale. Then

$$\varepsilon^2 \langle d\Upsilon^\varepsilon, d\Upsilon^\varepsilon \rangle_t = \left(Z(t) \exp\left(-\varepsilon \left(\Upsilon_t^\varepsilon + \int_0^t l^u(s)ds + \int_0^t m^u(s)d\langle B \rangle_s \right)\right)\right)^2 d\langle B \rangle_t .$$

Thus

$$d\Upsilon_t^\varepsilon = -l^u(t)dt - \left(m^u(t) + \frac{\varepsilon}{2} |N(t)|^2 \right) d\langle B \rangle_t + N(t)dB_t + dK_t,$$

where

$$\varepsilon N(t) := Z(t) \exp\left(-\varepsilon \left(\Upsilon_t^\varepsilon + \int_0^t l^u(s)ds + \int_0^t m^u(s)d\langle B \rangle_s \right)\right).$$

$$\varepsilon dK_t := \exp\left(-\varepsilon \left(\Upsilon_t^\varepsilon + \int_0^t l^u(s)ds + \int_0^t m^u(s)d\langle B \rangle_s \right)\right) d\overline{K}_t.$$

Indeed, $\exp(\varepsilon \Upsilon_T^\varepsilon) = \mathbb{E}_G [\exp(\varepsilon g(x_T^u)) | \mathcal{F}_T] = \exp(\varepsilon g(x_T^u))$. Then

$$\ln \exp(\varepsilon \Upsilon_T^\varepsilon) = \ln \exp(\varepsilon g(x_T^u)).$$

Finally, we get

$$\Upsilon_T^\varepsilon = g(x_T^u).$$

The second step in this proof is the other side. Suppose that (3.22), we have

$$d(\exp(\varepsilon \Upsilon_t^\varepsilon)) + \varepsilon \exp(\varepsilon \Upsilon_t^\varepsilon)(l^u(t)dt + m^u(t)d\langle B \rangle_t) = \varepsilon \exp(\varepsilon \Upsilon_t^\varepsilon)(N(t)dB_t + dK_t).$$

Multiplying the two sides of the above expression by $\exp \varepsilon \left(\int_0^t l^u(s)ds + \int_0^t m^u(s)d\langle B \rangle_s \right)$, yields that

$$\exp\left(\varepsilon\left(\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right)d\left[(\exp(\varepsilon\Upsilon_t^\varepsilon))\right.$$
$$\left.+\varepsilon\exp(\varepsilon\Upsilon_t^\varepsilon)\left(l^u(t)dt+m^u(t)d\langle B\rangle_t\right)\right]$$
$$=\varepsilon\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right)\left(N(t)dB_t+dK_t\right). \tag{3.26}$$

But, we have

$$d\left(\exp\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right)$$
$$=\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right)\left(\varepsilon d\Upsilon_t^\varepsilon+\varepsilon l^u(t)dt+\varepsilon m^u(t)d\langle B\rangle_t\right). \tag{3.27}$$

By (3.26) and (3.27) we notice that

$$d(\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right))$$
$$=\varepsilon\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(s)ds+\int_0^t m^u(s)d\langle B\rangle_s\right)\right)\left(N(t)dB_t+dK_t\right).$$

Then

$$\mathbb{E}_G\left[\exp\left(\varepsilon\left(\Upsilon_T^\varepsilon+\int_0^T l^u(r)dr+\int_0^T m^u(r)d\langle B\rangle_r\right)\right)|\mathcal{F}_t\right]$$
$$=\mathbb{E}_G\left[\left(\exp\left(\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(r)dr+\int_0^t m^u(r)d\langle B\rangle_r\right)\right)\right.\right.$$
$$+\varepsilon\int_t^T\exp\left(\varepsilon\left(\Upsilon_s^\varepsilon+\int_0^s l^u(r)dr+\int_0^s m^u(r)d\langle B\rangle_r\right)\right)N(s)dB_s$$
$$\left.\left.+\varepsilon\int_t^T\exp\left(\varepsilon\left(\Upsilon_s^\varepsilon+\int_0^s l^u(r)dr+\int_0^s m^u(r)d\langle B\rangle_r\right)\right)dK_s\right)|\mathcal{F}_t\right].$$

Using the second property in proposition 3.2.3 [33] and the Independence property of

$\mathbb{E}_G[.|\mathcal{F}_t]$, yields that

$$\begin{aligned}
&\mathbb{E}_G\left[\exp\varepsilon\left(\Upsilon_T^\varepsilon+\int_0^T l^u(r)dr+\int_0^T m^u(r)d\langle B\rangle_r\right)|\mathcal{F}_t\right]\\
&=\exp\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(r)dr+\int_0^t m^u(r)d\langle B\rangle_r\right)\\
&+\mathbb{E}_G\left[\varepsilon\int_t^T\exp\varepsilon\left(\Upsilon_s^\varepsilon+\int_0^s l^u(r)dr+\int_0^s m^u(r)d\langle B\rangle_r\right)N(s)dB_s\right.\\
&\left.+\varepsilon\int_t^T\exp\varepsilon\left(\Upsilon_s^\varepsilon+\int_0^s l^u(r)dr+\int_0^s m^u(r)d\langle B\rangle_r\right)dK_s\right].
\end{aligned}$$

Pay attention that G-Itô's integral is a symmetric G-martingale and for any $\varphi(.)\in\mathbb{M}_G^2(0,T)$, $\int_t^T\varphi(t)dK_t$ is a G-martingale if and only if $\varphi(.)\geq 0$ a.e., q.s.. Then we get

$$\begin{aligned}
&\mathbb{E}_G\left[\exp\varepsilon\left(\Upsilon_T^\varepsilon+\int_0^T l^u(r)dr+\int_0^T m^u(r)d\langle B\rangle_r\right)|\mathcal{F}_t\right]\\
&=\exp\varepsilon\left(\Upsilon_t^\varepsilon+\int_0^t l^u(r)dr+\int_0^t m^u(r)d\langle B\rangle_r\right).
\end{aligned}$$

By the positive homogeneity property of $\mathbb{E}_G$ and as $\Upsilon_T^\varepsilon=g(x_T^u)$, we have

$$\begin{aligned}
\exp\left(\varepsilon\Upsilon_t^\varepsilon\right)&=\exp\left(-\varepsilon\int_0^t l^u(r)dr-\varepsilon\int_0^t m^u(r)d\langle B\rangle_r\right)\mathbb{E}_G\left[\exp\varepsilon\left(\Upsilon_T^\varepsilon\right.\right.\\
&\left.\left.+\int_0^T l^u(r)dr+\int_0^T m^u(r)d\langle B\rangle_r\right)|\mathcal{F}_t\right]\\
&=\mathbb{E}_G\left[\exp\varepsilon\left(g(x_T^u)+\int_t^T l^u(r)dr+\int_t^T m^u(r)d\langle B\rangle_r\right)|\mathcal{F}_t\right].
\end{aligned}$$

We get the desired result. ■

3.5 Merton problem with power utility

Our system is governed by a G-Stochastic differential equation under the form for all $0 \leq t \leq T$

$$\begin{cases} dx_t^u = x_t^u u_t e^{3t} d\langle B\rangle_t + x_t^u u_t e^{-3t} dB_t, \\ x_0^u = 1. \end{cases} \tag{3.28}$$

The functional should be minimized, over the set of admissible controls U_{ad} as described in the beginning of section 3.1, is given under the G-expectation for all $u \in U_{ad}$

$$\mathcal{J}(u) = \mathbb{E}_G\left[3x_T^{\frac{1}{3}}\right]. \tag{3.29}$$

Using G-Itô's formula to $\ln x_t^u$, we get

$$d(\ln x_t^u) = \left(e^{3t}u_t - \frac{1}{2}e^{-6t}u_t^2\right) d\langle B\rangle_t + u_t e^{-3t} dB_t.$$

Then, the explicit solution of (3.28), can be given as

$$x_t^u = \exp\left\{\int_0^t \left(e^{3s}u_s - \frac{1}{2}e^{-6s}u_s^2\right) d\langle B\rangle_s + \int_0^t u_s e^{-3s} dB_s\right\}.$$

The cost functional with respect to G-expectation (3.29), gets the form

$$\begin{aligned} \mathcal{J}(u) &= \mathbb{E}_G\left[3\exp\left\{\frac{1}{3}\int_0^T \left(e^{3s}u_s - \frac{1}{2}e^{-6s}u_s^2\right) d\langle B\rangle_s + \frac{1}{3}\int_0^T u_s e^{-3s} dB_s\right\}\right] \\ &= 3\mathbb{E}_G\left[\exp\left\{-\frac{1}{18}\int_0^T e^{-6s}u_s^2 d\langle B\rangle_s + \frac{1}{3}\int_0^T u_s e^{-3s} dB_s\right\}\right. \\ &\quad \left.\times \exp\left\{\int_0^T \left(\frac{1}{3}e^{3s}u_s - \frac{1}{9}e^{-6s}u_s^2\right) d\langle B\rangle_s\right\}\right]. \end{aligned} \tag{3.30}$$

G-Girsanov's theorem for G-expectation and G-martingale allows us to put

$$\xi^{\varepsilon}(T) = \exp\left\{\frac{1}{3}\int_0^T e^{-3t}u_t dB_t - \frac{1}{18}\int_0^T e^{-6t}u_t^2 d\langle B\rangle_t\right\}.$$

By (1.9) we note that and $Z(t) = u_t e^{-3t}$. We can rewrite (3.30) with respect to the new G-expectation $\widetilde{\mathbb{E}}_G$ under the G-Brownian motion $B_t^{\varepsilon} = B_t - \frac{1}{3}\int_0^t e^{-3s} u_s d\langle B\rangle_s$ for every $t \in [0, T]$ as follows

$$\mathcal{J}^{\varepsilon}(u) = 3\widetilde{\mathbb{E}}_G\left[\exp\left\{\int_0^T \left(\frac{1}{3}e^{3s}u_s - \frac{1}{9}e^{-6s}u_s^2\right) d\langle B^{\varepsilon}\rangle_s\right\}\right]. \tag{3.31}$$

The minimum of the new cost functional with respect to G-expectation $\widetilde{\mathbb{E}}_G$, is given by

$$\mathcal{J}^{\varepsilon}(u) = 3\inf_{u\in U_{ad}} \widetilde{\mathbb{E}}_G\left[\exp\left\{\int_0^T \left(\frac{1}{3}e^{3s}u_s - \frac{1}{9}e^{-6s}u_s^2\right) d\langle B^{\varepsilon}\rangle_s\right\}\right].$$

Then, it is very easy to verify that the optimal control in a weaker sense is the solution of the function $u \longmapsto \frac{1}{3}e^{3t}u_t - \frac{1}{9}e^{-6t}u_t^2$, hence

$$\widehat{u}_t = \frac{3}{2}e^{9t}. \tag{3.32}$$

The G-adjoint equation related to optimal control (3.32) of a linear G-backward stochastic differential equation can be defined as a G-conditional expectation representation as follows

$$\begin{aligned}
\widehat{p}^G(t) &= (\widehat{x}_t)^{-1}\,\mathbb{E}_G\left[(\widehat{x}_T)^{\frac{1}{3}}\,|\mathcal{F}_t\right] \\
&= (\widehat{x}_t)^{-1}\,(\widehat{x}_t)^{\frac{1}{3}}\,\mathbb{E}_G\left[\exp\left\{-\frac{1}{18}\int_t^T e^{-6s}\widehat{u}_s^2 d\langle B\rangle_s + \frac{1}{3}\int_t^T \widehat{u}_s e^{-3s} dB_s\right\}\right. \\
&\times \left.\exp\left\{\int_t^T \left(\frac{1}{3}e^{3s}\widehat{u}_s - \frac{1}{9}e^{-6s}\widehat{u}_s^2\right) d\langle B\rangle_s\right\}|\mathcal{F}_t\right] \\
&= (\widehat{x}_t)^{-\frac{2}{3}}\,\widetilde{\mathbb{E}}_G\left[\exp\left\{\int_t^T \left(\frac{1}{3}e^{3s}\widehat{u}_s - \frac{1}{9}e^{-6s}\widehat{u}_s^2\right) d\langle B^{\varepsilon}\rangle_s\right\}|\mathcal{F}_t\right].
\end{aligned}$$

By replacing the optimal control with its value given in the expression (3.32) , the optimal G-adjoint process can be written as

$$\begin{aligned}\widehat{p}^G(t) &= (\widehat{x}_t)^{-\frac{2}{3}}\,\widetilde{\mathbb{E}}_G\left[\exp\left\{\int_t^T\left(\frac{1}{3}e^{3s}\widehat{u}_s-\frac{1}{9}e^{-6s}\widehat{u}_s^2\right)d\left\langle B^{\varepsilon}\right\rangle_s\right\}|\mathcal{F}_t\right]\\ &= (\widehat{x}_t)^{-\frac{2}{3}}\,\widetilde{\mathbb{E}}_G\left[\exp\left\{\int_t^T\frac{1}{4}e^{12s}d\left\langle B^{\varepsilon}\right\rangle_s\right\}|\mathcal{F}_t\right].\end{aligned}$$

We know that the distribution of the quadratic variation process of G-Brownian motion does not change under the transformation. Then

$$\begin{aligned}\widehat{p}^G(t) &= (\widehat{x}_t)^{-\frac{2}{3}}\,\mathbb{E}_G\left[\exp\left\{\int_t^T\frac{1}{4}e^{12s}d\left\langle B\right\rangle_s\right\}|\mathcal{F}_t\right]\\ &= (\widehat{x}_t)^{-\frac{2}{3}}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_t^T\frac{1}{4}e^{12s}d\left\langle B\right\rangle_s\right\}|\mathcal{F}_t\right]\\ &= (\widehat{x}_t)^{-\frac{2}{3}}\exp\left(\int_t^T\frac{1}{4}e^{12s}ds\right);\qquad \mathbb{P}\text{-a.s..}\end{aligned}$$

On one hand, by using G-Itô's formula to $(\widehat{x}_t)^{-\frac{2}{3}}\exp\left(\int_t^T\frac{1}{4}e^{12s}ds\right)$, and replacing the optimal control by its value in (3.32) , we have

$$\begin{aligned}d\widehat{p}^G(t) &= d\left((\widehat{x}_t)^{-\frac{2}{3}}\exp\left(\int_t^T\frac{1}{4}e^{12s}ds\right)\right)\\ &= -\widehat{p}^G(t)e^{6t}dB_t+\frac{1}{4}e^{12t}\widehat{p}^G(t)\left(d\left\langle B\right\rangle_t-dt\right).\end{aligned}\tag{3.33}$$

On the other hand, by using our result, found in the previous sections, the G-adjoint equation (2.26), we get

$$d\widehat{p}^G(t) = -u_t\left(e^{3t}\widehat{p}^G(t)+e^{-3t}\widehat{q}^G(t)\right)d\left\langle B\right\rangle_t+\widehat{q}^G(t)dB_t+dk_t.\tag{3.34}$$

By identifying (3.33) with (3.34), we first obtain

$$\widehat{q}^G(t) = -\widehat{p}^G(t)e^{6t}.$$

Second, and by replacing the value of $\widehat{q}^G$ in (3.34), we get

$$dk_t = \frac{1}{4}e^{12t}\widehat{p}^G(t)\left(d\left\langle B\right\rangle_t + dt\right). \tag{3.35}$$

Remark 3.4 *We have got that the process* $(dk_t)_{t\in[0,T]}$ *has the explicit form* (3.35)*, with respect to* $\langle B\rangle_t$ *and* t*, which is non trivial process.*

Conclusion and Perspectives

This thesis is concerned with recent advances in G-SMP and contains two main results, the first result is the Theorems 2.2 and 2.3, where the system is governed by G-SDE (2.2), establishing the necessary and sufficient optimality conditions respectively using an almost similar scheme as in Xu [46] while the control problem model studied is different from our model. The proof is based on the convexity condition of the set of admissible controls. The form of the maximum condition (2.27) is very similar to its counterpart (see Theorem 2.1, [46]) with the non-increasing G-martingale $(K(.))$ in the G-adjoint equation (2.26) there replaced by $(-k(.))$. In the G-framework, the situation is complicated by the presence of the third component of the G-adjoint equation which is the non-increasing G-martingale $(K(.))$, and the main difficulty is to get the stochastic maximum principle since the sublinear operator $\mathbb{E}_G$ in the main Theorem 2.2 can not be deleted. The second main result is the Theorem 3.2, suggests G-SMP for the system of type control given in the form of risk-sensitive performance, as our best acknowledgement that these results are a good extension of the results established by Chala in [5]. The main difference between our risk-sensitive optimal control problem (3.1), (3.2), (3.3) and usual risk-neutral problems is the exponential-of-integral type cost functional (3.2) which contains also the volatility uncertainty term $d\langle B\rangle_t$.

Note that if we substitute $\varepsilon = 0$ in (3.16), the risk-sensitive loss functional $\Psi(\varepsilon)$ is dominated by the risk-neutral cost functional $\mathbb{E}_G[\Theta(T)]$ and this is due to the subadditivity of $\mathbb{E}_G[.]$. Despite the success of proving the result of Theorem 3.2 which is a good

G-SMP for risk-neutral control problem, it is not a desirable one. Since augmenting the state process with the auxiliary process y yields a system of two G-adjoint equations that appears complicated to be solved in concrete situations, which are all left for our future exploration.

Following this study, several perspectives are considered. We plan to deal with the optimal control problem where the state equation is driven by G-Brownain motion.

- Stochastic maximum principle for risk-sensitive control problem.
- Maximum principle for G-stochastic control systems with controlled jump diffusions.

As future perspectives, It will be interesting to treat applications in finance in case of volatility uncertainty, controlled G-BSDEs and the problem of mean-field control.

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