People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Mohamed Khider University of Biskra

Faculty of Exact Sciences and Siences of Nature and Life

Department of Mathematics

Thesis Submitted in Partial Execution of the Requirements of the Degree of

Master in "**Applied Mathematics**"

Option: Probability

AHLAM Morghad

Title :

A Stochastic Maximum Principle With Dissipativity Conditions

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18/06/2023

Dedication

I dedicate this work, to my dearest family and loving parents for all sacrifices and efforts they made and still are making so I can thrive and prosper.

To my lovely fiance and closest friends for their help and support throughout this academic journey.

To myself and all 2023's students' maths promotion.

I thank them all for their unconditional love and unlimited trust and support.

AHLAM

Acknowledgment

All praises and thanks are due to Almighty Allah who guided me to achieve such work and success.

I would like to express my appreciation and gratitude to my honoured supervisor **Khalfallah Nabil** *for his continuous guidance and limitless pieces of advice.*

I would like to thank the members of the jury for devoting their time and efforts to read and correct my work.

I want to extend my thanks to ***Salima Doubbakh*** *for all her efforts and the support she gave me at every step.*

My thanks to all teachers of the mathematics department who I appreciate their contribution to my training.

Notations and Symbols

Initially, we need to define the abbreviatons and symbols employed throughout this recollection.

SDE	Stochastic differential equations
BSDE	Backward stochastic differential equations
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$	Filtered probability space
W_t	Brownian motion
$\mathbb{P}-a.s$	Almost sure
SMP	Stochastic maximmum principle
$\mathbb{E}[.]$	Expectation
$Var[.]$	Variance
$(.) \otimes (.)$	Tensor product
$Tr\,(.)$	The trace
$\mathcal{B}\,(.)$	Borel $\sigma-$algebra
$L^2_{\mathcal{F}}\,([0,T], \mathbb{R}^n)$	The set of all $\{\mathcal{F}\}_{t\geq 0}$ $-$progressive processes $x(.)$ such that $\mathbb{E}\int_0^T \lvert x(t)\rvert^2\, dt < \infty$
$\lvert . \rvert$	The Euclidean norm in $\mathbb{R}^n$.
$\mathbb{E}[./.]$	Conditional expectation

Contents

Introduction

A stochastic differential equation includes two parts, the first describes the deterministic part of the system and the second describes the random part of the system. The stochastic differential equation is typically represented by a wiener process or a Brownian motion, which is a continuous-time random process that has the properties of being non-differentiable and having independent and identically distributed increments. The existence and uniqueness of solutions to stochastic differential equations(SDEs) are important properties that must be established before we can use them to model real-word systems. The first results on the existence and uniqueness of solution to SDE are due to the work of Kiyoshi Itô, where the coefficients satisfy certain growth conditions on top of lipschitz continuity in 1944, in his paper"Stochastic integral" a lot researchers have tried to study how to weaken the assumptions, we can mention among them, Y Lin and X Bai [7]. Recently, in 2013, Orrieri studied the case where the coefficients of SDE satisfy dissipative assumption. Throughout this thesis, we will be interested in studying existence and uniqueness of solutions to SDE in case where the drift and diffusion terms satisfy respectively the dissipativity and lipschitz conditions as well as the stochastic maximum principle (SMP) for such systems. In 1940s engineers and mathematicians began to apply optimal control theory in order to solve some control problems related to engineering fields such as aircrafts, missiles, and other mechanical issues. The early pioneers in this field include Richard Bellman, who introduced the dynamic programming method for solving optimal control problems, and Lev Pontryagin, who developed the maximum principle, a powerful analytical tool for solving certain types of optimal control problems. That deals with finding the best control strategy for a given systems, with the aim of minimizing a certain cost or maximizing a certain performance measure. The stochastic control problem that we will focus on throughout this thesis is described by the following controlled SDE

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t))\, dW(t) & t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{1}$$

where $b : \Omega \times [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$, $\sigma : \Omega \times [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n \times d}$ and $x_0 \in \mathbb{R}$ and the functional cost is given by

$$J(u(.)) = \mathbb{E}\left[\int_0^T f(t, x(t), u(t))\, dt + h(x(T))\right]. \tag{2}$$

The aim of this thesis is to investigate the necessary as well as the sufficient conditions of optimality for the control problem (1,2), in the case where the drift of SDE (1) is dissipative with respect to the stat variable and the diffusion satisfies the lipschitz condition and the controls domain is convex. In other words, we first want to prove the necessary condition of optimality for the control problem (1,2) which claims that, if $\bar{u}$ is an optimal control minimizing a cost functional (2) in the sense that

$$J(\bar{u}) = \inf_{u(.) \in \mathcal{U}[0,T]} J(u(.)) \tag{3}$$

we have the following necessary condition for optimality

$$\frac{\partial H}{\partial u}\left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right)\left(u\left(t\right)-\bar{u}(t)\right) \leq 0 \tag{4}$$

where $H(t,x,u,p,q) = pb\left(t,x,u\right) + Tr\left[q^{T}\sigma\left(t,x,u\right)\right] - f(t,x,u)$ stands for the Hamiltonian functional, $(p(.), q(.))$ is the solution of the following backward stochastic differential equations arising as adjoint equations

$$\begin{cases} dp(t) = -\Big[D_x b(t,\bar{x}(t),\bar{u}(t))^T p(t) + \sum_{j=1}^{d} D_x\sigma^j(t,\bar{x}(t),\bar{u}(t))^T q_j(t) \\ \qquad\qquad -D_x f(t,\bar{x}(t),\bar{u}(t))\Big]\, dt + \sum_{j=1}^{d} q_j(t) dW(t) \\ p(T) = -D_x h(\bar{x}(T)) \end{cases}$$

such that $p(.)$ is the adjoint process and $\bar{x}(t), \bar{u}(t)$ are the optimal trajectory and the optimal control process, secondly we want to prove under some extra convexity conditions on the Hamiltonian and the cost functional, the sufficient condition for optimality every admissible control satisfies the necessary condition of optimality (4) is in fact an optimal control.

The content of this thesis is divided into three chapters.

Chapter 1 (General of Stochastic Calculus): This chapter is essentially an introductory sort, aiming to highlight the tools of our study, we will present a host of definitions, propositions,theorems made without proofs and basic results of stochastic calculus such as stochastic processes...etc.

Chapter 2 (Existence and Uniqueness Solutions of SDEs and BSDEs with Dissipative Coefficients): The purpose of this chapter is the definitions of SDE and BSDE, theorems of existence and uniqueness of the solutions along with their proofs. All this done in the in the dissipative framework.

Chapter 3 (A Stochastic Maximum Principle with Dissipativity Con-

ditions): In this chapter we study the stochastic maximum principal (SMP) represented by Necessary Conditions. We further state and prove the Sufficient Conditions of optimality.

Chapter 1

General of Stochastic Calculus

The aim of this chapter is to introduce the main tools of stochastic calculus and some of the results and theorems used throughout this thesis. We are particularly interested in some of the basic concepts of stochastic processes. The content of this chapter is mainly based on this references [12].

Let $W = \left\{W^1(t), ..., W^d(t)\right\}_{t\geq 0}$ be a standard $d-$dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t\}_{t\geq 0}$the natural filtration as associated to W, satisfying the usual conditions. We suppose that all the processes are defined for times $t \in [0, T]$. Then we denote by $\mathcal{P}$ the $\sigma-$algebra on $\Omega \times [0, T]$ generated by progressive processes.

1.1 Stochastic processes

Definition 1.1.1 *($\sigma-$**algebra**):* *If Ω is a given set, then a $\sigma-$algebra $\mathcal{F}$ on Ω is a family of subsets of Ω with the following properties.*

(i) $\emptyset \in \mathcal{F}$.

(ii) $F \in \mathcal{F} \Longrightarrow F^c \in \mathcal{F}$, where $F^c = \Omega / F$ is the complement of F in Ω.

(iii) $A_1, A_2, ... \in \mathcal{F} \Longrightarrow A = \overset{\infty}{\underset{i=1}{\cup}} A_i \in \mathcal{F}$.

Definition 1.1.2 ***(Random variables):*** *Let* $(\Omega, \mathcal{F})$ *and* $(\mathbf{E}, \mathcal{B})$ *be measurable spaces, then* $X : \Omega \longrightarrow \mathbf{E}$ *is a* $\mathbf{E}-$ *valued random variable if for all* $B \in \mathcal{B}$ *we have*

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in \mathcal{B}\} \in \mathcal{F}.$$

Definition 1.1.3 ***($\sigma-$algebra):*** *Given a random variable on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in a measurable space* $(\mathbf{E}, \mathcal{B})$ *we define the* $\sigma-$*algebra generated by the random variable* X *as*

$$\sigma(X) = \sigma(\{X^{-1}(B)/B \in \mathcal{B}\}).$$

Definition 1.1.4 ***(Filtration):*** *Given an indexing set* T*, a filtration of* $\sigma-$*algebras is a set sigma algebras* $\{\mathcal{F}_t\}_{t\in T}$ *such that for all* $t_1 < ... < t_m \in T$ *we have* $\mathcal{F}_{t_1} \subset ... \subset \mathcal{F}_{t_m}$.

Definition 1.1.5 ***(Stochastic processes):*** *A stochastic process is a parametrized collection of random variables* $\{X_t\}_{t\in T}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and assuming values in* $\mathbb{R}^n$.

Remark 1.1.1 *:*

1. *For each* $t \in T$ *fixed we have a random variable* $\omega \longrightarrow X_t(\omega)$, $\omega \in \Omega$,
2. *When we fixing* $\omega \in \Omega$ *we can consider the function* $t \longrightarrow X_t$, $t \in T$ *which a path of* X_t.

Definition 1.1.6 ***(Adapted):*** *Let* $\{\mathcal{F}_t\}_{t\geq 0}$*be an increasing family of* $\sigma-$*algebras of subsets of* Ω*. A process* $g\,(t,\omega):[0,\infty)\times\Omega\longrightarrow\mathbb{R}^n$ *is called* $\mathcal{F}_t-$*adapted if for each* $t\geq 0$ *the function*

$$\omega\longrightarrow g\,(t,\omega)$$

is $\mathcal{F}_t-$*measurable.*

Definition 1.1.7 ***(progressively measurable):*** *A stochastic process* $\{X_t\}_{t\geq 0}$ *on* $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}\,)$ *is called progressively measurable, if for any* $t\geq 0$*,* $X_t(\omega)$ *viewed as a function of two variables* (t,ω) *is* $\mathcal{B}_{[0,t]}\otimes\mathcal{F}_t-$*measurable, where* $\mathcal{B}_{[0,t]}$ *is the Borel* $\sigma-$*algebra on* $[0,t]$*.*

Definition 1.1.8 ***(Predictable process):*** *A stochastic process* $X=(X)_{t\geq 0}$ *on a filtered probability space* $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$ *is called predictable if it is measurable with respect to a* $\sigma-$*algebra* $\mathcal{P}$ *on* $\bar{\Omega}=\Omega\times[0,\infty)$ *generated by* $\mathcal{F}$*-adapted left-continuous processes when viewed as a mapping* $X:\bar{\Omega}\longrightarrow\mathbb{R}$*.*

1.1.1 Martingale:

Definition 1.1.9 ***(Martingale):*** $\{X_t\}$ *is a martingale with respect to a filtration* $\{\mathcal{F}_t\}$ *if for all* $t>s$ *we have*

i) X_t is $\mathcal{F}_t-$measurable.

ii) $\mathbb{E}\,[|X_t|]<\infty$.

iii) $\mathbb{E}\,[X_t/\mathcal{F}_s]=X_s$.

Similarly: X_t is a $\mathcal{F}_t-$supermartingale $[\mathcal{F}_t-$ submartingale$]$ is it satisfies condition

i) and ii) above, and

$$\mathbb{E}\left[X_t/\mathcal{F}_s\right] \leq X_s, \ \left[\mathbb{E}\left[X_t/\mathcal{F}_s\right] \geq X_s\right] \quad \mathbb{P}-a.s.$$

Proposition 1.1.1 *: Let X_t be a stochastic process such that for any stopping time T, X_T is integral and*

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_T\right],$$

then X_t is a martingale.

Definition 1.1.10 ***(Local martingale):*** *An adapted process X_t is a local martingale if there exists a sequence of stopping times $\{T_n\}$ such that*

$$\lim_{n \longmapsto \infty} T_n\left(\omega\right) = \infty \ \mathbb{P}-a.s,$$

and the stopped process $X_{T_n \wedge t}$ is a martingale for all n.

1.1.2 Brownian motion:

Definition 1.1.11 ***(Brownian motion):*** *Standard Brownian motion $\{W_t\}$ is a stochastic process on $\mathbb{R}$ such that*

1. $W_0 = 0$ almost surely $(i.e : \mathbb{P}(\{\omega \in \Omega : W_0 \neq 0\}) = 0)$.

2. W_t has independent increments for any $t_1 < t_2 < ... < t_n$ $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are independent.

3. The increments $W_t - W_s$ are Gaussian random variables with mean 0 and variance given by the length of the interval

$$Var(W_t - W_s) = \left|t - s\right|.$$

4. The paths $t \longrightarrow W_t(\omega)$ are continuous with probability one we define in particular a process satisfying assumption (3) above as continuos.

Theorem 1.1.1 *: Let W_t be a stochastic process such that the following conditions hold:*

i) $\mathbb{E}\left(W_1^2\right)$ *=constant.*

ii) $W_0 = 0$ *almost surely.*

iii) $W_{t+h} - W_t$ *is independent of* $\{W_s : s \leq t\}$.

iv) *The distribution of* $W_{t+h} - W_t$ *is independent of* $t \geq 0$ *(stationary increments).*

v) *(Continuity in probability) For all* $\delta > 0$:

$$\lim_{h \longrightarrow 0} \mathbb{P}\left[|W_{t+h} - W_t| > \delta\right] = 0.$$

then W_t *is Brownian motion. When* $\mathbb{E}[W(1)]^2 = 1$ *we call is standard Brownian motion.*

1.1.3 Itô processes

Definition 1.1.12 ***(Itô processes):*** $X_t(\omega)$ *is an Itô processes if there exist stochastic processes* $b(t, X_t)$ *and* $\sigma(t, X_t)$ *such that*

i) $b(t, X_t)$ *and* $\sigma(t, X_t)$ *are* $\mathcal{F}_t-$*measurable.*

ii) $\int_0^t |b|\, ds < \infty$ *and* $\int_0^t |\sigma|^2\, ds < \infty$ *almost surely.*

iii) $X_0(\omega)$ *is* $\mathcal{F}_0-$*measurable.*

iv) *With probability one the following holds*

$$X_t(\omega) = X_0(\omega) + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

Theorem 1.1.2 ***(Itô formula):***

The first formula*: Let* $f \in C^2(\mathbb{R})$ *(the set of twice continuously differentiable functions on* $\mathbb{R}$*) and* W *be a standard brownian motion, then for any****:*** $t > 0$,

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds.$$

The second formula: *Let* $(t, x) \longrightarrow f(t, x)$ *be a real function twice differentiable in* x *and once differentiable in* t *and* X *be an Itô process*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \textstyle\int_0^t f'_x(s, X_s)\, dX_s + \int_0^t f'_s(s, X_s)\, ds \\ &\quad + \tfrac{1}{2}\textstyle\int_0^t f''_{xx}(s, X_s)\, d < X, X >_s . \end{aligned}$$

Theorem 1.1.3 ***(Integration by parts):***

Suppose $f(s, \omega) = f(s)$ *only depends on* s *and that* f *is continuous and of bounded variation in* $[0, t]$ *. Then*

$$\int_0^t f(s)dW_s = f(t)W_t - \int_0^t W_s df_s.$$

1.2 Useful results:

Lemma 1.2.1 ***(Gronwall inequalities):*** *Let* $y(t)$ *be a non negative function that satisfies the following condition for some* $T \leq +\infty$ *there exist constants*

$A, B \geq 0$ *such that for all* $0 \leq t \leq T$

$$y(t) \leq A + B \int_0^t y(s)ds < +\infty,$$

then

$$y(t) \leq A \exp(Bt).$$

Theorem 1.2.1 ***(Fixed Point):*** *Let f be a contraction on a complete metric space X. Then f has a unique fixed point $x \in X$ (such that $f(x) = x$).*

Theorem 1.2.2 ***(Hölder's Theorem for integrals):*** *Let $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$. Let $f^k, g^{k'} : I \longrightarrow \mathbb{R}$ be integrable. Then fg is integrable and*

$$\int fgdx \leq \left(\int f^k dx\right)^{\frac{1}{k}} \left(\int g^{k'} dx\right)^{\frac{1}{k'}},$$

equality holds if and only if there are two constants A, B, not both zero, such that $Af^k \equiv Bg^{k'}$.

Chapter 2

Existence and Uniqueness of Solutions of SDEs and BSDEs with Dissipative Coefficients

In this chapter we consider the result of existence and uniqueness solution of SDE and BSDE. Under some assumptions including the dissipative ([13],[11]). We prove this result.

2.1 Notation:

- $\mathcal{S}^2(R^n)$: Denotes the set of R^n-valued, adapted and cadlag process $\{X_t\}_{t\in[0,T]}$ such that

$$||X||_{S^2} = \mathbb{E}[\sup_t |X_t|^2]^{1/2} < +\infty.$$

- $M^2(R^n)$: Denotes the set of (equivalent classes of) predictable processes $\{X_t\}_{t\in[0,T]}$

with values in R^n such that

$$||X||_{M^2} = \mathbb{E}[(\int_0^T |X_r|^2 \, dr)]^{1/2} < +\infty.$$

► $C([0,T]; L^p(\Omega, \mathbb{R}^n))$: The set of all$\{\mathcal{F}_t\}_{t\geq 0}$ −progressive processes $x(.)$ such that the map $t \longmapsto x(t) \in L^p(\Omega)$ is continuous and

$$\sup_{t\in[0,T]} \mathbb{E}\,|x(t)|^p < \infty.$$

2.2 Stochastic Differential Equation (SDEs):

An SDE is essentially a classical differential equation which is perturbed by a random noise. This type of stochastic differential equations is used as a modeling tool in several sciences such as telecommunication, economics, finance, biology, and quantum field theory. The Ornstem-Uhlenbeck process [8] and the Bessel processes [5] can be defined as solution to stochastic differential equation with drift and diffusion coefficients. The general form of such an equation is: For any $t \in [0\ T] \quad 0 \leq s \leq t$

$$\begin{cases} dx(t) = b(t, x(t))dt + \sigma(t, x(t))dW_t \\ x(0) = x_0 \end{cases}, \tag{2.1}$$

or equivalently

$$x(t) = x_0 + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dW(s),$$

with $b : \Omega \times [0\ T] \times \mathbb{R}^n \longmapsto \mathbb{R}^n$ and $\sigma : \Omega \times [0\ T] \times \mathbb{R}^n \longmapsto \mathbb{R}^{n\times d}$ and $x_0 \in \mathbb{R}$.

It is well known that the strongest condition ensure the existence and uniqueness for SDEs of Itô's type is Lipschitz condition along with the linear growth condition (i.e. $b(t,x)$ and $\sigma(t,x)$ are lipschitz and of linear growth). In the sequel, we will trade of the Lipschitz condition by a weaker one called a dissipative condition.

2.2.1 Existence and Uniqueness of Solution of SDE with Dissipative Condition:

We will discuss the existence and the uniqueness of a solution of SDE via the following assumptions.

Assumptions 1

i) The drift term $b : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable where $\mathcal{P}$ is the progressive $\sigma-$algebra. The map $x \to b(t,x)$ satisfies the so-called $\alpha-$dissipativity condition in the sense that there exists a constant $\alpha \in \mathbb{R}$, such that

$$(b(t,x) - b(t,x'))(x - x') \leq \alpha |x - x'|^2, \qquad t \in [0,T],\ x, x' \in \mathbb{R}^n.$$

ii) The diffusion coefficient $\sigma : \Omega \times [0\ T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$ moreover the map $x \longrightarrow \sigma(t,x)$ there exists a constant $c \geqslant 0$ such that $\mathbb{P} - a.s.$

$$|\sigma(t,x) - \sigma(t,x')| \leq c\,|x - x'|, \qquad t \in [0,T],\ x, x' \in \mathbb{R}^n.$$

Theorem 2.2.1 *Under assumptions i),ii) SDE (2.1) has a unique solution in*

$C\left([0,T]\,;L^2\left(\Omega,\mathbb{R}\right)\right)$, *i.e a progressive process $x(t)$ satisfying*

$$\sup_{t\in[0,T]} \mathbb{E}\left|x(t)\right|^2 < \infty.$$

Proof. To simplify the notation we drop the dependence on the control, the case of controlled equation can be treated exactly in the same way, by fixing $\gamma \in C([0,T]\,;L^2\left(\Omega,\mathbb{R}^n\right))$ we want to show that the problem

$$dx(t) = b(t,x(t))dt + \sigma(t,\gamma(t))dW(t) \qquad\qquad x(0) = x_0,$$

admits a unique solution $J(\gamma)$ which belongs to $C\left([0,T]\,;L^2\left(\Omega,\mathbb{R}^n\right)\right)$. The existence part follows from the fact that the initial problem can easily reformulated as a differential equation with random coefficients of the from

$$\frac{d}{dt}\eta(t) = b(t,\eta(t)) + d\omega^\gamma(t).$$

Where the quantity

$$\omega^\gamma(t) := \int_0^t \sigma(s,\gamma(s))dW(s),$$

is well defined thanks to the linear growth assumption. Since $b(.)$ is continuous, we know that there is a local s olution which can be easily extended to the whole $[0,T]$, by the dissipativity assumptions. Now we have to verify that the operator $J : C\left([0,T_0];L^2\left(\Omega,\boldsymbol{R}^n\right)\right) \longrightarrow C\left([0,T_0];L^2\left(\Omega,\mathcal{R}^n\right)\right)$ is a contraction.

- **Step01:** T_0 is small enough for any γ_1, $\gamma_2 \in C([0\ T];L(\Omega,\mathbb{R}^n))$, $t\ \epsilon[0\ T]$

$$\begin{aligned} d\mid J_t(\gamma_1) - J_t(\gamma_2)\mid \ &= \left|(b(t,J_t(\gamma_1)) - b(t,J_t(\gamma_2)))dt\ + (\sigma(t,\gamma_1(t))\right.\\ &\left.-\sigma(t,\gamma_2(t)))dW_t\right|. \end{aligned}$$

Applying Itô's formula to $\mid J_t(\gamma_1) - J_t(\gamma_2) \mid^2$, we obtain

$$\begin{aligned} d \mid J_t(\gamma_1) - J_t(\gamma_2) \mid^2 &= 2 \mid J_t(\gamma_1) - J_t(\gamma_2) \mid d \mid J_t(\gamma_1) - J_t(\gamma_2) \mid \\ &+ d < J(\gamma_1) - J(\gamma_2) >_t . \end{aligned}$$

By taking the expectation we get

$$\begin{aligned} \mathbb{E} \mid J_t(\gamma_1) - J_t(\gamma_2) \mid^2 &= 2\mathbb{E} \int_0^t (b(s, J_s(\gamma_1)) - b(s, J_s(\gamma_2)) \\ &(J_s(\gamma_1) - J_s(\gamma_2))) \, ds + \mathbb{E} \int_0^t \parallel \sigma(s, \gamma_1(s)) - \sigma(s, \gamma_2(s)) \parallel_2^2 ds \\ &\leq 2\alpha \int_0^t \mathbb{E} \mid J_s(\gamma_1) - J_s(\gamma_2) \mid^2 ds \\ &+ c^2 \int_0^t \mathbb{E} \mid \gamma_1(s) - \gamma_2(s) \mid^2 ds \end{aligned} .$$

Thanks to Gronwall's lemma, we have

$$\mathbb{E} \mid J_t(\gamma_1) - J_t(\gamma_2) \mid^2 \leq c^2 e^{2\alpha t} \int_0^t \mathbb{E} \mid \gamma_1(s) - \gamma_2(s) \mid^2 ds.$$

Finally,

$$\sup_{t \epsilon [0,T]} \mathbb{E} \mid J_t(\gamma_1) - J_t(\gamma_2) \mid^2 \leq c^2 e^{2\alpha T} T \sup_{t \epsilon [0,T]} \mathbb{E} \mid \gamma_1(s) - \gamma_2(s) \mid^2 .$$

So if we get T_0 such that $c \sqrt{T_0} e^{\alpha T_0} < 1$ we prove that J is a contraction then there exists a fixed point such that this point is the solution.

- **Step02:** In this step, we assume that T is an arbitrary large time duration. Firstly, let $([T_i, T_{i+1}])_{i=0}^{i=k}$ be a subdivision of $[0, T]$ such that for any $0 \leq i \leq k$, $|T_{i+1} - T_i| \leq \delta$, where δ is a strictly positive number.

For $t \in [0, T_0]$

$$x(t) = x_0 + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dW(s). \tag{2.2}$$

It is obvious from step01 that (2.2) remains valid on the small interval time $[0, T_0]$. Next, for $t \in [T_0, T_1]$ we consider the following SDE

$$x(t) = x_{T_0} + \int_{T_0}^{t} b(s, x(s))ds + \int_{T_0}^{t} \sigma(s, x(s))dW(s). \tag{2.3}$$

Since $[T_0, T_1]$ is small enough according to step 1 x is solution to (2.3). Repeating this procedure forwardly for $i = 0, ..., n$, we obtain the desired result on the whole time interval $[0, T]$. ■

2.3 Backward Stochastic Differential Equation:

Backward stochastic differential equation (BSDEs) are stochastic differential equation with terminal value. The theory of BSDEs has found wide application in several areas such as stochastic optimal control, theoretical economics and mathematical finance problems. A general BSDE can be written as: $\forall t \in [0, T] \quad 0 \leq s \leq t$

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t \\ Y_T = \xi \end{cases}, \tag{2.4}$$

or equivalently

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{2.5}$$

where ξ is the terminal condition and f the coefficient (also called the generator), linear backward stochastic differential equations were first studies by Bismut and the general non linear backward stochastic differential equations have been introduced by Pardoux and Peng (1990), they proved an existence and uniqueness result under the following assumption f is lipschitz continuous in both variables Y and Z and the data ξ and the process $\{f(t, 0, 0)\}_{t \in [0,T]}$ are square integrable.

From the article by Pardoux and Peng, many researchers tried to study how to make the assumptions concerning the regularity of the generator with respect to (y, z), one of these results Orrieri and al. [13] who studied BSDE whose generator is dissipative.

2.3.1 Existence and Uniqueness of Solution of BSDE with Dissipative Condition:

Now we will proof this results based on the following assumptions:

Assumptions 2:

A coefficient $f : \Omega \times [0, T], \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$ real numbers μ and $K > 0$

H1) $f(., y, z)$ is progressively measurable, $\forall y, z$.

H2) For $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(|y'|), \forall t, y, a.s.$$

H3)

$$\mathbb{E}\int_0^T |f(t, 0, 0)|^2 \, dt < \infty.$$

H4) $|f(t, y, z) - f(t, y, z')| \leqslant K \, ||z - z'||, \; \forall t, y, z, z', \mathbb{P}{-}a.s.$, where $||z|| = [Tr(zz^*)]^{1/2}$.

H5) $\langle y - y', f(t, y, z) - f(t, y', z)\rangle \leqslant \mu |y - y'|^2, \forall t, y, y', z, \mathbb{P}{-}a.s.$

H6) $y \longrightarrow f(t, y, z)$ is continuous, $\forall t, z, \mathbb{P}{-}a.s.$

A solution of the BSDE(2.5) is a pair $\{(Y_t, Z_t); 0 \leq t \leq T\}$ of progressively measurable processes with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that

(j) $\mathbb{E}\int_0^T \|Z_t\|^2 dt < \infty \left(i.e.\ Z \in \left(M^2(0,T)\right)^{k\times d}\right)$, $\mathbb{E}\left(\sup\limits_{0\leq t\leq T} |Y_t|^2\right) < \infty.$

(jj) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s.$

Note that the progresive measurability of $\{Y_t\}$ implies in particular that Y_0 is deterministie.

Proposition 2.3.1 *Given $V \in \left(M^2(0,T)\right)^{k\times d}$, there exists a unique pair of progressively measurable processes $\{(Y_t, Z_t); 0 \leq t \leq T\}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ satisfying (j),and*

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)\, ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T.$$

Proof. Pages (512-516) in [11]. ∎

Theorem 2.3.1 *Under the assumptions (H1)-(H6), the BSDE (2.4) has a unique solution satisfying (j),(jj).*

Proof. Uniqueness part:

Let (Y, Z) and (Y', Z') be two solution of (2.4)

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s \\ Y'_t &= \xi + \int_t^T f(s, Y'_s, Z'_s)\, ds - \int_t^T Z'_s dW_s, \end{aligned}$$

It follows from Itô's formula for $|Y_t - Y'_t|^2$ we get

$$d|Y_t - Y'_t|^2 = 2|Y_t - Y'_t|\, d|Y_t - Y'_t| + d < Y - Y', Y - Y' >_t .$$

So

$$\begin{aligned} d\left|Y_t - Y_t'\right|^2 &= 2\left|Y_t - Y_t'\right|\left[f\left(t, Y_t, Z_t\right) - f\left(t, Y_t', Z_t'\right)\right]dt \\ &-2\left|Y_t - Y_t'\right|\left|\left|Z_t - Z_t'\right|\right|dW_t + \left|\left|Z_t - Z_t'\right|\right|^2 dt. \end{aligned}$$

Then, we take the expectation to obtain

$$\begin{aligned} \mathbb{E}\left|Y_t - Y_t'\right|^2 + \mathbb{E}\int_t^T \left|\left|Z_s - Z_s'\right|\right|^2 ds &= 2\mathbb{E}\int_t^T \left(\left(Y_s - Y_s'\right)\right. \\ &\left.\left(f\left(s, Y_s, Z_s\right) - f\left(s, Y_s', Z_s'\right)\right)\right) ds \\ &\leq 2\mathbb{E}\int_t^T \left[\mu\left|Y_t - Y_t'\right|^2\right. \\ &\left.+K\left|Y_t - Y_t'\right|\left|\left|Z_s - Z_s'\right|\right|\right] ds \\ &\leq \left(2\mu + K^2\right)\mathbb{E}\int_t^T \left|Y_t - Y_t'\right|^2 ds \\ &+\mathbb{E}\int_t^T \left|\left|Z_s - Z_s'\right|\right|^2 ds. \end{aligned}$$

Hence

$$\mathbb{E}\left|Y_t - Y_t'\right|^2 \leq \left(2\mu + K^2\right)\mathbb{E}\int_t^T \left|Y_t - Y_t'\right|^2 ds,$$

and from Gronwall's lemma we get

$$\mathbb{E}\left|Y_t - Y_t'\right|^2 = 0, \qquad 0 \leq t \leq T,$$

and then we have also that

$$\mathbb{E}\int_0^T \left|\left|Z_t - Z_t'\right|\right|^2 dt = 0.$$

Existence part: We first note that (Y, Z) solves the BSDE (2.4) if

$$\left(\bar{Y}_t, \bar{Z}_t\right) := \left(e^{\lambda t}Y_t, e^{\lambda t}Z_t\right).$$

Solve the BSDE (2.4) where

$$f'(t,y,z) := e^{\lambda t} f(t, e^{\lambda t} y, e^{-\lambda t} z) - \lambda y,$$

if we choose $\lambda = \mu$, then f' satisfies the same assumptions as f, but with (H5) replaced by

(H5') $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq 0.$

Hence we shall assume until the end of this proof that f satisfies (H1)-(H5) and (H6), let us admit for a moment the using proposition (2.3.1), we can construct a mapping Φ from $\mathcal{B}^2 = \mathcal{S}^2 \otimes M^2$ into itself as follows. For any $(U,V) \in \mathcal{B}^2$, $(Y,Z) = \Phi(U,V)$ is the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)\, ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Let $(U,V), (U',V') \in \mathcal{B}^2$,

$$(Y,Z) = \Phi(U,V),$$
$$(Y',Z') = \Phi(U',V').$$

We shall use the notations

$$\left(\bar{U}, \bar{V}\right) = (U - U', V - V')$$
$$\left(\bar{Y}, \bar{Z}\right) = (Y - Y', Z - Z').$$

It follows from Itô's formula for $e^{\gamma t}\left|\bar{Y}_t\right|^2$ for each $\gamma \in \mathbb{R}$ we have

$$de^{\gamma t}\left|\bar{Y}_t\right|^2 = \gamma e^{\gamma t}\left|\bar{Y}_t\right|^2 dt + 2e^{\gamma t}\left|\bar{Y}_t\right| d\left|\bar{Y}_t\right| + d < \bar{Y} >_t .$$

Then we get

$$
\begin{aligned}
& e^{\gamma t}\mathbb{E}\left|\bar{Y}_t\right|^2 + \mathbb{E}\int_t^T e^{\gamma s}(\gamma\left|\bar{Y}_s\right|^2 + \left|\left|\bar{Z}_s\right|\right|^2)ds \\
& = 2\mathbb{E}\int_t^T e^{\gamma s}\bar{Y}_s\left(f\left(Y_s, V_s\right) - f\left(Y_s', V_s'\right)\right)ds \\
& \leq 2K\mathbb{E}\int_t^T e^{\gamma s}\left|\bar{Y}_s\right| \times \left|\left|\bar{V}_s\right|\right| ds \\
& \leq \mathbb{E}\int_t^T e^{\gamma s}(2K^2\left|\bar{Y}_s\right|^2 + \frac{1}{2}\left|\left|\bar{V}_s\right|\right|^2)ds.
\end{aligned}
$$

Hence, if we choose $\gamma = 1 + 2K^2$, we have that

$$
\begin{aligned}
\mathbb{E}\int_0^T e^{\gamma t}(\left|\bar{Y}_t\right|^2 + \left|\left|\bar{Z}_s\right|\right|^2)dt & \leq \frac{1}{2}\mathbb{E}\int_0^T e^{\gamma t}\left|\left|\bar{V}_t\right|\right|^2 dt \\
& \leq \frac{1}{2}\mathbb{E}\int_0^T e^{\gamma t}\left(\left|\bar{U}_t\right|^2 + \left|\left|\bar{V}_t\right|\right|^2\right)dt.
\end{aligned}
$$

Consequently, Φ is a strict contraction on $\mathcal{B}^2$ equipped with the norm

$$
\left|\left|(Y, Z)\right|\right|_\gamma = \left[\mathbb{E}\int_0^T e^{\gamma t}(\left|\bar{Y}_t\right|^2 + \left|\left|\bar{Z}_s\right|\right|^2)dt\right]^{1/2}.
$$

So

$$
\left|\left|\left(\bar{Y}, \bar{Z}\right)\right|\right|_\gamma = \frac{1}{2}\left|\left|\left(\bar{U}, \bar{V}\right)\right|\right|_\gamma,
$$

and it has a unique fixed point, which is the unique solution of our BSDE. ∎

Chapter 3

A Stochastic Maximum Principle with Dissipativity Conditions

In this chapter we will study the necessary condition of optimality (often called the stochastic maximum principal) as well as the sufficient condition of optimality for a controlled stochastic differential equation in the case where both the drif and diffusion coefficients are controlled and the control domain is convex. We get the main results of this chapter under the dissipativity condition on the drift instead of the Lipschitz one.

3.1 Preliminaries and Problem Formulation

Let $W = \left\{W^1(t), ..., W^d(t)\right\}_{t\geq 0}$ be a standard $d-$dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t\}_{t\geq 0}$the natural filtration associated to W, satisfying the usual conditions. We suppose that all the processes are defined for times $t \in [0, T]$. Then, we denote by $\mathcal{P}$ the $\sigma-$algebra on $\Omega \times [0, T]$ generated by progressive processes. For any $p \geq 1$ we define the following spaces

- $L^1\left([0\ T], R^d\right)$: The space of adapted real valued process $(x(t))_{t\in[0,T]}$ such as

$$\mathbb{E}\left[\int_0^T |x_t|\, dt\right] < +\infty.$$

- $L^2\left([0\ T], R^d\right)$: The set of all $\{\mathcal{F}\}_{t\geq 0}$ $-$progressive processes $x(.)$ such that

$$\mathbb{E}\int_0^T |x(t)|^2\, dt < \infty.$$

- $C\left([0,T]; L^p(\Omega, \mathbb{R}^n)\right)$: The set of all$\{\mathcal{F}_t\}_{t\geq 0}$ $-$progressive processes $x(.)$ such that the map $t \longmapsto x(t) \in L^p(\Omega)$ is continuous and

$$\sup_{t\in[0,T]} \mathbb{E}\,|x(t)|^p < \infty.$$

- The class of admissible controls is defined by requiring that they are progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, more precisely

$$\mathcal{U}\,[0,T] := \left\{u(.) : [0,T] \times \Omega \longrightarrow U : u\,(.) \text{ is } \{\mathcal{F}_t\}_{t\geq 0} - \text{progressive}\right\}.$$

We will denote by $|.|$ the Euclidean norm in $\mathbb{R}^n$.

Now, we define the following stochastic control problem

$$\begin{cases} dx\,(t) = b\,(t, x\,(t)\,, u\,(t))\, dt + \sigma\,(t, x\,(t)\,, u\,(t))\, dW\,(t) & t \in [0,T] \\ x\,(0) = x_0 \end{cases}, \quad (3.1)$$

with a cost functional given by

$$J(u(.)) = \mathbb{E}\left[\int_0^T f(t, x(t), u(t))\, dt + h(x(T))\right]. \tag{3.2}$$

If $x(.)$ is a solution of (3.1) and $u(.) \in \mathcal{U}[0,T]$ then we call $(x(.), u(.))$ an admissible pair. The control problem can be formulated as a minimization of the cost over $\mathcal{U}[0,T]$, more precisely a control $\bar{u}$ is optimal if

$$J(\bar{u}) = \inf_{u(.)\in\mathcal{U}[0,T]} J(u(.)). \tag{3.3}$$

Assumptions 3:

(A1) The drift term $b : \Omega\times[0,T]\times\mathbb{R}^n\times U \longrightarrow \mathbb{R}^n$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(U)-$measurable, where $\mathcal{P}$ is the progressive $\sigma-$algebra. The map $x \longmapsto b(t,x,u)$ is $C^1(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies an $\alpha-$dissipativity condition in the sense that there exists a constant $\alpha \in \mathbb{R}$ such that $\mathbb{P}-a.s$

$$(b(t,x,u) - b(t,x',u))(x - x') \leq \alpha |x - x'|^2, \quad u \in U,\ t \in [0,T],\ x, x' \in \mathbb{R}^n.$$

(A2) The diffusion coefficient $\sigma : \Omega \times [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$ is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(U)$. Moreover the map $x \longmapsto \sigma(t,x,u)$ is $C^1(\mathbb{R}^n, \mathbb{R}^{n\times d})$ and there exists a constant $C_1 > 0$ such that, $\mathbb{P}-a.s.$

$$|\sigma(t,x,u) - \sigma(t,x',u)| \leq C_1 |x - x'|, \quad u \in U,\ t \in [0,T],\ x, x' \in \mathbb{R}^n.$$

(A3) (Polynomial Growth) There exist $h \geq 0$, $C_2 > 0$ such that, for $j = 0, 1$,

$$\mathbb{P}{-}a.s. \sup_{u \in U} \sup_{t \in [0,T]} \left| D_x^\beta b\left(t, x, u\right) \right| \leq C_2 \left(1 + \left| x \right|^h \right), \qquad \left| \beta \right| = j.$$

In addition we shall assume there exist $C_3 > 0$, such that, $\mathbb{P} - a.s$

$$\left| \sigma\left(t, 0, u\right) \right| \leq C_3 \qquad u \in U,\ t \in [0, T]$$

(A4) $f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ are measurable and the maps $x \longmapsto f(t, x, u)$ and $x \longmapsto h\left(x\right)$ are $C^1\left(\mathbb{R}^n, \mathbb{R}\right)$. Moreover there exists $C_5 > 0$, $m \geq 0$, $l \geq 0$ such that for $j = 0, 1$, we have,

$$\mathbb{P} - a.s. \sup_{u \in U} \sup_{t \in [0,T]} \left| D_x^\beta f\left(t, x, u\right) \right| \leq C_5 \left(1 + \left| x \right|^l \right), \qquad \left| \beta \right| = j.$$

$$\left| D_x^\beta h\left(x\right) \right| \leq C_5 \left(1 + \left| x \right|^m\right), \qquad \left| \beta \right| = j.$$

(A5) The control domain U is a convex subset of $\mathbb{R}^n$. If $\varphi = b, \sigma, f$, the maps $u \longmapsto \varphi\left(t, x, u\right)$ are $C^1(U)$ and their derivatives satisfy a polynomial growth such as

$$\left| D_u \varphi\left(t, x, u\right) \right| \leq C \left(1 + \left| x \right|^k \right), \text{ for some} \qquad k \in \mathbb{N}.$$

Remark 3.1.1 **(i)** *It is easy to see that the assumption* $(A1)$ *implies that the derivative of* $b\left(t, x, y\right)$ *with respect to* x *satisfies a dissipativity condition,* $\mathbb{P}$−a.s*, for all* y *in* $\mathbb{R}^n$

$$\left(D_x b\left(t, x, y\right) y\right) y \leq \alpha \left| y \right|^2, \qquad t \in [0, T],\ u \in U.$$

(ii) *It is worth mentioning that under assumptions (A1)-(A3), the assumptions i), ii) of Theorem*(2.2.1) *in chapter 2 are also satisfied. Therefore, the SDE* (3.1) *admits a unique solution which belongs to* $C\left([0,T];L^2(\Omega,\mathbb{R})\right)$

3.2 Stochastic Maximum Principle (SMP)

We have to deal with the backward stochastic differential equations arising as an adjoint equation with terminal condition in the formulation of the SMP. The adjoint equation has the following form

$$\begin{cases} dp(t) = -\Big[D_x b(t,\bar{x}(t),\bar{u}(t))^T p(t) + \sum_{j=1}^{d} D_x \sigma^j(t,\bar{x}(t),\bar{u}(t))^T q_j(t) \\ \quad -D_x f(t,\bar{x}(t),\bar{u}(t))\Big]\, dt + \sum_{j=1}^{d} q_j(t) dW(t) \\ p(T) = -D_x h(\bar{x}(T)) \end{cases}, \tag{3.4}$$

where $p(.)$ is the adjoint process and $\bar{x}(t)$, $\bar{u}(t)$ are the optimal trajectory and the optimal control process. Where H is the Hamiltonian and it is defined by

$$H(t,x,u,p,q) = pb\,(t,x,u) + Tr\left[q^T\sigma\,(t,x,u)\right] - f(t,x,u). \tag{3.5}$$

It is worth mentioning that assuming conditions (A1)-(A3), the assumptions (H1)-(H6) of theorem(2.3.1) presented in chapter 2 are also met. As a result, the adjoint equation (3.4) possesses a unique solution that falls under the described set $(p(.),q(.)) \in L^2_{\mathcal{F}}\left([0,T];\mathbb{R}^n\right) \times \left(L^2_{\mathcal{F}}\left([0,T];\mathbb{R}^n\right)\right)^d$.

Now, we will discuss a version of the SMP where controls take values in a closed convex subset U of $\mathbb{R}^n$.

3.2.1 Necessary Condition of Optimality

Let $u(.)$ be a arbitrary element of U, then for $0 \leq \theta \leq 1$, we define a perturbed control as follows

$$\bar{u}(.) + \theta \left(u(.) - \bar{u}(.) \right), \tag{3.6}$$

since the actions space being convex, it is clear that (3.6) is an admissible control and we set $x_\theta(t)$ the trajectory corresponding to the perturbed control. The optimality of $\bar{u}(.)$ guarantees that

$$J\left(\bar{u}(.) + \theta(u(.) - \bar{u}(.))\right) \geq J(\bar{u}(.)).$$

We have to prove that $J(.)$, considered as a functional on $L^1_{\mathcal{F}}([0,T];\mathbb{R}^n)$, is Gâteaux differentiable. Then, we will write

$$J'(\bar{u})\left(u(.) - \bar{u}(.)\right) \geq 0 \qquad\qquad \forall u(.) \in \mathcal{U}[0,T].$$

And we will deduce a form of the SMP. If we define a new process y as a solution of the stochastic differential equation

$$\begin{cases} dy(t) = \left[D_x b(t,\bar{x}(t),\bar{u}(t))y(t) + D_u b(t,\bar{x}(t),\bar{u}(t))u(t)\right] dt \\ + \left[D_x \sigma(t,\bar{x}(t),\bar{u}(t))y(t) + D_u \sigma(t,\bar{x}(t),\bar{u}(t))u(t)\right] dW(t) \\ y(0) = 0 \end{cases}. \tag{3.7}$$

We can state the following

Lemma 3.2.1 *The functional $J(.)$ is Gâteaux differentiable, moreover the deriv-*

ative has the form

$$\frac{d}{d\theta}J\left(\bar{u}(.)+\theta u(.)\right)|_{\theta=0}=\mathbb{E}\left[D_x h\left(\bar{x}(T)\right)y(T)+\zeta(T)\right], \tag{3.8}$$

where ζ *is the solution to*

$$\begin{cases} \frac{d\zeta}{dt}=D_x f(t,\bar{x}(t),\bar{u}(t))y(t)+D_u f(t,\bar{x}(t),\bar{u}(t))u(t) \\ \zeta(0)=0 \end{cases}.$$

Proof. We denote x_θ the trajectory corresponding to the perturbed control and set

$$\bar{x_\theta}(t)=\frac{x_\theta(t)-\bar{x}(t)}{\theta}-y(t).$$

The idea of the proof is to show that $\left|\bar{x_\theta}(t)\right|^2_{L^2(\Omega)}\longrightarrow 0$ when $\theta\longrightarrow 0$. In fact, this is crucial in order to show that

$$\frac{1}{\theta}\mathbb{E}\left[h\left(x_\theta(T)\right)-h\left(x(T)\right)\right]\longrightarrow\mathbb{E}\left[D_x h\left(\bar{x}(T)\right)y(T)\right].$$

We start by writing the equation for $\bar{x_\theta}(t)$

$$\begin{cases} d\bar{x_\theta}(t)=\frac{1}{\theta}\left[b\left(t,\bar{x}(t)+\theta y(t)+\theta\bar{x_\theta}(t),\bar{u}(t)+\theta u(t)\right)\right. \\ \left.-b(t,\bar{x}(t),\bar{u}(t))-\theta D_x b(t)y(t)-\theta D_u b(t)u(t)\right]dt \\ +\frac{1}{\theta}\left[\sigma(t,\bar{x}(t)+\theta y(t)+\theta\bar{x_\theta}(t),\bar{u}(t)+\theta u(t))\right. \\ \left.-\sigma(t,\bar{x}(t),\bar{u}(t))-\theta D_x\sigma(t)y(t)-\theta D_u\sigma(t)u(t)\right]dW(t) \\ \bar{x_\theta}(0)=0 \end{cases}.$$

Then using Taylor expansion, we obtain

$$
\begin{aligned}
d\bar{x}_\theta(t) = & \int_0^1 D_x b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) \bar{x}_\theta(t) d\lambda dt \\
& + \int_0^1 \left[D_x \sigma\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right)\right. \\
& \left.\bar{x}_\theta(t) d\lambda\right] dW(t) \\
& + \int_0^1 \left[D_x b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_x b(t)\right. \\
& \left. y(t) d\lambda\right] dt \\
& + \int_0^1 \left[D_x \sigma\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_x \sigma(t)\right. \\
& \left. y(t) d\lambda\right] dW(t) \\
& + \int_0^1 \left[D_u b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_u b(t)\right] \\
& u(t) d\lambda dt \\
& + \int_0^1 \left[D_u \sigma\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_u \sigma(t)\right. \\
& \left. u(t) d\lambda\right] dW(t),
\end{aligned}
$$

applying Itô's formula to the function $\bar{x}_\theta \longmapsto \left|\bar{x}_\theta\right|^2$, we have

$$
d\left|\bar{x}_\theta\right|^2 = 2|\bar{\bar{x}}_\theta| d\bar{x}_\theta + d\left\langle\bar{x}_\theta\right\rangle.
$$

Taking the expectation we get

$$
\begin{aligned}
\mathbb{E}\left|\bar{x}_\theta\right|^2 &\leq K\mathbb{E}\int_0^t \left|\bar{x}_\theta(s)\right|^2 ds \\
&+ K\mathbb{E}\left[\int_0^T \Big[|y(t)|^2 \right. \\
&\left.\int_0^1 \left|D_x b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_x b(t)\right|^2 d\lambda\Big]\, dt\right] \\
&+ K\mathbb{E}\left[\int_0^T \Big[|u(t)|^2 \right. \\
&\left.\int_0^1 \left|D_u b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_u b(t)\right|^2 d\lambda\Big]\, dt\right] \\
&+ K\mathbb{E}\left[\int_0^T \Big[|y(t)|^2 \right. \\
&\left.\int_0^1 [D_x \sigma\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_x\sigma(t)]^2 d\lambda\Big]\, dt\right] \\
&+ K\mathbb{E}\left[\int_0^T \Big[|u(t)|^2 \right. \\
&\left.\int_0^1 \left|D_u \sigma\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_u\sigma(t)\right|^2 d\lambda\Big]\, dt\right] \\
&= K\mathbb{E}\int_0^t \left|\bar{x}_\theta(s)\right|^2 ds + \rho_\theta,
\end{aligned}
$$

thanks to the polynomial growth of $D_x b$, $D_x\sigma$, $D_u b$, $D_u\sigma$ and the Young inequality. Now, let us estimate the second term of the right hand side of the above inequality. If $\theta \longmapsto 0$ then also

$$
\mathbb{E}\int_0^1 \left|D_x b\left(t, \bar{x}(t) + \lambda\theta(y(t) + \bar{x}_\theta(t)), \bar{u}(t) + \lambda\theta u(t)\right) - D_x b(t)\right|^2 d\lambda \longrightarrow 0
$$

due to the polynomial growth and the continuity of $D_x b$ with respect to (x, u). For the remaining terms, we apply the same argument, so we can conclude that if $\theta \longrightarrow 0$ also $\rho_\theta \longrightarrow 0$. Finally, by applying Gronwall's inequality we get

$$\mathbb{E}|\bar{x_\theta}|^2 \leq K\rho_\theta \longrightarrow 0 \qquad \text{if } \theta \longrightarrow 0.$$

Then, in order to prove formula (3.8) we compute the following

(i) $\mathbb{E}\frac{1}{\theta}\left[h(x_\theta(T)) - h(\bar{x}(T))\right] \longrightarrow \mathbb{E}D_x h(\bar{x}(T))y(T)$,

(ii) $\mathbb{E}\left(\frac{1}{\theta}\int_0^T \left[f\left(t, x_\theta, \bar{u} + \theta u\right) - f(t, \bar{x}, \bar{u})\right] dt\right) \longrightarrow \mathbb{E}\zeta(T)$.

But (i) can be rewritten in the form

$$\begin{aligned}
&\mathbb{E}\int_0^1 \left(D_x h(\bar{x}(T) + \lambda\left(x_\theta(T) - \bar{x}(T)\right))\right)\left(\bar{x}_\theta(T) + y(T)\right) d\lambda \\
&\leq \int_0^1 \mathbb{E}(\left|D_x h(\bar{x}(T) + \lambda\left(x_\theta(T) - \bar{x}(T)\right))\right|^2)^{\frac{1}{2}}(\mathbb{E}|x_\theta\bar{(T)}|^2)^{\frac{1}{2}} d\lambda \\
&+\mathbb{E}\int_0^1 D_x h(\bar{x}(T) + \lambda\left(x_\theta(T) - \bar{x}(T)\right))y(T)d\lambda,
\end{aligned}$$

where we have used Hölder inequality. Passing to the limit with $\theta \longrightarrow 0$ we can conclude. ■

Now we can state the maximum principle also in this particular case, where controls assume their values in a convex subset.

Theorem 3.2.1 *Suppose (A1)-(A5) hold and let* $\left(\bar{x}(.), \bar{u}(.)\right)$ *be an optimal pair for the control problem* (3.3)*. Then there exist* $p, q_j \in L^2_{\mathcal{F}}\left([0, T]; \mathbb{R}^n\right)$ *which are a solution of the BSDE* (3.4)*, such that*

$$\frac{\partial H}{\partial u}\left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right)\left(u\left(t\right) - \bar{u}(t)\right) \leq 0 \qquad d\mathbb{P}\times dt \ a.e.,\ u \in U, \tag{3.9}$$

where H *is the Hamiltonian* (3.5)*.*

Proof. *The existence and uniqueness of a solution to the BSDE* (3.4) *is guaranteed, due to* (2.3.1) *of chapter 2. Moreover, thanks to* (3.7) *and lemma* (3.2.1) *we*

Applying Itô formula on $p(t)y(t)$ we get

$$\begin{aligned}
dp(t)y(t) &= y(t)dp(t) + p(t)dy(t) + d < p, y >_t \\
&= \big[-y(t)D_x b\left(t, \bar{x}(t), \bar{u}(t)\right) p(t) - D_x \sigma \left(t, \bar{x}(t), \bar{u}(t)\right) q(t)y(t) \\
&\quad + D_x f(t, \bar{x}(t), \bar{u}(t))y(t) + p(t)D_x b(t, \bar{x}(t), \bar{u}(t))y(t) \\
&\quad + D_u b(t, \bar{x}(t), \bar{u}(t))u(t)p(t) + q(t)D_x \sigma \left(t, \bar{x}(t), \bar{u}(t)\right) y(t) \\
&\quad q(t)D_u \sigma \left(t, \bar{x}(t), \bar{u}(t)\right) u(t)\big] dt + \big[y(t)q(t) + p(t)D_x \sigma \left(t, \bar{x}(t), \bar{u}(t)\right) y(t) \\
&\quad + D_u \sigma \left(t, \bar{x}(t), \bar{u}(t)\right) u(t)p(t)\big] dW(t).
\end{aligned}$$

Then

$$\mathbb{E}\left[d\left\langle p(t), y(t)\right\rangle\right] = \mathbb{E}\left[D_x f(t)y(t) + p(t)D_u b(t)u(t) + \sum_{j=1}^{d} q_j(t)D_u \sigma^j(t)u(t)\right] dt.$$

And we know that with $\langle p(0), y(0)\rangle = 0$

$$\begin{aligned}
-\mathbb{E}D_x h(\bar{x}(T))y(T) &= \mathbb{E}p(T)y(T) \\
&= \mathbb{E}\int_0^T \Big[D_x f(t)y(t) + p(t)D_u b(t)u(t), \\
&+ \sum_{j=1}^{d} q_j(t)D_u \sigma^j(t)u(t)\Big] dt.
\end{aligned}$$

Hence from lemma (3.2.1)

$$\begin{aligned}
0 &\leq \frac{d}{d\theta} J(\bar{u} + \theta u) \mid_{\theta=0} \\
&= \mathbb{E} \int_0^T \left[D_u f(t) u(t) - p(t) D_u b(t) u(t) - q(t) D_u \sigma(t) u(t) \right] \\
&= \mathbb{E} \int_0^T \Big[\Big(\frac{\partial}{\partial u} \left[f\left(t, \bar{x}(t), \bar{u}(t)\right) - p(t) b(t, \bar{x}(t), \bar{u}(t)) \right. \\
&\quad - Tr\left(q(t) \sigma^T \left(t, \bar{x}(t), \bar{u}(t)\right)\right) \Big] \Big) u(t) \Big] \, dt \\
&= -\mathbb{E} \int_0^T \frac{\partial H}{\partial u} \left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right) u(t) dt.
\end{aligned}$$

And we have

$$\mathbb{E} \int_0^T -\frac{\partial H}{\partial u} \left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right) u(t) dt \geq 0, \tag{3.10}$$

as the admissible control $u(.)$ check condition $\bar{u}(.) + u(.) \in \mathcal{U}_{ad}$, then there is an admissible control $v\,(.)$ such that $v\,(.) = \bar{u}(.) + u(.)$. The inequality (3.10) is then written

$$\mathbb{E} \int_0^T -\frac{\partial H}{\partial u} \left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right) \left(v\,(.) - \bar{u}\,(.)\right) dt \geq 0, \forall v\,(.) \in \mathcal{U}_{ad}.$$

Now, let F be an arbitrary element of $\mathcal{F}_t$ and we set

$$\pi\,(t) = v\,(t)\, \mathbf{1}_F - \bar{u}\,(t)\, \mathbf{1}_{\Omega - F},$$

it is clear that $\pi\,(.)$ is an admissible control. Applying the same technique as in [2],[3],[4] one can easily get the necessary condition of optimality

$$\frac{\partial H}{\partial u} \left(t, \bar{x}(t), \bar{u}(t), p(t), q(t)\right) u(t) \leq 0$$

■

3.2.2 Sufficient Condition of Optimality

We have to make the following assumption:

(A6) $h(.)$ is a convex function and the Hamiltonian $H\left(t,.,.,p(t),q(t)\right)$ is concave for all $t\in\left[0,T\right]$, $\mathbb{P}-a.s.$

Theorem 3.2.2 *Let assumptions (A1)-(A6) hold. Let* $\left(\bar{x}(.),\bar{u}(.)\right)$ *be an admissible pair,* $(p(.),q(.))$ *be solution to* (3.4) *such that the necessary condition* (3.9) *is satisfied. Then,* $\left(\bar{x}(.),\bar{u}(.)\right)$ *is an optimal pair of the control problem* $(3.1)and(3.3)$.

Proof. Let $\left(\bar{x}(.),\bar{u}(.)\right)$ be an admissible pair candidate to be optimal which satisfy (3.9) and $(x(.),u(.))$ is any adimissible pair. From the definition of the cost function (3.2), we have

$$J\left(\bar{u}\left(.\right)\right)=\mathbb{E}\left[\int_0^T f\left(t,\bar{x}\left(t\right),\bar{u}\left(t\right)\right)dt+h\left(\bar{x}\left(T\right)\right)\right],$$

and

$$J\left(u\left(.\right)\right)=\mathbb{E}\left[\int_0^T f\left(t,x\left(t\right),u\left(t\right)\right)dt+h\left(x\left(T\right)\right)\right].$$

It follows, using the fact that $h\left(.\right)$ is a convex function and $p(T)=-D_xh(\bar{x}(T))$

$$\begin{aligned} J(u)-J(\bar{u}) &= \mathbb{E}\left[\int_0^T\left[f\left(t,x(t),u(t)\right)-f\left(t,\bar{x}(t),\bar{u}(t)\right)\right]dt\right] \\ &+\mathbb{E}\left[h(x(T))-h(\bar{x}(T))\right] \\ &\geq \mathbb{E}\left[\int_0^T\left[f\left(t,x(t),u(t)\right)-f\left(t,\bar{x}(t),\bar{u}(t)\right)\right]dt\right] \\ &+\mathbb{E}\left[-p(T)(x(T)-\bar{x}(T))\right] \end{aligned} \tag{3.11}$$

Applying Itô's formula to $p(.)(x(.)-\bar{x}(.))$, we get

$$d\left[p(t)\left(x(t)-\bar{x}(t)\right)\right]=p(t)d\left(x(t)-\bar{x}(t)\right)+(x(t)-\bar{x}(t))dp(t)+d<p,(x-\bar{x})>_t.$$

Then, by taking the expectation and using the fact that $p(0)(x(0)-\bar{x}(0))=0$, we have

$$\begin{aligned}\mathbb{E}\left[p(T)(x(T)-\bar{x}(T))\right] &= \mathbb{E}\Big[\textstyle\int_0^T b(t,x(t),u(t))p(t)dt - \int_0^T b(t,\bar{x}(t),\bar{u}(t))p(t)dt \\ &- \textstyle\int_0^T (x(t)-\bar{x}(t))D_x b(t,\bar{x}(t),\bar{u}(t))p(t)dt \\ &- \textstyle\int_0^T (x(t)-\bar{x}(t))D_x\sigma\left(t,\bar{x}(t),\bar{u}(t)\right)q(t)dt \\ &+ \textstyle\int_0^T (x(t)-\bar{x}(t))D_x f\left(t,\bar{x}(t),\bar{u}(t)\right)dt \\ &+ \textstyle\int_0^T q(t)\sigma\left(t,x(t),u(t)\right)dt - \int_0^T q(t)\sigma(t,\bar{x}(t),\bar{u}(t))dt\Big]\end{aligned} \quad . \tag{3.12}$$

By replacing (3.12) in (3.11), we obtain

$$\begin{aligned}J(u)-J(\bar{u}) &\geq \mathbb{E}\textstyle\int_0^T \left[f\left(t,x(t),u(t)\right) - b(t,x(t),u(t))p(t) - q(t)\sigma(t,x(t),u(t))\right]dt \\ &+\mathbb{E}\Big[\textstyle\int_0^T \left[-f\left(t,\bar{x}(t),\bar{u}(t)\right) + b(t,\bar{x}(t),\bar{u}(t))p(t) + q(t)\sigma(t,\bar{x}(t),\bar{u}(t))\right]dt\Big] \\ &+\mathbb{E}\Big[\textstyle\int_0^T \big[D_x f(t,\bar{x}(t),\bar{u}(t))(x(t)-\bar{x}(t)) + (x(t)-\bar{x}(t))D_x b(t,\bar{x}(t),\bar{u}(t)) \\ &p(t) + (x(t)-\bar{x}(t))D_x\sigma\left(t,\bar{x}(t),\bar{u}(t)\right)q(t)\big]dt\Big] \\ &= \mathbb{E}\Big[\textstyle\int_0^T \big[H(t,\bar{x}(t),\bar{u}(t),p(t),q(t)) - H(t,x(t),\bar{u}(t),p(t),q(t)) \\ &+H(t,x(t),\bar{u}(t),p(t),q(t)) - H(t,x(t),u(t),p(t),q(t)) \\ &+\tfrac{\partial H}{\partial x}\left(t,\bar{x}(t),\bar{u}(t),p(t),q(t)\right)(x(t)-\bar{x}(t))\big]dt\Big].\end{aligned}$$

Now, by invoking Assumption (A7),

$$J(u)-J(\bar{u}) \geq -\mathbb{E}\left[\int_0^T \frac{\partial H}{\partial u}\left(t,\bar{x}(t),\bar{u}(t),p(t),q(t)\right)\left((u(t)-\bar{u}(t))\right)dt\right]dt.$$

By taking account of the necessary condition (3.9), we get

$$J(u)-J(\bar{u}) \geq 0.$$

■

Conclusion

The main objective of this thesis is to present two results. The first is related to the existence and uniqueness of the solution for both (backward-forward) stochastic differential equations in the case when we change the Lipschitz condition with a weaker condition called dissipative condition, our work relied heavily on the references ([11],[13]). The second result is related to the study of the stochastic maximum principle (SMP) which is a fundamental concept in stochastic optimal control theory. We provide necessary and sufficient conditions for the optimal control of (SDEs) with control-dependent coefficients b and σ under some conditions.

In the last years the researchers concentrated on the easy and applicable methods to deal with the problems of optimal control, they were able to link this subject with deep learning techniques. This later can be applied to approximate the maximum principle and solve optimal control problems by learning control policies, value functions, or Hamiltonian functions from data. This integration of deep learning with the maximum principle provides a promising avenue for addressing complex control problems in various domains. In our future researches, we will focus on the use of the deep learning techniques to solve numerically some control problems via stochastic maximum principle [9] [16].

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Abstract

Our aim of this thesis is to study two results. The first is the existence and uniqueness of solutions for both (backward-forward) stochastic differential equations in the dissipative case. The second result focuses on investigating the stochastic maximum principle (SMP), we establish necessary and sufficient conditions for optimal control of stochastic differential equations (SDE) with dissipative drift

Key words: Dissipative condition , stochastic differential equations, stochastic maximum principle, necessary and sufficient conditions

الملخص

هدف هذه الأطروحة هو دراسة نتيجتين. النتيجة الأولى تتعلق بوجود و وحدانية الحل لكل من المعادلات التفاضلية العشوائية و المعادلات التفاضلية العشوائية التراجعية في حالة الشرط التزايدي. النتيجة الثانية تركز على المبدأ الأقصى الاحتمالي العشوائي، حيث نحدد الشروط الضرورية والكافية للتحكم الأمثل في المعادلات التفاضلية العشوائية .

الكلمات المفتاحية : الشرط التزايدي ، المعادلات التفاضلية العشوائية ، المعادلات التفاضلية العشوئية التراجعية، المبدأ الأقصى الاحتمالي العشوائي .

Résumé

Notre objectif dans cette thèse est d'étudier deux résultats. Le premier concerne l'existence et l'unicité des solutions pour les équations différentielles stochastiques (rétrograde) dans le cas dissipatif. Le deuxième résultat se concentre sur l'étude du principe maximum stochastique. Nous établissons les conditions nécessaires et suffisantes pour le contrôle optimal des EDS avec une dérive dissipative

Mot clés :condition de dissipative , les équations différentielles stochastiques, principe maximum stochastique, conditions nécessaires et suffisantes