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RELATIVISTIC GAUGE FIELD THEORY OF DEFECTS IN RIEMANN-CARTAN SPACE-TIME FABRIC

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Résumé

Nous présentons dans notre travail la théorie d'Einstein-Cartan de la gravitation et la théorie de jauge des défauts dans un milieu élastique du tissu de l'espace-temps en comparant la structure de ces deux théories. La première est une extension de la relativité générale et fait référence à l'espace-temps à quatre dimensions, tandis que nous introduisons le second comme une description de l'état d'un continuum d'espace-temps à quatre dimensions en analysant ces déformations résultant de sa courbure et sa torsion. Malgré ces différences importantes, une analogie formelle est construite sur des bases géométriques communes, et il est montré qu'un espace-temps avec courbure et torsion peut être considéré comme un état d'un continuum à quatre dimensions contenant des défauts en étendant la théorie de jauge des défauts à quatre dimensions.

Dans notre travail, les points fondamentaux sous-jacents à la géométrisation des défauts du continuum sur l'espace-temps de Riemann-Cartan sont discutés et un modèle de jauge des défauts est formulé en analogie avec les théories de jauge de la gravitation avec torsion plutôt que les théories de jauge de Yang-Mills de la physique des hautes énergies. L'énergie invariante de jauge contenant les termes de torsion et de courbure avec le second gradient d'élasticité est construite et leurs équations de lois de conservation sont obtenues.

Mots-clés : Théorie de jauge, dislocations, disclinations, élasticity, géométrie différentielle.

Abstract

We present in our work the Einstein-Cartan theory of gravitation and the gauge theory of defects in an elastic medium of the space-time fabric by comparing the structure of these two theories. The first is an extension of general relativity and refers to four-dimensional space-time, while we introduce the second as a description of the state of a continuum of four-dimensional space-time by analyzing these deformations resulting from its curvature and its torsion. Despite these important differences, a formal analogy is built on common geometric foundations, and it is shown that a spacetime with curvature and torsion can be considered as a state of a four-dimensional continuum containing defects by extending the gauge theory of defects to four dimensions.

In our work, the basic points underlying the geometrization of the continuum defects on Riemann-Cartan Space-Time are discussed and a gauge model for them is formulated in analogy with gauge theories of gravitation with torsion rather than the Yang-Mills gauge theories of high-energy physics. Gauge-invariant energy containing torsion and curvature terms with the second gradient elasticity are constructed and their conservation laws equations are obtained.

Keywords : Gauge theory, dislocation, disclination, elasticity, differential geometry.

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Contents

Chapter 1

Introduction

The correspondence between the space-time mathematical description of the Einstein's 1915 general relativity theory (GR) [1] and the geometrical concepts applied to a flat three-dimensional space of continuous medium has remote roots in the ether theories of the XIXth century (see some interesting references, for example A. Unzicker [2]). More specifically a formal link between moving dislocations and special relativity was pointed out by Frank in 1949 [3], then variously discussed by a number of other authors (cited in sect. 2.1 of [2]).

This geometrization of space-time suggests immediately a correspondence with material continua, their metric properties, and the theory of elasticity. This longly known analogy has been, and is now and then, revived, but has never been taken too seriously and/or used as a constitutive theory of space-time [5].

Therefore, there is a specific motivation to try and explore again the present description of space-time and to seriously take space-time as an existing real entity, most in the sense of what Einstein said in an address delivered on May 5th, 1920, in the University of Leyden: " *... according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. ... But this ether may not be thought of as endowed with the quality characteristic of ponderable media, as consisting of parts which may be tracked through time...*"[4].

More than forty years later, in 1952, Einstein wrote in Appendix V, *Relativity and the Problem of Space*, of his book on Relativity [6]: "*There is no such thing as an empty space, i.e. a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field*".

Given that the spacetime continuum behaves as a deformable continuum medium, there is no reason not to expect dislocations, disclinations and other defects to be present in the spacetime continuum. Dislocations and disclinations of the spacetime continuum represent the fundamental displacement processes that occur in its structure. These fundamental displacement processes should correspond to basic quantum phenomena and provide a framework for the description of quantum physics in STCED [7].

Howevere, the geometrization of a such physical theory, i.e., expressing it in a differential geometric form, requires the identification of differential geometric concepts with certain physically measurable quantities, and the specification of how the metric, curvature, and torsion of the space corresponding to the underlying continuum are generated or determined by the physical objects (particles, defects, etc.) of the theory and their behaviour in the corresponding mathematical space.

Therefore, the basic geometric identifications made in the continuum mechanics of defects are that:

1. The underlying continuum used for the description of physical phenomena related to defects is a differentiable manifold (body manifold).

2. The dislocation and disclination line densities are identified with the torsion and curvature of that manifold, respectively.

In the present work, following a close analogy with gravitational gauge theories with torsion, I shall show that the Einstein-Cartan theory of gravitation (ECT) which turns out to be an extension of general relativity to Riemann-Cartan spaces in which both the metric and the torsion determine the geometry of space-time, and the gauge theory with Cartan differential geometry treatment introduced in the 1980s, also referred to as the gauge field theory of defects, have similar fundamental equations. The treatment is based on Kleinert ([8],[9]) who is a key reference on this thesis (as applied to condensed matter). This theory is popular for the investigation of defects in condensed matter physics and in string theory due to the mathematical elegance and popularity of gauge theories and Cartan differential geometry.

Then, we shall stress the analogies and differences in their underlying geometric structure. Because the two theories describe very different physical phenomena, it is clear that the comparison can only be formal. However, we believe that this way of presenting the subject will give deeper insight into the geometrical tools used in both theories.

The remainder of this thesis is outlined as follows:

Chapter 2, gives a quick overview of classical elasticity and defect densities.

Chapter 3 introduces the basic theoretical concepts related to the gauge theory of defects in second gradient elasticity.

Chapter 4 reviews the basic concepts related to the Einstein-Cartan theory of gravitation based on Cartan's differential forms formulation.

In Chapter 5, we have formulated the Einstein-Cartan Theory as a Gauge Theory of Defects.

In chapter 6 is the final content chapter, we constructe a four dimensional gauge field theory of defect which governs ensemble of dislocations and disclinations, including their higher gradient elastic interactions.

Chapter 7 is reserved for presenting conclusions, outlook and also briefly illustrate some possible applications of the theory of defects to contemporary physics.

Chapter 2

Classical Elasticity and Defect Densities

2.1 Classical Elasticity

At each point in space, any change in position can be described by a *displacement field* $\vec{u}(\vec{x})$ which is defined by

$$\vec{x}' = \vec{x} + \vec{u}(\vec{x}) \tag{2.1}$$

During the deformation of the media, the displacement vector $d\vec{x} = \vec{x} - \vec{y}$ between two infinitely neighboring points spaced at $\vec{x}$ and $\vec{y}$ becomes

$$dx_i' = dx_i + \partial_j u_i dx_j \tag{2.2}$$

and its length before deformation $dl = \sqrt{d\vec{x}^2}$ becomes

$$dl' = (dl + 2u_{ij}dx_i dx_j)^{1/2} \tag{2.3}$$

where u_{ij} is an symmetric matrix

$$u_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_l \partial_j u_l) \tag{2.4}$$

called the *strain* tensor. To linear approximation, this strain tensor is just

$$u_{ij} \simeq \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{2.5}$$

In an elastic media, the *elastic energy* density is given by

$$e(\vec{x}) = \frac{1}{2} c_{ijkl} u_{ij}(\vec{x}) u_{kl}(\vec{x}) \,, \tag{2.6}$$

where c_{ijkl} is an symmetric tensor under the exchanges $i \leftrightarrow j, k \leftrightarrow l, ij \leftrightarrow kl$; qnd is called *elastic tensor.*

As a quadratic form, the elastic energy density can be expressed by

$$e(\vec{x}) = \frac{1}{2} c_{ab} u_a(\vec{x}) u_b(\vec{x}) \,, \tag{2.7}$$

where c_{ab} is an elastic constant.

In terms of the two rotational invariants, u_{ij}^2 and u_{ii}, the elastic energy density can be expressed as

$$e(\vec{x}) = \mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2 \,, \tag{2.8}$$

where μ is called the *shear modulus* and λ the *Lamé constant*.

In an isotropic media,

$$c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl} \,. \tag{2.9}$$

The total elastic energy is given by

$$E = \int d^3x e(\vec{x}) \,, \tag{2.10}$$

By performing partial integrations we can bring the energy density (2.8) to the equivalent form

$$e(\vec{x}) = \frac{\mu}{2}(\partial_i u_j)^2 + \frac{1}{2}(\lambda + \mu)(\partial_i u_i)^2 \,, \tag{2.11}$$

The strain tensor can be represented by the sum of rotational invariants consisting of traceless part of U_{ij} of spin 2 and the trace itself of spin 0 as follow

$$\begin{aligned} u_{ij} &= u_{ij}^{(2)} + u_{ij}^{(0)} \\ &= (u_{ij} - \frac{1}{3}\delta_{ij} u_{kk}) + \frac{1}{3}\delta_{ij} u_{kk} \,, \end{aligned} \tag{2.12}$$

where the projection matrices are

$$P_{ijkl}^{(2)} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl} \,, \tag{2.13}$$

$$P_{ijkl}^{(0)} = \frac{1}{3}\delta_{ij}\delta_{kl} \,, \tag{2.14}$$

They are orthonormal

$$(P^{(2)2})_{ijkl} = P_{ijmn}^{(2)} P_{mnkl}^{(2)} = P_{ijkl}^{(2)} \,, \tag{2.15}$$

$$(P^{(0)2})_{ijkl} = P_{ijmn}^{(0)} P_{mnkl}^{(0)} = P_{ijkl}^{(0)} \,, \tag{2.16}$$

$$(P^{(2)}P^{(0)})_{ijkl} = P_{ijmn}^{(2)} P_{mnkl}^{(0)} = (P^{(0)}P^{(2)})_{ijkl} = P_{ijmn}^{(0)} P_{mnkl}^{(2)} = 0 \,, \tag{2.17}$$

The sum of these projections gives the unit matrix

$$(P^{(2)} + P^{(0)})_{ijkl} = \mathbb{I}_{ijkl}^s \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \,. \tag{2.18}$$

Therefore, the decomposition of u_{ij} can be written as

$$u_{ij} = \mathbb{I}_{ijkl}^s u_{kl} = (P^{(2)} + P^{(0)})_{ijkl} u_{kl} = u_{ij}^{(2)} + u_{ij}^{(0)} \,. \tag{2.19}$$

and the elastic tensor may be decomposed into spin-2 and spin-0 parts

$$c_{ijkl} = c^{(2)} P^{(2)}_{ijkl} + c^{(0)} P^{(0)}_{ijkl}\,, \tag{2.20}$$

whith

$$c^{(2)} = 2\mu, \qquad c^{(0)} = 3(\lambda + 2/3\mu) \equiv 3\kappa\,. \tag{2.21}$$

The spin-0 combination κ is called the *modulus of compression*.

By inserting the decomposition (2.20) into the energy (2.6) and using the projection property (2.17) we can write

$$e(\vec{x}) = \frac{1}{2} u(c^{(2)} P^{(2)} + c^{(0)} P^{(0)}) u = \frac{1}{2} c^{(2)} u^{(2)2} + \frac{1}{2} c^{(0)} u^{(0)2} = \mu u^{(2)2} + \frac{3}{2} \kappa u^{(0)2}\,. \tag{2.22}$$

The change of u_{ij} by a small increment δu_{ij}, the energy density is changed by

$$\delta e(\vec{x}) = c_{ijkl} u_{kl} \delta u_{ij}\,. \tag{2.23}$$

The quantity

$$\sigma_{ij} \equiv \frac{\delta e}{\delta u_{ij}} = c_{ijkl} u_{kl}\,, \tag{2.24}$$

is called *stress*. For isotropic media, we may insert (2.21) and obtain

$$\sigma_{ij} = c^{(2)} u^{(2)}_{ij} + c^{(0)} u^{(0)}_{ij} = 2\mu u^{(2)}_{ij} + 3\kappa u^{(0)}_{ij} = 2\mu u_{ij} + \lambda \delta_{ij} u_{kk}\,. \tag{2.25}$$

By using the orthonormality properties (2.17) we can invert the stress strain (2.25) and obtain

$$\begin{aligned} u_{ij} &= (c^{-1})_{ijkl}\sigma_{kl} = (c^{(2)^{-1}} P^{(2)} + c^{(0)^{-1}} P^{(0)})_{ijkl}\sigma_{kl} = c^{(2)^{-1}} \sigma^{(2)}_{kl} + c^{(0)^{-1}} \sigma^{(0)}_{kl} \\ &= \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\right) + \frac{1}{3\kappa}\frac{1}{3}\delta_{ij}\sigma_{kk} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\kappa}\delta_{ij}\sigma_{kk}\right)\,. \end{aligned} \tag{2.26}$$

The ratio $\lambda/3\kappa$ is usually expressed in terms of the *Poisson ratio*

$$\nu \equiv \frac{\lambda}{2(\lambda + \mu)}\,, \tag{2.27}$$

as

$$\frac{\lambda}{3\kappa} \equiv \frac{\nu}{2(\nu + 1)}\,, \tag{2.28}$$

so that

$$u_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\nu}{\nu + 1}\delta_{ij}\sigma_{kk}\right)\,. \tag{2.29}$$

So, the elastic energy density may be written in terms of stresses as follow

$$\begin{aligned} e &= \frac{1}{2} u(c^{(2)} P^{(2)} + c^{(0)} P^{(0)}) u = \frac{1}{2}\sigma(c^{(2)^{-1}} P^{(2)} + c^{(0)^{-1}} P^{(0)})\sigma = \frac{1}{4\mu}\sigma^{(2)^2} + \frac{1}{6\kappa}\sigma^{(0)^2} \\ &= \frac{1}{4\mu}\left(\sigma_{ij} - \frac{\nu}{\nu + 1}\delta_{ij}\sigma_{kk}\right)^2 + \frac{1}{18\kappa}\sigma^2_{kk} = \frac{1}{4\mu}\left(\sigma^2_{ij} - \frac{\nu}{\nu + 1}\sigma^2_{kk}\right)\,. \end{aligned} \tag{2.30}$$

2.2 Defects Densities

2.2.1 Dislocation and Disclination Densities

Considering the displacement field $u_i(\vec{x})$ in the 3-dimensional media. Plastic deformations introduce defects. In linear approximation, dislocation and disclination densities are respectively given by the following tensor:

$$\alpha_{ij}(\vec{x}) \equiv \varepsilon_{ikl}\partial_k\partial_l u_j(\vec{x})\,, \tag{2.31}$$
$$\theta_{ij}(\vec{x}) \equiv \varepsilon_{ijk}\partial_k\partial_l \omega_j(\vec{x})\,. \tag{2.32}$$

where

$$\omega_i(\vec{x}) = \frac{1}{2}\,\varepsilon_{ijk}\omega_{jk}(\vec{x}) = \frac{1}{2}(\vec{\nabla}\times\vec{u})_i = \frac{1}{2}\,\varepsilon_{ijk}(\partial_j u_k - \partial_k u_j)\,. \tag{2.33}$$

is the *local rotation field* and

$$\omega_{ij}(\vec{x}) \equiv \frac{1}{2}(\partial_i u_j - \partial_j u_i) \tag{2.34}$$

the antisymmetric tensor field.

For the general defect line along L, these densities have the form

$$\alpha_{ij}(\vec{x}) = \delta_i(\vec{x};L)(b_i - \Omega_{jk}x_k)\,, \tag{2.35}$$
$$\theta_{ij}(\vec{x}) = \delta_i(\vec{x};L)\Omega_j\,, \tag{2.36}$$

where

$$\Omega_i = \frac{1}{2}\,\varepsilon_{ijk}\Omega_{jk}\,, \tag{2.37}$$

is the *Frank vector*.

In terms of the tensor

$$S_{ijk}(\vec{x}) \equiv \frac{1}{2}(\partial_k\partial_m - \partial_m\partial_k)u_i(\vec{x})\,, \tag{2.38}$$

the dislocation density (2.31) reads

$$\alpha_{ij}(\vec{x}) = \varepsilon_{ikl}S_{lkj}\,. \tag{2.39}$$

Due to the identity

$$\partial_i\delta_i(\vec{x};L) = 0\,, \tag{2.40}$$

for closed lines L, the disclination density satisfies the conservation law

$$\partial_i\theta_{ij}(\vec{x}) = 0\,, \tag{2.41}$$

which implies that disclination lines are always closed. Differentiating (2.35) we find the conservation law for disclination lines

$$\partial_i\alpha_{ij}(\vec{x}) = -\Omega_{jk}\delta_i(\vec{x};L)\,, \tag{2.42}$$

which, in turn, can be expressed in the form

$$\partial_i \alpha_{ij}(\vec{x}) = -\,\varepsilon_{jkl}\theta_{kl}(\vec{x})\,. \tag{2.43}$$

In terms of the tensor S_{ijk}, this becomes

$$\varepsilon_{jkl}(\partial_i S_{kli} + \partial_k S_{lnm} - \partial_l S_{knm}) = -\,\varepsilon_{jkl}\theta_{kl}(\vec{x})\,. \tag{2.44}$$

Note that the tensors u_{ij}, $\partial_k \partial_i u_j$ and $\partial_k \omega_{ij}(\vec{x})$ must satisfy the integrability condition

$$(\partial_l \partial_n - \partial_n \partial_l)\partial_k u_{ij}(\vec{x}) = 0 \tag{2.45}$$
$$(\partial_l \partial_n - \partial_n \partial_l)\partial_k \partial_i u_j(\vec{x}) = 0 \tag{2.46}$$
$$(\partial_l \partial_n - \partial_n \partial_l)\partial_k \omega_{ij}(\vec{x}) = 0\,. \tag{2.47}$$

If we write down this relation three times, each time with l, n, k exchanged cyclically, we find

$$\partial_l R_{nkij} + \partial_n R_{klij} + \partial_k R_{lnij} = 0\,, \tag{2.48}$$

where R_{nkij} is an abbreviation for the expression,

$$R_{nkij} = (\partial_n \partial_k - \partial_k \partial_n)\partial_i u_j(\vec{x})\,. \tag{2.49}$$

and which is anti-symmetric not only in n and k but also in i and j.

The information contained in α_{ij} and θ_{ij} can be combined into a single symmetric tensor, called the defect density $\eta_{ij}(x)$,

$$\eta_{ij}(\vec{x}) \equiv \theta_{ij}(\vec{x}) - \frac{1}{2}\partial_m(\,\epsilon_{min}\alpha_{jn} + (ij) -\,\epsilon_{ijn}\alpha_{mn}) =\,\epsilon_{ijk}\,\epsilon_{jmn}\partial_k \partial_m u_{ln}(\vec{x})\,. \tag{2.50}$$

2.2.2 Constructing Defect Densities

For a general defect line, *translating and rotating* a piece of continuum volume media, the full displacement field is

$$\begin{aligned} u_l(\vec{x}) &= -\delta(\vec{x};V)[b_l + (\vec{\Omega} \times \vec{x})_l] \\ &= -\delta(\vec{x};V)(b_l +\,\varepsilon_{lqr}\Omega_q x_r) \qquad (2.51)\\ &= -\delta(\vec{x};V)B_l \qquad (2.52) \end{aligned}$$

where

$$B_l \equiv b_l +\,\varepsilon_{lqr}\Omega_q x_r\,, \tag{2.53}$$

is the *total Burgers vector*. The full displacement field is not defined for an open surface due to the $\delta(\vec{x};V)$ term. It is multi-valued.

The distortion tensor

$$\partial_k u_l(\vec{x}) = \delta_k(\vec{x};S)B_l - \delta(\vec{x};V)\,\varepsilon_{lqk}\Omega_q\,. \tag{2.54}$$

The symmetric combination of $\partial_k u_l(\vec{x})$, gives the *plastic strain* and denote it by u^p_{kl}

$$u^p_{kl} = \frac{1}{2}(\partial_k u_l + \partial_l u_k) = \frac{1}{2}[\delta_k(\vec{x};S)B_l + (k \leftrightarrow l)]\,. \tag{2.55}$$

The field

$$\beta^p_{kl} \equiv \delta_k(\vec{x};S)B_l\,, \tag{2.56}$$

is usually called a *plastic distortion*. It is a single valued field (i.e., derivatives in front of it commute). In terms of β^p_{kl}, the plastic strain is simply

$$u^p_{kl} = \frac{1}{2}(\beta^p_{kl} + \beta^p_{lk})\,. \tag{2.57}$$

The dislocation density, however, is single valued. Indeed, we can easily calculate

$$\alpha_{il} = \varepsilon_{ijk}\partial_j\partial_k u_l = \varepsilon_{ijk}\partial_j[\delta_k(\vec{x};S)B_l - \delta(\vec{x};V)\,\varepsilon_{lqk}\Omega_q] = \delta_i(\vec{x};L)B_l\,. \tag{2.58}$$

The disclination density

$$\begin{aligned}
\theta_{pj} &= \varepsilon_{pmn}\partial_m\partial_n\omega_j = \varepsilon_{pmn}\partial_m\partial_n\frac{1}{2}\,\varepsilon_{jkl}\partial_k u_l \\
&= \varepsilon_{pmn}\partial_m\frac{1}{2}\,\varepsilon_{jkl}\partial_n[\delta_k(\vec{x};S)(b_l + \varepsilon_{lqr}\Omega_q x_r) - \delta(\vec{x};V)\,\varepsilon_{lqk}\Omega_q] \\
&= \varepsilon_{pmn}\partial_m[\frac{1}{2}\,\varepsilon_{jkl}\partial_n\beta^p_{kl} + \delta_n(\vec{x};S)\Omega_j] \\
&= \varepsilon_{pmn}\partial_m[\frac{1}{2}\,\varepsilon_{jkl}\partial_n\beta^p_{kl} + \phi^p_{nj}) \\
&= \frac{1}{2}\,\varepsilon_{jkl}\,\varepsilon_{pmn}\partial_m\partial_n\beta^p_{kl} + \varepsilon_{pmn}\partial_m\phi^p_{nj} \\
&= \varepsilon_{pmn}\partial_m\phi^p_{nj}\,,
\end{aligned} \tag{2.59}$$

since the derivatives commute in front of β^p_{kl}, and

$$\phi^p_{nj} \equiv \delta_n(\vec{x};S)\Omega_j\,, \tag{2.60}$$

is the *plastic rotation*. Use of Stokes' theorem on the second term gives

$$\theta_{pj} = \varepsilon_{pmn}\partial_m\phi^p_{nj} = \delta_p(\vec{x};L)\Omega_j\,. \tag{2.61}$$

The field od *plastic bend-twist* is defined by

$$\kappa^p_{nj} \equiv \partial_n\omega_j = \frac{1}{2}\,\varepsilon_{jkl}\partial_n\beta^p_{kl} + \phi^p_{nj}\,. \tag{2.62}$$

Note that the dislocation density can also be expressed in terms of β^p_{kl} and ϕ^p_{nj} as

$$\alpha_{il} = \epsilon_{ijk}\partial_j\beta^p_{kl} + \delta_{il}\phi^p_{kk} - \phi^p_{li}\,. \tag{2.63}$$

A combination of the two

$$\eta_{ij}(\vec{x}) \equiv \theta_{ij}(\vec{x}) + \frac{1}{2}\partial_m[\,\varepsilon_{min}\alpha_{jn}(\vec{x}) + (i \leftrightarrow j) - \varepsilon_{ijn}\alpha_{mn}(\vec{x})] \tag{2.64}$$

forms the *defect tensor*

$$\eta_{ij}(\vec{x}) \equiv \varepsilon_{ikl}\,\varepsilon_{jmn}\partial_k\partial_m u^p_{ln}(\vec{x})\,. \tag{2.65}$$

Due to the conservation laws (2.41,2.42), the tensor η_{ij} is symmetric and conserved

$$\partial_i\eta_{ij}(\vec{x}) = 0\,. \tag{2.66}$$

Chapter 3

Gauge Theory of Defects In Second Gradient Elasticity

The purpose of this chapter is to deduce the general form of the defect (plastic) energy in the presence of defects, in terms of plastic gauge fields if we are to account for the higher gradient terms in linear elasticity.

3.1 Defect Gauge Invariance

In the previous chapter, we saw that dislocation and disclination densities, $\alpha_{ij}(\vec{x})$ and $\theta_{ij}(\vec{x})$ in terms of plastic fields β^p_{kl}, ϕ^p_{kl} are given by the following expressions

$$\alpha_{il}(\vec{x}) = \varepsilon_{ijk}\partial_j\beta^p_{kl}(\vec{x}) + \delta_{il}\phi^p_{kk} - \phi^p_{li}\,, \quad (3.1)$$
$$\theta_{ij}(\vec{x}) = \varepsilon_{ikl}\partial_k\phi^p_{lj}(\vec{x})\,. \quad (3.2)$$

The freedom in choosing the Volterra surfaces for the construction of the defect lines allows us to derive the expressions for the dislocations and disclinations densities. It is the freedom of the gauge transformations corresponding to the change of the form of the Volterra cutting surface.

Translating and rotating a piece of continuum media in general Volterra procedure corresponds to the following gauge transformation

$$u_l(\vec{x}) \to u_l(\vec{x}) - \delta(\vec{x};V)(b_l + \varepsilon_{lqr}\Omega_q x_r) \equiv u_l(\vec{x}) - \delta(\vec{x};V)B_l\,. \quad (3.3)$$

By making the following abbreviations

$$u_l(\vec{x}) \to u_l(\vec{x}) + N_l(\vec{x}) + \varepsilon_{iqr}M_q(\vec{x})x_r \equiv u_l(\vec{x}) + \tilde{N}_l(\vec{x})\,, \quad (3.4)$$

with

$$N_l(\vec{x}) \equiv -\delta(\vec{x};V)b_l\,, \quad (3.5)$$
$$M_l(\vec{x}) \equiv -\delta(\vec{x};V)\Omega_l\,, \quad (3.6)$$
$$\tilde{N}_l(\vec{x}) \equiv -\delta(\vec{x};V)B_l\,. \quad (3.7)$$

We note that, under the change $S \to S'$, we have

$$\delta_i(S') = \delta_i(S) - \partial_i\delta(V)\,, \quad (3.8)$$

where V is the volume over which the surface S has swept.

The pure gauge transformation of the rotation angle is

$$\begin{aligned} \omega'_i &= \frac{1}{2}\,\varepsilon_{jkl}\partial_k u'_l \\ &= \frac{1}{2}\,\varepsilon_{jkl}\partial_k(u_l+\tilde{N}_l) \\ &= \frac{1}{2}\,\varepsilon_{jkl}\partial_k u_l+\frac{1}{2}\,\varepsilon_{jkl}\partial_k\tilde{N}_l \\ &= \omega_i+\frac{1}{2}\,\varepsilon_{jkl}\partial_k\tilde{N}_l \\ &= \omega_i+M_i\,, \end{aligned} \tag{3.9}$$

The pure gauge transformation of the plastic distortions as

$$\begin{aligned} \beta'^p_{kl} &= \delta_k(\vec{x};S')B_l \\ &= [\delta_k(\vec{x};S)-\partial_k\delta(\vec{x};V)]B_l \\ &= \delta_k(\vec{x};S)B_l-[\partial_k\delta(\vec{x};V)]B_l \\ &= \beta^p_{kl}-\partial_k[\delta(\vec{x};V)B_l]+\varepsilon_{lqk}\Omega_q\delta(\vec{x};V) \\ &= \beta^p_{kl}+\partial_k\tilde{N}_l(\vec{x})-\varepsilon_{lqk}M_q(\vec{x}) \\ &= \beta^p_{kl}+\partial_k\tilde{N}_l(\vec{x})-\varepsilon_{klq}M_q(\vec{x})\,. \end{aligned} \tag{3.10}$$

The pure gauge transformation of the plastic rotations is

$$\begin{aligned} \phi'^p_{kl} &= \delta_k(\vec{x};S')\Omega_l \\ &= [\delta_k(\vec{x};S)-\partial_k\delta(\vec{x};V)]\Omega_l \\ &= \delta_k(\vec{x};S)\Omega_l-\partial_k\delta(\vec{x};V)\Omega_l \\ &= \phi^p_{kl}+\partial_k M_l(\vec{x})\,. \end{aligned} \tag{3.11}$$

Hence, The defect densities (3.2;3.1) are invariant under the following defect gauge transformations

$$u_l(\vec{x}) \;\to\; u_l(\vec{x})+\tilde{N}_l(\vec{x})\,, \tag{3.12}$$

$$\omega_i(\vec{x}) \;\to\; \omega_i(\vec{x})+M_i(\vec{x})\,, \tag{3.13}$$

$$\beta^p_{kl}(\vec{x}) \;\to\; \beta^p_{kl}(\vec{x})+\partial_k\tilde{N}_l(\vec{x})-\varepsilon_{klq}M_q(\vec{x})\,, \tag{3.14}$$

$$\phi^p_{kl}(\vec{x}) \;\to\; \phi^p_{kl}(\vec{x})+\partial_k M_l(\vec{x})\,. \tag{3.15}$$

Thus

$$h_{ij} \equiv \beta^p_{ij}+\varepsilon_{ijk}M_k\,, \tag{3.16}$$

$$A_{ijk} \equiv \varepsilon_{jkl}\phi^p_{ij}\,, \tag{3.17}$$

are translational and rotational defect gauge fields in the continuum media[13].

3.2 Defect Energy In Second Gradient Elasticity

In linear elasticity with second gradients of the strain tensor, the elastic energy is given by

$$E_{el}=\int d^3x\left\{\mu u_{ij}^2+\frac{\lambda}{2}u_{ii}^2+\frac{2\mu+\lambda}{2}l'^2(\partial_i u_{ii})^2+2\mu l^2(\partial_i\omega_j)^2\right\}. \tag{3.18}$$

where the parameters l and l' are two length scales over which the space-time fabric is rotationally stiff.

In the stress representation, the canonical form of (3.18) takes the following expression

$$\begin{aligned} E_{el} &= \int d^3x \left\{ \frac{1}{4\mu}\left({\sigma^s_{ij}}^2 - \frac{\nu}{1+\nu}{\sigma^s_{ii}}^2\right) + \frac{1}{2(2\mu+\lambda)l'^2}\tau'^2_i \right. \\ &\left. + \frac{1}{8\mu l^2}\tau^2_{ii} + \sigma^s_{ij}(\partial_i u_j + \partial_j u_i)/2 + \tau'_i \partial_i \partial_l u_l + \tau_{ij}\partial_i \omega_j \right\} . \\ &= \int d^3x \left\{ \frac{1}{4\mu}\left({\sigma^s_{ij}}^2 - \frac{\nu}{1+\nu}{\sigma^s_{ii}}^2\right) + \frac{1}{2(2\mu+\lambda)l'^2}\tau'^2_i \right. \\ &\left. + \frac{1}{8\mu l^2}\tau^2_{ii} + \sigma_{ij}(\partial_i u_j - \varepsilon_{ijk}\omega_k) + \tau'_i \partial_i \partial_l u_l + \tau_{ij}\partial_i \omega_j \right\} . \end{aligned} \tag{3.19}$$

where the canonical momenta are given by

$$\sigma^s_{ij} = \frac{\partial e}{\partial u_{ij}} = 2\mu u_{ij} + \lambda \delta_{ij} u_{ll}\,, \tag{3.20}$$

$$\tau_i = \frac{\partial e}{\partial \partial_i u_{ll}} = (2\mu+\lambda) l'^2 \partial_i u_{ll}\,, \tag{3.21}$$

$$\tau_{ij} = \frac{\partial e}{\partial \partial_i \omega_j} = 4\mu l^2 \partial_i \omega_j\,. \tag{3.22}$$

and the superscript "s" indicates the symmetric part of the "momentum variable" tensor

$$\sigma_{ij} = \sigma^s_{ij} + \sigma^a_{ij}\,, \tag{3.23}$$

with the conservation laws

$$\partial_i \tau_{ij} = -\,\varepsilon_{jkl}\sigma_{kl}, \qquad \partial_i \sigma_{ij} = 0\,, \tag{3.24}$$

which are the stress analogue of the equations (2.41;2.42)

$$\partial_i \theta_{ij} = 0, \qquad \partial_i \alpha_{ij} = -\,\varepsilon_{jkl}\theta_{kl}\,. \tag{3.25}$$

Now, in the presence of defects (plastic deformations), the "*defect energy*" is measured by the deviations of the total strain u_{ij} and the total gradient $\partial_i\omega_j$ from the plastic strain u^p_{ij} and the plastic gradient $\partial_i\omega^p_j$, respectively

$$\begin{aligned} E_{def} &= \int d^3x \left\{ \mu(u_{ij} - u^p_{ij})^2 + \frac{\lambda}{2}(u_{ii} - u^p_{ii})^2 + \frac{2\mu+\lambda}{2} l'^2 [\partial_i(u_{ii} - u^p_{ii})]^2 \right. \\ &\left. + 2\mu l^2 [(\partial_i\omega_j - \partial_i\omega^p_j)^2 + \varepsilon(\partial_i\omega_j - \partial_i\omega^p_j)(\partial_j\omega_i - \partial_j\omega^p_i)] \right\} . \end{aligned} \tag{3.26}$$

In terms of plastic gauge fields, β^p_{ij} and ϕ^p_{ij}, of dislocations and disclinations

$$\begin{aligned} E_{def} = \int d^3x &\left\{ \mu[u_{ij} - (\beta^p_{ij} + \beta^p_{ji})]^2 + \frac{\lambda}{2}(u_{ii} - \beta^p_{ii})^2 + \frac{2\mu+\lambda}{2} l'^2 [\partial_i(u_{ii} - u^p_{ii})]^2 \right. \\ &\left. + 2\mu l^2 [(\partial_i\omega_j - 1/2\,\varepsilon_{ikl}\partial_j\beta^p_{kl} - \phi^p_{ji})^2 + \varepsilon(\partial_i\omega_j - 1/2\,\varepsilon_{ikl}\partial_j\beta^p_{kl} - \phi^p_{ji})(\partial_j\omega_i - 1/2\,\varepsilon_{jkl}\partial_i\beta^p_{kl} - \phi^p_{ij})] \right\} \end{aligned} \tag{3.27}$$

Inserting the transformation laws (3.12),(3.13),(3.14) and (3.15)

$$\begin{aligned} u_l(\vec{x}) &\to u_l(\vec{x}) + \tilde{u}^p_l(\vec{x}) + \varepsilon_{iqr}\omega^p_q(\vec{x})x_r\,, \\ \omega_i(\vec{x}) &\to \omega_i(\vec{x}) + \omega^p_i(\vec{x})\,, \\ \beta^p_{kl}(\vec{x}) &\to \beta^p_{kl}(\vec{x}) + \partial_k u^p_l(\vec{x}) - \varepsilon_{klq}\omega^p_q(\vec{x})\,, \\ \phi^p_{kl}(\vec{x}) &\to \phi^p_{kl}(\vec{x}) + \partial_k\omega^p_l(\vec{x})\,, \\ u^p_{kl}(\vec{x}) &\to u^p_{kl}(\vec{x}) + \frac{1}{2}(\partial_k\tilde{N}_l + \partial_l\tilde{N}_k)\,. \end{aligned}$$

into the energy (3.27) we see that it is invariant under defect gauge transformations.

Now, it is useful to rewrite (3.27) in canonical form, and we obtain the following expression

$$\begin{aligned} E_{def} &= \int d^3x \left\{ \frac{1}{4\mu}\left(\sigma^{s\,2}_{ij} - \frac{\nu}{1+\nu}\sigma^{s\,2}_{ii}\right) + \frac{1}{2(2\mu+\lambda)l'^2}\tau'^2_i + \frac{1}{8\mu l^2}(\delta_1\tau^2_{ii} + \delta_2\tau^2_{ll}) \right. \\ &\left. +\sigma_{ij}(\partial_i u_j - \varepsilon_{ijk}\omega_k - \beta^p_{ij}) + (\tau'_i\partial_i\partial_l u_l - \partial_i\beta^p_{ll}) + \tau_{ij}(\partial_i\omega_j - \phi^p_{ij})\right\}\,. \quad (3.28) \end{aligned}$$

where

$$\delta_1 \equiv 1/(1-\varepsilon^2), \qquad \delta_2 \equiv -\varepsilon\delta_1\,. \quad (3.29)$$

By integrating out ω_i and u_i in (3.28), we find the conservation laws (3.24)

$$\partial_i\sigma_{ij} = 0, \qquad \partial_i\tau_{ij} = -\varepsilon_{jkl}\sigma_{kl}\,. \quad (3.30)$$

They are solved in terms of the *stress gauge fields* A_{lj}, h_{lj} in the same way in the case of defect gauge fields

$$\begin{aligned} \sigma_{ij} &= \varepsilon_{ikl}\partial_k A_{lj}\,, \quad (3.31) \\ \tau_{ij} &= \epsilon_{ikl}\partial_k h_{lj} + \delta_{ij}A_{ll} - A_{ji}\,. \quad (3.32) \end{aligned}$$

which are invariant under the following *stresses local gauge transformations*

$$\begin{aligned} A_{lj} &\to A_{lj} + \partial_l\Lambda_j\,, \quad (3.33) \\ h_{lj} &\to h_{lj} + \partial_l\xi_j - \varepsilon_{ljk}\Lambda_k\,. \quad (3.34) \end{aligned}$$

which have the same structure as the gauge transformations on the defect fields (3.14) and (3.15).

Relations (3.31,3.32) allows us to reexpress the defect energy as

$$\begin{aligned} E &= \int d^3x \left\{ \frac{1}{4\mu}\left(\sigma^{s\,2}_{ij} - \frac{\nu}{1+\nu}\sigma^{s\,2}_{ii}\right) + \frac{1}{8l^2}(\delta_1{\tau_{ij}}^2 + \delta_2{\tau_{ll}}^2) \right. \\ &\left. +(A_{ij}\alpha_{ij} + h_{ij}\theta_{ij}) + (c_\alpha\alpha^2_{ij} + c_\theta\theta^2_{ij})\right\}\,. \quad (3.35) \\ &= E_{stress} + E_{int} + E_{def} \quad (3.36) \end{aligned}$$

with

$$\begin{aligned} E_{stress} &= \int d^3x \frac{1}{4\mu}\left(\sigma^{s\,2}_{ij} - \frac{\nu}{1+\nu}\sigma^{s\,2}_{ii}\right)\,, \quad (3.37) \\ E_{def} &= \int d^3x c_\alpha\alpha^2_{ij} + c_\theta\theta^2_{ij}\,, \quad (3.38) \\ E_{int} &= \int d^3x A_{ij}\alpha_{ij} + h_{ij}\theta_{ij}\,. \quad (3.39) \end{aligned}$$

We note that, in (3.35) we have ignored the $\tau_i'^2$ term since it produces only small quantitative corrections to linear elasticity.

The stress gauge fields couple locally to the defect densities which are singular on the boundary lines of the Volterra surfaces. In the limit of a vanishing length scale, τ_{ij} is forced to be identically zero and (3.31,3.32) allows us to express A_{ij} in terms of h_{ij}. Then the energy becomes

$$E = \int d^3x \left\{ \frac{1}{4\mu} \left(\sigma_{ij}^{s\,2} - \frac{\nu}{1+\nu} \sigma_{ii}^{s\,2} \right) + h_{ij}\eta_{ij} \right\} . \tag{3.40}$$

where the defect density η_{ij} contains dislocation and disclination lines.

Using the stress gauge fields, the fundamental identity (**??**) takes the *double gauge* forme

$$E = \int d^3x \left\{ \frac{1}{4\mu} \left(\sigma_{ij}^{s\,2} - \frac{\nu}{1+\nu} \sigma_{ii}^{s\,2} \right) + \frac{1}{8\mu l^2} \tau_{ij}^2 + \sigma_{ij}\beta_{ij}^p + \tau_{ij}\phi_{ij}^p \right\} . \tag{3.41}$$

Chapter 4

Einstein-Cartan Theory of Gravitation

The Einstein-Cartan (EC) theory of gravity, also known as the Einstein–Cartan–Sciama–Kibble theory, is a simplest possible extension of General Theory of Relativity (GR), formulated within the framework of Riemann–Cartan geometry that allowing space-time to have torsion field, in addition to curvature, and relating torsion to the density of intrinsic angular momentum. The theory was first introduced by Elie Cartan (1922), in an attempt to propose torsion as the macroscopic manifestation of the intrinsic angular momentum (spin) of the matter [10].

According to the EC theory, the antisymmetric part of the affine connection is non-vanishing (affine connection is asymmetric), in contrast to the symmetric Riemann connection of Christoffel symbol [10]. Therefore, in addition to the metric tensor sourced by the stress-energy tensor of the matter fields, there is also an independent torsional field, sourced via the spin density tensor. Physically, just as the presence of matter is responsible for the spacetime curvature, the intrinsic angular momentum (spin) of the matter is responsible for the presence of torsion.

4.1 Riemann Space-time (V_4) of GR

Einstein's general theory of relativity is described as a geometric property of space-time continuum, where the background space-time is Riemann manifold denoted as V_4 which is torsion less.

We use coordinates x^a $(a = 0, 1, 2, 3)$ to specify the points in Minkowski flat space-time M_4 with the four-dimensional basis vectors e_a, and an arbitrary four-dimensional vector $x = x^a e_a$. The basis vectors are orthonormal with respect to the Minkowski metric g_{ab}:

$$g_{ab} = e_a e_b \,. \tag{4.1}$$

with the matrix elements

$$g_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{4.2}$$

Let us now reparametrize this Minkowski spacetime by a new set of coordinates x^μ whose values are given by a mapping

$$x^a \to x^\mu = x^\mu(x^a)\,. \tag{4.3}$$

The general coordinate transformation (4.3) and their inverse $x^a(x^\mu)$ will satisfy the integrability conditions of Schwartz:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)x^a(x^\kappa) = 0\,, \tag{4.4}$$
$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\partial_\lambda x^a(x^\kappa) = 0\,, \tag{4.5}$$

The conditions $x^\mu(x^a) = const$ define a network of new coordinate hypersurfaces whose normal vectors are given by :

$$e_\mu = \frac{\partial x^a}{\partial x^\mu} e_a \equiv e^a{}_\mu(x) e_a\,. \tag{4.6}$$

These are called *local basis vectors.* Their components $e^a{}_\mu(x)$ are called *local basis tetrads.*

The metric tensor in the curvilinear coordinates can be expressed as a scalar product of the local basis vectors:

$$g_{\mu\nu}(x) = e_\mu(x)e_\nu(x) = g_{ab}e^a{}_\mu(x)e^b{}_\nu(x)\,. \tag{4.7}$$

The *Riemann connection of Christoffel symbols* are:

$$\bar{\Gamma}_{\mu\nu\lambda} \equiv \{\mu\nu,\lambda\} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})\,, \tag{4.8}$$

The *modified Christoffel* symbols are

$$\bar{\Gamma}_{\mu\nu}{}^\kappa \equiv \left\{{\kappa \atop \mu\nu}\right\} = g^{\kappa\lambda}\bar{\Gamma}_{\mu\nu\lambda}\,, \tag{4.9}$$

The *Riemann covariant derivative*

$$\bar{D}_\nu v^\mu = \partial_\nu v^\mu + \bar{\Gamma}_{\nu\lambda}{}^\mu v^\lambda\,, \tag{4.10}$$

The *Riemann curvature tensor* is

$$\bar{R}_{\mu\nu\lambda}{}^\sigma = \partial_\mu\bar{\Gamma}_{\nu\lambda}{}^\sigma - \partial_\nu\bar{\Gamma}_{\mu\lambda}{}^\sigma - (\bar{\Gamma}_{\mu\lambda}{}^\delta\bar{\Gamma}_{\nu\delta}{}^\sigma - \bar{\Gamma}_{\nu\lambda}{}^\delta\bar{\Gamma}_{\mu\delta}{}^\sigma)\,. \tag{4.11}$$

From the curvature tensor we get the Ricci tensor, $\bar{R}_{\mu\nu}$, and scalar, R:

$$\bar{R}_{\mu\nu} = g^{\rho\sigma}\bar{R}_{\rho\mu\nu\sigma}\,, \tag{4.12}$$
$$\bar{R} = g^{\mu\nu}\bar{R}_{\mu\nu}\,. \tag{4.13}$$

The Einstein field equations are

$$\bar{G}^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{4.14}$$

where $\bar{G}_{\mu\nu}$ is the Einstein tensor formed from the Riemann curvature tensor and Ricci scalar

$$\bar{G}_{\mu\nu} \equiv \bar{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\bar{R}\,, \tag{4.15}$$

and $T^{\mu\nu}$ is the energy-momentum tensor of matter. $\kappa = 8\pi G/c^4$.

4.2 Riemann-Cartan Space-time (U_4)

Cartan generalised Riemann's geometry by simply not imposing symmetry on the connection symbols which was considered in GR. The resulting antisymmetric part, a tensor, commonly known as *torsion*,

$$S_{\mu\nu}{}^{\lambda} \equiv \frac{1}{2}(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}) \,, \tag{4.16}$$

It is a third rank tensor that is antisymmetric in its first two indices and has 24 independent components.

The covariant derivative D_ν with the nonsymmetric connection $\Gamma_{\mu\nu}{}^{\lambda}$, is given by,

$$D_\nu v^\mu = \partial_\nu v^\mu + \Gamma_{\nu\lambda}{}^{\mu} v^\lambda \,, \tag{4.17}$$

where

$$\Gamma_{\mu\nu}{}^{\lambda} \equiv e_a{}^{\lambda} \partial_\mu e^a{}_\nu \equiv -e^a{}_\nu \partial_\mu e_a{}^{\lambda} \,, \tag{4.18}$$

is called the *affine connection* of the Riemann-Cartan manifold U_4.

The physical observability requires the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma_{\mu\nu}{}^{\lambda}$ to be a smooth single-valued functions, so that they satisfie the integrability condition

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) g_{\lambda\kappa} = 0 = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) \Gamma_{\lambda\kappa}{}^{\delta} \,. \tag{4.19}$$

Minkowski spacetime has no torsion so that the tensor nature of $S_{\mu\nu}{}^{\lambda}$ implies that it vanishes in all curvilinear coordinates[1].

The affine connection of Riemann Cartan manifold (U_4), $\Gamma_{\mu\nu\lambda}$, can be decomposed into a sum of

$$\Gamma_{\mu\nu\lambda} = \bar{\Gamma}_{\mu\nu\lambda} + K_{\mu\nu\lambda} \,, \tag{4.20}$$

where $\bar{\Gamma}_{\mu\nu\lambda}$ is the Riemann connection and $K_{\mu\nu\lambda}$ is the *contortion tensor* defined as:

$$K_{\mu\nu\lambda} \equiv S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} \,. \tag{4.21}$$

In Riemann-Cartan Space-time (U_4), quantities such as covariant derivative, Riemannian-Cartan curvature tensor, Ricci tensor, Ricci scalar and Einstein tensor, are defined in a similar fashion as in GR, the only difference being that the Riemann connections are replaced by the total connection as defined in equation (4.20),

$$D_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma_{\mu\nu}{}^{\lambda} v^\nu \,, \tag{4.22}$$

$$R_{\mu\nu\lambda}{}^{\sigma} = \partial_\mu \Gamma_{\nu\lambda}{}^{\sigma} - \partial_\nu \Gamma_{\mu\lambda}{}^{\sigma} - (\Gamma_{\mu\lambda}{}^{\delta}\Gamma_{\nu\delta}{}^{\sigma} - \Gamma_{\nu\lambda}{}^{\delta}\Gamma_{\mu\delta}{}^{\sigma}) \,. \tag{4.23}$$

$$R_{\mu\nu} = R_{\kappa\mu\nu}{}^{\kappa} \,, \tag{4.24}$$

$$R = g^{\mu\nu} R_{\mu\nu} \,, \tag{4.25}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,. \tag{4.26}$$

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[1]If we describe Minkowski space in terms of coordinates x^μ which coincide with the inertial coordinates x^a, then the basis tetrads $e^a{}_\mu = \partial_\mu x^a$ a are unit matrices, so that the connection vanishes, and so does its antisymmetric part, the torsion. If we now perform a general coordinate transformation to curvilinear coordinates $x^\mu(x^a)$ the connection will in general become nonzero. The torsion, however, being a tensor, remains zero for all coordinate transformations of Minkowski space.

However it must be noted that $R_{\mu\nu}$ and $G_{\mu\nu}$ are no longer symmetric, and $G_{\mu\nu}$ can also be written as

$$G^{\nu\mu} = \frac{1}{4} e^{\mu\alpha\beta\gamma} e^{\nu}{}_{\alpha}{}^{\sigma\tau} R_{\beta\gamma\sigma\tau} \,, \tag{4.27}$$

where $e^{\mu\nu\lambda\kappa}$ is the contravariant version of the Levi-Civita.

In terms of the basic tetrads $e_a{}^{\mu}$, the curvature tensor takes the form:

$$R_{\mu\nu\lambda}{}^{\sigma} = e_a{}^{\sigma}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda \,. \tag{4.28}$$

Just as this Minkowski spacetime had a vanishing torsion tensor for any curvilinear parametrization, it also has a vanishing curvature tensor. In fact, the representation (4.28) shows that a space x^μ can have curvature only if the derivatives of the mapping functions $x^a \to x^\mu$ are not integrable in the Schwarz sense. Expressed differently, the vanishing of $R_{\mu\nu\lambda}{}^{\sigma}$ follows from the obvious fact that for the trivial choice of the basis tetrad $e_a{}^{\kappa} = \delta_a{}^{\kappa}$

$$R_{\mu\nu\lambda}{}^{\kappa} = e_a{}^{\kappa}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda \equiv 0 \,, \tag{4.29}$$

which remains identically zero in any curvilinear parametrization of Minkowski space.

The full curvature tensor $R_{\mu\nu\lambda}{}^{\sigma}$ can be decomposed into

$$R_{\mu\nu\lambda}{}^{\kappa} = \bar{R}_{\mu\nu\lambda}{}^{\kappa} + \bar{D}_\mu K_{\nu\lambda}{}^{\kappa} - \bar{D}_\nu K_{\mu\lambda}{}^{\kappa} - (K_{\mu\lambda}{}^{\rho} K_{\nu\rho}{}^{\kappa} - K_{\nu\lambda}{}^{\rho} K_{\mu\rho}{}^{\kappa}) \,, \tag{4.30}$$

Spaces with curvature and torsion are known as *Riemann-Cartan spaces*, while spaces with no torsion are known as *Riemann spaces*.

It should be pointed out that a nonvanishing curvature tensor has the consequence that covariant derivatives no longer commute:

$$(D_\nu D_\mu - D_\nu D_\mu) v_\lambda = -R_{\nu\mu\lambda}{}^{\kappa} v_\kappa - 2 S_{\nu\mu}{}^{\rho} D_\rho v_\lambda \,, \tag{4.31}$$

$$(D_\nu D_\mu - D_\nu D_\mu) v^\kappa = R_{\nu\mu\lambda}{}^{\kappa} v^\lambda - 2 S_{\nu\mu}{}^{\rho} D_\rho v^\kappa \,. \tag{4.32}$$

4.3 Einstein-Cartan Theory

The Einstein-Cartan field equations can be derived by the usual procedure where the action of space-time is constructed and varied with respect to the metric tensor field $g_{\mu\nu}$ and contorsion tensor field $K_{\alpha\beta\mu}$ in the Action. This action is given by:

$$S = \frac{1}{2K} \int d^4x \sqrt{-g} R(g, \partial g, K) \,. \tag{4.33}$$

Here $\kappa = 8\pi G/c^4$ and R denotes the Lagrangian density due to the gravitational field.

The variation with respect to the metric tensor fields $g_{\mu\nu}$ and $K_{\alpha\beta\mu}$ yields

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} = G^{\mu\nu} - \nabla = \kappa T^{\mu\nu} \,, \tag{4.34}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = T^{\alpha\beta\mu} = \kappa \Sigma_{\mu\kappa}{}^{,\tau} \,, \tag{4.35}$$

$$\tag{4.36}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensorand $\Sigma_{\mu\kappa}{}^{,\tau}$ is the spin-current-density tensor of matter.

Using the definition of the curvature tensor and torsion tensor defined in the earlier section, field equations of EC theory can be written as :

$$G^{\mu\nu} - \frac{1}{2}D^*_\lambda(S^{\mu\nu,\lambda} - S^{\nu\lambda,\mu} + S^{\lambda\mu,\nu}) = \kappa T^{\mu\nu}\,, \tag{4.37}$$

$$S_{\mu\kappa}{}^{,\tau} = \kappa\Sigma_{\mu\kappa}{}^{,\tau}\,, \tag{4.38}$$

where $S_{\mu\kappa}{}^{,\tau}$ is the following combination of torsion tensors:

$$\frac{1}{2}S_{\mu\kappa}{}^{,\tau} = 2(S_{\mu\kappa}{}^{\tau} + \delta_\mu{}^\tau S_\kappa - \delta_\kappa{}^\tau S_\mu) \equiv P_{\mu\kappa}{}^{\tau}\,. \tag{4.39}$$

with

$$S_\kappa \equiv S_{\kappa\lambda}{}^{\lambda}, \qquad S^\kappa \equiv S^\kappa{}_\lambda{}^\lambda\,. \tag{4.40}$$

This tensor is referred to as the *Palatini tensor*.

Here D^*_μ is defined as

$$D^*_\mu \equiv D_\mu + 2S_{\mu\kappa}{}^{\kappa}\,. \tag{4.41}$$

with

$$D_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma_{\mu\nu}{}^{\lambda} v^\nu\,. \tag{4.42}$$

The conservation laws read

$$D^*_\mu G_\lambda{}^\mu = -2S^{\nu\kappa\lambda}G_{\kappa\nu} + \frac{1}{2}S^{\nu\kappa,\mu}R_{\lambda\mu\nu\kappa}\,, \tag{4.43}$$

$$D^{*\mu}S_{\lambda\kappa,\mu} = G_{\lambda\kappa} - G_{\kappa\lambda}\,. \tag{4.44}$$

They are Bianchi identities ensuring the single-valuedness of observables, connection $\Gamma_{\mu\nu}{}^\lambda$ and metric $g_{\mu\nu}$, via the integrability conditions $[\partial_\sigma, \partial_\tau]\Gamma_{\mu\nu}{}^\lambda = 0$ and $[\partial_\sigma, \partial_\tau]g_{\mu\nu} = 0$.

For a set of spinless point particles EC equations reduce to

$$G^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{4.45}$$

$$S_{\mu\kappa}{}^{,\tau} = 0\,. \tag{4.46}$$

Chapter 5

Einstein-Cartan Theory as a Theory of Defects

In the previous chapters we have outlined the main features of theory of defects and ECT. In this chapter we will show the similarities between the two theories.

5.1 Defects Theory in 3D Riemann-Cartan Space-Time

A Minkowski space-time has neither torsion nor curvature. The absence of torsion follows from its tensor property, which was a consequence of the commutativity of derivatives in front of the infinitesimal translation field

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\xi^\kappa(x) = 0 \,. \tag{5.1}$$

The absence of curvature, on the other hand, was a consequence of the integrability condition of the transformation matrices

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\lambda \xi^\kappa(x) = 0 \,. \tag{5.2}$$

A general affine spaces with torsion or curvature or both from a Minkowski spacetime can be constructed by performing *infinitesimal multivalued coordinate transformations*

$$e_a{}^\mu = \delta_a{}^\mu - \partial_a \xi^\mu, \qquad e^a{}_\mu = \delta^a{}_\mu + \partial_\mu \xi^a \,. \tag{5.3}$$

which do not satisfy (5.1), (5.2), where the metric is

$$g_{\mu\nu} = e^a{}_\mu e_{a\nu} = \eta_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \,. \tag{5.4}$$

Here $\eta_{\mu\nu}$ denotes the Minkowski spacetime metric where Greek subscripts refer to curvilinear coordinates.

Inserting the basis tetrads (5.3) into Eq. (4.18) we find

$$\Gamma_{\mu\nu\lambda} = \partial_\mu \partial_\nu \xi_\lambda \,, \tag{5.5}$$

$$S_{\mu\nu\lambda} = \frac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\xi_\lambda \,, \tag{5.6}$$

$$R_{\mu\nu\lambda\kappa} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\lambda \xi_\kappa \,. \tag{5.7}$$

We note that we must postulated that the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}{}^{\lambda}$ should be smooth enough to permit two differentiations which commute with each other, and so we consider only a singular coordinate transformations which satisfy the condition

$$
\begin{aligned}
(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(\partial_\lambda \xi^\kappa + \partial_\kappa \xi_\lambda) &= 0\,, &(5.8)\\
(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\sigma \partial_\lambda \xi_\kappa &= 0\,. &(5.9)
\end{aligned}
$$

Now, we consider an intrinsic description of an infinitesimal displacements in the fabric (or crystal displacements) is given by

$$x_i \to x'_i = x_i - u_i(\vec{x})\,, \tag{5.10}$$

Hence the non-commutativity of derivatives in front of singular coordinate changes $\xi^a(x^\lambda)$ is completely analogous to that in front of crystal displacements $u_i(\vec{x})$.

In three-dimensional Euclidian subspace of the Minkowski space, we have to identify the physical coordinates of continuum space points x^a for $a = 1, 2, 3$ with the previous spatial coordinates x^i for $i = 1, 2, 3$. The infinitesimal translations $\xi^{(a=i)}(\vec{x})$ are equal to the displacements $u_i(\vec{x})$ such that the basis tetrads are

$$e^i_a = \delta^i_a - \partial_a u_i, \qquad e^a_i = \delta^i_a + \partial_i u_a\,, \tag{5.11}$$

and the metric becomes, to linear approximation,

$$g_{ij} = e_{ai} e^a_j = \delta_{ij} + \partial_i u_j + \partial_j u_i\,. \tag{5.12}$$

Apart from the trivial unit matrix it coincides with twice the strain tensor

$$u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)\,. \tag{5.13}$$

Hence

$$
\begin{aligned}
\Gamma_{ijk} &= \partial_i \partial_j u_k\,, &(5.14)\\
S_{ijk} &= \frac{1}{2}(\partial_i \partial_j - \partial_j \partial_i) u_k\,, &(5.15)\\
R_{ijkl} &= (\partial_i \partial_j - \partial_j \partial_i)\partial_k u_l\,. &(5.16)
\end{aligned}
$$

and the integrability conditions read

$$
\begin{aligned}
(\partial_i \partial_j - \partial_j \partial_i)(\partial_k u_l + \partial_l u_k) = 0\,, &&(5.17)\\
(\partial_i \partial_j - \partial_j \partial_i)\partial_n(\partial_k u_l + \partial_l u_k) = 0\,, &&(5.18)\\
(\partial_i \partial_j - \partial_j \partial_i)\partial_k(\partial_k u_l - \partial_l u_k) = 0\,. &&(5.19)
\end{aligned}
$$

Now, if we suggest, in three dimensions, the introduction of a tensor of second rank analogous to (4.27)

$$G_{ij} = \frac{1}{4} e_{ikl} e_{jmn} R^{klmn}\,, \tag{5.20}$$

to linear approximation becomes:

$$G_{ij} = \varepsilon_{ikl}\partial_k\partial_l(\frac{1}{2}e_{jmn}\partial_m u_n)\,. \tag{5.21}$$

where the second factor is the local rotation

$$\omega_j = \frac{1}{2}\,\varepsilon_{jmn}\partial_m u_n\,, \tag{5.22}$$

Hence we see that the Einstein curvature tensor can be written as

$$G_{ij} = \varepsilon_{ikl}\partial_k\partial_l\omega_j\,. \tag{5.23}$$

and the Einstein tensor $\bar{G}_{ij}$ associated with the Riemann curvature $\bar{R}_{ijkl}$ is

$$\bar{G}_{ij} = \varepsilon_{ikl}\,\varepsilon_{jmn}\partial_k\partial_m\frac{1}{2}(\partial_l u_n + \partial_n u_l)\,. \tag{5.24}$$

Comparing the dislocation, disclination and the defect densities

$$\alpha_{ij} = \varepsilon_{ikl}\partial_k\partial_l u_j\,, \tag{5.25}$$
$$\theta_{ij} = \varepsilon_{ikl}\partial_k\partial_l\omega_j\,, \tag{5.26}$$
$$\eta_{ij} = \varepsilon_{ikl}\,\varepsilon_{jmn}\partial_k\partial_m u_{lm}\,. \tag{5.27}$$

we find that

$$\alpha_{ij} \equiv \varepsilon_{ikl}\Gamma_{klj} \equiv \varepsilon_{ikl}S_{klj}\,, \tag{5.28}$$
$$\theta_{ij} \equiv G_{ji}\,, \tag{5.29}$$
$$\eta_{ij} = \bar{G}_{ij}\,. \tag{5.30}$$

Hence we can conclude:
"*A spacetime with torsion and curvature can be generated from a Minkowski spacetime via singular coordinate transformations and is com- pletely equivalent to a crystal which has undergone plastic deformation and is filled with dislocations and disclinations.*"

Now, in three dimensions, the linearized version of (4.37,4.38) reads[1]

$$G_{ij} - \frac{1}{2}\partial_k(S_{ij,k} - S_{jk,i} + S_{ki,j}) = \kappa T_{ij}\,. \tag{5.31}$$
$$S_{ij,k} = S_{ijk} + \delta_{ik}S_j - \delta_{jk}S_i = \kappa\Sigma_{ij,k}\,. \tag{5.32}$$

where

$$S_{ij,k} = S_{ijk} + \delta_{ik}S_j - \delta_{jk}S_i\,. \tag{5.33}$$

If we insert the dislocation density according to

$$S_{ijk} = \frac{1}{2}(\partial_i\partial_j - \partial_j\partial_i)u_k = \frac{1}{2}\,\varepsilon_{ijl}\alpha_{lk}\,. \tag{5.34}$$

[1]In three dimensions, T_{ij} and $\Sigma_{ij,k}$ coincide, respectively, with the force stress field and the moment stress field.

then the spin density reads

$$S_{ij,k} = \varepsilon_{ijk}\alpha_{lk} + \delta_{ik}\,\varepsilon_{jpl}\alpha_{lp} - \delta_{jk}\,\varepsilon_{ipl}\alpha_{lp}\,. \tag{5.35}$$

Since both sides are antisymmetric in ij, we can contract them with ϵ_{ijn},

$$\begin{aligned} \epsilon_{ijn}S_{ij,k} &= 2\alpha_{nk} + \varepsilon_{kjn}\,\varepsilon_{jpl}\alpha_{lp} - \varepsilon_{ikn}\,\varepsilon_{ipl}\alpha_{lp}\,, && (5.36)\\ &= 2\alpha_{nk} - 2(\delta_{kp}\delta_{nl} - \delta_{kl}\delta_{np})\alpha_{lp}\,, && (5.37)\\ &= 2\alpha_{nk}\,. && (5.38) \end{aligned}$$

and see that $S_{ij,k}$ becomes simply

$$S_{ij,k} = \varepsilon_{ijl}\alpha_{kl} \equiv \alpha_{ij,k}\,. \tag{5.39}$$

Thus the spin density is equal to the dislocation density, which has a vanishing divergence

$$\partial_k S_{ij,k} = \varepsilon_{ijl}\partial_k\alpha_{kl} = 0\,. \tag{5.40}$$

Therefore, this fact lets us identify the moment stress tensor with the density of dislocations:

$$\alpha_{ijk} = \kappa\Sigma_{ij,k}\,. \tag{5.41}$$

The three combinations of

$$\frac{1}{2}(S_{ij,k} - S_{jk,i} - S_{ki,j}) = \frac{1}{2}(\,\varepsilon_{ijl}\alpha_{kl} - \varepsilon_{jkl}\alpha_{ij} + \varepsilon_{kil}\alpha_{jl})\,. \tag{5.42}$$

By contracting the identity

$$\varepsilon_{ijl}\delta_{km} + \varepsilon_{jkl}\delta_{im} + \varepsilon_{kil}\delta_{jm} = \varepsilon_{ijk}\delta_{lm}\,, \tag{5.43}$$

with α_{ml}, we find

$$\varepsilon_{ijl}\alpha_{kl} + \varepsilon_{jkl}\alpha_{il} + \varepsilon_{kil}\alpha_{jl} = \varepsilon_{ijk}\alpha_{ll}\,, \tag{5.44}$$

so that

$$\frac{1}{2}(S_{ij,k} - S_{jk,i} - S_{ki,j}) = \varepsilon_{jkl}\alpha_{il} + \frac{1}{2}\,\varepsilon_{ijk}\alpha_{ll} = \varepsilon_{jkl}K_{li}\,, \tag{5.45}$$

where

$$\mathcal{K}_{lj} = -\alpha_{jl} + \frac{1}{2}\delta_{lj}K_{kk}\,, \tag{5.46}$$

is Nye's contortion tensor.

With this notation, and using the fact that $\theta_{ij} \equiv G_{ij}$, equation (5.31) becomes

$$\theta_{ij} - \varepsilon_{jhl}\partial_n K_{li} = \kappa T_{ij}\,. \tag{5.47}$$

Equation (5.47) allows us to identify the total defect density tensor η_{ij} with the force stress tensor times κ,

$$\eta_{ij} = \kappa T^{ij} \,. \tag{5.48}$$

Hence, Einstein-Cartan field equations describing defect states of a three-dimensional continuum are :

$$\theta_{ij} - \varepsilon_{jhl}\partial_n K_{li} = \kappa T_{ij} \,. \tag{5.49}$$
$$\alpha_{ij,k} = \kappa \Sigma_{ij,k} \,. \tag{5.50}$$

Thus, the Einstein-Cartan field equations describe the defect state of a three-dimensional continuum, at least when the defects are small so that we can use a linear approximation. The analogy is completed by the conservation equations, which, stated as geometric identities, give the correct conservation laws for dislocations and disclinations. This analogy should not be surprising, because, as we said before, it is based on the common geometric structure of the two theories. In particular, it is evident that the comparison cannot be done with general relativity, where the torsion is zero [11].

5.2 Defects Theory in 4D Riemann-Cartan Space-Time

In four dimensional Einstein-Cartan space-time, dislocation and disclination densities are functions of position x^ξ and are defined by

$$\alpha^{\mu\nu}(x^\xi) = \epsilon^{\mu\sigma\tau} u^{\nu}_{;\sigma\tau}(x^\xi) \,, \tag{5.51}$$
$$\theta^{\mu\nu}(x^\xi) = \epsilon^{\mu\sigma\tau} \omega^{\nu}_{;\sigma\tau}(x^\xi) \,, \tag{5.52}$$

respectively, where u^ν is the displacement vector and ω^ν is the rotation vector. Here the semicolon (;) denotes covariant differentiation. For a defect line along L, (5.51,5.54) becomes

$$\alpha^{\mu\nu}(x^\xi) = \delta^\mu(L)(b^\nu - \Omega^{\nu\tau} x_\tau) \,, \tag{5.53}$$
$$\theta^{\mu\nu}(x^\xi) = \delta^\mu(L)\Omega^\nu \,, \tag{5.54}$$

respectively, where b^ν is the Burgers vector and Ω^ν is the Frank vector defined as in section (2.2.1), and the defect line L has a core discontinuity in the displacement and rotation fields, represented by the delta function $\delta(L)$.

These defect density tensors satisfy the conservation laws

$$\alpha^{\mu\nu}{}_{;\mu} = -\,\epsilon^{\nu\sigma\tau}\theta_{\sigma\tau} \,, \tag{5.55}$$
$$\theta^{\mu\nu}{}_{;\mu} = 0 \,. \tag{5.56}$$

Now, for small displacements in Minkowski spacetime:

$$g_{\mu\nu} = e^{a}{}_{\mu} e_{a\nu} = \eta_{\mu\nu} + (\partial_\mu u_\nu + \partial_\nu u_\mu) \,. \tag{5.57}$$

Then the connection, torsion tensor and curvature tensor are given by

$$\Gamma_{\mu\nu\lambda} = \partial_\mu \partial_\nu u_\lambda \,, \tag{5.58}$$

$$S_{\mu\nu\lambda} = \frac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) u_\lambda \,, \tag{5.59}$$

$$R_{\mu\nu\lambda\kappa} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\lambda u_\kappa \,. \tag{5.60}$$

The dislocation density tensor is related to the torsion tensor (contortion tensor) according to

$$\alpha_{\mu\nu} = \epsilon_\mu{}^{\kappa\lambda} S_{\kappa\lambda\nu} = \epsilon_\mu{}^{\kappa\lambda} K_{\kappa\lambda\nu} \,, \tag{5.61}$$

where the contortion tensor $K^{\mu\nu}$ defined as

$$K^{\mu\nu} = S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} \,, \tag{5.62}$$

Nye's contortion tensor of rank two is obtained from the contortion tensor using

$$K_{\mu\nu} = \frac{1}{2}\, \epsilon^{\kappa\lambda}{}_\nu K_{\mu\kappa\lambda} \,. \tag{5.63}$$

and substituting into (5.61), we obtain

$$\alpha^{\mu\nu} = -K^{\nu\mu} + \frac{1}{2}\delta^{\mu\nu} K^\lambda{}_\lambda \,, \tag{5.64}$$

Nye's contortion tensor can also be written in terms of the displacement and rotation fields as

$$K_{\mu\nu} = \partial_\mu \omega_\nu - \epsilon_\nu{}^{\sigma\tau} u_{\tau\mu} \,, \tag{5.65}$$

The disclination density tensor is related to the Einstein tensor of

$$\theta_{\mu\nu} = G_{\mu\nu} \,. \tag{5.66}$$

Now we extend the analogy to a four-dimensional continuum, and we can write four-dimensional equations which characterize the state of the medium as:

$$\theta^{\mu\nu} - \frac{1}{2} D^*_\lambda (S^{\mu\nu,\lambda} - S^{\nu\lambda,\mu} + S^{\lambda\mu,\nu}) = \kappa T^{\mu\nu} \,, \tag{5.67}$$

$$\alpha_{\mu\kappa}{}^\tau = \kappa \Sigma_{\mu\kappa}{}^{,\tau} \,, \tag{5.68}$$

where

$$\alpha_{\mu\nu} \equiv \frac{1}{2}\, \epsilon_{\mu\gamma\sigma} \alpha_{\gamma\sigma\beta} \,, \tag{5.69}$$

and the conservation laws of defect densities read

$$D^*_\mu \theta^\mu{}_\alpha = -2 S_{\alpha\mu}{}^\gamma \theta^\mu{}_\gamma + \frac{1}{2} \alpha^\mu{}_\gamma{}^\beta R_{\alpha\mu\beta}{}^\gamma \,, \tag{5.70}$$

$$D^*_\mu \alpha^\mu{}_\beta{}^\alpha = \theta^\alpha{}_\beta - \theta_\beta{}^\alpha \,. \tag{5.71}$$

Chapter 6

Gauge Theory of defects in 4D Riemann-Cartan Space-time Fabric

In this chapter we propose to construct a new unified non-linear gauge theory of defect in four dimensional Riemann-Cartan space-time fabric. We have to find an appropriate way of generalisation defect energy and incorporating the correct elastic interactions between the defects into the theory. The technic for doing so has been developed and explained before in detail in reference [8].

We take the three dimensional elastic energy (3.28) of a given defect distribution β^p_{ij}, ϕ^p_{ij}

$$\begin{aligned} E_{el} &= \int d^3x \left\{ \frac{1}{4\mu} \left({\sigma^s_{ij}}^2 - \frac{\nu}{1+\nu} {\sigma^s_{ii}}^2 \right) + \frac{1}{2(2\mu+\lambda)l'^2} \tau'^2_i + \frac{1}{8\mu l^2} (\delta_1 \tau^2_{ii} + \delta_2 \tau^2_{ll}) \right. \\ & \left. + \sigma_{ij} (\partial_i u_j - \varepsilon_{ijk}\omega_k - \beta^p_{ij}) + (\tau'_i \partial_i \partial_l u_l - \partial_i \beta^p_{ll}) + \tau_{ij} (\partial_i \omega_j - \phi^p_{ij}) \right\} . \quad (6.1) \end{aligned}$$

For simplicity, we assume that $\delta_1 = 1$, $\delta_2 = 0 = \tau'_i$, then

$$\begin{aligned} E_{el} &= \int d^3x \left\{ \frac{1}{4\mu} \left(\sigma_{ij}\sigma^{ij} - \frac{\nu}{1+\nu} {\sigma_i}^{i2} \right) + \frac{1}{8\mu l^2} {\tau_i}^{i2} \right. \\ & \left. + \sigma_{ij} (\partial_i u_j - \varepsilon_{ijk}\omega_k - \beta^p_{ij}) + \tau_{ij} (\partial_i \omega_j - \phi^p_{ij}) \right\} . \quad (6.2) \end{aligned}$$

Now, according to ([14],[15]), defects move under stresses as if they were spinning external particles in a curved space-time described by a four-dimensional Riemann-Cartan spacetime where the corresponding geometry is described by the direct generalizations of translational and rotational defect gauge fields h_{ij} and A_{ijk}, which are here the orthonormal vierbein field ${h^\alpha}_\mu$, and the spin connection ${A_{\mu\alpha}}^\beta$.

For this we go to the nonholonomic coordinates definded deffencially via

$$\partial_\mu \equiv {h_\mu}^\alpha \partial_\alpha \, , \quad (6.3)$$

where ${h_\mu}^\alpha$ are orthonormal vierbein field, just as ${e_\mu}^\alpha$, and

$$\overset{h}{g}_{\mu\nu} = {h_\mu}^\alpha {h_\nu}^\beta g_{\alpha\beta} \, , \quad (6.4)$$

just as $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta g_{\alpha\beta}$. The only difference between $e_\mu{}^\alpha$ and $h_\mu{}^\alpha$ is that, when calculating connections $\overset{h}{\Gamma}_{\alpha\beta}{}^\gamma$ using $h_\mu{}^\alpha$, we allow only the torsion to be nonzero, while the curvature vanishes, $\overset{h}{R}_{\alpha\beta\gamma}{}^\delta = 0$.

The covariant derivative is defined by

$$D_\lambda h_\beta{}^\mu = \partial_\lambda h_\beta{}^\mu - A_{\lambda\beta}{}^\gamma h_\gamma{}^\mu + \Gamma_{\lambda\nu}{}^\mu h_\beta{}^\nu \equiv \overset{L}{D}\lambda h_\beta{}^\mu + \Gamma_{\lambda\nu}{}^\mu h_\beta{}^\nu \,. \tag{6.5}$$

The field strength of $A_{\mu\alpha}{}^\beta \equiv (A_\mu)_\alpha{}^\beta$

$$F_{\mu\nu\beta}{}^\gamma \equiv (\partial_\mu A_\nu - \partial_\mu A_\mu - [A_\mu, A_\nu])_\beta{}^\gamma \,, \tag{6.6}$$

determines the Cartan curvature

$$R_{\mu\nu\lambda}{}^\kappa \equiv h^\beta{}_\lambda h_\gamma{}^\kappa F_{\mu\nu\beta}{}^\gamma \,, \tag{6.7}$$

The field strength of $h^\lambda{}_\nu$ is the torsion :

$$S_{\alpha\beta}{}^\gamma \equiv \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu [\overset{L}{D}_\mu h^\gamma{}_\nu - (\mu \leftrightarrow \nu)] \,. \tag{6.8}$$

At first sight, the generalization of the elastic strain tensors coupled to the stress tensors in (6.2) seems to be $[\omega_\mu{}^\alpha = h_\mu{}^\beta \omega_\beta{}^\alpha]$

$$u^{el}_{ij} \equiv \partial_i u_j - \varepsilon_{ijk}\omega_k - \beta^p_{ij} \longrightarrow u^{el}{}_\mu{}^\alpha = D_\mu u^\alpha - \omega_\mu{}^\alpha - h^\alpha{}_\mu \,, \tag{6.9}$$

$$\omega^{el}_{ij} \equiv \partial_i \omega_j - \phi^p_{ij} \longrightarrow \omega^{el}{}_{\mu\alpha}{}^\beta = D_\mu \omega_\alpha{}^\beta - A_{\mu\alpha}{}^\beta \,. \tag{6.10}$$

where we have exchanged the vector ω_γ by the antisymmetric tensor $\omega_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\omega_\gamma$ and introduced, similarly, $\phi^p_{\mu\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\phi^p_{\mu\alpha}$.

In (6.9) and (6.10), u^α, $\omega_\alpha{}^\beta$ are the displacement and the rotation fields in non-holonomic coordinates dx^α, and $h_\mu{}^\alpha$, $A_{\mu\alpha}{}^\beta$ are the gauge fields of local translations and rotations in affine space which are the non linear generalizations of the plastic dislocation and disclination $\beta^p_{\mu\nu}$ and $\phi^p_{\mu\nu}$. The covariant derivatives D_μ are formed using the gauge field of rotations $A_{\mu\alpha}{}^\beta$, for instance,

$$D_\mu u^\alpha = \partial_\mu u^\alpha + A_{\mu\beta}{}^\alpha u^\beta \,. \tag{6.11}$$

Elastic distortions of the space-time fabric must correspond precisely to the local Einstein translations and Lorentz rotations of the coordinate system (which do not change the defect content of the geometry):

$$\begin{aligned} \delta_E x^\alpha &\equiv \xi^\alpha \\ \delta_E h_\mu{}^\alpha &\equiv D_\mu \xi^\alpha - (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha)\xi^\beta \,, \\ \delta_E A_{\mu\alpha}{}^\beta &\equiv D_\mu(\xi^\gamma A_{\gamma\alpha}{}^\beta) - \xi^\gamma F_{\mu\gamma\alpha}{}^\beta \,, \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} \delta_L x^\alpha &\equiv \Delta\omega^\alpha{}_\beta x^\beta \\ \delta_L h_{\alpha\mu} &\equiv \Delta\omega_\alpha{}^\beta h_{\beta\mu} \,, \\ \delta_L A_{\mu\alpha}{}^\beta &\equiv D_\mu \Delta\omega_\alpha{}^\beta \,, \end{aligned} \tag{6.13}$$

where ξ^α and $\Delta\omega^\alpha{}_\alpha$ are the local displacements and rotations of the coordinates [1]. The elastic strain tensors are found from transformation laws (6.12, 6.13), replacing, on the right-hand sides,

$$\xi^\alpha \quad\to\quad -u^\alpha \tag{6.14}$$
$$\Delta\omega_\alpha{}^\alpha \quad\to\quad -\omega_\alpha{}^\alpha \tag{6.15}$$

hence

$$h_\mu{}^\alpha \quad\longrightarrow\quad h_\mu{}^\alpha - D_\mu u^\alpha + (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha)u^\beta\,, \tag{6.16}$$
$$A_{\mu\alpha}{}^\beta \quad\longrightarrow\quad A_{\mu\alpha}{}^\beta - D_\mu(u^\gamma A_{\gamma\alpha}{}^\beta) + u^\gamma F_{\mu\gamma\alpha}{}^\beta\,. \tag{6.17}$$

Then

$$u^{el}{}_\mu{}^\alpha \quad=\quad D_\mu u^\alpha - \omega_\mu{}^\alpha - h^\alpha{}_\mu - (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha)u^\beta\,, \tag{6.18}$$
$$\omega^{el}{}_{\mu\alpha}{}^\beta \quad=\quad D_\mu\omega_\alpha{}^\beta - A_{\mu\alpha}{}^\beta + D_\mu(u^\gamma A_{\gamma\alpha}{}^\beta) - u^\gamma F_{\mu\gamma\alpha}{}^\beta\,. \tag{6.19}$$

These non linear tensors are manifestely gauge invariant under the transformations (6.12,6.13) if the total distortions in the space-time fabric, u^α, $\omega_\alpha{}^\beta$, are simultaneously changed by

$$u^\alpha \quad\longrightarrow\quad u^\alpha + \xi^\alpha\,, \tag{6.20}$$
$$\omega_\alpha{}^\beta \quad\longrightarrow\quad \omega_\alpha{}^\beta + \Delta\omega_\alpha{}^\beta\,. \tag{6.21}$$

The stress gauge transformations (6.12,6.13) are absorbed in a corresponding transformations of the displacement field

$$\delta_E u^\alpha = \xi^\alpha, \qquad \delta_L u^\alpha = \omega^\alpha{}_\beta u^\beta\,. \tag{6.22}$$

making the following nonlinear geometric description of gauge invariant defect energy,

$$\begin{aligned} E \quad=\quad & \int d^4x^\mu\sqrt{-g}\left\{\frac{1}{4\mu}\left(\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{\nu}{1+\nu}\sigma_\mu{}^{\mu^2}\right) + \frac{1}{16\mu l^2}\tau_\mu{}^\alpha{}_\beta\tau^\mu{}_\alpha{}^\beta\right.\\ & +\sigma^\mu{}_\alpha\left[D_\mu u^\alpha - \omega_\mu{}^\alpha - h^\alpha{}_\mu - (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha)u^\beta\right]\\ & \left.+\frac{1}{2}\tau_{\mu\alpha}{}^\beta\left[D_\mu\omega_\alpha{}^\beta - A_{\mu\alpha}{}^\beta + D_\mu(u^\gamma A_{\gamma\alpha}{}^\beta) - u^\gamma F_{\mu\gamma\alpha}{}^\beta\right]\right\}\,. \end{aligned} \tag{6.23}$$

where the conjugate variables $\sigma^\mu{}_\alpha$ and $\tau_{\mu\alpha}{}^\beta$ are again the stress and torque stress fields carrie non-holonomic index α, β and Einstein index μ. Energy (6.23) describes the fluctuations of dislocations and disclinations under the effect of external stresses and torque stresses carried by the gauge fields $h_\mu{}^\alpha$ and $A_{\mu\alpha}{}^\beta$.

From (6.23) we calculate the tensor $\sigma^\mu{}_\nu$ in the stress energy as

$$\sigma^\mu{}_\nu = h^\alpha{}_\nu\sigma^\mu{}_\alpha\,. \tag{6.24}$$

[1] We have written $\Delta\omega_\alpha{}^\beta$ insted of $\omega_\alpha{}^\beta$ to distinguish the local rotations of the coordinates from the total rotational distortions of the space-time fabric.

Integrating out u^α and $\omega_\alpha{}^\beta$ gives the *stress conservation laws*,

$$D^*_\mu \sigma^\mu{}_\alpha = -2S_{\alpha\mu}{}^\gamma \sigma^\mu{}_\gamma - \frac{1}{2}\tau^\mu{}_\gamma{}^\beta R_{\alpha\mu\beta}{}^\gamma \tag{6.25}$$

$$D^*_\mu \tau^\mu{}_\beta{}^\alpha = \sigma^\alpha{}_\beta - \sigma_\beta{}^\alpha . \tag{6.26}$$

which have the same form of those of defect densities (5.70,5.71)

$$D^*_\mu \theta^\mu{}_\alpha = -2S_{\alpha\mu}{}^\gamma \theta^\mu{}_\gamma - \frac{1}{2}\alpha^\mu{}_\gamma{}^\beta R_{\alpha\mu\beta}{}^\gamma \tag{6.27}$$

$$D^*_\mu \alpha^\mu{}_\beta{}^\alpha = \theta^\alpha{}_\beta - \theta_\beta{}^\alpha . \tag{6.28}$$

where the dislocation density,

$$\alpha_{\mu\nu} \longrightarrow \alpha_{\mu\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma}\alpha_{\mu\gamma} , \tag{6.29}$$

being equal to the Palatini tensor via

$$P_{\mu\nu,\kappa} = \varepsilon_{\mu\nu l}\alpha_{kl} \stackrel{\Lambda}{=} \tau_{\mu\nu\kappa} , \tag{6.30}$$

and the disclination density, $\theta_{\mu\nu}$, to Einstein tensor

$$G_{\mu\nu} = \theta_{\mu\nu} \stackrel{\Lambda}{=} \sigma_{\mu\nu} . \tag{6.31}$$

The same conservation laws were obtained for the energy-momentum tensor $\theta^\mu{}_\alpha$ and the spin density $\Sigma_\beta{}^{\alpha,\mu}$ of arbitrary matter moving in the affine space. One merely has to replace

$$\sigma^\mu{}_\alpha \longrightarrow \theta_\alpha{}^\mu, \qquad \tau^\mu{}_\beta{}^\alpha \longrightarrow \Sigma_\beta{}^{\alpha,\mu} . \tag{6.32}$$

In this way we arrive at the total energy expression

$$\begin{aligned} E'_{el} + E_{def} + E_{int} = & \int d^4x^\mu \sqrt{-g} \left\{ \frac{1}{4\mu}\left(G_{\mu\nu}G^{\mu\nu} - \frac{\nu}{1+\nu} G_\mu{}^{\mu^2} \right) + \frac{1}{16\mu l^2} S_\mu{}^\alpha{}_\beta S^\mu{}_\alpha{}^\beta \right. \\ & + G^\mu{}_\alpha \left[D_\mu u^\alpha - \omega_\mu{}^\alpha - h^\alpha{}_\mu - (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha) u^\beta \right] \\ & + \frac{1}{2} S_{\mu\alpha}{}^\beta \left[D_\mu \omega_\alpha{}^\beta - A_{\mu\alpha}{}^\beta + D_\mu (u^\gamma A_{\gamma\alpha}{}^\beta) - u^\gamma F_{\mu\gamma\alpha}{}^\beta \right] \\ & \left. + c_\alpha \alpha_\mu{}^\alpha \alpha^\mu{}_\alpha + c_\theta \theta_\mu{}^\alpha \theta^\mu{}_\alpha \right\} , \end{aligned} \tag{6.33}$$

where

$$E'_{el} = \frac{1}{4\mu} \int d^4x^\mu \sqrt{-g} \left[\frac{1}{4\mu}\left(G_{\mu\nu}G^{\mu\nu} - \frac{\nu}{1+\nu} G_\mu{}^{\mu^2} \right) + \frac{1}{16\mu l^2} S_\mu{}^\alpha{}_\beta S^\mu{}_\alpha{}^\beta \right] \tag{6.34}$$

$$E_{def} = \int d^4x^\mu \sqrt{-g} (c_\alpha \alpha_\mu{}^\alpha \alpha^\mu{}_\alpha + c_\theta \theta_\mu{}^\alpha \theta^\mu{}_\alpha) , \tag{6.35}$$

$$\begin{aligned} E_{int} = & \frac{1}{4\mu} \int d^4x^\mu \sqrt{-g} \left\{ G^\mu{}_\alpha \left[D_\mu u^\alpha - \omega_\mu{}^\alpha - (A_{\beta\mu}{}^\alpha - 2S_{\beta\mu}{}^\alpha) u^\beta - h^\alpha{}_\mu \right] \right. \\ & \left. - \frac{1}{2} S^\mu{}_\beta{}^\alpha \left[D_\mu \omega_\alpha{}^\beta - A_{\mu\alpha}{}^\beta + D_\mu (u^\gamma A_{\gamma\alpha}{}^\beta) - u^\gamma F_{\mu\gamma\alpha}{}^\beta \right] \right\} . \end{aligned} \tag{6.36}$$

In this way, we obtain a complete non-linear gauge field descreption of defects with their interactions.

Conclusion and outlook

We constructed a four dimensional gauge field theory of defect which governs ensemble of dislocations and disclinations, including their higher gradient elastic interactions. The defects appearing in this theory are *idealised* objects. Of particular interest also is that the dislocation density tensor is related to the torsion tensor while the disclination density tensor is related to the Einstein tensor. The elastic, defect and interaction actions can be expressed entirely in terms of ${h_\mu}^\alpha$ and ${A_{\alpha\beta}}^\gamma$, and the significance of the gauge fields lies in its having a *local* coupling with curvature via Einstein-cartan tensor, and torsion. Theses couplings correspond to the non linear gauge theory of defect on the Riemann-Cartan space-time.

Due to the identical form of stress and Einstein-Cartan conservation laws, one may consider the gauge theory of stresses as defining the differential geometry of the Riemann-Cartan space-time, where plastic deformations introduce torsion and curvature into this geometry and the defects both of dislocations and disclinations are being the extra matter fields describing *idealised* objects. However, at this level, the physical interpretation of these idealised defect in this formulation is not very transparent and this will not contribute to a better understanding of the present approach.

In the future we plan to investigate in a deeper fashion the Riemann-Cartan space-time fabric containing defects of both dislocations and disclinations at the quantum level based on dislocations where dislocations, as translational processes, correspond to bosons, while disclinations, as rotational processes, correspond to fermions. This identification of the quantum particles based on their associated spacetime defects developed and explained before in detail in reference [7].

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