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**************TITRE***************

Sur certaine propriétés des équations différentielles stochastiques doublement rétrogrades.

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par

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Contents

Abstract

This thesis is composed of two parts. The first part is devoted the study of backward stochastic differential equations (BSDEs) under Lipschitz conditions in both multidimensional case, we prove an existence and uniqueness result for BSSDE's in this case. After, we give an existence result for a BSDE where the coefficients are assumed to be only continuous. In The second part, we prove the existence of the solution where is forced to stay above a given stochastic process, called the obstacle. We prove the existence and uniqueness under Lipischitz coefficient. In the case where the coefficient is continuous we prove the existence only.

In the second part , we present some new results in the theory of backward doubly stochastic differential equations (BDSDEs), In first, we give the result the existence and uniqueness under some Lipshitz assumption on the coefficients, finally we establish existence and uniqueness results for a reflected BDSDE with one barriers and with two barriers, study the case Lipschitz and Continuous.

Résumé

Cette thèse se compose de deux parties. La première partie est consacré à l'étude des équations différentielles stochastiques rétrogrades (EDSR), sous les conditions de Lipischitz en prouve l'existence et l'unicité, mais avec coefficient continu, nous donnons un résultat d'existence pour une EDSR. Nous étudions dans chapitre 2, une classe des équations différentielles stochastiques rétrogrades.
Dans la deuxième partie, nous présentons de nouveaux résultats dans la théorie des équations différentielles doublement stochastiques rétrogrades (EDDSR), on commence par les résultats classique sur cette classe des équations. Et enfin en définie les équations différentielles doublement stochastiques rétrogrades réfléchies (EDDSRR). D'abord, sous la condition de Lipshitz sur les coefficients nous établissons l'existence et l'unicité pour une EDDSR réfléchie. Ensuite, un résultat d'existence de la solution dans le cas où les coefficients sont continus à croissance linéaire.

Introduction

It was mainly during the last decade that the theory of backward stochastic differential equation took shape as a distinct mathematical discipline. This theory has found a wide field of application as in mathematical finance, the theory of hedging and non-linear pricing theory for imperfect markets (see El Karoui, Peng and Quenez [14]) and at the same time, in stochastic optimal control and stochastic games (see Hamadène and Lepeltier [23]), and they provide probabilistic formulae for solutions to partial differential equations (see Pardoux and Peng [34]).
The Linear backward stochastic differential equation :

$$Y_t = \xi + \int_t^T (Y_s\beta_s + Z_s\gamma_s + \phi_s)ds - \int_t^T Z_s dW_s. \tag{1}$$

have been introduced by Bismut [8] and [9] when he was studying the adjoint equations associated with the stochastic maximum principle in optimal control. However, the first published paper on nonlinear BSDE's (see Pardoux and Peng [34])

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{2}$$

In [34], Pardoux and Peng have established the existence and uniqueness of the solution of equation 2 under the uniform Lipschitz condition, i.e. there exists a constant $K > 0$ such that

$$|f(t,y,z) - f(t,y',z')| \leq K(|y-y'| + |z-z'|).$$

for all $y, y' \in \mathbb{R}^d,\ z, z' \in \mathbb{R}^{d\times n}$. Note that, since the boundary condition is given at the terminal time T, it is not really natural for the solution Y_t be to adapted at each time t to the past of the Brownian motion W_s before time t. The presence of the process Z_t seems superfluous. However, we point out that it is the presence of this process that makes it possible to find adapted process Y_t to satisfy equation (2). Hence, a solution of BSDE (2) on the probability space of Brownian motion , as mentioned above, is a pair (Y, Z) of adapted processes that satisfies (2) almost surely.

In [27], Lepeltier and San Martin have prove the existence of a solution for one-dimensional BSDE's where the coefficient is continuous, it has a linear growth, i.e assume the for fixed $t,\ \omega,\ f(t,.,.)$ os continuous, and there exists a constant $K>0$ such that for all t,y,z we have

$$|f(t,y,z)| \leq K(1+|y|+|z|).$$

Kobylanski in [26] prove existence and uniqueness results for BSDE's (2) in one dimensional when the generator $(f(t,y,z)$ has a quadratic growth in Z.

$$|f(t,y,z)| \leq C(1+|y|+|z|^2).$$

These results are inspired by the analogous one for quasilinear partial differential equations and hold for processes $(Y_t,Z_t)_{0\leq t\leq T}$ such that $(Y_t)_{0\leq t\leq T}$ is bounded. K.Bahlali in [2] we deal with multidimensional BSDE's with locally Lipschitz coefficient and a square integrable terminal data. We study the existence and uniqueness, as well as the stability of solutions. We show that if the coefficient f is locally Lipschitz in both variables y,z and the Lipschitz constant L_N in the ball $B(0,N)$ is such that $L_N = o(\sqrt{\log N})$, then the BSDE 2 has a unique solution.
In the case where the solution is forced to remain above an obstacle, El Karoui et *al.* [15] have derived an existence and uniqueness result for Reflected BSDE's with Lipschitz coefficient by Picard iteration method as well as a penalization argument. In this case, the solution is a triple (Y,Z,K), where K is an increasing process, satisfying

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s. \tag{3}$$

Note that the study of Reflected BSDE's on one barriers has been primarily motivated by the evaluation of an American option in a market constrained, which may be a market where interest rates are not the same if we wants to borrow or invest money. Indeed, it has been proved that the price of a American contingent action is a solution of a reflected BSDE's that the barrier is given by the payoff and the optimal time of exercise is the first time when the price reaches the payoff (see [16]). Other application is in mixed stochastic control see [20]. In the paper of Matoussi [29] he will be inspired by the works of El Karoui [15] and Lepeltier [27] to establish existence of Reflected solution of one-dimensional BSDE's with continuous and linear growth coefficient.
Recently, a new class of BSDE's, called doubly stochastic, has been considered by Pardoux and Peng (1994) (see [35]). This new kind of backward SDE's seems to be suitable to give a probabilistic representation for a system of parabolic stochastic partial differential equations (SPDE's). We refer to Pardoux and Peng (1994) [35] for the link between

SPDE's and BDSDE's in the particular case where solutions of SPDE's are regular. In [35] Pardoux-Peng study existence and uniqueness of the solution to a backward doubly stochastic differential equations as follows

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s. \tag{4}$$

where the dW integral is a forward Itô's integral and the $d\overleftarrow{B}$ integral is a backward Itô's integral, we prove under the conditions f and g are Lipschitz, the BDSDE 4 has a unique solution.

In [41]. Shi et *al.*, we shall prove the comparison theorem of BDSDE's (4). Then we study the case where the generator are continuous with linear growth. We show the existence of the minimal solution of (4). This method is due to [27].

In this thesis, we present some new results in the theory of Backward Doubly Stochastic Differential Equations.

First, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued reflected backward doubly stochastic differential equation :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s + K_T - K_t - \int_t^T Z_s dW_s. \tag{5}$$

We establish the existence and uniqueness of solutions for equation (5) under uniformly Lipschitz condition on the coefficients [4]. In contrast to classical reflected BSDEs, the section theorem can not be easily used for RBDSDEs. Indeed, it is not possible to prove that the solution stays above the obstacle for all time, by only using the classical BSDEs technics. This is due to the fact that the solution should be adapted to a family $(\mathcal{F}_t)$ which is not a filtration. We give here a method which allows us to overcome this difficulty in the Lipschitz case. The idea consists to start from the penalized basic RBDSDE with f and g independent from (y, z). We transform it to a RBDSDE with $f = g = 0$, for which we prove the existence and uniqueness of solution by penalization method. The section theorem is then only used in the simple context where $f = g = 0$ to prove that the solution of the RBDSDE (with $f = g = 0$) stays above the obstacle for each time. A new type of comparison theorem is also established and used in this context. The (general) case, where the coefficients f, g depend on (y, z), is treated by a Picard type approximations.

In the case where the coefficient f is only continuous, we establish the existence of a maximal and a minimal solutions. In this case, we approximate f by a sequence of Lipschitz functions (f_n) and use a comparison theorem which is established here for RBDSDEs.

Other new result, we generalize the above result to the case of two reflecting barrier processes, we obtain the real valued double reflected backward doubly stochastic differential equation (in short DRBDSDE):

$$Y_t = \xi+\int_t^T f(s, Y_s, Z_s)\, ds+\int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s+\int_t^T dK_s^+-\int_t^T dK_s^- -\int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{6}$$

We establish the existence and uniqueness of solutions for equation (6) under uniformly Lipschitz condition on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of a solutions.

The thesis is organized as follows.
The first part of this thesis is on the Backward Stochastic Differential Equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{7}$$

In Chapter 1, we present, an existence and uniqueness theorem for solution of BSDE's (7) where the coefficient is Lipuschitz, and we give the comparison result in this case and we prove the existence it the case where the generator are continuous with linear growth.
In Chapter 2, we prove existence and uniqueness results of solution of reflected backward stochastic differential equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s. \tag{8}$$

where the coefficient is Lipischitz and existence only in the case where the generator are continuous with linear growth.
The second part of this thesis is on the Backward Doubly Stochastic Differential Equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s. \tag{9}$$

In Chapter 3, we give a background on BDSDE's, we prove existence and uniqueness results of solution of BDSDE (9) where f and g are Lipischitz, and the generalize this case where f is continuous with linear growth and we prove also the result comparison for BDSDE's.
In Chapter 4, in this chapter we establish a new result of BDSDE's, is when the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real

valued reflected BDSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s + K_T - K_t - \int_t^T Z_s dW_s. \tag{10}$$

We prove the existence and uniqueness of solutions for equation (10) under uniformly Lipschitz condition on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of a maximal and a minimal solutions.
In Chapter 5, we give other new result is for double reflected backward doubly stochastic differential equation (in short DRBDSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{11}$$

We establish the existence and uniqueness of solutions for equation (11) under uniformly Lipschitz condition on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of a solutions.

Part I

BACKGROUND ON BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS.

Chapter 1

Existence result of backward stochastic differential equation

We prove the existence and uniqueness of solution for backward stochastic differential equation with lipschitz generator and squared integrable terminal condition, we prove moreover the existence of a solution when the generator is merely continuous.

1.1 Introduction

Backward stochastic differential equations (BSDEs) have been first introduced by E. Pardoux and S. Peng [39] who proved existence and uniqueness of adapted solutions for these equations under suitable Lipschitz and linear growth conditions on the coefficients. The principal interest of BSDE is that they provide a useful framewark for formulating many problems as in finance theory, stochastic control and in the games theory.
Following the idea of J.P. Lepeltier and J. San Martin [27], we use an approximation argument to prove the existence of a solution of one dimensional BSDE with a continuous coefficient.

1.2 BSDE with Lipschitz coefficient

Let consider a filtered space $(\Omega, \mathfrak{F}, P, \mathfrak{F}_t, W_t, t \in [0,1])$ be a complete Wiener space in $\mathbb{R}^n$,i.e. $(\Omega, \mathfrak{F}, P)$ is a complete probability space, $(\mathfrak{F}_t, t \in [0,1])$ is a right continuous increasing family of complete sub $\sigma - algebras$ of $\mathfrak{F}$, $(W_t, t \in [0,1])$ is a standard Wiener process in $\mathbb{R}^n$ with respect to $(\mathfrak{F}_t, t \in [0,1])$. We assume that

$$\mathfrak{F}_t = \sigma[W_s, s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of P-null sets. Now, we define the following two objects :
(H1.1) A terminal value $\xi \in L^2(\Omega, \mathfrak{F}_1, P)$.
(H1.2) A function process f defined on $\Omega \times [0,1] \times \mathbb{R}^k \times \mathbb{R}^{k\times n}$ with values in $\mathbb{R}^k$ and satisfies the following assumptions :
(i) for all $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times n}$: $(\omega, t) \rightarrow f(\omega, t, y, z)$ is $\mathfrak{F}_t$-progressively measurable.
(ii) $E\int_0^1 |f(t,0,0)|^2 dt < \infty$
(iii) for some $K > 0$ and all $y, y' \in \mathbb{L}^k$, $z, z' \in \mathbb{R}^{k\times n}$, and $(\omega, t) \in \Omega \times [0,1]$

$$|f(\omega,t,y,z) - f(\omega,t,y',z')| \leq K(|y-y'| + |z-z'|).$$

Let us consider the following kind of space of variable or processes.
(1) $\mathbb{L}^2_T(\mathbb{R}^k)$ is the space of all $\mathfrak{F}_T$-measurable random variables $X : \Omega \rightarrow \mathbb{R}^k$ satisfying $\|X\|^2 = E(|X|^2) < \infty$.
(2) $\mathbb{H}^2_T(\mathbb{R}^k)$ is the space of all the predictable process $\phi : \Omega \times [0,T] \rightarrow \mathbb{R}^k$ such that $\|\phi\|^2 = E(\int_0^T |\phi|^2 dt) < \infty$. Such processes are said to be square integrable.
Let us now introduce our BSDE : Given a data (f, ξ) we want to solve the following backward stochastic differential equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{1.1}$$

Definition 1.2.1. *A solution of equation (1.1) is a pair of processes (Y, Z) progressively measurable and satisfying : $\int_0^T (|f(s, Y_s, Z_s)| + \|Z_s\|^2) ds < \infty$, and equation (1.1).*

We now make more precise the dependence of the norm of the solution (Y, Z) upon the data (ξ, f).

Proposition 1.2.2. *Let assumptions (H1.1) and (H1.2) hold. Then there exists a constant C, which depend only on K, such that*

$$E(sup_{0\leq t\leq T}|Y_t|^2) + E(\int_0^T |Z_t|^2 dt) \leq CE(|\xi|^2 + \int_0^T |f(t,0,0)|^2 dt)$$
$$|Y_t|^2 \leq E(e^{\alpha(1-t)}|\xi|^2 + \int_0^T e^{\alpha(s-t)}|f(s,0,0)|^2 ds/\mathfrak{F}_t),$$

where $\alpha = 1 + 2K + 2K^2$.

Before proving proposition (1.2.2), let us first prove the inequality

$$Esup_{0\leq s\leq T}|Y_s|^2+E(\int_0^T|Z_s|^2ds)<\infty. \tag{1.2}$$

Define for each $n\in\mathbb{N}$, the stopping time

$$\tau_n=inf\{0\leq t\leq T;|Y_t|\geq n\},$$

and the processes

$$Y_t^n=Y_{t\wedge\tau_n}.$$

By noting

$$Z_t^n=1_{[0,\tau_n]}(t)Z_t,$$

we have

$$Y_t^n=\xi+\int_t^T 1_{[0,\tau_n]}(s)f(s,Y_s^n,Z_s^n)ds-\int_t^T Z_s^n dW_s,\ 0\leq t\leq T.$$

If we apply Itô's formula to the process $|Y_t^n|^2$, then

$$|Y_t^n|^2+\int_t^T|Z_s^n|^2ds=|\xi|^2+2\int_t^T 1_{[0,\tau_n]}(s)(Y_s^n)^*f(s,Y_s^n,Z_s^n)ds-\int_t^T\langle Y_s^n,Z_s^n dW_s\rangle,$$

which implies

$$E(|Y_t^n|^2+\int_t^T|Z_s^n|^2ds)\leq E|\xi|^2+E\int_t^T(|f(s,0,0)|^2ds+(1+2K+2\varepsilon^2)|Y_s^n|^2)ds \\ +\frac{K}{2\varepsilon^2}|Z_s^n|^2ds.$$

If we take $\frac{K}{2\varepsilon^2}\leq\frac{1}{2}$, we get

$$E|Y_t^n|^2+\frac{1}{2}E\int_t^T|Z_s^n|^2ds\leq C(1+E\int_t^T|Y_s^n|^2)ds.$$

Now it follows from Gronwall's lemma that

$$\sup_{n\in\mathbb{N}}\sup_{0\leq t\leq T}E|Y_t^n|^2\leq C.$$

On the other hand,

$$\sup_{n\in\mathbb{N}}E\int_t^T|Z_s^n|^2ds\leq\infty.$$

From Fatou's lemma, we can see that

$$\sup_{0\leq t\leq T}E|Y_t|^2\leq\infty.$$

Burkholder-Davis-Gundy inequality implies that

$$Esup_{0\leq t\leq T}|Y_t|^2 \leq \infty.$$

It follows that $\tau \uparrow T$ $a.s.$ Using again Fatou's lemma, we obtain

$$E\int_0^T |Z_s|^2 ds \leq \infty.$$

Proof. (of proposition 1.2.2) Since (Y, Z) satisfies equation (1.1) and (2.12), $E\int_t^T \langle Y_s, Z_s dW_s\rangle = 0$, because the local martingale, $\{E\int_t^T \langle Y_s, Z_s dW_s\rangle,\ 0 \leq t \leq T\}$ is uniformly integrable from the Burkholder-Davis-Gundy's inequality for stochastic integrals see (barlow and protter) and the fact that

$$E\sup_{0\leq t\leq T}|\int_t^T (Y_s)^* Z_s dW_s| \leq C(E\sup_{0\leq t\leq T}|Y_t|^2)^{1/2}(E\sup_{0\leq t\leq T}|\int_0^t Z_s dW_s|^2)^{1/2},$$

and from Doob's inequality, we obtain

$$E\sup_{0\leq t\leq T}|\int_t^T (Y_s)^* Z_s dW_s| \leq C(E\sup_{0\leq t\leq T}|Y_t|^2)^{1/2}(E\sup_{0\leq t\leq T}\int_0^t |Z_s|^2 ds)^{1/2} < \infty,$$

From Itô's formula, (H1.2)(iii) and schwartz's inequality,

$$\begin{aligned}
|Y_t|^2 + \int_t^T |Z_s|^2 ds &\leq |\xi|^2 + 2\int_t^T (Y_s)^* f(s, Y_s, Z_s) ds - 2\int_t^T \langle Y_s, Z_s dW_s\rangle \\
&\leq |\xi|^2 + \int_t^T (|f(s,0,0)|^2 + (1 + 2K + 2K^2)|Y_s|^2 + \frac{1}{2}|Z_s|^2) ds \\
&- 2\int_t^T \langle Y_s, Z_s dW_s\rangle.
\end{aligned}$$

Taking expectation and using Gronwall's lemma we get

$$\sup_{0\leq t\leq T} E|Y_t|^2 + E\int_0^T |Z_s|^2 dt \leq CE(|\xi|^2 + \int_0^T |f(s,0,0)|^2 dt) < \infty.$$

Then the result follows from the Burkholder-Davis-Gundy inequality. The second result follows by taking the conditional expectation in the following inequality

$$e^{at}|Y_t|^2 + \frac{1}{2}\int_t^T e^{as}|Z_s|^2 ds \leq e^{aT}|\xi|^2 + 2\int_t^T e^{as}|f(s,0,0)|^2 ds - 2\int_t^T e^{as}\langle Y_s, Z_s dW_s\rangle.$$

□

We shall now prove existence and uniqueness for BSDE (1.1) under conditions (H1.1) and (H1.2).

Theorem 1.2.3. ***(Pardoux-Peng)*** *Under conditions (H1.1), (H1.2), there exists a unique solution for equation (1.1).*

Proof. **Existence.**First, let us prove that the BSDE

$$Y_t = \xi + \int_t^T f(s)ds - \int_t^T Z_s dW_s,$$

has one solution. Let

$$Y_t = E(\xi + \int_0^T f(s)ds/\mathfrak{F}_t),$$

and $\{Z_t,\ 0 \le t \le T\}$ is given by Itô's martingales representation theorem applied to the square integrable random variable $\xi + \int_0^T f(s)ds$, that is

$$\xi + \int_0^T f(s)ds = E(\xi + \int_0^T f(s)ds) + \int_0^T Z_s dW_s, \quad 0 \le t \le T,$$

Taking the conditional expectation with respect to $\mathfrak{F}_t$, we deduce that

$$Y_t = \xi + \int_t^T f(s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T,$$

i.e. (Y, Z) is a solution of our BSDE. Let us define the following sequence $(Y^n, Z^n)_{n\in\mathbb{N}}$ such that $Y^0 = Z^0 = 0$ and (Y^{n+1}, Z^{n+1}) is the unique solution of the BSDE

$$\textbf{(1)}\ Z^{n+1}\texttt{is a predictable process and}E(\int_0^T |Z_t^{n+1}|^2 dt) < \infty,$$

$$\textbf{(2)}\ Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^{n+1} dW_s,\ 0 \le t \le T.$$

We shall prove that the sequence (Y^n, Z^n) is Cauchy. Using Itô's formula, we obtain for every $n > m$

$$\begin{aligned} e^{\alpha t}|Y_t^{n+1} - Y_t^{m+1}|^2 &+ \int_t^T e^{\alpha s}|Z_s^{n+1} - Z_s^{m+1}|^2 ds + \alpha \int_t^T e^{\alpha s}|Y_s^{n+1} - Y_s^{m+1}|^2 ds \\ &= 2\int_t^T e^{\alpha s}(Y_s^{n+1} - Y_s^{m+1})^*[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)]ds \\ &+ 2\int_t^T e^{\alpha s}(Y_s^{n+1} - Y_s^{m+1})^*(Z_s^{n+1} - Z_s^{m+1})dW_s, \end{aligned}$$

and then,

$$\begin{aligned} Ee^{\alpha t}|Y_t^{n+1} - Y_t^{m+1}|^2 &+ E\int_t^T e^{\alpha s}|Z_s^{n+1} - Z_s^{m+1}|^2 ds + \alpha E\int_t^T e^{\alpha s}|Y_s^{n+1} - Y_s^{m+1}|^2 ds \\ &\le 2KE\int_t^T e^{\alpha s}|Y_s^{n+1} - Y_s^{m+1}|[|Y_s^n - Y_s^m| + |Z_s^n - Z_s^m|]ds \end{aligned}$$

which implies

$$
\begin{aligned}
&Ee^{\alpha t}|Y_t^{n+1} - Y_t^{m+1}|^2 + E\int_t^T e^{\alpha s}|Z_s^{n+1} - Z_s^{m+1}|^2 ds \\
&\qquad \le (K^2\varepsilon^2 - \alpha)E\int_t^T e^{\alpha s}|Y_s^{n+1} - Y_s^{m+1}|^2 ds + \frac{2}{\varepsilon^2}E\int_t^T e^{\alpha s}|Y_s^n - Y_s^m|^2 ds \\
&\qquad + \frac{2}{\varepsilon^2}E\int_t^T e^{\alpha s}|Z_s^n - Z_s^m|^2] ds.
\end{aligned}
$$

Choosing α and ε such that $\frac{2}{\varepsilon^2} = \frac{1}{2}$ and $\alpha - 4K^2 = 1$, then

$$
\begin{aligned}
&Ee^{\alpha t}|Y_t^{n+1} - Y_t^{m+1}|^2 + E\int_t^T e^{\alpha s}|Z_s^{n+1} - Z_s^{m+1}|^2 ds \\
&\le \frac{1}{2}(E\int_t^T e^{\alpha s}|Y_s^n - Y_s^m|^2 ds + E\int_t^T e^{\alpha s}|Z_s^n - Z_s^m|^2] ds).
\end{aligned}
$$

It follows immediately that

$$
E\int_0^T e^{\alpha s}|Y_s^n - Y_s^m|^2 ds + E\int_0^T e^{\alpha s}|Z_s^n - Z_s^m|^2] ds \le \frac{C}{2^n}.
$$

Consequently, $(Y^n, Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let

$$
Y = \lim_{n\to\infty} Y^n, \quad \texttt{and} \quad Z = \lim_{n\to\infty} Z^n.
$$

It is easy to see that (Y, Z) is a solution of our BSDE.
Uniqueness. Let $\{(Y_t, Z_t);\ 0 \le t \le T\}$ and $\{(Y_t^{'}, Z_t^{'});\ 0 \le t \le T\}$ denote two solutions of our BSDE, and define

$$
\{(\Delta Y_t, \Delta Z_t),\ 0 \le t \le T\} = \{(Y_t - Y'_t, Z_t - Z'_t),\ 0 \le t \le T\}.
$$

It follows from Itô's formula that

$$
E[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds] = 2E\int_t^T \langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s^{'}, Z_s^{'})\rangle ds.
$$

Hence

$$
E[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds] \le CE\int_t^T |\Delta Y_s|^2 ds + frac12E\int_t^T |\Delta Z_s|^2 ds.
$$

the result follows from Gronwall's lemma. □

The following proposition shows, in particular, the existence and uniqueness result for linear backward stochastic differential equation. Such a way is well-known in mathematical finance, where the solution of a linear BSDE is in fact the pricing and hedging strategy of the contingent claim ξ (see [14]).

Proposition 1.2.4. *Let (β, γ) be a bounded $(\mathbb{R}, \mathbb{R}^n)$-valued progressively measurable process, Φ be an element of $\mathbb{H}_T^2(\mathbb{R})$, and ξ be an element of $\mathbb{L}_T^2(\mathbb{R}^k)$. Then the Linear BSDE*

$$Y_t = \xi + \int_t^T (\Phi_s + Y_s\beta_s + Z_s^*\gamma_s)ds - \int_t^T Z_s dW_s \tag{1.3}$$

has a unique solution (Y, Z) in $\mathbb{H}_T^2(\mathbb{R}) \times \mathbb{H}_T^2(\mathbb{R}^n)$ given explicitly by :

$$\Lambda_t Y_t = E[\xi\Lambda_T + \int_t^T \Lambda_s\Phi_s ds/\mathfrak{F}_t], \tag{1.4}$$

where Λ_t is the adjoint process defined by the forward linear stochastic differential equation,

$$\Lambda_t = 1 + \int_0^t \Lambda_s\beta_s ds + \int_0^t \Lambda_s\gamma_s^* dW_s,$$

Proof. By theorem1.2.3, there exists a unique solution to the BSDE (1.3). Using Itô's formula we obtain

$$\Lambda_t Y_t + \int_0^t \Lambda_s\Phi_s ds = Y_0 + \int_0^t \Lambda_s Y_s\gamma_s^* dW_s + \int_0^t \Lambda_s Y_s Z_s^* dW_s.$$

Since $\sup_{s\leq T}|Y_s|$ and $\sup_{s\leq T}|\Lambda_s|$ are square integrable, therefore the local martingale $\{\Lambda_t Y_t + \int_0^t \Lambda_s\Phi_s ds,\ t \in [0, T]\}$ is a uniformly integrable martingale, whose t-time value is the $\mathfrak{F}_t$-conditional expectation of its terminal value. □

1.3 Comparison theorem

We state in the one dimensional case a comparison theorem (first obtained by Peng,[38]).

Theorem 1.3.1. ***(Comparison theorem)****Let (f^1, ξ^1) and (f^2, ξ^2) be two data of BSDE's, and let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions. We suppose that $\xi^1 \leq \xi^2$ P a.s., and $f^1(t, y, z) \leq f^2(t, y, z)$ $dt \times P$ a.s. Then we have $Y_t^1 \leq Y_t^2$ $P - a.s.$*

Proof. We use the follows notation

$$\delta(Y_t) = Y_t^1 - Y_t^2,\ \delta(Z_t) = Z_t^1 - Z_t^2,\ \delta(\xi_t) = \xi_t^1 - \xi_t^2.$$

We obtain the follows BSDE

$$\delta(Y_t) = \delta(\xi_t) + \int_t^T (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))ds - \int_t^T \delta(Z_s)dW_s,\ 0 \leq t \leq T,$$

we can write

$$\begin{aligned}\delta(Y_t) = \delta(\xi_t) &+ \int_t^T (\alpha_s\delta(Y_s) + \beta_s\delta(Z_s) + f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2))ds \\ &- \int_t^T \delta(Z_s)dW_s, \quad 0 \leq t \leq T,\end{aligned}$$

where $\{\alpha_t;\ 0 \leq t \leq T\}$ is defined by

$$\alpha_t = \begin{cases} (f^1(t,y^1,z^1) - f^1(t,y^2,z^2))(\delta(Y_t))^{-1} & \text{if } Y_t^1 \neq Y_t^2, \\ 0, & \text{if } Y_t^1 = Y_t^2. \end{cases}$$

and the $\mathbb{R}^n$ valued process $\{\beta_t,\ 0 \leq t \leq T\}$ as follows. For $1 \leq i \leq n$, let $Z_t^{(i)}$ denote the n-dimensional vector whose components are equal to those of Z_t^2, and whose $n-i$ last components are equal to those of Z_t^1. With this notation, we define for each $1 \leq i \leq n$,

$$\beta_t^i = \begin{cases} (f^1(t,Y_t^2,Z_t^{(i)}) - f^1(t,Y_t^2,Z_t^{i-1})(\delta(Z_t^i))^{-1} \text{ if } Z_t^{1i} \neq Z_t^{2i}, \\ 0, \text{if } Z_t^{1i} = Z_t^{2i}. \end{cases}$$

Since f is a Lipschitz function, α and β are bounded processes, for $0 \leq s \leq t \leq T$, let

$$\Gamma_{s,t} = \exp[\int_s^t \langle \beta_r, dW_r\rangle + \int_s^t (\alpha_r - \frac{|\beta_r|^2}{2})dr].$$

We have for $0 \leq s \leq t \leq T$,

$$\delta(Y_s) = \Gamma_{s,t}\delta(Y_t) + \int_s^t \Gamma_{s,r}(f^1(r,Y_r^2,Z_r^2) - f^2(r,Y_r^2,Z_r^2))dr - \int_s^t \Gamma_{s,r}(\delta(Z_r) + \beta_r\delta(Y_r))dW_r.$$

Hence

$$\delta(Y_s) = E(\Gamma_{s,t}\delta(Y_t) + \int_s^t \Gamma_{s,r}(f^1(r,Y_r^2,Z_r^2) - f^2(r,Y_r^2,Z_r^2))dr/\mathfrak{F}_s)$$

The result follows from this formula and the negativity of $\delta(\xi)$ and $(f^1(r,Y_r^2,Z_r^2) - f^2(r,Y_r^2,Z_r^2))$. $\square$

1.4 BSDE with continuous coefficient

Now we prove the existence of a solution for one dimensional backward stochastic differential equations where the coefficient is continuous, it has a linear growth , and the terminal condition is squared integrable. we also obtain the existence of a minimal solution. First let us consider the following assumption :

(H1.3) (i) The function f is $\mathbb{R}$-valued.

(ii) there exists a constant $C \geq 0$ such that $P-a.s.$, $|f(t,y,z) \leq C(1+|y|+|z|)$ for any $(t,y,z) \in [0,T] \times \mathbb{R}^{1+d}$

(iii) $P-a.s.$ for any $t \in [0,T]$, the function which with $(y,z) \to (t,y,z)$ is continuous.

Then we have the following result:

Theorem 1.4.1. ***(Lepeltier-San Martin [27]).*** *The BSDE associated with (f, ξ) has a maximal solution (Y, Z), i.e., we have*

$$\begin{cases} Y \in \mathcal{S}^2, \;\; Z \in \mathbb{H}^2(\mathbb{R}^d), \\ Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{1.5}$$

and if (Y', Z') is another solution for (1.5) then $P - a.s.,\; Y \geq Y'$.

The idea of the proof is to give an approximation of the coefficient f by a Lipschitz sequence of functions f_n, and establish that the limit of the sequence (Y^n, Z^n) of corresponding solutions for BSDE (ξ, f_n) is a solution of BSDE with parameter (ξ, f) .

Lemma 1.4.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with linear growth, that is there exists a constant $K < \infty$ such that $\forall x \in \mathbb{R}^p$ $|g(x)| \leq K(1 + |x|)$. Then the sequence of functions*

$$\{g_n)_{(n \geq 1)}(x) = \inf_{y \in \mathbb{Q}} \{g(y) + n|x - y|\}$$

is well defined for $n \geq K$ and it satisfies :

(i) linear growth :$\forall x \in \mathbb{R},\; |g_n(x)| \leq K(1 + |x|)$;

(ii) monotonicity in n:$\forall x \in \mathbb{R}$, $g_n(x) \uparrow$;

(iii) Lipschitz condition : $\forall\; x\;, y \in \mathbb{R},\; |g_n(x) - g_n(y)| \leq n|x - y|$,

(iv) Strong convergence : if x_n which converge to $x \in \mathbb{R}$ we have,
$\lim_{n\to\infty} g_n(x_n) = g(x),\; ds\; a.e,$

Proof. By the linear growth hypothesis on g, g_n is well defined for $n \geq K$. Also it follows at once that $g_n \leq g$. Again, from the linear growth condition on g, we obtain

$$g_n(x) \geq \inf_{y \in \mathbb{Q}} -K - K|y| + K|x - y| = -K(1 + |x|)$$

from which (i) holds. Property (ii) is evident from the definition of the sequence (g_n). Take $\varepsilon > 0$ and consider $y_\varepsilon \in \mathbb{Q}$ such that

$$\begin{aligned} g_n(x) &\geq g(y_\varepsilon) + n|x - y_\varepsilon| - \varepsilon \\ &\geq g(y_\varepsilon) + n|y - y_\varepsilon| + n|x - y_\varepsilon| - n|y - y_\varepsilon| - \varepsilon \\ &\geq g(y_\varepsilon) + n|y - y_\varepsilon| - n|x - y_\varepsilon| - \varepsilon \\ &\geq g(y_\varepsilon) - n|x - y| - \varepsilon. \end{aligned}$$

Therefore, interchanging the roles of x and y, and since ε is arbitrary we deduce that

$$|g_n(x) - g_n(y)| \le n|x-y|.$$

In order to prove (iv), consider $\lim_{n\to\infty} x_n = x$. Take for every n, $y_n \in \mathbb{Q}$ such that $g(x_n) \ge g_n(x_n) \ge g(y_n) + n|x_n - y_n| - \frac{1}{n}$. Since $(g(x_n))$ is bounded and g has linear growth, we deduce that (y_n) is bounded, and so is $(g(y_n))$. Consequently, $\limsup n|y_n - x_n| < \infty$, and in particular $y_n \to x$. Moreover

$$g(x_n) \ge g_n(x_n) \ge g(y_n) - \frac{1}{n}.$$

from which the result follows. □

Proof. **(of theorem 1.4.1)** For $n \ge 0$, let (Y^n, Z^n) be the solution of the BSDE associated with (f_n, ξ). On the other hand let $(\widetilde{Y}, \widetilde{Z})$ be the solution of the BSDE associated with $(-C(1+|y|+|z|), \xi)$. The comparison theorem (1.3.1), implies that for any $n \ge 0$, $Y^n \ge Y^{n+1} \ge \widetilde{Y}$. It follows that $P-a.s.$ for any $t \le T$, $Y_t^n \to Y_t$ and the sequence $(Y^n)_{n\ge 0}$ converge in $\mathbb{H}^2(\mathbb{R})$ to a process Y which also upper semi-continuous.
Now by Itô's formula with $(Y^n)^2$, using the inequality $|ab| \le \varepsilon|a|^2 + \frac{1}{\varepsilon}|b|^2$ for any $\varepsilon > 0$ and $a, b \in \mathbb{R}$ and since $|f_n(t,y,z)| \le C(1+|y|+|z|)$ we deduce that the sequence $(Z^n)_{n\ge 0}$ is uniformly bounded in $\mathbb{H}^2(\mathbb{R})$.
Next let $n, m \ge 0$. Then using Itô's formula yields:

$$\begin{aligned}(Y_t^n - Y_t^m)^2 + \int_t^T |Z_s^n - Z_s^m|^2 ds = 2\int_t^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds \\ - 2\int_t^T (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)dW_s, \quad t \le T.\end{aligned}$$

Then the sequence $(Z^n)_{n\ge 0}$ is of Cauchy type in $\mathbb{H}^2(\mathbb{R}^d)$ and converges to a process $(Z_t)_{t\le}$ which belongs to $\mathbb{H}^2(\mathbb{R}^d)$. Now the pair of processes satisfies,

$$\begin{aligned}Y_t^n &= \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s \\ &= Y_0^n - \int_0^t f_n(s, Y_s^n, Z_s^n) + \int_0^t Z_s^n dW_s, \ t \le T.\end{aligned}$$

But for any stopping time ν we have $\lim_{n\to\infty} E(|Y_\nu^n - Y_\nu|) = 0$, $\lim_{n\to\infty} E(|\int_0^\nu (Z_s^n - Z_s)dW_s|) = 0$ through Bulkhoder-Davis-Gundy inequality. Finally

$$\begin{aligned}E(|\int_0^\nu (f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s))ds|) \le E(\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|1_{[|Y_s^n|+|Z_s^n|\le k]}ds) \\ + E(\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|1_{[|Y_s^n|+|Z_s^n|>k]}ds)\end{aligned}$$

But there exists a subsequence which we still denote by n such that the first term in the rights converges to 0 as $n \to \infty$ since $P-a.s.$ and for any $t \leq T$, $(f_n(t, y, z))_{n \geq 0}$ converges uniformly to $f(t, y, z)$ on compact subsets of $\mathbb{R}^{1+d}$. The second term is majorities by a constant δ_k which converges to 0 as $k \to \infty$. Henceforth the left converges to 0 as $n \to \infty$. It follows that for any ν a stopping time we have :

$$Y_\nu = Y_0 - \int_0^\nu f(s, Y_s, Z_s)ds + \int_0^\nu Z_s dW_s.$$

Finally the optional section theorem (see [10], page220) implies that (Y, Z) is a solution for (1.5). Now if (Y', Z') is another solution for (1.5), then comparison theorem implies that $Y^n \geq Y'$ since $f_n \geq f$. Therefore taking the limit we obtain $Y \geq Y'$ and then Y is maximal. □

Remark 1.4.3. *In the same way as previously has we used an increasing approximation of f we would have constructed the minimal solution of (1.5).*

Chapter 2

Reflected Backward Stochastic Differential Equations

We study reflected solutions of one-dimensional backward stochastic differential equations. We prove uniqueness and existence by approximation via penalization, we show that when the coefficient has Lipschitz, and we prove the existence of a solution of RBSDE with continuous and linear growth coefficient.

2.1 Introduction

In this section, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. An increasing process is introduced which pushes the solution upwards, so that it may remain above the obstacle. The existence is established via approximation is constructed by penalization of the constraint, we prove also a comparison theorem, similar to that in [14], for non-reflected BSDE's. Finally we prove the existence of a reflected solution of one-dimensional backward stochastic differential equations with continuous and linear growth coefficient.

2.2 Reflected BSDE's with Lipschitz coefficient

Along with this section the dimension is equal to 1. So we are going to deal with solutions of BSDE's whose components Y are forced to stay above a given barrier.
Let $\{B_t, 0 \leq t \leq T\}$ be a d-dimensional brownian motion defined on a probability space

$(\Omega, \mathfrak{F}, P)$. Let $\{\mathfrak{F}_t,\ 0 \le t \le T\}$ be the natural filtration of $\{B_t\}$, where $\mathfrak{F}_0$ contains all $P-$null sets of $\mathfrak{F}$ and let $\mathfrak{P}$ be the $\sigma-$algebra of predictable subsets of $\Omega \times [0, T]$.
Let us introduce some notation.

$$\mathbb{L}^2 = \{\xi \texttt{ is an } \mathfrak{F}_T - \texttt{measurable random variable s.t. } E(|\xi|^2) < \infty\}$$

$$\mathbb{H}^2 = \{\{\varphi_t, 0 \le t \le T\} \texttt{ is a predictable process s.t. } E(\int_0^T |\varphi_t|^2 dt) < \infty\}$$

$$\mathcal{S}^2 = \{\{\varphi_t, 0 \le t \le T\} \texttt{ is a predictable process s.t. } E(\sup_{0 \le t \le T} |\varphi_t|^2) < \infty\}$$

and we define :
(I) a terminal value $\xi \in \mathbb{L}^2$.
(II) a coefficient f, which is a map : $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$, $f(., y, z) \in \mathbb{H}^2$,
(III) for some $K > 0$ and all $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d, a.s.$

$$|f(t, y, z) - f(t, y', z')| \le K(|y - y'| + |z - z'|)$$

(IV) An obstacle $\{S_t, 0 \le t \le T\}$, which is a continuous progressively measurable real-valued process satisfying : $E(\sup_{0 \le t \le T}(S_t^+)^2) < \infty$.
We shall always assume that $S_T \le \xi$ $a.s.$.

Definition 2.2.1. *The solution of RBSDE is a triple* $\{(Y_t, Z_t, K_t),\ 0 \le t \le T\}$ *of* $\mathfrak{F}_t$ *progressively measurable processes taking values in* $\mathbb{R}, \mathbb{R}^d$ *and* $\mathbb{R}_+$*, respectively, and satisfying:*
(V) $Z \in \mathbb{H}^2$*, in particular* $E\int_0^T |Z_t|^2 dt < \infty$
(V') $Y \in \mathcal{S}^2$ *and* $K_T \in \mathbb{L}^2$
(VI) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s,\ 0 \le t \le T$
(VII) $Y_t \ge S_t,\ 0 \le t \le T$
(VIII) $\{K_t\}$ *is continuous and increasing,* $K_0 = 0$ *and* $\int_0^T (Y_t - S_t) dK_t = 0$.

Our main result in this section is

Theorem 2.2.2. *(**EL karoui et al.[15]**) Under the above assumptions, in particular (I), (II), (III) and (IV), the RBSDE with (V), (VI), (VII), (VIII) has a unique solution* (Y, Z, K).

Our prove based on approximation via penalization. In the following c will denote a constant whose value can vary from line to line.

Proof. For each $n \in \mathbb{N}$, let $\{(Y^n, Z^n); 0 \le t \le T\}$ denote the unique pair of $\mathfrak{F}_t$ progressively measurable processes with values in $\mathbb{R} \times \mathbb{R}^d$ satisfying $E\int_0^T |Z_t^n|^2 dt < \infty$ and

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s \tag{2.1}$$

where ξ and f satisfy the above assumptions. We define

$$K_t^n = n\int_0^t (Y_s^n - S_s)^- ds,\ 0 \le t \le T.$$

We now establish a priori estimates, uniform in n, on the sequence (Y^n, Z^n, K^n). Indeed using Itô's formula with $(Y^n)^2$ and taking expectation yields :

$$\begin{aligned} E|Y_t^n|^2 + E\int_t^T |Z_s^n|^2 ds &= E|\xi|^2 + 2E\int_t^T f(s, Y_s^n, Z_s^n) Y_s^n ds + 2E\int_t^T Y_s^n dK_s^n \\ &\le E|\xi|^2 + 2E\int_t^T (f(s,0,0) + K|Y_s^n| + K|Z_s^n|)|Y_s^n| ds + 2E\int_t^T S_s dK_s^n \\ &\le c(1 + E\int_t^T |Y_s^n|^2 ds) + \frac{1}{3}E\int_t^T |Z_s^n|^2 ds \\ &+ \frac{1}{\alpha}E(\sup_{0\le t\le T}(S_t^+)^2) + \alpha E(K_T^n - K_t^n)^2, \end{aligned}$$

but for any $t \le T$ we have,

$$K_T^n - K_t^n = Y_t^n - \xi - \int_t^T f(s, Y_s^n, Z_s^n)ds + \int_t^T Z_s^n dB_s.$$

Hence

$$E|(K_T^n - K_t^n)^2| \le c\{E(|Y_t^n|^2) + E|\xi|^2 + 1 + E(\int_t^T (|Y_s^n|^2 + |Z_s^n|^2)ds)\},$$

choosing $\alpha = (1/3c)$, we have

$$\frac{2}{3}E(|Y_t^n|^2) + \frac{1}{3}E\int_t^T |Z_s^n|^2 ds \le c(1 + E\int_t^T |Y_s^n|^2 ds).$$

From Gronwall's lemma it follows that

$$\sup_{0\le t\le T} E(|Y_t^n|^2) + E\int_t^T |Z_s^n|^2 ds + E|(K_T^n)^2| \le c,\ \forall n \in \mathbb{N}.$$

Using again equation (2.1) and the Burkholder-Davis-Gundy inequality, we deduce that

$$E(\sup_{0\le t\le T}|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + |(K_T^n)^2|) \le c,\ \forall n \in \mathbb{N}. \tag{2.2}$$

We define

$$f_n(t, y, z) = f(t, y, z) + n(y - S_t)^-,$$

$$f_n(t, y, z) \leq f_{n+1}(t, y, z),$$

and it follows from the comparison theorem (for nonreflected BSDE's see ([34])) that $Y_t^n \leq Y_t^{n+1},\ 0 \leq t \leq T,\ a.s.$ Hence $Y_t^n \uparrow Y_t,\ 0 \leq t \leq T,\ a.s.$
and from (2.2) and Fatou's lemma,

$$E(\sup_{0 \leq t \leq T} Y_t^2) \leq c.$$

It then follows by dominated convergence that

$$E \int_0^T (Y_t - Y_t^n)^2 dt \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Now we prove $\lim_{n\to\infty} E(\sup_{t \leq T} |(Y_t^n - S_t)^-|^2 = 0$, this property is the key point in the proof of our result. Let $(\widetilde{Y}_s^n, \widetilde{Z}_s^n)$ be the solution of the following BSDE :

$$\widetilde{Y}_t^n = \xi + \int_t^T (f(s, Y_s^n, Z_s^n) - n(\widetilde{Y}_s^n - S_s))ds - \int_t^T \widetilde{Z}_s^n dB_s$$

By comparison we have $P - a.s.\ \forall t \leq T,\ Y_t^n \geq \widetilde{Y}_t^n$, for any $n \geq 0$. Now let ν be an $\mathfrak{F}_t-$stopping time such that $\nu \leq T$. Then,

$$\widetilde{Y}_\nu^n = E[\xi \exp(-n(T - \nu)) + \int_\nu^T \{f(s, Y_s^n, Z_s^n) + nS_s\} \exp(-n(s - \nu))ds/\mathfrak{F}_\nu\}]$$

Since S is continuous then $\xi \exp(-n(T - \nu)) + \int_\nu^T nS_s \exp(-n(s - \nu))ds \to \xi 1_{\{\nu=T\}} + S_\nu 1_{\{\nu<T\}}\ P - a.s.$ and in mean square. On the other hand

$$|\int_\nu^T \exp(-n(s - \nu))f(s, Y_s^n, Z_s^n)ds| \leq \frac{1}{\sqrt{2n}}\{\int_0^T (f(s, Y_s^n, Z_s^n))^2 ds\}^{1/2}.$$

Therefore,

$$\widetilde{Y}_\nu^n \to \xi 1_{\{\nu=T\}} + S_\nu 1_{\{\nu<T\}} \in \mathbb{L}^2 \text{ as } n \to \infty$$

and then $Y_\nu \geq S_\nu\ a.s.$ From that and the section theorem (see [10]), it follows that $P - a.s.,\ \forall t \in [0, T],\ Y_t \geq S_t$ and then $(Y_t^n - S_t)^- \searrow 0,\ t \leq T,\ P - a.s.$ Now Dini's theorem implies that $\sup_{t \leq T}(Y_t^n - S_t)^- \to 0$ as $n \to \infty$. Finally the conclusion stems from the dominated convergence theorem since for any $n \geq 0,\ Y_t^1 - S_t^+ \leq Y_t^n - S_t$ and

then $(Y_t^n - S_t)^- \leq |Y_t^1| + S_t^+$.
Now it follows from Itô's formula that for any $m \geq n \geq 0$

$$\begin{aligned}
E(|Y_t^n - Y_t^m|^2) + E\int_t^T |Z_s^n - Z_s^m|^2 ds &= 2E\int_t^T [f(s, Y_s^n, Z_s^m) - f(s, Y_s^m, Z_s^m)](Y_s^n - Y_s^m)ds \\
&\quad + 2E\int_t^T (Y_s^n - Y_s^m)d(K_s^n - K_s^m) \\
&\leq 2KE\int_t^T (|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m||Z_s^n - Z_s^m|)ds \\
&\quad + 2E\int_t^T (Y_s^n - S_s)^- dK_s^m + 2E\int_t^T (Y_s^m - S_s)^- dK_s^n
\end{aligned} \tag{2.4}$$

Since $m \geq n$, then $\int_t^T (Y_s^n - Y_s^m)(dK_s^n - dK_s^m) \leq \sup_{t\leq T}(Y_s^n - Y_s^m)^- K_T^m$, and using the estimate (2.2) and the fact $\lim_{n\to\infty} E(\sup_{t\leq T} |(Y_t^n - S_t)^-|^2 = 0$ yield

$$E\int_t^T |Z_s^n - Z_s^m|^2 ds \leq cE\int_t^T |Y_s^n - Y_s^m|^2 ds + 2E\sup_{t\leq T}(Y_s^n - S_s)^- K_T^m \to 0 \texttt{ as } n \to \infty$$

henceforth there exists a process $(Z_t)_{t\leq T}$ which belong of $\mathbb{H}^2(\mathbb{R}^d)$ and which is the $\mathbb{H}^2(\mathbb{R}^d)-$limit of $(Z^n)_n$.
Next going back to (2.4), taking the supermum and using Burkholder-Davis-Gundy inequality to obtain

$$\begin{aligned}
E\sup_{t\leq s\leq T}(Y_s^n - Y_s^m)^2 + E\int_t^T |Z_s^n - Z_s^m|^2 ds &\leq 2E[\sup_{t\leq s\leq T}(Y_s^n - S_s)^- K_T^m] + \varepsilon E\sup_{t\leq s\leq T}|Y_s^n - Y_s^m|^2 \\
&\quad + \frac{1}{\varepsilon}E\int_t^T |Z_s^n - Z_s^m|^2 ds + cE\int_t^T |Y_s^n - Y_s^m|^2 ds
\end{aligned}$$

where $\varepsilon > 0$. We chose $\varepsilon < \frac{1}{2}$ implies that $E\sup_{0\leq t\leq T}(Y_t^n - Y_t^m) \to 0$ as $n, m \to \infty$ and then $E\sup_{0\leq t\leq T}(Y_t^n - Y_t) \to 0$ as $n \to \infty$, moreover $Y = (Y_t)_{t\leq T}$ is a continuous process.
Now since for any $n \geq 0$ and $t \leq T$

$$K_t^n = Y_0^n - Y_t^n - \int_0^t f(s, Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s$$

then we have also, $E\sup_{0\leq t\leq T}|K_s^n - K_s^m|^2$ as $n, m \to \infty$. Hence there exists an $\mathfrak{F}_t-$adapted non-decreasing and continuous process $(K_t)_{t\leq T}$, $K_0 = 0$ such that $E\sup_{0\leq s\leq T}|K_s^n - K_s|^2 \to 0 \texttt{ as } n \to \infty$.
Finally we prove the limiting process $(Y, Z, K) = (Y_t, Z_t, K_t)_{t\leq T}$ is the solution of the reflected BSDE associated with (f, ξ, S).
Obviously the processes $(Y_t, Z_t, K_t)_{t\leq T}$ satisfy :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad \forall t \leq T.$$

On the other hand since $\lim_{n\to\infty} E[\sup_{t\le T}((Y_t - S_t)^-)^2] = 0$ then $P-a.s.$, $\forall t \le T,\ Y_t \ge S_t$. We have also $\int_0^T (Y_t - S_t)dK_t = 0$ since the sequences $(Y^n)_{n\ge 0}$ and $(K^n)_{n\ge 0}$ converge uniformly respectively to Y and K and thanks to

$$\int_0^T (Y_s^n - S_s)dK_s^n = -n\int_0^T ((Y_s^n - S_s)^-)^2 ds \le 0.$$

□

2.3 Comparison theorem for RBSDE

We prove a comparison theorem, similar to that of [34] for non-reflected BSDE's.

Theorem 2.3.1. *Let (ξ, f, S) and $(\xi^{'}, f', S')$ be two Reflected BSDE's, each one satisfying all the assumptions (I), (II), (III) and (IV), and suppose in addition the following :*
1) $\xi \le \xi^{'}$ a.s.
2) $f(t,y,z) \le f'(t,y,z)$ $dP\times dta.e.,\ \forall (y,z)\in \mathbb{R}\times\mathbb{R}^d$,
3) $S_t \le S_t^{'},\ 0\le t\le T,$ a.s.
Let (Y,Z,K) be a solution of the RBSDE (ξ, f, S) and (Y',Z',K') a solution of the RBSDE $(\xi^{'}, f', S')$. Then $Y_t \le Y_t',\ 0\le t\le T$ a.s.

Proof. Applying Itô's formula to $|(Y_t - Y_t')^+|^2$, and taking expectation, we get

$$\begin{aligned} E|(Y_t - Y_t')^+|^2 + E\int_t^T 1_{\{Y_s > Y_s'\}}|Z_s - Z_s'|^2 ds &\le 2E\int_t^T (Y_s - Y_s')^+[f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')]ds \\ &\quad + 2E\int_t^T (Y_s - Y_s')^+(dK_s - dK_s'). \end{aligned}$$

Since on $\{Y_t > Y_t'\},\ Y_t > S_t' \le S_t$, we have

$$\int_t^T (Y_s - Y_s')^+(dK_s - dK_s') = -\int_t^T (Y_s - Y_s')^+ dK_s'.$$

Assume now that the Lipschitz condition to f. Then

$$\begin{aligned} E|(Y_t - Y_t')^+|^2 &+ E\int_t^T 1_{\{Y_s > Y_s'\}}|Z_s - Z_s'|^2 ds \\ &\le 2E\int_t^T (Y_s - Y_s')^+[f(s,Y_s,Z_s) - f(s,Y_s',Z_s')]ds \\ &\le 2KE\int_t^T (Y_s - Y_s')^+(|(Y_s - Y_s')| + |Z_s - Z_s'|)ds \\ &\le E\int_t^T 1_{\{Y_s > Y_s'\}}|Z_s - Z_s'|^2 ds + cE\int_t^T |(Y_s - Y_s')^+|^2 ds. \end{aligned}$$

Hence

$$E|(Y_t - Y'_t)^+|^2 \leq cE \int_t^T |(Y_s - Y'_s)^+|^2 ds.$$

and from Gronwall's lemma, $(Y_t - Y'_t)^+ = 0,\ 0 \leq t \leq T.$ $\square$

2.4 Reflected BSDE with continuous coefficient

We are given three objects :
1) a terminal value $\xi \in \mathbb{L}^2(\Omega, \mathfrak{F}_T, P)$.
2) a coefficient f which is a map

$$f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

such that :
i) $\forall (y,z) \in \mathbb{R} \times \mathbb{R}^d,\ (t,\omega) \mapsto f(t,\omega,y,z)$ is $\mathcal{P}-$measurable.
ii) $P-a.s.\ \forall t \in [0,T],\ f(t,\omega,y,z)$ is continuous in (y,z) on $\mathbb{R}\times\mathbb{R}^d$. Moreover there exists a constant $C > 0$ such that for any $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d,\ |f(t,\omega,y,z)| \leq C(1+|y|+|z|)$
3) An obstacle $\{S_t,\ 0 \leq t \leq T\}$, which is continuous and $\mathfrak{F}_t-$ progressively measurable process satisfying :

$$E(\sup_{0\leq t\leq T}(S_t^-)^2) < \infty.$$

We shall always assume that $S_T \leq \xi\ a.s.$

Theorem 2.4.1. ***(A.Matoussi)[29].*** *Let (ξ, f, S) be a triple satisfying the above assumptions 1)-3). Then there exists a $\mathfrak{F}_t-$progressively measurable triple$\{(Y_t, Z_t, K_t),\ 0 \leq t \leq T\}$ solution of the reflected BSDE :*

$$Y_t = \xi + \int_t^T f(s,y,z)ds + K_T - K_t - \int_t^T Z_s dB_s;\ 0 \leq t \leq T. \tag{2.5}$$

such that :
i') $E\int_0^T (Y_t^2 + |Z_t|^2)dt < \infty$,
ii') $Y_t \geq S_t,\ 0 \leq t \leq T$,
iii') $\{K_t,\ 0 \leq t \leq T\}$ is a continuous and increasing process, $K_0 = 0$, and

$$\int_0^T (Y_t - S_t)dK_t = 0.$$

To prove theorem (2.4.1), we need an important result which gives an approximation of continuous functions by Lipschitz function (see [27]) $f_n(x) = \inf_{y\in\mathbb{Q}}\{f(y) + n|x-y|\}$.

Proof. Consider, f_n define below. Then f_n is a measurable function as well as a Lipschitz function. Moreover, since $\xi \in \mathbb{L}^2$ and $\{S_t,\ 0 \leq t \leq T\}$ satisfy 3), we get from El karoui et al. [15] that there is a unique solution $(Y_t^n, Z_t^n, K_t^n),\ 0 \leq t \leq T$ for RBSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s \tag{2.6}$$

satisfying equation 2.6 and:

$$\begin{aligned}
&E\int_0^T (|Y_t^n|^2 + |Z_t^n|^2)dt < \infty,\\
&Y_t^n \geq S_t,\ 0 \leq t \leq T,\\
&\{K_t^n,\ 0 \leq t \leq T\} \texttt{ is a continuous and increasing process, } K_0 = 0\\
&\texttt{and } \int_0^T (Y_t^n - S_t)dK_t^n = 0.
\end{aligned}$$

Using the comparison theorem of RBSDE's in El karoui et al. (1996)[15], we obtain

$$\forall n \geq m \geq K,\ Y^n \geq Y^m,\ dt \otimes dP - a.s. \tag{2.7}$$

The idea of the proof of theorem (2.4.1) is to establish that the limit of the sequence (Y^n, Z^n, K^n) is a solution of the RBSDE (2.6) with parameters (f, ξ, S).
From now on the proof will be divided into four steps.
Step 1: There exists a constant C, such that $\forall n \geq K,\ E(\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + (K_T^n)^2) \leq C$.
$C > 0$ denote a constant, whose value may vary from line to line. From Itô's formula applied to $(Y_t^n)^2$, it follows that

$$(Y_t^n)^2 + \int_t^T |Z_s^n|^2 = \xi^2 + 2\int_t^T f_n(s, Y_s^n, Z_s^n)Y_s^n ds + 2\int_t^T Y_s^n dK_s^n - 2\int_t^T Y_s^n Z_s^n dB_s,$$

Taking expectation, and using the fact that $\int_0^T Y_t^n Z_t^n dB_t$ is uniformly integrable(see [15]) and used the identity $\int_0^T (Y_t^n - S_t)dK_t^n = 0$, we deduce

$$E|Y_t^n|^2 + E\int_t^T |Z_s^n|^2 = E\xi^2 + 2E\int_t^T f_n(s, Y_s^n, Z_s^n)Y_s^n ds + 2E\int_t^T S_s dK_s^n.$$

using the uniform linear growth of f_n and the inequality $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2,\ \forall \varepsilon > 0,$

$$\begin{aligned}
E|Y_t^n|^2 + E\int_t^T |Z_s^n|^2 \leq C(1 + E\int_t^T |Y_s^n|^2 ds) + \frac{1}{3}E\int_t^T |Z_s^n|^2 ds\\
+ \frac{1}{\varepsilon}E(\sup_{0\leq t\leq T} (S_s^+)^2) + \varepsilon E((K_T^n - K_t^n)^2).
\end{aligned}$$

Now from (2.6), we get

$$E((K_T^n - K_t^n)^2) \leq C(E(Y_t^n)^2 + E\xi^2 + 1 + E\int_t^T ((Y_s^n)^2 + |Z_s^n|^2)ds).$$

Choosing $\alpha = \frac{1}{3C}$, we obtain

$$\frac{2}{3}E(Y_t^n)^2 + \frac{1}{3}E\int_t^T |Z_s^n|^2 \leq C(1 + E\int_t^T (Y_s^n)^2 ds), \;\; E(Y_t^n)^2 \leq C(1 + E\int_t^T (Y_s^n)^2 ds). \tag{2.8}$$

It then follows from Gronwall's lemma that : $\sup_{0\leq t\leq T} E|Y_t^n|^2 \leq C$, and from (2.8) and the last inequality we get

$$E\int_0^T |Z_s^n|^2 ds \leq C, \;\; E(K_T^n)^2 \leq C.$$

The result of step 1 then from equation (2.6), the above estimates and the Burkholder-Davis-Gundy inequality.

Step 2. We should prove that the sequence of processes Z^n converge in $\mathbb{H}^2(\mathbb{R})$.

We have from (2.7) and the result of step 1. $Y_t^n \leq Y_t^{n+1}, \;\; 0 \leq t \leq T, \; P-a.s.$ and $E\sup_{0\leq t\leq T}(|Y_t^n|^2) \leq C$. Hence $Y_t^n \uparrow Y_t, \; 0 \leq t \leq T, \; P-a.s.$, and from Fatou's lemma, we have $E(\sup_{0\leq t\leq T}|Y_t|^2) \leq C$.

It then follows by the dominated convergence theorem that

$$E\int_0^T |Y_t - Y_t^n|^2 dt \to 0 \;\; \texttt{as} \; n \to \infty. \tag{2.9}$$

For all $n \geq p \geq n_0 \geq K$, from Itô's formula for $t = 0$, and using the fact $Y_t^n \geq S_t$, we obtain

$$\begin{aligned} E|Y_0^n - Y_0^p|^2 + E\int_0^T |Z_t^n - Z_t^p|^2 dt &\leq 2E\int_0^T (Y_t^n - Y_t^p)(f_n(t, Y_t^n, Z_t^n) - f_p(t, Y_t^p, Z_t^p))dt \\ &\quad + 2E\int_0^T (Y_t^n - S_t)dK_t^n + 2E\int_0^T (Y_t^p - S_t)dK_t^p, \end{aligned}$$

From the identity $\int_0^T (Y_t^n - S_t)dK_t^n = 0$, and using the Hölder inequality, we have

$$E\int_0^T |Z_t^n - Z_t^p|^2 dt \leq 2(E\int_0^T |Y_t^n - Y_t^p|^2 dt)^{1/2}(E\int_0^T |f_n(t, Y_t^n, Z_t^n) - f_p(t, Y_t^p, Z_t^p)|^2 dt)^{1/2},$$

Now, using the uniform linear growth on the sequence $(f_n)_n$ and the fact $\|(Y^n, Z^n)\|$ is bounded, we obtain the existence of a constant C depending only on $K, \; T, \; E\xi^2$ and $E(\sup_{0\leq t\leq T}(S_t^+)^2)$ such that

$$\forall n, p \geq n_0, \;\; \|Z^n - Z^p\| \leq C\|Y^n - Y^p\|^{1/2}.$$

Then from (2.9), (Z^n) is a Cauchy sequence in $\mathbb{H}^2(\mathbb{R}^d)$, and there exists a $\mathfrak{F}_t-$progressively measurable process Z such that $Z^n \to Z$ in $\mathbb{H}^2(\mathbb{R}^d)$, as $n \to \infty$. Hence,

$$E\int_0^T (|Y_t^n - Y_t^p|^2 + |Z_t^n - Z_t^p|^2)dt \to 0 \;\; \texttt{as} \; n,p \to \infty. \tag{2.10}$$

Step 3. We prove that $E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) \to 0 \;\; \texttt{as} \; n,p \to \infty.$
From Itô's formula, we have

$$\begin{aligned}|Y_t^n - Y_t^p|^2 + \int_t^T |Z_s^n - Z_s^p|^2 ds &= 2\int_t^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds \\ &+ 2\int_t^T (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) - 2\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)dB_s.\end{aligned}$$

From the above proof, we have $\forall n \geq p, \; \int_t^T (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) \leq 0$. Then

$$|Y_t^n - Y_t^p|^2 \leq 2\int_t^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds - 2\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)dB_s,$$

from which we deduce

$$\begin{aligned}E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) &\leq 2(E\int_0^T |Y_s^n - Y_s^p|^2 ds)^{1/2}(E\int_0^T f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))^2 ds)^{1/2} \\ &+ 2E(\sup_{0\leq t\leq T}|\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)dB_s|).\end{aligned}$$

Using again the uniform linear growth on the sequence (f_n) and the fact that $\|(Y_n, Z_n)\|$ is bounded, we deduce

$$(E\int_0^T f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))^2 ds)^{1/2} \leq C. \tag{2.11}$$

Afterwards, from the Burkholder-Davis-Gundy inequality, we obtain

$$2E(\sup_{0\leq t\leq T}|\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)dB_s|) \leq \frac{1}{2}E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) + CE\int_0^T |Z_s^n - Z_s^p|^2 ds.$$

Hence, from (2.11) and the above inequality

$$E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) \leq C\left((E\int_0^T |Y_s^n - Y_s^p|^2 ds)^{1/2} + E\int_0^T |Z_s^n - Z_s^p|^2 ds\right).$$

Then from (2.10), we have

$$E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) \to 0, \;\; \texttt{as} \; n,p \to \infty, \tag{2.12}$$

from which we deduce that $P-a.s.$, Y^n converges uniformly in t to Y and that Y is a continuous process.

Step 4. Now according to (2.6), we have for all $n, p \geq n_0 \geq K$,

$$\begin{aligned} E(\sup_{0\leq t\leq T}|K_t^n - K_t^p|^2) \leq E|Y_0^n - Y_0^p|^2 + E(\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2) \\ + CE\int_0^T |f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p)|^2 ds \\ + E(\sup_{0\leq t\leq T}|\int_t^T (Z_s^n - Z_s^p)dB_s|^2). \end{aligned} \tag{2.13}$$

We need to show that the sequence of processes $(f_n(., Y^n, Z^n))_n$ converge to $f(., Y, Z)$ in $\mathbb{H}^2(\mathbb{R})$. This is deduce from the following facts :

a) $Y^n \uparrow Y$ in $\mathbb{H}^2(\mathbb{R})$ and $dt \otimes dP - a.s.$

b) Since $Z^n \to Z$ in $\mathbb{H}^2(\mathbb{R})$ then there exists a process Z' in $\mathbb{H}^2(\mathbb{R}^d)$ and a subsequence such that $\forall n,\ |Z^n| \leq Z',\ Z^n \to Z,\ dt \otimes dP - a.s.$

Therefore, from the lemma (1.4.2) we get $f_n(t, Y_t^n, Z_t^n) \to f(t, Y_t, Z_t),\ dt{-}a.s.$ as $n \to \infty$ and $|f_n(t, Y_t^n, Z_t^n)| \leq K(1 + |Y_t| + |Y_t^{n_0}| + Z_t')$.

Thus, it follows by the dominated convergence theorem that

$$E\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 ds \to 0 \text{ as } \ n \to \infty. \tag{2.14}$$

From Burkholder-Davis-Gundy inequality and (2.12)-(2.14) we obtain

$$E(\sup_{0\leq t\leq T}|K_t^n - K_t^p|^2) \to 0 \text{ as } \ n, p \to \infty.$$

Consequently, there exists a progressively measurable process K with value in $\mathbb{R}$ such that

$$E(\sup_{0\leq t\leq T}|K_t^n - K_t|^2) \to 0 \text{ as } \ n \to \infty, \tag{2.15}$$

and then $\{K_t,\ 0 \leq t \leq T\}$ is clearly an increasing (with $K_0 = 0$) and a continuous process.

Taking limit in the RBSDE (2.6) we obtain that the triple $\{(Y_t, Z_t, K_t),\ 0 \leq t \leq T\}$ is a solution of the RBSDE (2.5). Now from Step 1., we have $E\int_0^T (|Y_t^n|^2 + |Z_t^n|^2)dt \leq C$, taking limit in this inequality, we obtain $i')$ $E\int_0^T (|Y_t|^2 + |Z_t|^2)dt \leq C$.

On the other hand, we have $\forall n \geq K,\ Y_t^n \geq S_t,\ \forall t \in [0, T]$, taking limit we have clearly $ii')$.

From (2.12) and (2.15) we have $\int_0^T (Y_t^n - S_t)dK_t^n \to \int_0^T (Y_t - S_t)dK_t,\ P_a.s.$ as $n \to \infty$, using the identity $\int_0^T (Y_t^n - S_t)dK_t^n = 0$, we obtain

$$\int_0^T (Y_t - S_t)dK_t = 0.$$

□

Part II

BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS.

Chapter 3

Background on backward doubly stochastic differential equations

In this chapter, a new class of backward stochastic differential equations is investigated. which allows us to produce a probabilistic representation of certain quasi-linear stochastic partial differential equation, we prove the existence and uniqueness of a solution where the coefficient is Lipschitz, after we obtain a comparison theorem of these Backward Doubly SDE's. As one of its applications, we prove the existence of a solution for BDSDE's with continuous coefficients.

3.1 Introduction

This new kind of backward SDEs seems to be suitable to give a "probabilistic" representation for a system of parabolic stochastic partial differential equations (SPDE). We refer to Pardoux and Peng (1994)[35] for the link between SPDEs and BDSDEs in the particular case where solutions of SPDEs are regular. In section 1, we study existence and uniqueness of the solution where the coefficient is Lipschitz, in section 2 and 3, we shall prove the comparison theorem of BDSDE's. Then we study BDSDE's with continuous coefficients as an application of the comparison theorem.

3.2 Backward doubly stochastic differential equations with Lipschitz coefficients

3.2.1 Notation and assumptions

Let T be a fixed final time. Throughout this part $\{W_t,\ 0 \leq t \leq T\}$ and $\{B_t,\ 0 \leq t \leq T\}$ will denote two independent d-dimensional Brownian motions ($d \geq 1$), defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \otimes \mathcal{F}_{t,T}^B \vee \mathcal{N},$$

In other words the σ-fields $\mathcal{F}_t$, $0 \leq t \leq T$, are $\mathbb{P}$-complete. We notice that the family of σ-algebras $\mathrm{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is neither increasing nor decreasing; in particular, it is not a filtration. For any $n \geq 1$, we consider the following spaces of processes:

1. The Banach space $\mathcal{M}^2(\mathrm{F}, [0, T]; \mathbb{R}^n)$ of all equivalence classes (with respect to the measure $d\mathbb{P} \times dt$) where each equivalence class contains an n-dimensional jointly measurable random process $\{\varphi_t,\ t \in [0, T]\}$ which satisfies:

   (i) $\mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty$;

   (ii) φ_t is $\mathcal{F}_t$-measurable, for dt-almost all $t \in [0, T]$. Usually an equivalence class will be identified with (one of) its members.

2. $\mathcal{S}^2(\mathrm{F}, [0, T];\ \mathbb{R}^n)$ is the set of continuous n-dimensional random processes which satisfy:

   (i) $\mathbb{E} \sup\limits_{0 \leq t \leq T} |\varphi_t|^2 < \infty$;

   (ii) φ_t is $\mathcal{F}_t$-measurable, for a.e $t \in [0, T]$.

We consider coefficients (f, g) with the following properties:

$$f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \longrightarrow \mathbb{R}^n,$$
$$g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \longrightarrow \mathbb{R}^{n \times d},$$

such that there exist $\mathcal{F}_t$-adapted processes $\{f_t, g_t :\ 0 \leq t \leq T\}$ with values in $[1, +\infty)$ and with the property that for any $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, the following hypotheses are satisfied for some strictly positive finite constant L and $0 < \alpha < 1$ such that for any

$(y_1, z_1),\ (y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d},$:

$$\text{(H3.1)} \qquad \begin{cases} \quad f\,(t,y,z)\ ,\ g\,(t,y,z) \text{ are } \mathcal{F}_t\text{-measurable processes}\,, \\ \text{(i)} \quad |f\,(t,y_1,z_1) - f\,(t,y_2,z_2)\,|^2 \le c\,\left(|y_1-y_2|^2 + \|z_1-z_2\|^2\right), \\ \text{(ii)} \quad |g\,(t,y_1,z_1) - g\,(t,y_2,z_2)\,|^2 \le c\,|y_1-y_2|^2 + \alpha\|z_1-z_2\|^2. \end{cases}$$

We point out that by C we always denote a finite constant whose value may change from one line to the next, and which usually is (strictly) positive.

3.2.2 Existence and uniqueness theorem

Suppose that we are given a terminal condition $\xi \in L^2\,(\Omega, \mathcal{F}_T, \mathbb{P})$

Definition 3.2.1. *By definition a solution to a BDSDE $(\xi, f, g,)$ is a pair $(Y, Z) \in \mathcal{S}^2\,(\mathrm{F}, [0,T];\ \mathbb{R}^n) \times \mathcal{M}^2\left(\mathrm{F}, [0,T];\ \mathbb{R}^{n\times d}\right)$, such that for any $0 \le t \le T$*

$$Y_t = \xi + \int_t^T f\,(s, Y_s, Z_s)\ ds + \int_t^T g\,(s, Y_s, Z_s)\ \overleftarrow{dB_s} - \int_t^T Z_s\, dW_s. \tag{3.1}$$

Here $\overleftarrow{dB_s}$ denotes the classical backward Itô integral with respect to the Brownian motion B.

Our main goal in this section is to prove the following theorem.

Theorem 3.2.2. *Under the above hypothesis (H3.1) there exists a unique solution for the BDSDE (3.1).*

Let us first establish the result in Theorem 3.2.2 for BDSDEs, where the coefficients f, g do not depend on (y, z). More precisely, let f, and $g\ :\ \Omega \times [0,T] \longrightarrow \mathbb{R}^{n\times d}$ satisfy (H3.1), and let ξ be as before. Consider the equation:

$$Y_t = \xi + \int_t^T f\,(s)\, ds + \int_t^T g\,(s)\, \overleftarrow{dB_s} - \int_t^T Z_s dW_s. \tag{3.2}$$

Then we have the following result.

Theorem 3.2.3. *Under the hypothesis (H3.1), there exists a unique solution to equation (3.2).*

Proof. **Existence.** To show the existence, we consider the filtration $\mathcal{G}_t = \mathcal{F}_t^W \otimes \mathcal{F}_T^B$ and the martingale

$$M_t = \mathbb{E}\left[\xi + \int_0^T f\,(s)\ ds + \int_0^T g\,(s)\, \overleftarrow{dB_s}/\mathcal{G}_t\right], \tag{3.3}$$

which is clearly a square integrable martingale by (H3.1). An extension of Itô's martingale representation theorem yields the existence of a $\mathcal{G}_t$-progressively measurable process (Z_t) with values in $\mathbb{R}^{n\times d}$ such that

$$\mathbb{E}\int_0^T \|Z_t\|^2 dt < \infty \quad \text{and} \quad M_T = M_t + \int_t^T Z_s dW_s. \tag{3.4}$$

We subtract the quantity $\int_0^t f(s)\, ds + \int_0^t g(s)\, \overleftarrow{dB_s}$ from both sides of the martingale in (3.3) and we employ the martingale representation in (3.4) to obtain

$$Y_t = \xi + \int_t^T f(s)\, ds + \int_t^T h(s)\, dk_s + \int_t^T g(s)\, \overleftarrow{dB_s} - \int_t^T Z_s dW_s,$$

where

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s)\, ds + \int_t^T g(s)\, \overleftarrow{dB_s}/\mathcal{G}_t\right].$$

It remains to show that (Y_t) and (Z_t) are in fact $\mathcal{F}_t$-adapted. For Y_t, this is obvious since for each t,

$$Y_t = \mathbb{E}(\Theta/\mathcal{F}_t \vee \mathcal{F}_t^B)$$

Where Θ is $\mathcal{F}_t \vee \mathcal{F}_t^B$ measurable. Hence $\mathcal{F}_t^B$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = \mathbb{E}(\Theta/\mathcal{F}_t).$$

Now

$$\int_t^T Z_s dW_s = \xi + \int_t^T f(s)\, ds + \int_t^T g(s)\, \overleftarrow{dB_s} - Y_t,$$

and the right side is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ measurable. Hence, from Itô's martingale representation theorem, $Z_s, t < s < T$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ adapted. Consequently Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable, for any $t < s$, so it is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable.

Uniqueness. Is immediate, since if $(\overline{Y}, \overline{Z})$ is the difference of two solutions,

$$\overline{Y_t} + \int_t^T \overline{Z_s} dW_s = 0, \quad 0 \le t \le T.$$

Hence by orthogonality

$$E(|\overline{Y_t}|^2) + E\int_t^T Tr[\overline{Z_s Z_s^*}]ds = 0,$$

and $\overline{Y_t} \equiv 0$ $Pa.s.$, $\overline{Z_t} = 0$ $dtdPa.e.$ □

We will also need the following Itô-formula.

Lemma 3.2.4. *Let* $\alpha \in \mathcal{S}^2(\mathrm{F}, [0,T]; \mathbb{R}^n)$, $\beta \in \mathcal{M}^2(\mathrm{F}, [0,T]; \mathbb{R}^n)$, $\gamma \in \mathcal{M}^2\left(\mathrm{F}, [0,T]; \mathbb{R}^{n\times d}\right)$, *and* $\delta \in \mathcal{M}^2\left(\mathrm{F}, [0,T];\ \mathbb{R}^{n\times d}\right)$ *be such that:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s \overleftarrow{dB_s} + \int_0^t \delta_s dW_s.$$

Then, for any function $\phi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$

$$\begin{aligned}\phi(\alpha_t) &= \phi(\alpha_0) + \int_0^t \langle \nabla\phi(\alpha_s), \beta_s\rangle ds + \int_0^t \left\langle \nabla\phi(\alpha_s), \gamma_s \overleftarrow{dB_s}\right\rangle \\ &\quad + \int_0^t \langle \nabla\phi(\alpha_s), \delta_s dW_s\rangle - \frac{1}{2}\int_0^t \mathrm{Tr}[\phi''(\alpha_s)\ \gamma_s\,\gamma_s^*]\,ds + \frac{1}{2}\int_0^t \mathrm{Tr}[\phi''(\alpha_s)\ \delta_s\,\delta_s^*]\,ds.\end{aligned}$$

In particular,

$$\begin{aligned}|\alpha|_t^2 = |\alpha_0|^2 &+ 2\int_0^t \langle \alpha_s, \beta_s\rangle ds + 2\int_0^t \left\langle \alpha_s, \gamma_s \overleftarrow{dB_s}\right\rangle + 2\int_0^t \langle \alpha_s, \delta_s dW_s\rangle \\ &- \int_0^t \|\gamma_s\|^2 ds + \int_0^t \|\delta_s\|^2 ds.\end{aligned}$$

Proof. See [35]. □

Next, we establish an a priori estimate for the solution of the BSDE in (3.1). for that sake, we need an additional assumption on g.

$$\text{(H3.2)} \qquad \begin{cases} \texttt{there exists } c \texttt{ such that for all} (t,y,z) \in [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}, \\ gg^*(t,y,z) \le zz^* + c(\|g(t,0,0)\|^2 + |y|^2)I. \end{cases}$$

Proposition 3.2.5. *Assume, in addition to the condition of Theorem(3.2.3), that (H3.2) holds and for some* $p > 2$, $\xi \in (\Omega, \mathfrak{F}, P, \mathbb{R}^k)$ *and*

$$E\int_0^T (|f(t,0,0)|^p + \|g(t,0,0)\|^p)dt < \infty.$$

Then

$$E(\sup_{0\le t\le T} |Y_t|^p + (\int_0^T \|Z_t\|^2)^{p/2}) < \infty.$$

Proof. By lemma 3.2.4 applied to $\varphi(x) = |x|^p$, we obtain that

$$\begin{aligned}&|Y_t|^p + \frac{p}{2}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds + \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}(Z_sZ_s^*Y_s, Y_s)ds \\ &= |\xi|^p + p\int_t^T |Y_s|^{p-2}\langle f(s,Y_s,Z_s), Y_s\rangle ds + p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s,Y_s,Z_s)dB_s\rangle \\ &+ \frac{p}{2}\int_t^T |Y_s|^{p-2}\|g(s,Y_s,Z_s)\|^2 ds \\ &+ \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}\langle gg^*(s,Y_s,Z_s)Y_s, Y_s\rangle ds - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s dW_s\rangle.\end{aligned}$$

Taking the expectation, we get

$$\mathbb{E}(|Y_t|^p) + \frac{p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds + \frac{p}{2}(p-2)\mathbb{E}\int_t^T |Y_s|^{p-4}\langle Z_s Z_s^* Y_s, Y_s\rangle ds$$
$$\leq \mathbb{E}(|\xi|^p) + p\mathbb{E}\int_t^T |Y_s|^{p-2}\langle f\left(s, Y_s, Z_s\right), Y_s\rangle ds + \frac{p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\|g\left(s, Y_s, Z_s\right)\|^2 ds$$
$$+ \frac{p}{2}(p-2)\mathbb{E}\int_t^T |Y_s|^{p-4}\langle gg^*(s, Y_s, Z_s)Y_s, Y_s\rangle ds.$$

We can conclude from (H3.1) that for any $\alpha < \alpha' < 1$, there exists $c(\alpha')$ such that

$$\| g(t,y,z) \| \leq c(\alpha')(| y |^2 + \| g(t,0,0) \|^2) + \alpha' \| z \|^2 .$$

But from (H3.1), (H3.2) and the fact that $2\,ab \leq \dfrac{1-\alpha}{2c}a^2 + \dfrac{2c}{1-\alpha}b^2$, $c > 0$, it follows that there exists a constant $\theta > 0$ and c such that

$$\mathbb{E}(|Y_t|^p) + \theta\mathbb{E}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds$$
$$\leq \mathbb{E}(|\xi|^p) + c\mathbb{E}\int_t^T (|Y_s|^p + |f\left(s,0,0\right)|^p + \|g\left(s,0,0\right)\|^p) ds$$

Then, from Gronwall‘s Lemma we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left(|Y_t|^p + \int_0^T |Y_t|^{p-2}\|Z_t\|^2 dt\right) < \infty$$

Applying the same inequalities we have already used to the first identity of the proof, we deduce that

$$|Y_t|^p \leq |\xi|^p + c\int_t^T (|Y_s|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p) ds$$
$$+ p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s,Y_s,Z_s)dB_s\rangle - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s dW_s\rangle$$

from the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}(sup_{0\leq t\leq T}|Y_t|^p) \leq \mathbb{E}|\xi|^p + c\mathbb{E}\int_0^T (|Y_s|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p) ds$$
$$+ c\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle gg^*(s,Y_s,Z_s)Y_s, Y_s\rangle ds}$$
$$+ c\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle Z_s Z_s^* Y_s, Y_s\rangle ds}$$

We estimate the last term as follows :

$$\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle Z_sZ_s^*Y_s, Y_s\rangle ds} \leq \mathbb{E}(Y_t^{p/2}\sqrt{\int -0^T|Y_t|^{p-2}\|Z_t\|^2dt}$$
$$\leq \frac{1}{3}\mathbb{E}(sup_{0\leq t\leq T}|Y_t|^p) + \frac{1}{4}\mathbb{E}\int_0^T |Y_t|^{p-2}\|Z_t\|^2dt$$

we deduce that

$$\mathbb{E}(sup_{0\leq t\leq T}|Y_t|^p) < \infty$$

Now we have

$$\int_0^T \|Z_t\|^2dt = |\xi|^2 - |Y_0|^2 + 2\int_0^T \langle f(t,Y_t,Z_t),Y_t\rangle dt + 2\int_0^T \langle Y_t, g(t,Y_t,Z_t)dB_t\rangle$$
$$+ \int_0^T \|g(t,Y_t,Z_t)\|^2dt - 2\int_0^T \langle Y_t, Z_tdW_t\rangle$$

Hence for any $\delta > 0$,

$$(\int_0^T \|Z_t\|^2dt)^{p/2} \leq (1+\delta)(\int_0^T \|g(t,Y_t,Z_t)\|^2dt)^{p/2} + c(\delta,p)[|\xi|^p + |Y_0|^p + |\int_0^T \langle f(t,Y_t,Z_t),Y_t\rangle dt|^{p/2}$$
$$+ |\int_0^T \langle Y_t, g(t,Y_t,Z_t)dB_t\rangle|^{p/2} + |\int_0^T \langle Y_t, Z_tdW_t\rangle|^{p/2}]$$

Passing to expectation

$$E(\int_0^T \|Z_t\|^2dt)^{p/2} \leq (1+\delta)^2\alpha E(\int_0^T \|Z_t\|^2dt)^{p/2} + c'(\delta,p)$$
$$+ c(\delta,p)E[\int_0^T \|Z_t\||Y_t|dt|^{p/2}] + c(\delta,p)E[(\int_0^T |Y_t|^2\|Z_t\|^2dt)^{p/4}]$$
$$\leq (1+\delta)^2\alpha E(\int_0^T \|Z_t\|^2dt)^{p/2} + c'(\delta,p)$$
$$+ c(\delta,p)E\{(sup_{0\leq t\leq T}|Y_t|^{p/2})[(\int_0^T \|Z_t\|dt)^{p/2} + (\int_0^T \|Z_t\|^2dt)^{p/4}]\}$$
$$\leq [(1+\delta)^2\alpha + (1+\delta)]E[(\int_0^T \|Z_t\|^2dt)^{p/2}] + c"(\delta,p).$$

The second part of the result now follows, if we choose $\delta > 0$ small enough such that

$$(1+\delta)^2\alpha + (1+\delta) < 1$$

$\square$

We can now turn to the proof of theorem 3.2.2

Proof. 3.2.2 **Uniqueness.**Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be two solutions. Define

$$\overline{Y}_t = Y_t^1 - Y_t^2, \quad \overline{Z}_t = Z_t^1 - Z_t^2, \quad 0 \leq t \leq T$$

Then

$$\overline{Y}_t = \int_t^T [f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)]ds + \int_t^T [g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)]dB_s - \int_t^T \overline{Z}_s dW_s.$$

Applying Itô's formula to $|\overline{Y}_t|^2$ yields :

$$E|\overline{Y}_t|^2 + E\int_t^T \|\overline{Z}_s\|^2 ds = 2E\int_t^T \langle f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2), \overline{Y}_s\rangle ds$$
$$+ E\int_t^T \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2 ds.$$

Hence from (H3.1) and the inequality $ab \leq \frac{1}{2(1-\alpha)}a^2 + \frac{1-\alpha}{2}b^2$,

$$E|\overline{Y}_t|^2 + E\int_t^T \|\overline{Z}_s\|^2 ds \leq c(\alpha)E\int_t^T |\overline{Y}_s|^2 ds + \frac{1-\alpha}{2}E\int_t^T \|\overline{Z}_s\|^2 ds + \alpha E\int_t^T \|\overline{Z}_s\|^2 ds.$$

where $0 < \alpha < 1$ is the constant appearing in (H3.1). Consequently

$$E|\overline{Y}_t|^2 + \frac{1-\alpha}{2}E\int_t^T \|\overline{Z}_s\|^2 ds \leq c(\alpha)E\int_t^T |\overline{Y}_s|^2 ds.$$

From Gronwall's lemma, $E(|\overline{Y}_t|^2) = 0, \quad 0 \leq t \leq T$, and hence $E\int_0^T \|\overline{Z}_s\|^2 = 0$.

Existence. We define recursively a sequence $(Y_t^n, Z_t^n)_{n=0,1,\ldots}$ as follows. Let $Y_t^0 \equiv 0$, $Z_t^0 \equiv 0$. Given (Y_t^n, Z_t^n), (Y_t^{n+1}, Z_t^{n+1}) is the unique solution, constructed as in theorem (3.2.3), of the following equation :

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T Z_s^{n+1} dW_s. \tag{3.5}$$

Let $\overline{Y}_t^{n+1} \triangleq Y_t^{n+1} - Y_t^n$, $\overline{Z}_t^{n+1} \triangleq Z_t^{n+1} - Z_t^n$, $0 \leq t \leq T$. The same computation as in the proof of uniqueness yield :

$$E(|\overline{Y}_t^{n+1}|^2) + E\int_t^T \|\overline{Z}_t^{n+1}\|^2 ds = 2E\int_t^T \langle f(s, Y_s^n, Z_s^n) - f(s, Y_s^{n-1}, Z_s^{n-1}), \overline{Y}_t^{n+1}\rangle$$
$$+ E\int_t^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^{n-1}, Z_s^{n-1})\|^2 ds.$$

Let $\beta \in \mathbb{R}$. By integration by parts, we deduce

$$E(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + \beta E\int_t^T |\overline{Y}_t^{n+1}|^2 e^{\beta s} ds + E\int_t^T \|\overline{Z}_t^{n+1}\|^2 e^{\beta s} ds$$
$$= 2E\int_t^T \langle f(s, Y_s^n, Z_s^n) - f(s, Y_s^{n-1}, Z_s^{n-1}), \overline{Y}_t^{n+1}\rangle e^{\beta s} ds$$
$$+ E\int_t^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^{n-1}, Z_s^{n-1})\|^2 e^{\beta s} ds.$$

There exists $c, \gamma > 0$ such that

$$E(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + (\beta - \gamma) E \int_t^T |\overline{Y}_t^{n+1}|^2 e^{\beta s} ds + E \int_t^T \|\overline{Z}_t^{n+1}\|^2 e^{\beta s} ds$$
$$\leq E \int_t^T (c|\overline{Y}_s^n|^2 + \frac{1+\alpha}{2} \|Z_s^n\|^2) e^{\beta s} ds.$$

Now choose $\beta = \gamma + \frac{2c}{1+\alpha}$, and define $\overline{c} = \frac{2c}{1+\alpha}$,

$$E(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + E \int_t^T (\overline{c}|\overline{Y}_t^{n+1}|^2 + \|\overline{Z}_t^{n+1}\|^2) e^{\beta s} ds$$
$$\leq \frac{1+\alpha}{2} E \int_t^T (\overline{c}|\overline{Y}_s^n|^2 + \|Z_s^n\|^2) e^{\beta s} ds.$$

It follows immediately that

$$E \int_t^T (\overline{c}|\overline{Y}_t^{n+1}|^2 + \|\overline{Z}_t^{n+1}\|^2) e^{\beta s} ds \leq (\frac{1+\alpha}{2})^n E \int_t^T (\overline{c}|\overline{Y}_s^n|^2 + \|Z_s^n\|^2) e^{\beta s} ds.$$

and, since $\frac{1+\alpha}{2} < 1$, $(Y_t^n, Z_t^n)_{n=0,1,\ldots}$ is a Cauchy sequence in $M^2(0,T;\mathbb{R}^k) \times M^2(0,T;\mathbb{R}^{k\times l})$. It is then easy to conclude $(Y_t^n)_{n=0,1,\ldots}$ is also Cauchy in $S^2([0,T];\mathbb{R}^k)$, and that

$$(Y_t, Z_t) = lim_{n\to\infty}(Y_t^n, Z_t^n)$$

solves equation (3.1). $\square$

3.3 Comparison Theorems of Backward doubly stochastic differential equations

In this section, we only consider one-dimensional BDSDEs. We consider the following BDSDEs : $(0 \leq t \leq T)$

$$Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1) ds + \int_t^T g(s, Y_s^1, Z_s^1) dB_s - \int_t^T Z_s^1 dW_s \tag{3.6}$$
$$Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^2, Z_s^2) ds + \int_t^T g(s, Y_s^2, Z_s^2) dB_s - \int_t^T Z_s^2 dW_s \tag{3.7}$$

where BDSDEs (3.6) and (3.7) satisfy the conditions of theorem (3.2.2). Then there exist two pairs of measurable processes (Y^1, Z^1) and (Y^2, Z^2) satisfying BDSDEs (3.6) and (3.7), respectively. Assume

$$\text{(H3.3)} \qquad \begin{cases} \xi^1 \geq \xi^2, & a.s., \\ f^1(t,Y,Z) \geq f^2(t,Y,Z), & a.s., \end{cases}$$

Then we have the following comparison theorem.

Theorem 3.3.1. *Assume BDSDEs (3.6) and (3.7) satisfy the conditions of theorem (3.2.2), let* (Y^1, Z^1) *and* (Y^2, Z^2) *be solutions of BDSDEs (3.6) and (3.7), respectively. If (H3.3) holds, then* $Y_t^1 \geq Y_t^2,\ a.s,\ \forall t \in [0, T]$.

Proof. The pair $(Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)$ satisfies the following BDSDE.

$$\begin{aligned} Y_t^1 - Y_t^2 = (\xi^1 - \xi^2) &+ \int_t^T (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))ds \\ &+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s \\ &- \int_t^T (Z_s^1 - Z_s^2)dW_s, \quad 0 \leq t \leq T. \end{aligned}$$

Applying Itô's formula to $|(Y_t^1 - Y_t^2)^-|^2$, we get

$$\begin{aligned} |(Y_t^1 - Y_t^2)^-|^2 = |(\xi^1 - \xi^2)^-|^2 &- 2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^1, Z_s^1) - f^2(t, Y_s^2, Z_s^2))ds \\ &- 2\int_t^T (Y_s^1 - Y_s^2)^-(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s \\ &+ \int_t^T 1_{Y_s^1 \leq Y_s^2} |g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 ds \\ &+ 2\int_t^T (Y_s^1 - Y_s^2)^-(Z_s^1 - Z_s^2)dW_s - \int_t^T 1_{Y_s^1 \leq Y_s^2} |Z_s^1 - Z_s^2|^2 ds. \end{aligned} \tag{3.8}$$

From (H3.3), we have $\xi^1 - \xi^2 \geq 0$, so

$$|(\xi^1 - \xi^2)^-|^2 = 0.$$

Since (Y^1, Z^1) and (Y^2, Z^2) are in $\mathcal{S}^2([0, T]; \mathbb{R}) \times \mathcal{M}^2(0, T; \mathbb{R}^d)$ it easily follows that

$$\begin{aligned} &E\int_t^T (Y_s^1 - Y_s^2)^-(Z_s^1 - Z_s^2)dW_s = 0, \\ &E\int_t^T (Y_s^1 - Y_s^2)^-(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s = 0. \end{aligned}$$

Let

$$\begin{aligned} \Delta &= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^1, Z_s^1) - f^2(t, Y_s^2, Z_s^2))ds \\ &= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^1, Z_s^1) - f^1(t, Y_s^2, Z_s^2))ds \\ &- 2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^2, Z_s^2) - f^2(t, Y_s^2, Z_s^2))ds \\ &= \Delta_1 + \Delta_2, \end{aligned}$$

where

$$\Delta_1 = -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^1, Z_s^1) - f^1(t, Y_s^2, Z_s^2))ds$$
$$\Delta_2 = -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(t, Y_s^2, Z_s^2) - f^2(t, Y_s^2, Z_s^2))ds \leq 0.$$

From (H3.1) and Young's inequality, it follows that

$$\begin{aligned}\Delta \leq \Delta_1 &\leq 2C\int_t^T (Y_s^1 - Y_s^2)^-(|Y_s^1 - Y_s^2| + |Z_s^1 - Z_s^2|)ds \\ &\leq (2C + \frac{C^2}{1-\alpha})\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds + (1-\alpha)\int_t^T 1_{Y_s^1 \leq Y_s^2}|Z_s^1 - Z_s^2|^2 ds,\end{aligned}$$

Using the assumption (H3.1), again, we deduce

$$\begin{aligned}&\int_t^T 1_{Y_s^1 \leq Y_s^2}|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 ds \\ &\leq \int_t^T 1_{Y_s^1 \leq Y_s^2}[C|Y_s^1 - Y_s^2|^2 + \alpha|Z_s^1 - Z_s^2|^2]ds \\ &= C\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds + \alpha\int_t^T 1_{Y_s^1 \leq Y_s^2}|Z_s^1 - Z_s^2|^2 ds.\end{aligned}$$

Taking expectation on both sides of (3.8), we get

$$E|(Y_t^1 - Y_t^2)^-|^2 \leq CE\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds.$$

By Gronwall's inequality, it follows that

$$E|(Y_t^1 - Y_t^2)^-|^2 = 0 \quad \forall t \in [0, T].$$

That is, $Y_t^1 \geq Y_t^2,\ a.s.,\ \forall t \in [0, T]$. □

3.4 Backward doubly stochastic differential equations with continuous coefficients

In this section we study BDSDEs with continuous coefficient. We consider coefficients (f, g) with the following properties:

$$f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R},$$
$$g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^l,$$

such that there exist $\mathcal{F}_t$-adapted processes $\{f_t, g_t :\ 0 \le t \le T\}$ with values in $[1, +\infty)$ and with the property that for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, the following hypotheses are satisfied for some strictly positive finite constant K, L and $0 < \alpha < 1$ such that for any (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$, :

$$\text{(H3.4)} \qquad \begin{cases} \quad f\,(t, y, z)\ ,\ g\,(t, y, z) \text{ are } \mathcal{F}_t\text{-measurable processes}\,, \\ \text{(i)}\ \ |f\,(t, y, z)\,| \le K\ (1 + |y| + |z|)\,, \\ \text{(ii)}\ \ |g\,(t, y_1, z_1) - g\,(t, y_2, z_2)\,|^2 \le c\,|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{cases}$$

Theorem 3.4.1. *Under the above hypothese (H3.4) and if $\xi \in \mathcal{L}^2$, there exists a solution for the BDSDE (3.1). Moreover, there is a minimal solution $(\underline{Y}, \underline{Z})$ of BDSDE (3.1) in the sense that, for any other solution (Y, Z) of BDSDE (3.1), we have $\underline{Y} \le Y$.*

We still assume that $l = d = 1$. Before giving the proof of Theorem 3.4.1, we define, as the classical approximation can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [1], the sequence $f_n(t, y, z)$ associated to f,

$$f_n(t, y, z) = \inf_{y', z' \in \mathcal{Q}} [f(t, y, z) + n(|y - y'| + |z - z'|)],$$

then for $n \ge N$,f_n is jointly measurable and uniformly linear growth in y, z with constant N. We also define the function.

$$F(t, y, z) = N(1 + |y| + |Z|)$$

Given $\xi \in \mathbb{L}^2$, by theorem (3.2.2), there exist two pair of processes (Y^n, Z^n) and (U, V), which are the solutions to the following BDSDEs, respectively,

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T Z_s^n dW_s \tag{3.9}$$

$$U_t = \xi + \int_t^T F(s, U_s, V_s)ds + \int_t^T g(s, U_s, V_s)dB_s - \int_t^T V_s dW_s \tag{3.10}$$

From theorem (3.3.1) and lemma 1 (see [27]), we get

$$\forall\ n \ge m \ge N,\ Y^m \le Y^n \le U,\ dt \otimes dP - a.s. \tag{3.11}$$

Lemma 3.4.2. *There exists a constant $A > 0$ depending only on N, C, α, T and ξ, such that*

$$\forall\ n \ge N, \|Y^n\| \le A,\ \ \|Z^n\| \le A;\ \ \|U\| \le A,\ \ \|V\| \le A$$

Proof. First of all, we prove that $\|U\|$ and $\|V\|$ are all bounded. Clearly, from 3.11 there exist a constant B depending only on $N,\ C,\ \alpha,\ T$ and ξ, such that

$$(E\int_0^T |Y_s^n|^2 ds)^{1/2} \leq B,\ (E\int_0^T |U_s|^2 ds)^{1/2} \leq B,\ \|V\| \leq B.$$

Applying Itô's formula to $|U_s|^2$, we have

$$\begin{aligned} |U_t|^2 &= |\xi|^2 + 2\int_t^T U_s F(s,U_s,V_s)ds + 2\int_t^T U_s g(s,U_s,V_s)dB_s \\ &\quad - 2\int_t^T U_s V_s dW_s + \int_t^T |g(s,U_s,V_s)|^2 ds - \int_t^T |V_s|^2 ds \end{aligned} \tag{3.12}$$

From (H3.1), for all $\alpha < \alpha^{'} < 1$, there exists a constant $C(\alpha^{'}) > 0$ such that

$$|g(t,u,v)|^2 \leq C(\alpha^{'})(|u|^2 + |g(t,0,0)|^2) + \alpha^{'}|v|^2 \tag{3.13}$$

From 3.12 and 3.13, it follows that

$$\begin{aligned} |U_t|^2 + \int_t^T |V_s|^2 ds &\leq |\xi|^2 + 2N\int_t^T |U_s|(1+|U_s|+|V_s|)ds + 2\int_t^T U_s g(s,U_s,V_s)dB_s \\ &\quad - 2\int_t^T U_s V_s dW_s + C(\alpha^{'})\int_t^T (|U_s|^2 + |g(t,0,0)|^2)ds + \alpha^{'}\int_t^T |V_s|^2 ds \\ &\leq |\xi|^2 + N^2(T-t) + C(\alpha^{'})\int_t^T |g(t,0,0)|^2 ds \\ &+ \frac{1+\alpha^{'}}{2}\int_t^T |V_s|^2 ds \\ &+ (1 + 2N + C(\alpha^{'}) + \frac{2N^2}{1-\alpha^{'}})\int_t^T |U_s|^2 ds \\ &+ 2\int_t^T U_s g(s,U_s,V_s)dB_s - 2\int_t^T U_s V_s dW_s. \end{aligned}$$

Taking expectation, we get by Young's inequality,

$$\begin{aligned} \|U_t\|^2 + \frac{1-\alpha^{'}}{2}\int_t^T \|V_s\|^2 ds &\leq E(|\xi|^2 + N^2 T + C(\alpha^{'})\int_t^T |g(t,0,0)|^2 ds) \\ &\quad + (1 + 2N + C(\alpha^{'}) + \frac{2N^2}{1-\alpha^{'}})E\int_t^T |U_s|^2 ds \\ &\quad + 2E\sup_{0\leq t\leq T}|\int_t^T U_s g(s,U_s,V_s)dB_s| + 2E\sup_{0\leq t\leq T}|\int_t^T U_s V_s dW_s|. \end{aligned} \tag{3.14}$$

By BDG's inequality, we deduce

$$\begin{aligned}
E(sup_{0\leq t\leq T}|\int_t^T U_s g(s,U_s,V_s)dB_s|) &\leq cE(\int_0^T |U_s|^2|g(s,U_s,V_s)|^2ds)^{1/2} \\
&\leq cE((sup_{0\leq t\leq T}|U_t|^2)^{1/2}(\int_0^T |g(s,U_s,V_s)|^2ds)^{1/2}) \\
&\leq 2c^2C(\alpha^{'})E(\int_0^T |U_s|^2ds + \int_0^T |g(s,0,0)|^2ds) \\
&+ \frac{1}{8}\|U\|_s^2 + 2c^2\alpha^{'}\|V\|_m^2. \qquad (3.15)
\end{aligned}$$

In the same, way, we have

$$E(sup_{0\leq t\leq T}|\int_t^T U_sV_sdW_s|) \leq \frac{1}{8}\|U\|_s^2 + 2c^2\alpha^{'}\|V\|_m^2. \qquad (3.16)$$

From 3.15, 3.16 and 3.14, it follows that

$$\begin{aligned}
\|U\|_s^2 + \frac{1-\alpha^{'}}{2}\|V|_m^2 &\leq 2(E|\xi|^2 + N^2T + C(\alpha^{'})(1+4c^2)E\int_0^T |g(t,0,0)|^2ds) \\
&+ 2(1+2N+\frac{2N^2}{1-\alpha^{'}} + C(\alpha^{'})(1+4c^2))E\int_0^T |U_s|^2ds \\
&+ 8c^2(1+\alpha^{'})\|V\|_m^2 \\
&\leq 2(E|\xi|^2 + N^2T + C(\alpha^{'})(1+4c^2)E\int_0^T |g(t,0,0)|^2ds) \\
&+ 2(1+2N+\frac{2N^2}{1-\alpha^{'}} + C(\alpha^{'})(1+4c^2) + 4c^2(1+\alpha^{'})B^2 \\
&= \frac{1-\alpha^{'}}{2}(B')^2.
\end{aligned}$$

that is

$$\|U\|_s^2 \leq B', \quad \|V|_m^2 \leq B'.$$

From 3.11, it easily follows that

$$\|Y^n\|_s \leq B'.$$

Next, we prove that bound of $\|Z^n\|_m$. Applying Itô's formula to $|Y_t^n|^2$, it follows that

$$\begin{aligned}
|Y_t^n|^2 &= |\xi|^2 + 2\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n)ds \\
&+ 2\int_t^T Y_s^n g(s,Y_s^n,Z_s^n)dB_s - 2\int_t^T Y_s^n Z_s^n dW_s \\
&+ \int_t^T |g(s,Y_s^n,Z_s^n)|^2ds - \int_t^T |Z_s^n|^2ds.
\end{aligned}$$

Taking expectation, we have

$$E(|Y_t^n|^2) + E\int_t^T |Z_s^n|^2 ds = E|\xi|^2 + 2E\int_t^T Y_s^n f_n(s, Y_s^n, Z_s^n)ds$$
$$+ E\int_t^T |g(s, Y_s^n, Z_s^n)|^2 ds.$$

From the well-known Young's inequality, it follows that

$$E(|Y_t^n|^2) + E\int_t^T |Z_s^n|^2 ds \leq E|\xi|^2 + C^{'}E\int_t^T |Y_s^n|^2 ds + \frac{1-\alpha^{'}}{2}E\int_t^T |Z_s^n|^2 ds$$
$$+ K^2(T-t) + C(\alpha^{'})E\int_t^T |g(s,0,0)|^2 ds + \alpha^{'}E\int_t^T |Z_s^n|^2 ds.$$

where $C^{'} = 1 + 2K + C(\alpha^{'}) + \frac{2K^2}{1-\alpha^{'}}$, and we know $0 < \alpha^{'} < 1$ from 3.13. Then

$$\begin{aligned}\|Z^n\|_M^2 &\leq \frac{2}{1-\alpha^{'}}(C^{'}T(B^{'})^2 + K^2T + E|\xi|^2 + C(\alpha^{'})E\int_0^T |g(s,0,0)|^2 ds)\\ &= (A)^2.\end{aligned}$$

□

Lemma 3.4.3. *The sequence (Y^n, Z^n) converge in $\mathcal{S}^2([0,T];R) \times \mathcal{M}^2(0,T;R)$.*

Proof. Let $n_0 \geq K$. Since (Y^n) is increasing and bounded in $\mathcal{S}^2([0,T];R)$, we deduce from the dominated convergence theorem that Y^n converges in $\mathcal{S}^2([0,T];R)$. We shall denote by Y the limit of Y^n. Applying Itô's formula to $|Y_t^n - Y_t^m|^2$, we get for $n, m \geq n_0$,

$$\begin{aligned}&E(|Y_0^n - Y_0^m|^2) + E\int_0 T|Z_s^n - Z_s^m|^2 ds\\ &= 2E\int_0^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds\\ &+ E\int_0^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)|^2 ds\\ &\leq 2(E\int_0^T |Y_s^n - Y_s^m|^2 ds)^{1/2}(E\int_0^T |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|^2 ds)^{1/2}\\ &+ E\int_0^T (C|Y_s^n - Y_s^m|^2 + \alpha|Z_s^n - Z_s^m|^2)ds\end{aligned}$$

Since f_n and f_m are uniformly linear and (Y^n, Z^n) is bounded, similarly to lemma 3.4.2, there exists a constant $\overline{K} > 0$ depending only on K, C, α, T and ξ, such that

$$E(|Y_0^n - Y_0^m|^2) + E\int_0 T|Z_s^n - Z_s^m|^2 ds \leq E\int_0^T (\overline{K}|Y_s^n - Y_s^m|^2 + \alpha|Z_s^n - Z_s^m|^2)ds$$

So

$$E\int_0^T \|Z_s^n - Z_s^m\|^2 \leq \frac{T\overline{K}}{1-\alpha} E \sup_{0\leq t\leq T} |Y_t^n - Y_t^m|^2.$$

Thus (Z^n) is a Cauchy sequence in $\mathcal{M}^2(0,T;R)$, from which the result follows. □

Proof. (of theorm 3.4.1) For all $n \geq n_0 \geq K$, we have $Y^{n_0} \leq Y^n \leq U$, and (Y^n) converges in $\mathcal{S}^2([0,T];R)$, $dt \otimes dP - a.s.$ to $Y \in \mathcal{S}^2([0,T];R)$. On the other hand, since (Z^n) converges in $\mathcal{M}^2(0,T;R)$ to Z, we can assume, choosing a subsequence if needed, that $Z^n \longrightarrow Z,\ dt \otimes dP - a.s.$ and $\overline{G} = sup_n|Z^n|$ is $dt \otimes dP$ integrable. Therefore, from (i) and (iv) of lemma 1 in [27], we get for almost all ω,

$$f_n(t, Y_t^n, Z_t^n) \rightarrow f(t, Y_t, Z_t), \quad n \rightarrow \infty \ dt - a.e.$$

$$\begin{aligned}|f_n(t, Y_t^n, Z_t^n)| &\leq K(1 + sup_n|Y_t^n| + sup_n|Z_t^n|) \\ &= K(1 + sup_n|Y_t^n| + \overline{G}_t) \in L^1([0,T];dt).\end{aligned}$$

Thus, for almost all ω and uniformly in t, it holds that

$$\int_t^T f_n(s, Y_s^n, Z_s^n)ds \rightarrow f(s, Y_s, Z_s), \quad n \rightarrow \infty$$

From the continuity properties of the stochastic integral, it follows that

$$\sup_{0\leq t\leq T} |\int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s| \rightarrow 0 \ \ in\ probability,$$

$$\sup_{0\leq t\leq T} |\int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T g(s, Y_s, Z_s)dB_s| \rightarrow 0 \ \ in\ probability.$$

Choosing, again, a subsequence, we can assume that the above convergence is $P - a.s.$ Finally,

$$\begin{aligned}|Y_t^n - Y_t^m| \leq &\int_t^T |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|ds \\ &+ |\int_t^T g(s, Y_s^n, Z_s^n)dB_s - g(s, Y_s^m, Z_s^m)dB_s| \\ &+ |\int_t^T Z_s^n dW_s - \int_t^T Z_s^m dW_s|,\end{aligned}$$

and taking limits on m and superemum over t, we get

$$\begin{aligned}\sup_{0\leq t\leq T} |Y_t^n - Y_t| \leq &\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|ds \\ &+ \sup_{0\leq t\leq T} |\int_t^T g(s, Y_s^n, Z_s^n)dB_s - g(s, Y_s, Z_s)dB_s| \\ &+ \sup_{0\leq t\leq T} |\int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s|, \ \ P - a.s.\end{aligned}$$

from which it follows that Y^n converges uniformly in t to Y (in particular, Y is a continuous process). Note that Y^n is monotone; therefore, we actually have the uniform convergence for the entire sequence and not just for a subsequence. Taking limits in equation (3.9), we deduce that (Y, Z) is a solution of equation (3.1).
Let $(\overline{Y}, \overline{Z}) \in \mathcal{S}^2([0,T]; R) \times \mathcal{M}^2(0,T; R)$ be any solution of equation (3.1). From comparison theorem 3.3.1, we get that $Y^n \leq \overline{Y}$, $\forall n$ and therefore $Y \leq \overline{Y}$ proving that Y is the minimal solution. $\square$

Chapter 4

Reflected Backward doubly stochastic differential equations.

In this chapter, we prove existence and uniqueness of a solution for Reflected Backward Doubly Stochastic Differential Equations (RBDSDEs) with one continuous barrier and uniformly Lipschitz coefficients. We establish moreover the existence of a maximal and a minimal solution when the generator is merely continuous.

4.1 Introduction

In this section, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued reflected backward doubly stochastic differential equation :

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right) ds + \int_t^T g\left(s, Y_s, Z_s\right) dB_s + K_T - K_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{4.1}$$

We establish the existence and uniqueness of solutions for equation (4.1) under uniformly Lipschitz condition on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of a maximal and a minimal solutions. We give here a method which allows us to overcome this difficulty in the Lipschitz case. The idea consists to start from the penalized basic RBDSDE where f and g do not depend on (y, z). We transform it to a RBDSDE with $f = g = 0$, for which we prove the existence and uniqueness of a solution by penalization method. The section theorem is then only used in the simple context where $f = g = 0$ to prove that the solution of the RBDSDE (with $f = g = 0$) stays above the obstacle for each time. A new type of comparison theorem is also established and used in this context. The (general) case, where the coefficients f, g depend on (y, z), is treated by a Picard type approximation.

In the case where the coefficient f is continuous with linear growth, we approximate f by a sequence of Lipschitz functions (f_n) and use a comparison theorem which is established here for RBDSDEs.

4.2 Assumptions and Definitions

We consider the following conditions,

H4.1) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are two measurable functions such that for every $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $f(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R})$ and $g(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R})$.

H4.2) There exist constants $L > 0$ and $0 < \alpha < 1$, such that for every $(t,\omega) \in \Omega \times [0,T]$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t,y,z) - f(t,y',z')| \leq L(|y-y'| + |z-z'|) \\ |g(t,y,z) - g(t,y',z')|^2 \leq L|y-y'|^2 + \alpha|z-z'|^2 . \end{cases}$$

H4.3) The terminal value ξ is a square integrable random variable which is $\mathcal{F}_T-$mesurable.

H4.4) The obstacle $\{S_t, 0 \leq t \leq T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying $E\left(\sup_{0\leq t\leq T}(S_t)^2\right) < \infty$ and $S_T \leq \xi$ *a.s.*

Definition 4.2.1. *A solution of equation (4.1) is a* $\left(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+\right)$ *-valued* $\mathcal{F}_t-$*progressively measurable process* $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ *which satisfies equation (4.1) and such that*
i) $(Y, Z, K_T) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathbb{L}^2(\Omega)$.
ii) $Y_t \geq S_t$.
iii) (K_t) *is continuous nondecreasing,* $K_0 = 0$ *and* $\int_0^T (Y_t - S_t)\,dK_t = 0$.

4.3 Existence of a solution of the RBDSDE with Lipschitz condition

Theorem 4.3.1. *Under conditions,* ***H4.1)***, ***H4.2)***, ***H4.3)*** *and* ***H4.4)***, *the RBDSDE (4.1) has unique solution.*

Remark 4.3.2. *In the sequel C will denotes a constant which may changes from line to line.*

Lemma 4.3.3. *Let (η^1), (η^2) be two square integrable and $\mathcal{G}_T$-measurable random variables and h^1, $h^2 : [0,T] \times \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ be two measurable functions. For $i \in \{1,2\}$, let (Y^i, Z^i) be a solution of the following BSDE :*

$$\begin{cases} Y_t^i = \eta^i + \int_t^T h^i(s, Y_s^i)ds - \int_t^T Z_s^i dW_s \\ E(\sup_{t\leq T} |Y_t^i|^2 + \int_0^T |Z_s^i|^2 ds) < \infty. \end{cases}$$

Assume that,
i) For every $\mathcal{G}_t$-adapted process $\{Y_t, 0 \leq t \leq T\}$ satisfying $E(sup_{t\leq T} {Y_t}^2) < \infty$, $h^i(t, Y_t)$ is $\mathcal{G}_t$-adapted and satisfies $E\int_0^T (h^i(s, Y_s))^2 ds < \infty$.
ii) h^1 is uniformly Lipschitz in the variable y, uniformly with respect (t,ω).
iii) $\eta^1 \leq \eta^2$ a.s.
iv) $h^1(t, Y_t^2) \leq h^2(t, Y_t^2)$ $dP \times dt$ a.e.
Then,

$$Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t^1 - Y_t^2)^+\right|^2$ and using the fact that $\eta^1 \leq \eta^2$, we obtain

$$\begin{aligned} &\left|(Y_t^1 - Y_t^2)^+\right|^2 + \int_t^T 1_{\{Y_s^1 > Y_s^2\}} \left|Z_s^1 - Z_s^2\right|^2 ds \\ &\leq 2\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(h^1\left(s, Y_s^1\right) - h^2\left(s, Y_s^2\right)\right) ds - 2\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(Z_s^1 - Z_s^2\right) dW_s. \end{aligned}$$

Using the fact that h^1 is Lipschitz and Gronwall's lemma, we get $(Y_t^1 - Y_t^2)^+ = 0$, for all $0 \leq t \leq T$ a.s. Which implies that $Y_t^1 \leq Y_t^2$, for all $0 \leq t \leq T$, a.s. □

We first consider the following simple RBDSDE, with f, g independent from (Y, Z).

$$\begin{cases} Y_t = \xi + \int_t^T f(s)\, ds + K_T - K_t + \int_t^T g(s)\, dB_s - \int_t^T Z_s dW_s \\ Y_t \geq S_t, \quad \forall t \leq T, \quad a.s. \\ \int_0^T (Y_s - S_s)\, dK_s = 0. \end{cases} \tag{4.2}$$

Proposition 4.3.4. *There exists a unique process (Y, Z, K) which solves equation (4.2).*

Proof. By [35], for $n \in \mathbb{N}$, let $(Y_t^n, Z_t^n)_{0\leq t\leq T}$ denotes the unique pair of processes, with values in $\mathbb{R} \times \mathbb{R}^d$ satisfying: $(Y^n, Z^n) \in \mathcal{S}^2 \times \mathcal{M}^2$ and

$$Y_t^n := \xi + \int_t^T f(s)\, ds + n\int_t^T (S_s - Y_s^n)^+ ds + \int_t^T g(s)\, dB_s - \int_t^T Z_s^n dW_s.$$

We define

$$\begin{cases} \overline{\xi} := \xi + \int_0^T f(s)\, ds + \int_0^T g(s)\, dB_s \\ \overline{S}_t := S_t + \int_0^t f(s)\, ds + \int_0^t g(s)\, dB_s \\ \overline{Y}_t^n := Y_t^n + \int_0^t f(s)\, ds + \int_0^t g(s)\, dB_s. \end{cases}$$

we have,

$$\overline{Y}_t^n = \overline{\xi} + n\int_t^T \left(\overline{S_s} - \overline{Y}_s^n\right)^+ ds - \int_t^T Z_s^n dW_s. \tag{4.3}$$

Let $\Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right]$. Then there exists a $\mathcal{G}_t$-predictable process $\gamma \in \mathbb{L}^2([0,T]\times \Omega, \mathbb{R}^d)$ such that

$$\Lambda_t = \Lambda_T - \int_t^T \gamma_s dW_s. \tag{4.4}$$

Since $(\overline{S}_s - \Lambda_s)^+ = 0$, we have

$$\Lambda_t = \Lambda_T + n\int_t^T (\overline{S}_s - \Lambda_s)^+ ds - \int_t^T \gamma_s dW_s. \tag{4.5}$$

By Lemma 4.3.3, we have for all $n \in \mathbb{N}$

$$\overline{Y}_t^0 = E^{\mathcal{G}_t}\left[\overline{\xi}\right] \leq \overline{Y}_t^n \leq \overline{Y}_t^{n+1} \leq \Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right].$$

Set $\quad \overline{Y}_t := \sup_n \overline{Y}_t^n \quad$ and $\quad Y_t := \sup_n Y_t^n$.

Since $\Lambda_s \geq \overline{S}_s$, we then have for every n,

$$(\overline{S_s} - \overline{Y}_s^n)^+(\Lambda_s - \overline{Y}_s^n) = (\overline{S_s} - \overline{Y}_s^n)^+(\Lambda_s - \overline{S}_s) + (\overline{S_s} - \overline{Y}_s^n)^+(\overline{S}_s - \overline{Y}_s^n) \ \geq 0$$

Therefore, using Itô's formula, we obtain

$$\begin{aligned}
|\Lambda_t - \overline{Y}_t^n|^2 + \int_t^T |\gamma_s - Z_s^n|^2 ds &= |\Lambda_T - \overline{\xi}|^2 - 2n\int_t^T (\overline{S_s} - \overline{Y}_s^n)^+(\Lambda_s - \overline{Y}_s^n)ds \\
&\quad - 2\int_t^T (\Lambda_s - \overline{Y}_s^n)(\gamma_s - Z_s^n)dW_s \\
&\leq |\Lambda_T - \overline{\xi}|^2 - 2\int_t^T (\Lambda_s - \overline{Y}_s^n)(\gamma_s - Z_s^n)dW_s
\end{aligned}$$

Passing to expectation we get

$$E\int_0^T |\gamma_s - Z_s^n|^2 ds \leq E|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2 \tag{4.6}$$

Coming back to equation (4.3) and using equation (4.4) we obtain

$$\begin{aligned}
n\int_0^T (S_s - Y_s^n)^+ ds &= n\int_0^T (\overline{S_s} - \overline{Y}_s^n)^+ ds \\
&= (\overline{Y}_0^n - \overline{\xi}) + \int_0^T Z_s^n dW_s \\
&\leq (\overline{\Lambda}_0 - \overline{\xi}) + \int_0^T Z_s^n dW_s \\
&\leq (\overline{\Lambda}_T - \overline{\xi}) + \int_0^T (Z_s^n - \gamma_s) dW_s
\end{aligned}$$

which yield that

$$(n\int_0^T (S_s - Y_s^n)^+ ds)^2 \le 2(\overline{\Lambda}_T - \overline{\xi})^2 + 2\int_0^T (Z_s^n - \gamma_s)^2 ds$$

Passing to expectation

$$\begin{aligned} E(n\int_0^T (S_s - Y_s^n)^+ \, ds)^2 &= E(n\int_0^T \left(\overline{S_s} - \overline{Y}_s^n\right)^+ ds)^2 \\ &\le 2E(\overline{\Lambda}_T - \overline{\xi})^2 + 2E\int_0^T (Z_s^n - \gamma_s)^2 ds \\ &\le 4E|\sup_{s\le T}(\overline{S}_s - \overline{\xi})^+|^2. \end{aligned}$$

Hence, there exist a nondecreasing and right continuous process K satisfying $E(K_T^2) < \infty$ such that for a subsequence of n (which still denoted n) we have for all $\varphi \in \mathbb{L}^2(\Omega; \mathcal{C}([0,T]))$,

$$\lim_n E\int_0^T \varphi_s n(S_s - Y_s^n)^+ ds = E\int_0^T \varphi_s dK_s.$$

Let $N \in \mathbb{N}^*$ and $n, m \ge N$. We have

$$\begin{aligned} (Y_t^n - Y_t^m)^2 \le\; & 2\int_t^T \left(S_s - Y_s^N\right) n \left(S_s - Y_s^n\right)^+ ds + 2\int_t^T \left(S_s - Y_s^N\right) m \left(S_s - Y_s^m\right)^+ ds \\ & - 2\int_t^T \left(Z_s^n - Z_s^m\right)\left(Y_s^n - Y_s^m\right) dW_s - \int_t^T |Z_s^n - Z_s^m|^2 \, ds. \end{aligned}$$

By BDG inequality, there exists a constant C such that

$$\limsup_{n,m}\left(E\left(\sup_{t\le T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T |Z_s^n - Z_s^m|^2 \, ds\right) \le 2CE\int_0^T \left(S_s - Y_s^N\right) dK_s.$$

Letting N tends to ∞, by using a Lebesgue's theorem we obtain

$$\limsup_{n,m}\left(E\left(\sup_{t\le T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T |Z_s^n - Z_s^m|^2 \, ds\right) \le 2CE\int_0^T \left(S_s - Y_s\right) dK_s.$$

Let

$$\widetilde{Y}_t^n := \overline{S}_T + n\int_t^T \left(\overline{S}_s - \widetilde{Y}_s^n\right) ds - \int_t^T \widetilde{Z}_s^n dW_s.$$

Since $\overline{S}_T \le \overline{\xi}$, the comparison theorem (Lemma 4.3.3), shows that, for every n we have, $\forall t \in [0,T]$, $\overline{Y}_t^n \ge \widetilde{Y}_t^n$ *a.s.*

Let σ be a $\mathcal{G}_t-$stopping time, and $\tau = \sigma \wedge T$. We have

$$\widetilde{Y}_\tau^n = E^{\mathcal{G}_\tau}\left[\overline{S}_T e^{-n(T-\tau)} + n\int_\tau^T \overline{S}_s e^{-n(s-\tau)} ds\right].$$

It is not difficult to see that $\widetilde{Y}^n$ converges to $\overline{S}_\tau$ $a.s.$ Therefore $\overline{Y}_\tau \geq \overline{S}_\tau$ $a.s.$, and hence $Y_\tau \geq S_\tau$ $a.s.$

Using section theorem [10], we get, $a.s.$ for every $t \in [0,T]$, $Y_t \geq S_t$, which implies that

$$\limsup_{n,m} \left(E \left(\sup_{t \leq T} (Y_t^n - Y_t^m)^2 \right) + E \int_0^T |Z_s^n - Z_s^m|^2 \, ds \right) = 0$$
$$\text{and} \quad E \int_0^T (S_s - Y_s) \, dK_s = 0$$

We deduce that (Y, K) is continuous and there exists Z in $\mathbb{L}^2$ such that Z^n converges strongly in $\mathbb{L}^2$ to Z. Finally, it is not difficult to check that (Y, Z, K) satisfies equation (4.2) □

Proof of Theorem 4.3.1. **Existence.** We define a sequence $(Y_t^n, Z_t^n, K_t^n)_{0 \leq t \leq T}$ as follows. Let $Y_t^0 = S_t$, $Z_t^0 = 0$ and for $t \in [0,T]$ and $n \in \mathbb{N}^*$,

$$\begin{cases} Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n) \, ds + \int_t^T g(s, Y_s^n, Z_s^n) \, dB_s + \int_t^T dK_s^{n+1} - \int_t^T Z_s^{n+1} dW_s \\ Y_t^{n+1} \geq S_t \quad \text{a.s.} \\ \int_0^T (Y_s^{n+1} - S_s) \, dK_s^{n+1} = 0. \end{cases}$$

Such sequence $(Y^n, Z^n, K^n)_n$ exists by the previous step.

Put $\overline{Y}^{n+1} = Y^{n+1} - Y^n$. By Itô's formula, we have,

$$\begin{aligned} \left| \overline{Y}_t^{n+1} \right|^2 + \int_t^T \left| \overline{Z}_s^{n+1} \right|^2 ds &= 2 \int_t^T \overline{Y}_s^{n+1} \left(f(s, Y_s^n, Z_s^n) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right) ds \\ + \int_t^T \overline{Y}_s^{n+1} \left(dK_s^{n+1} - dK_s^n \right) &+ 2 \int_t^T \overline{Y}_s^{n+1} \left(g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right) dB_s \\ +2 \int_t^T \overline{Y}_s^{n+1} \overline{Z}_s^{n+1} dW_s &+ \int_t^T \left| g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right|^2 ds. \end{aligned}$$

Therefore, Itô's formula applied to $|y|^2 e^{\beta t}$ shows that :

$$\begin{aligned} &\left| \overline{Y}_t^{n+1} \right|^2 e^{\beta t} - \beta \int_t^T \left| \overline{Y}_s^{n+1} \right|^2 e^{\beta s} ds + \int_t^T e^{\beta s} \left| \overline{Z}_s^{n+1} \right|^2 ds \\ &= 2 \int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(f(s, Y_s^n, Z_s^n) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right) ds + \int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(dK_s^{n+1} - dK_s^n \right) \\ &+ 2 \int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right) dB_s + 2 \int_t^T e^{\beta s} \overline{Y}_s^{n+1} \overline{Z}_s^{n+1} dW_s \\ &+ \int_t^T e^{\beta s} \left| g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right) \right|^2 ds. \end{aligned}$$

Using the fact that $\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(dK_s^{n+1}-dK_s^n\right)\leq 0$ and taking expectation, we get for every $\delta>0$:

$$E\left(\left|\overline{Y}_t^{n+1}\right|^2\right)e^{\beta t}-\beta E\left(\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right)ds+E\int_t^T e^{\beta s}\left|\overline{Z}_s^{n+1}\right|^2 ds$$
$$\leq 2L\delta E\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}ds+\frac{2L}{\delta}E\int_t^T\left(\left|\overline{Y}_s^{n}\right|^2+\left|\overline{Z}_s^{n}\right|^2\right)e^{\beta s}ds$$
$$+LE\int_t^T e^{\beta s}\left|\overline{Y}_s^{n}\right|^2 ds+\alpha E\int_t^T\left|\overline{Z}_s^{n}\right|^2 e^{\beta s}ds$$

This implies that,

$$E\left(\left|\overline{Y}_t^{n+1}\right|^2\right)e^{\beta t}-(\beta+2L\delta)E\left(\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right)ds+E\int_t^T\left|\overline{Z}_s^{n+1}\right|^2 ds$$
$$\leq\left(L+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Y}_s^{n}\right|^2 e^{\beta s}ds+\left(\alpha+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Z}_s^{n}\right|^2 e^{\beta s}ds$$

Choose $\delta=\frac{4L}{(1-\alpha)}$, $\overline{C}=\frac{2}{1+\alpha}\left(L+\frac{1-\alpha}{2}\right)$, and $\beta=-2L\delta-\overline{C}$, we have

$$E\int_t^T\left(\overline{C}\left|\overline{Y}_s^{n+1}\right|^2+\left|\overline{Z}_s^{n+1}\right|^2\right)e^{\beta s}ds$$
$$\leq\left(\frac{1+\alpha}{2}\right)^n E\int_t^T\left(\overline{C}\left|\overline{Y}_s^{1}\right|^2+\left|\overline{Z}_s^{1}\right|^2\right)e^{\beta s}ds$$

Since $\frac{1+\alpha}{2}<1$, there exists (Y,Z) in $\mathcal{M}^2\times\mathcal{M}^2$ such that (Y^n,Z^n) converges to (Y,Z) in $\mathcal{M}^2\times\mathcal{M}^2$. It is not difficult to deduce that Y^n converges to Y in $\mathcal{S}^2$.

It remains to prove that (Y,Z,K) is a solution to RBDSDE (4.1). By Proposition 4.3.4, there exists $(\overline{Y},\overline{Z},K)$ which satisfies,

$$\overline{Y}_t=\xi+\int_t^T f\left(s,Y_s,Z_s\right)ds+K_T-K_t+\int_t^T g\left(s,Y_s,Z_s\right)dB_s-\int_t^T\overline{Z}_s dW_s, \tag{4.7}$$

$(\overline{Y},\overline{Z},K_T)\in S^2\times M^2\times L^2$, $\overline{Y}_t\geq S_t$, (K_t) is continuous nondecreasing, $K_0=0$ and $\int_0^T\left(\overline{Y}_t-S_t\right)dK_t=0$.

We shall prove that $(Y,Z)=(\overline{Y},\overline{Z})$. By Itô's formula we have

$$(Y_t^{n+1}-\overline{Y}_t)^2-\int_t^T|Z_s^{n+1}-\overline{Z}_s|^2ds$$
$$=2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{n+1}-dK_s)$$
$$+\int_t^T(Y_s^{n+1}-\overline{Y}_s)(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))dB_s+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(Z_s^{n+1}-\overline{Z}_s)dW_s$$
$$+\int_t^T|g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds.$$

Taking expectation and using the fact that $\int_t^T (Y_s^{n+1} - \overline{Y}_s)(dK_s^{n+1} - dK_s) \leq 0$, we get

$$
\begin{aligned}
&E(Y_t^{n+1} - \overline{Y}_t)^2 + E\int_t^T |Z_s^{n+1} - \overline{Z}_s|^2 ds \\
&\leq 2E\int_t^T (Y_s^{n+1} - \overline{Y}_s)(f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)ds + E\int_t^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|^2 ds \\
&\leq C\left(E\int_t^T |Y_s^{n+1} - \overline{Y}_s|^2 ds + E\int_0^T |Y_s^n - Y_s|^2 ds + E\int_0^T |Z_s^n - Z_s|^2 ds\right).
\end{aligned}
$$

Using Growall's lemma and letting n tends to ∞ we obtain $\overline{Y}_t = Y_t$ and $\overline{Z}_t = Z_t$, $dP \times dt$ *a.e.*

Uniqueness. It follows from the comparison theorem which will be established below.

□

4.4 RBDSDE's with continuous coefficient

In this section we prove the existence of a solution to RBDSDE's where the coefficient is only continuous.

We consider the following assumption

H4.5) i) for $a.e$ (t, ω), the map $(y, z) \mapsto f(t, y, z)$ is continuous.

ii) There exist constants $\kappa > 0$, $L > 0$ and $\alpha \in]0, 1[$, such that for every $(t, \omega) \in \Omega \times [0, T]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$
\begin{cases}
|f(t, y, z)| \leq \kappa(1 + |y| + |z|) \\
|g(t, y, z) - g(t, y', z')|^2 \leq L|y - y'|^2 + \alpha|z - z'|^2.
\end{cases}
$$

Theorem 4.4.1. *Under assumption **H4.1)**, **H4.3)**, **H4.4)** and **H4.5)**, the RBDSDE (4.1) has an adapted solution (Y, Z, K) which is a minimal one, in the sense that, if (Y^*, Z^*) is any other solution we have $Y \leq Y^*$, $P - a.s.$*

Before giving the proof of Theorem 4.4.1, we recall the following classical lemma. It can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [1].

Lemma 4.4.2. *Let $f : [0, T] \times \Omega \times \mathbb{R}^d \longmapsto \mathbb{R}$ be a measurable function such that:*
(a) For almost every $(t, \omega) \in [0, T] \times \Omega$, $x \longmapsto f(t, x)$ is continuous,
(b) There exists a constant $K > 0$ such that for every $(t, x) \in [0, T] \times \mathbb{R}^d$ $|f(t, x)| \leq K(1 + |x|)$ a.s.
Then, the sequence of functions

$$f_n(t, x) = \inf_{y \in \mathbb{Q}^d} \{f(t, y) + n|x - y|\}$$

is well defined for each $n \geq K$ and satisfies:
(1) for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $|f_n(t,x)| \leq K(1+|x|)$,
(2) for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $n \to f_n(t,x)$ is increasing,
(3) for every $n \geq K$, $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, $|f_n(t,x) - f_n(t,y)| \leq n|x-y|$,
(4) If $x_n \to x$, as $n \to \infty$ then for every $t \in [0,T]$ $f_n(t,x_n) \to f(t,x)$ as $n \to \infty$.

Since ξ satisfies **H4.3)**, we get from Theorem 4.3.1, that for every $n \in \mathbb{N}^*$, there exists a unique solution $\{(Y_t^n, Z_t^n, K_t^n),\ 0 \leq t \leq T\}$ for the following RBDSDE

$$\left\{ \begin{array}{c} Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n + \int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T Z_s^n dW_s,\ 0 \leq t \leq T, \\ Y_t^n \geq S_t,\ \forall t \leq T,\ a.s. \\ \int_0^T (Y_s^n - S_s)\, dK_s^n = 0. \end{array} \right. \tag{4.8}$$

We consider the function defined by

$$f^1(t,u,v) := \kappa(1+|u|+|v|).$$

Since, $|f^1(t,u,v) - f^1(t,u',v')| \leq \kappa(|u-u'|+|v-v'|)$, then similar argument as before shows that there exists a unique solution $((U_s, V_s, K_s), 0 \leq s \leq T)$ to the following RBDSDE:

$$\left\{ \begin{array}{c} U_t = \xi + \int_t^T f^1(s, U_s, V_s)ds + K_T - K_t + \int_t^T g(s, U_s, V_s)dB_s - \int_t^T V_s dW_s \\ U_t \geq S_t,\ \forall t \leq T,\ a.s. \\ \int_0^T (U_s - S_s)\, dK_s = 0. \end{array} \right. \tag{4.9}$$

We need also the following comparison theorem

Theorem 4.4.3. *Let (ξ, f, g, S) and (ξ', f', g, S') be two RBDSDEs. Each one satisfying all the previous assumptions **H4.1), H4.2), H4.3)** and **H4.4)**. Assume moreover that :*
i) $\xi \leq \xi'$ a.s.
ii) $f(t,y,z) \leq f'(t,y,z)$ $dP \times dt$ a.e. $\forall (y,z) \in \mathbb{R} \times \mathbb{R}^d$.
iii) $S_t \leq S'_t,\ 0 \leq t \leq T$ a.s.
Let (Y,Z,K) be a solution of RBDSDE (ξ, f, g, S) and (Y', Z', K') be a solution of RBDSDE (ξ', f', g, S'). Then,

$$Y_t \leq Y'_t, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t - Y_t')^+\right|^2$, and passing to expectation, we have

$$\begin{aligned}
&E\left|(Y_t - Y_t')^+\right|^2 + E\int_t^T 1_{\{Y_s > Y_s'\}}\left|Z_s - Z_s^{'}\right|^2 ds \\
&= 2E\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(f\left(s, Y_s, Z_s\right) - f'\left(s, Y_s^{'}, Z_s^{'}\right)\right) ds \\
&+ 2E\int_t^T \left(Y_s - Y_s^{'}\right)^+ (dK_s - dK_s') \\
&\quad + E\int_t^T \left|g\left(s, Y_s, Z_s\right) - g\left(s, Y_s^{'}, Z_s'\right)\right|^2 1_{\{Y_s > Y_s'\}} ds.
\end{aligned}$$

Since on the set $\{Y_s > Y_s'\}$, we have $Y_t > S_t' \geq S_t$, then

$$\int_t^T \left(Y_s - Y_s^{'}\right)^+ (dK_s - dK_s') = -\int_t^T \left(Y_s - Y_s^{'}\right)^+ dK_s' \leq 0$$

Since f is Lipschitz, we have on the set $\{Y_s > Y_s'\}$,

$$\begin{aligned}
&E\left|(Y_t - Y_t')^+\right|^2 + E\int_t^T 1_{\{Y_s > Y_s'\}}\left|Z_s - Z_s^{'}\right|^2 ds \\
&\leq \left(3L + \frac{1}{\varepsilon}L^2\right) E\int_t^T \left|Y_s - Y_s^{'}\right|^2 1_{\{Y_s > Y_s'\}} ds \\
&+ (\varepsilon + \alpha)\, E\int_t^T \left|Z_s - Z_s^{'}\right|^2 1_{\{Y_s > Y_s'\}} ds.
\end{aligned}$$

We now choose $\varepsilon = \frac{1-\alpha}{2}$, and $\overline{C} = 3L + \frac{1}{\varepsilon}L^2$, to deduce that

$$E\left|(Y_t - Y_t')^+\right|^2 \leq \overline{C}E\int_t^T \left|(Y_s - Y_s^{'})^+\right|^2 ds.$$

The result follows now by using Gronwall's lemma. □

Lemma 4.4.4. *Let (Y^n, Z^n) be the process defined by equation 4.8. Then,*
i) For every $n \in \mathbb{N}^*$, $Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t,\ \forall t \leq T,\ a.s.$
ii) There exists $Z \in \mathcal{M}^2$ such that Z^n converges to Z in $\mathcal{M}^2$.

Proof. Assertion i) follows from Theorem 4.3.1. We shall prove ii).
Itô's formula yields

$$\begin{aligned}
E|Y_0^n|^2 + E\int_0^T |Z_s^n|^2 ds =& E|\xi|^2 + 2E\int_0^T Y_s^n f_n(s, Y_s^n, Z_s^n) ds + 2E\int_0^T S_s dK_s^n \\
&+ E\int_0^T \left|g(s, Y_s^n, Z_s^n)\right|^2 ds
\end{aligned}$$

But, assumption H4.5) and the inequality $2ab \leq \dfrac{a^2}{r} + rb^2$ for $r > 0$, show that :

$$\begin{aligned} 2Y_s^n f_n(s, Y_s^n, Z_s^n) &\leq \frac{1}{r}|Y_s^n|^2 + r\,|f_n(s, Y_s^n, Z_s^n)|^2 \\ &\leq \frac{1}{r}|Y_s^n|^2 + r\,(\kappa(1 + |Y_s^n| + |Z_s^n|))^2\,. \end{aligned}$$

and

$$|g(s, Y_s^n, Z_s^n)|^2 \leq (1+\varepsilon)L|Y_s^n|^2 + (1+\varepsilon)\alpha\,|Z_s^n|^2 + (1 + \frac{1}{\varepsilon})\,|g(s,0,0)|^2\,.$$

Hence

$$\begin{aligned} E\int_0^T |Z_s^n|^2 ds &\leq C + \big(r\kappa^2 + (1+\varepsilon)\alpha\big)\,E\int_0^T |Z_s^n|^2 ds + 2E\int_0^T S_s dK_s^n \\ &\leq C + \big(r\kappa^2 + (1+\varepsilon)\alpha\big)\,E\int_0^T |Z_s^n|^2 ds + \beta E(K_T^n)^2. \end{aligned}$$

On the other hand, we have from (4.8)

$$K_T^n = Y_0^n - \xi - \int_0^T f_n(s, Y_s^n, Z_s^n)ds - \int_0^T g(s, Y_s^n, Z_s^n)dB_s + \int_0^T Z_s^n dW_s, \tag{4.10}$$

then

$$E(K_T^n)^2 \leq C\left(1 + E\int_0^T |Z_s^n|^2 ds\right)$$

which yield that

$$E\int_0^T |Z_s^n|^2 ds \leq C + \big(r\kappa^2 + (1+\varepsilon)\alpha + \beta C\big)\,E\int_0^T |Z_s^n|^2 ds.$$

Choosing $r = \varepsilon = \beta = \frac{1-\alpha}{2(\kappa^2+\alpha+C)}$, we obtain

$$E\int_0^T |Z_s^n|^2 ds \leq C.$$

For $n, p \geq K$, Itô's formula gives,

$$\begin{aligned} E(Y_0^n - Y_0^p)^2 + E\int_0^T |Z_s^n - Z_s^p|^2 ds =& 2E\int_0^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds \\ &+ 2E\int_0^T (Y_s^n - Y_s^p)dK_s^n + 2E\int_0^T (Y_s^p - Y_s^n)dK_s^p \\ &+ E\int_0^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)|^2\,.ds. \end{aligned}$$

But

$$E\int_0^T (Y_s^n - Y_s^p)dK_s^n = E\int_0^T (S_s - Y_s^p)dK_s^n \leq 0.$$

Similarly, we have $E\int_0^T (Y_s^p - Y_s^n)dK_s^p \leq 0$.
Therefore,

$$\begin{aligned} E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq & 2E\int_0^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds \\ & + E\int_0^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)|^2 \, ds. \end{aligned}$$

By Hölder's inequality and the fact that g is Lipschitz, we get

$$\begin{aligned} & E\int_0^T |Z_s^n - Z_s^p|^2 ds \\ & \leq \left(E\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}} \left(E\int_0^T (f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))^2 ds\right)^{\frac{1}{2}} \\ & + CE\int_0^T |Y_s^n - Y_s^p|^2 ds + \alpha E\int_0^T |Z_s^n - Z_s^p|^2 ds. \end{aligned}$$

Since $\sup_n E\int_0^T |f_n(s, Y_s^n, Z_s^n)|^2 \leq C$, we obtain,

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq C\left(E\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}}.$$

Hence

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \longrightarrow 0; \;\; as \;\; n, p \to \infty.$$

Thus $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}^2(\mathbb{R}^d)$, which end the proof of this Lemma. $\square$

Proof of Theorem 4.3.5. Thanks to Lemma 4.4.4, we can define $Y_t := sup_n Y_t^n$. The arguments used in the proof of the previous Lemma allow us to show that $(Y^n, Z^n) \to (Y, Z)$ in $\mathcal{M}^2 \times \mathcal{M}^2$. Then, along a subsequence which we still denote (Y^n, Z^n), we get

$$(Y^n, Z^n) \to (Y, Z), \quad dt \otimes d\mathrm{P} \; a.e$$

then, using Lemma 4.4.2, we get $f_n(t, Y_t^n, Z_t^n) \to f(t, Y_t, Z_t) \quad dPdt \; a.e.$
On the other hand, since $Z^n \longrightarrow Z$ in $\mathcal{M}^2(\mathbb{R}^d)$, then there exists $\Lambda \in \mathcal{M}^2(\mathbb{R})$ and a subsequence which we still denote Z^n such that $\forall n, |Z^n| \leq \Lambda$, $Z^n \longrightarrow Z, \; dt \otimes dP \; a.e.$
Moreover from **H4.5)**, and Lemma 4.4.4 we have

$$|f_n(t, Y_t^n, Z_t^n)| \leq \kappa(1 + \sup_n |Y_t^n| + \Lambda_t) \in \mathrm{L}^2([0, T], \; dt), \quad P - a.s.$$

It follows from the dominated convergence theorem that,

$$E\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|^2\,ds \longrightarrow 0, \quad n\to\infty. \tag{4.11}$$

We have,

$$\begin{aligned}&\mathbb{E}\int_0^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds\\ &\leq C\mathbb{E}\int_0^T |Y_s^n-Y_s|^2 ds+\alpha\mathbb{E}\int_0^T |Z_s^n-Z_s|^2 ds \longrightarrow 0, \quad \text{as } n\to\infty.\end{aligned}$$

Let

$$\overline{Y}_t=\xi+\int_t^T f\left(s,Y_s,Z_s\right)ds+K_T-K_t+\int_t^T g\left(s,Y_s,Z_s\right)dB_s-\int_t^T \overline{Z}_s dW_s, \tag{4.12}$$

$\overline{Z}\in M^2$, $\overline{Y}\in S^2$, $K_T\in L^2$, $\overline{Y}_t\geq S_t$, (K_t) is continuous and nondecreasing, $K_0=0$ and $\int_0^T\left(\overline{Y}_t-S_t\right)dK_t=0$. By Itô's formula we have

$$\begin{aligned}(Y_t^n-\overline{Y}_t)^2&=2\int_t^T (Y_s^n-\overline{Y}_s)(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds+2\int_t^T (Y_s^n-\overline{Y}_s)(dK_s^n-dK_s)\\ &\quad+\int_t^T (Y_s^n-\overline{Y}_s)(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))dB_s+2\int_t^T (Y_s^n-\overline{Y}_s)(Z_s^n-\overline{Z}_s)dW_s\\ &\quad+\int_t^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds-\int_t^T |Z_s^n-\overline{Z}_s|^2 ds\end{aligned}$$

Passing to expectation and using the fact that $\int_t^T (Y_s^n-\overline{Y}_s)(dK_s^n-dK_s)\leq 0$, we get

$$\begin{aligned}E(Y_t^n-\overline{Y}_t)^2+E\int_t^T |Z_s^n-\overline{Z}_s|^2 ds&\leq 2E\int_t^T (Y_s^n-\overline{Y}_s)(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds\\ &\quad+E\int_t^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds\end{aligned}$$

Letting n goes to ∞, we have $\overline{Y}_t=Y_t$ and $\overline{Z}_t=Z_t$ $dP\times dt$ *a.e.*
Let (Y^*,Z^*,K^*) be a solution of (4.1). Then, by Theorem 4.3.1, we have for every $n\in\mathbb{N}^*$, $Y^n\leq Y^*$. Therefore, $\overline{Y}$ is a minimal solution of (4.1) □

Remark 4.4.5. *Using the same arguments and the following approximating sequence*

$$f_n(t,x)=\sup_{y\in\mathbb{Q}^p}\left\{f(y)-n|x-y|\right\},$$

one can prove that the RBDSDE (4.1) has a maximal solution.

Chapter 5

Existence result of Double barriers Reflected backward doubly stochastic differential equation

We prove the existence and uniqueness result for solution of Backward Doubly Stochastic Differential Equations with two reflecting barriers and uniformly Lipschitz coefficients. We prove moreover the existence of a solution when the generator is merely continuous.

5.1 Introduction

In this Chapter, we generalize the result of K. Bahlali et al. ([4]) to the case of two reflecting barrier processes, we obtain the real valued double reflected backward doubly stochastic differential equation (in short DRBDSDE):

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right) ds + \int_t^T g\left(s, Y_s, Z_s\right) dB_s + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{5.1}$$

We establish the existence and uniqueness of solutions for equation (5.1) under uniformly Lipschitz condition on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of a solutions.

5.2 Assumptions and Definitions

We consider the following conditions,

H5.1) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are two measurable functions such that for every $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $f(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R})$ and $g(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R})$.

H5.2) There exist constants $L > 0$ and $0 < \alpha < 1$, such that for every $(t,\omega) \in \Omega \times [0,T]$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t,y,z) - f(t,y',z')| \leq L(|y-y'| + |z-z'|) \\ |g(t,y,z) - g(t,y',z')|^2 \leq L|y-y'|^2 + \alpha|z-z'|^2. \end{cases}$$

H5.3) The terminal value ξ is a square integrable random variable which is $\mathcal{F}_T-$mesurable.

H5.4) The obstacles $\{L_t, U_t, 0 \leq t \leq T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying $E\left(\sup_{0\leq t\leq T}(L_t)^2\right) < \infty$, $E\left(\sup_{0\leq t\leq T}(U_t)^2\right) < \infty$ and $L_T \leq \xi \leq U_T a.s.$

Definition 5.2.1. *A solution of equation (5.1) is a* $\left(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+\right)$ *-valued* $\mathcal{F}_t-$*progressively measurable process* $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{0\leq t\leq T}$ *which satisfies equation (5.1) and such that*
i) $(Y, Z, K_T^+, K_T^-) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$.
ii) $U_t \geq Y_t \geq L_t$.
iii) $\left(K_t^+, K_t^-\right)$ *is continuous nondecreasing,* $K_0^{\pm} = 0$ *and* $\int_0^T (Y_t - L_t)\, dK_t^+ = \int_0^T (U_t - Y_t)\, dK_t^- = 0$,

5.3 Existence of a solution of the DRBDSDE with Lipschitz condition

Theorem 5.3.1. *Under conditions,* ***H5.1)****,* ***H5.2)****,* ***H5.3)*** *and* ***5.4)****, the DRBDSDE (5.1) has unique solution.*

Remark 5.3.2. *In the sequel C will denotes a constant which may changes from line to line.*

We first consider the following simple RBDSDE, with f, g independent from (Y,Z).

$$\begin{cases} Y_t = \xi + \int_t^T f(s)\, ds + \int_t^T dK_s^+ - \int_t^T dK_s^- + \int_t^T g(s)\, dB_s - \int_t^T Z_s dW_s \\ U_t \geq Y_t \geq L_t, \quad \forall t \leq T, \quad a.s. \\ \int_0^T (Y_s - L_s)\, dK_s^+ = \int_0^T (U_s - Y_s)\, dK_s^- = 0. \end{cases} \tag{5.2}$$

Proposition 5.3.3. *There exists a unique process (Y, Z, K^+, K^-) which solves equation (5.2).*

Proof. By ([4]), for $n \in \mathbb{N}$, let $\left(Y_t^n, Z_t^n, K_t^+\right)_{0\leq t\leq T}$ denotes the unique pair of processes, with values in $\mathbb{R} \times \mathbb{R}^d$ satisfying: $(Y^n, Z^n, K_t^+) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathbb{L}^2$ and

$$Y_t^n := \xi + \int_t^T f(s)\,ds + \int_t^T dK_s^+ ds - n\int_t^T (Y_s^n - U_s)^+ ds + \int_t^T g(s)\,dB_s - \int_t^T Z_s^n dW_s.$$

We define

$$\begin{cases} \overline{\xi} := \xi + \int_0^T f(s)ds + \int_0^T dK_s^+ + \int_0^T g(s)\,dB_s \\ \overline{L}_t := L_t + \int_0^t dK_s^+ + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s \\ \overline{U}_t := U_t + \int_0^t dK_s^+ + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s \\ \overline{Y}_t^n := Y_t^n + \int_0^t dK_s^+ + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s. \end{cases}$$

we have,

$$\overline{Y}_t^n = \overline{\xi} - n\int_t^T (\overline{Y}_s^n - \overline{U}_s)^+ ds - \int_t^T Z_s^n dW_s. \tag{5.3}$$

By ([4]) we get

$$\begin{cases} \overline{Y}_t^n = \overline{\xi} - n\int_t^T (\overline{Y}_s^n - \overline{U}_s)^+ ds - \int_t^T Z_s^n dW_s \\ \overline{L} \leq \overline{Y}_t^n \\ \int_t^T (\overline{Y}_s^n - \overline{L}_s)(dK_s)^+ = 0 \end{cases}$$

have the solution.

Let $\Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \wedge inf_{s\leq T}\overline{U}_s\right]$. Then there exists a $\mathcal{G}_t$-predictable process $\gamma \in \mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^d)$ such that

$$\Lambda_t = \Lambda_T - \int_t^T \gamma_s dW_s. \tag{5.4}$$

Since $(\Lambda_s - \overline{U}_s)^+ = 0$, we have

$$\Lambda_t = \Lambda_T - n\int_t^T (\Lambda_s - \overline{U}_s)^+ - \int_t^T \gamma_s dW_s. \tag{5.5}$$

By comparison result of ([4]), we have for all $n \in \mathbb{N}$

$$\overline{Y}_t^0 = E^{\mathcal{G}_t}\left[\overline{\xi}\right] \geq \overline{Y}_t^n \geq \overline{Y}_t^{n+1} \geq \Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \wedge inf_{s\leq T}\overline{U}_s\right].$$

Set $\quad \overline{Y}_t := \inf_n \overline{Y}_t^n \quad$ and $\quad Y_t := \inf_n Y_t^n$.

$$(\overline{Y}_t^n - \Lambda_t) = (\overline{\xi} - \Lambda_T) - n\int_t^T (\overline{Y}_s^n - \overline{U}_s)^+ ds - \int_t^T (Z_s^n - \gamma_s)dW_s \tag{5.6}$$

Using Itô's formula, we obtain

$$|\overline{Y}_t^n - \Lambda_t|^2 + \int_t^T |Z_s^n - \gamma_s|^2 ds = |\overline{\xi} - \Lambda_T|^2 - 2n\int_t^T (\overline{Y}_s^n - \overline{U}_s)^+ (\overline{Y}_t^n - \Lambda_t)ds - \int_t^T (Z_s^n - \gamma_s)(\overline{Y}_t^n - \Lambda_t)dW_s \tag{5.7}$$

Since $\Lambda_s \leq \overline{U}_s$, we then have for every n,

$$(\overline{Y}_s^n - \overline{U_s})^+(\overline{Y}_s^n - \Lambda_s) = (\overline{Y}_s^n - \overline{U_s})^+(\overline{U}_s - \Lambda_s) + (\overline{Y}_s^n - \overline{U_s})^+(\overline{Y}_s^n - \overline{U}_s) \ \geq 0$$

Therefore

$$|\overline{Y}_t^n - \Lambda_t|^2 + \int_t^T |Z_s^n - \gamma_s|^2 ds \leq |\overline{\xi} - \Lambda_T|^2 - 2\int_t^T (Z_s^n - \gamma_s)(\overline{Y}_t^n - \Lambda_t) dW_s$$

Passing to expectation we get

$$E\int_t^T |Z_s^n - \gamma_s|^2 ds \leq E|\sup_{s\leq T}(\overline{\xi} - \overline{U}_s)^+|^2 \tag{5.8}$$

Coming back to equation (5.3) and using equation (5.4) we obtain

$$\begin{aligned}
n\int_0^T (Y_s^n - U_s)^+ ds &= n\int_0^T (\overline{Y}_s^n - \overline{U_s})^+ ds \\
&= (\overline{\xi} - \overline{Y}_0^n) - \int_0^T Z_s^n dW_s \\
&\leq (\overline{\xi} - \overline{\Lambda}_0) - \int_0^T Z_s^n dW_s \\
&\leq (\overline{\xi} - \overline{\Lambda}_T) - \int_0^T (Z_s^n - \gamma_s) dW_s
\end{aligned}$$

Passing to expectation

$$\begin{aligned}
E(n\int_0^T (Y_s^n - U_s))^+ ds)^2 &= E(n\int_0^T (\overline{Y}_s^n - \overline{U}_s))^+ ds)^2 \\
&\leq 2E(\overline{\xi} - \overline{\Lambda}_T)^2 + 2E\int_0^T (Z_s^n - \gamma_s)^2 ds \\
&\leq 4E|\sup_{s\leq T}(\overline{\xi} - \overline{U}_s)^+|^2.
\end{aligned}$$

Hence, there exist a nondecreasing and right continuous process K satisfying $E(K_T^2) < \infty$ such that for a subsequence of n (which still denoted n) we have for all $\varphi \in \mathbb{L}^2(\Omega; \mathcal{C}([0,T]))$,

$$\lim_n E\int_0^T \varphi_s n(Y_s^n - U_s)^+ ds = E\int_0^T \varphi_s dK_s.$$

Let $N \in \mathbb{N}^*$ and $n, m \geq N$. We have

$$\begin{aligned}
(Y_t^n - Y_t^m) = &-n\int_t^T (Y_s^n - U_s)^+ ds \\
&+ m\int_t^T (Y_s^m - U_s)^+ ds - \int_t^T (Z_s^n - Z_s^m) dW_s
\end{aligned}$$

by Itô's formula, we get

$$
\begin{aligned}
(Y_t^n - Y_t^m)^2 = &-2n\int_t^T (Y_s^n - U_s)^+ |Y_s^n - Y_s^m| ds \\
&+ 2m\int_t^T (Y_s^m - U_s)^+ |Y_s^n - Y_s^m| ds - 2\int_t^T |Y_s^n - Y_s^m|(Z_s^n - Z_s^m) dW_s \\
&- \int_t^T |Z_s^n - Z_s^m|^2 ds \\
&\leq 2\int_t^T \left(Y_s^N - U_s\right) n \left(Y_s^n - U_s\right)^+ ds + 2\int_t^T \left(Y_s^N - U_s\right) m \left(Y_s^m - U_s\right)^+ ds \\
&- 2\int_t^T \left(Z_s^n - Z_s^m\right)\left(Y_s^n - Y_s^m\right) dW_s - \int_t^T \left|Z_s^n - Z_s^m\right|^2 ds.
\end{aligned}
$$

By BDG inequality, there exists a constant C such that

$$
\limsup_{n,m}\left(E\left(\sup_{t\leq T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds\right) \leq 2CE\int_0^T \left(Y_s^N - U_s\right) dK_s.
$$

Letting N tends to ∞, by using a Lebesgue's theorem we obtain

$$
\limsup_{n,m}\left(E\left(\sup_{t\leq T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds\right) \leq 2CE\int_0^T \left(Y_s - U_s\right) dK_s.
$$

Let

$$
\widetilde{Y}_t^n := \overline{U}_T + n\int_t^T \left(\widetilde{Y}_s^n - \overline{U}_s\right) ds - \int_t^T \widetilde{Z}_s^n dW_s.
$$

Since $\overline{\xi} \leq \overline{U}_T$, the comparison theorem ([4]), shows that, for every n we have, $\forall t \in [0,T]$, $\overline{Y}_t^n \leq \widetilde{Y}_t^n$ $a.s.$

Let σ be a $\mathcal{G}_t-$stopping time, and $\tau = \sigma \wedge T$. We have

$$
\widetilde{Y}_\tau^n = E^{\mathcal{G}_\tau}\left[\overline{U}_T e^{-n(T-\tau)} + n\int_\tau^T \overline{U}_s e^{-n(s-\tau)} ds\right].
$$

It is not difficult to see that $\widetilde{Y}^n$ converges to $\overline{U}_\tau$ $a.s.$ Therefore $\overline{Y}_\tau \leq \overline{U}_\tau$ $a.s.$, and hence $Y_\tau \leq U_\tau$ $a.s.$

Using section theorem, we get, $a.s.$ for every $t \in [0,T]$, $Y_t \leq U_t$, which implies that

$$
\limsup_{n,m}\left(E\left(\sup_{t\leq T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds\right) = 0
$$

$$
\texttt{and} \quad E\int_0^T \left(Y_s - U_s\right) dK_s = 0
$$

We deduce that (Y,K) is continuous and there exists Z in $\mathbb{L}^2$ such that Z^n converges strongly in $\mathbb{L}^2$ to Z. Finally, it is not difficult to check that (Y,Z,K) satisfies equation (5.2) $\square$

Proof of Theorem 5.3.1. **Existence.** We define a sequence $\left(Y_t^n, Z_t^n, K_t^{n+}, K_t^{n-}\right)_{0\le t\le T}$ as follows. Let $Y_t^0 = L_t,\ Z_t^0 = 0$ and for $t \in [0,T]$ and $n \in \mathbb{N}^*$,

$$\begin{cases} Y_t^{n+1} = \xi + \int_t^T f\left(s, Y_s^n, Z_s^n\right) ds + \int_t^T g\left(s, Y_s^n, Z_s^n\right) dB_s + \int_t^T dK_s^{(n+1)+} \\ -\int_t^T dK_s^{(n+1)-} - \int_t^T Z_s^{n+1} dW_s \\ U_t \ge Y_t^{n+1} \ge L_t \quad \text{a.s.} \\ \int_0^T \left(Y_s^{n+1} - L_s\right) dK_s^{(n+1)+} = \int_0^T \left(U_s - Y_s^{n+1}\right) dK_s^{(n+1)-} = 0. \end{cases}$$

Such sequence $\left(Y^n, Z^n, K^{n+}, K^{n-}\right)_n$ exists by the previous step.
Put $\overline{Y}^{n+1} = Y^{n+1} - Y^n$. By Itô's formula, we have,

$$\begin{aligned} \left|\overline{Y}_t^{n+1}\right|^2 + \int_t^T \left|\overline{Z}_s^{n+1}\right|^2 ds &= 2\int_t^T \overline{Y}_s^{n+1}\left(f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds \\ &\quad + 2\int_t^T \overline{Y}_s^{n+1}\left(dK_s^{(n+1)+} - dK_s^{n+}\right) - 2\int_t^T \overline{Y}_s^{n+1}\left(dK_s^{(n+1)-} - dK_s^{n-}\right) \\ &\quad + 2\int_t^T \overline{Y}_s^{n+1}\left(g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s \\ +2\int_t^T \overline{Y}_s^{n+1}\overline{Z}_s^{n+1} dW_s &+ \int_t^T \left|g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds. \end{aligned}$$

Therefore, Itô's formula applied to $|y|^2 e^{\beta t}$ shows that :

$$\begin{aligned} &\left|\overline{Y}_t^{n+1}\right|^2 e^{\beta t} - \beta\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s} ds + \int_t^T e^{\beta s}\left|\overline{Z}_s^{n+1}\right|^2 ds \\ &= 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds + 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(dK_s^{(n+1)+} - dK_s^{n+}\right) \\ &- 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(dK_s^{(n+1)-} - dK_s^{n-}\right) \\ &+ 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s + 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\overline{Z}_s^{n+1} dW_s \\ &+ \int_t^T e^{\beta s}\left|g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds. \end{aligned}$$

Using the fact that $\int_t^T e^{\beta s}\overline{Y}_s^{n+1}(dK_s^{n+1} - dK_s^n) \le 0$ and $\int_t^T e^{\beta s}\overline{Y}_s^{n+1}(dK_s^{(n+1)-} - dK_s^{n-}) \ge 0$ and taking expectation, we get for every $\delta > 0$:

$$\begin{aligned} &E\left(\left|\overline{Y}_t^{n+1}\right|^2\right) e^{\beta t} - \beta E\left(\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right) ds + E\int_t^T e^{\beta s}\left|\overline{Z}_s^{n+1}\right|^2 ds \\ &\le 2L\delta E\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s} ds + \frac{2L}{\delta} E\int_t^T \left(\left|\overline{Y}_s^n\right|^2 + \left|\overline{Z}_s^n\right|^2\right) e^{\beta s} ds \\ &+ LE\int_t^T e^{\beta s}\left|\overline{Y}_s^n\right|^2 ds + \alpha E\int_t^T \left|\overline{Z}_s^n\right|^2 e^{\beta s} ds \end{aligned}$$

This implies that,

$$
\begin{aligned}
&E\left(\left|\overline{Y}_t^{n+1}\right|^2\right)e^{\beta t}-(\beta+2L\delta)\,E\left(\int_t^T\left|\overline{Y}_s^{n+1}\right|^2e^{\beta s}\right)ds+E\int_t^T\left|\overline{Z}_s^{n+1}\right|^2ds\\
&\le\left(L+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Y}_s^{n}\right|^2e^{\beta s}ds+\left(\alpha+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Z}_s^{n}\right|^2e^{\beta s}ds
\end{aligned}
$$

Choose $\delta=\frac{4L}{(1-\alpha)}$, $\overline{C}=\frac{2}{1+\alpha}\left(L+\frac{1-\alpha}{2}\right)$, and $\beta=-2L\delta-\overline{C}$, we have

$$
\begin{aligned}
&E\int_t^T\left(\overline{C}\left|\overline{Y}_s^{n+1}\right|^2+\left|\overline{Z}_s^{n+1}\right|^2\right)e^{\beta s}ds\\
&\le\left(\frac{1+\alpha}{2}\right)^nE\int_t^T\left(\overline{C}\left|\overline{Y}_s^{1}\right|^2+\left|\overline{Z}_s^{1}\right|^2\right)e^{\beta s}ds
\end{aligned}
$$

Since $\frac{1+\alpha}{2}<1$, there exists (Y,Z) in $M^2\times M^2$ such that (Y^n,Z^n) converges to (Y,Z) in $M^2\times M^2$. It is not difficult to deduce that Y^n converges to Y in S^2.

It remains to prove that (Y,Z,K^+,K^-) is a solution to RBDSDE (5.1). By Proposition 5.3.3, there exists $(\overline{Y},\overline{Z},K^+,K^-)$ which satisfies,

$$
\overline{Y}_t=\xi+\int_t^Tf\left(s,Y_s,Z_s\right)ds+(K_T^+-K_t^+)-(K_T^--K_t^-)+\int_t^Tg\left(s,Y_s,Z_s\right)dB_s-\int_t^T\overline{Z}_sdW_s,
\tag{5.9}
$$

$(\overline{Y},\overline{Z},K^+,K^-)\in S^2\times M^2\times L^2\times L^2$, $\overline{Y}_t\ge S_t$, $\left(K_t^+\right)$and $\left(K_t^-\right)$ is continuous nondecreasing, $K_0^+=0$, $K_0^-=0$ and $\int_0^T\left(\overline{Y}_t-L_t\right)dK_t^+=\int_0^T\left(U_t-\overline{Y}_t\right)dK_t^-=0$.
We shall prove that $(Y,Z)=(\overline{Y},\overline{Z})$. By Itô's formula we have

$$
\begin{aligned}
&(Y_t^{n+1}-\overline{Y}_t)^2-\int_t^T|Z_s^{n+1}-\overline{Z}_s|^2ds\\
&=2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{(n+1)+}-dK_s^+)\\
&-2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{(n+1)-}-dK_s^-)\\
&+\int_t^T(Y_s^{n+1}-\overline{Y}_s)(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))dB_s+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(Z_s^{n+1}-\overline{Z}_s)dW_s\\
&+\int_t^T|g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds.
\end{aligned}
$$

Taking expectation and using the fact that $\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{(n+1)+}-dK_s^+)\le 0$, and

$\int_t^T e^{\beta s}\overline{Y}_s^{n+1}(dK_s^{(n+1)-} - dK_s^-) \geq 0$ we get

$$E(Y_t^{n+1} - \overline{Y}_t)^2 + E\int_t^T |Z_s^{n+1} - \overline{Z}_s|^2 ds$$
$$\leq 2E\int_t^T (Y_s^{n+1} - \overline{Y}_s)(f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)ds + E\int_t^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|^2 ds$$
$$\leq C\left(E\int_t^T |Y_s^{n+1} - \overline{Y}_s|^2 ds + E\int_0^T |Y_s^n - Y_s|^2 ds + E\int_0^T |Z_s^n - Z_s|^2 ds\right).$$

Using Growall's lemma and letting n tends to ∞ we obtain $\overline{Y}_t = Y_t$ and $\overline{Z}_t = Z_t$, $dP \times dt$ *a.e.*

Uniqueness. It follows from the comparison theorem from [4]. □

5.4 Double barrier BDSDE with continuous coefficient

We consider the following assumption

H5.5) i) for *a.e* (t, ω), the map $(y, z) \mapsto f(t, y, z)$ is continuous.

ii) There exist constants $\kappa > 0$, $L > 0$ and $\alpha \in]0, 1[$, such that for every $(t, \omega) \in \Omega \times [0, T]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t, y, z)| \leq \kappa(1 + |y| + |z|) \\ |g(t, y, z) - g(t, y', z')|^2 \leq L|y - y'|^2 + \alpha|z - z'|^2. \end{cases}$$

Theorem 5.4.1. *Under assumption* ***H5.1)***, ***H5.3)***, ***H5.4)*** *and* ***H5.5)***, *the RBDSDE (5.1) has an adapted solution* (Y, Z, K^+, K^-).

We now give a comparison theorem, which allows us to compare the solution of double barriers reflected backward doubly stochastic differential equations.

Theorem 5.4.2. *Let* (ξ, f, g, L, U) *and* (ξ', f', g, L', U') *be two DRBDSDEs. Each one satisfying all the previous assumptions* ***H5.1)***, ***H5.2)***, ***H5.3)*** *and* ***H5.4)***. *Assume moreover that :*

i) $\xi \leq \xi'$ *a.s.*

ii) $f(t, y, z) \leq f'(t, y, z)$ $dP \times dt$ *a.e.* $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$.

iii) $L_t \leq L'_t$, $0 \leq t \leq T$ *a.s.*

iv) $U_t \leq U'_t$, $0 \leq t \leq T$ *a.s.*

Let (Y, Z, K^+, K^-) *be a solution of DRBDSDE* (ξ, f, g, L, U) *and* $(Y', Z', K^{+\prime}, K^{-\prime})$ *be a solution of DRBDSDE* (ξ', f', g, L', U'). *Then,*

$$Y_t \leq Y'_t, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t - Y_t')^+\right|^2$, and passing to expectation, we have

$$
\begin{aligned}
&E\left|(Y_t - Y_t')^+\right|^2 + E\int_t^T 1_{\{Y_s > Y_s'\}}\left|Z_s - Z_s^{'}\right|^2 ds \\
&= 2E\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(f\left(s, Y_s, Z_s\right) - f'\left(s, Y_s^{'}, Z_s^{'}\right)\right) ds \\
&+ 2E\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(dK_s^+ - dK_s^{+\prime}\right) \\
&- 2E\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(dK_s - dK_s^{-\prime}\right) \\
&+ E\int_t^T \left|g\left(s, Y_s, Z_s\right) - g\left(s, Y_s^{'}, Z_s^{'}\right)\right|^2 1_{\{Y_s > Y_s'\}} ds.
\end{aligned}
$$

Since on the set $\{Y_s > Y_s'\}$, we have $Y_t > L_t' \geq L_t$, then

$$
\int_t^T \left(Y_s - Y_s^{'}\right)^+ (dK_s - dK_s') = -\int_t^T \left(Y_s - Y_s^{'}\right)^+ dK_s' \leq 0
$$

Since on the set $\{Y_s > Y_s'\}$, we have $U_t' \geq U_t > Y_t'$, then

$$
\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(dK_s^- - dK_s^{-\prime}\right) = \int_t^T (U_s - Y_s')^+ dK_s^- \geq 0
$$

Since f is Lipschitz, we have on the set $\{Y_s > Y_s'\}$,

$$
\begin{aligned}
&E\left|(Y_t - Y_t')^+\right|^2 + E\int_t^T 1_{\{Y_s > Y_s'\}}\left|Z_s - Z_s^{'}\right|^2 ds \\
&\leq \left(3L + \frac{1}{\varepsilon}L^2\right) E\int_t^T \left|Y_s - Y_s^{'}\right|^2 1_{\{Y_s > Y_s'\}} ds \\
&+ (\varepsilon + \alpha)\, E\int_t^T \left|Z_s - Z_s^{'}\right|^2 1_{\{Y_s > Y_s'\}} ds.
\end{aligned}
$$

We now choose $\varepsilon = \frac{1-\alpha}{2}$, and $\overline{C} = 3L + \frac{1}{\varepsilon}L^2$, to deduce that

$$
E\left|(Y_t - Y_t')^+\right|^2 \leq \overline{C}E\int_t^T \left|(Y_s - Y_s^{'})^+\right|^2 ds.
$$

The result follows now by using Gronwall's lemma. □

we get from Theorem 5.3.1, that for every $n \in \mathbb{N}^*$, there exists a unique solution $\{(Y_t^n, Z_t^n, K_t^n),\ 0 \leq t \leq T\}$ for the following RBDSDE

$$
\begin{cases}
Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + (K_T^{n+} - K_t^{n+}) - (K_T^{n-} - K_t^{n-}) \\
\qquad + \int_t^T g(s, Y_s^n, Z_s^n) dB_s - \int_t^T Z_s^n dW_s,\ 0 \leq t \leq T, \\
\qquad U_t \geq Y_t^n \geq L_t,\ \forall t \leq T,\ a.s. \\
\int_0^T \left(Y_s^n - L_s\right) dK_s^{n+} = \int_0^T \left(U_s - Y_s^n\right) dK_s^{n-} = 0.
\end{cases}
\tag{5.10}
$$

And $(\rho_s, \theta_s, \Pi_s^+, \Pi_s^-)$ the solution of the following DRBDSDE

$$\begin{cases} \rho_t = \xi - \int_t^T \kappa(1+|\rho_s|+|\theta_s|)ds + (\Pi_T^+ - \Pi_t^+) - (\Pi_T^- - \Pi_t^-) \\ \qquad + \int_t^T g(s,\rho_s,\theta_s)dB_s - \int_t^T \theta_s dW_s,\ 0 \le t \le T, \\ \qquad \rho_t^n \ge L_t,\ \forall t \le T,\ a.s. \\ \qquad \int_0^T (\rho_s - L_s)\, d\Pi_s = 0. \end{cases} \tag{5.11}$$

Lemma 5.4.3. *(1)For every $n \in \mathbb{N}, Y_t^0 \ge Y_t^n \ge Y_t^{n+1} \ge \rho_t, \forall t \le T,\ a.s.$*
(2) *There exists $Z \in \mathcal{M}^2$ such that Z^n converge to Z in $\mathcal{M}^2$.*

Proof. (1)By comparison theorem (5.4.2) implies that $(Y^n)_{n\ge1}$ (resp. $(dK^n)_{n\ge1}$) is a non-increasing (resp. non-decreasing) sequence of processes and $\forall n \ge 1$, $Y_t^0 \ge Y_t^n \ge Y_t^{n+1} \ge \rho_t, \forall t \le T,\ a.s.$
Applying Itô's formula to $|Y^n|^2$, we get

$$\begin{aligned} |Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds = |\xi|^2 &+ 2\int_t^T \langle f_n(s,Y_s^n,Z_s^n), Y_s^n\rangle ds + 2\int_t^T Y_s^n (dK_s^{n+} - dK_s^{n-}) \\ &+ \int_t^T |g(s,Y_s^n,Z_s^n)|^2 ds \\ &+ 2\int_t^T \langle g(s,Y_s^n,Z_s^n), Y_s^n\rangle dB_s + \int_t^T \langle Z_s^n, Y_s^n\rangle dW_s \end{aligned} \tag{5.12}$$

Since $\int_t^T \langle g(s,Y_s^n,Z_s^n), Y_s^n\rangle dB_s$ and $\int_t^T \langle Z_s^n, Y_s^n\rangle dW_s$ is a martingale, we have

$$\begin{aligned} E|Y_t^n|^2 + E\int_t^T |Z_s^n|^2 ds = E|\xi|^2 &+ 2E\int_t^T \langle f_n(s,Y_s^n,Z_s^n), Y_s^n\rangle ds + 2E\int_t^T Y_s^n (dK_s^{n+} - dK_s^{n-}) \\ &+ E\int_t^T |g(s,Y_s^n,Z_s^n)|^2 ds \end{aligned} \tag{5.13}$$

But, assumption (H5.5) and the inequality $2ab \le \frac{a^2}{\varepsilon} + \varepsilon b^2$, and g is lipischitz, show that :

$$\begin{aligned} 2\langle f_n(s,Y_s^n,Z_s^n), Y_s^n\rangle &\le \frac{|Y_s^n|^2}{\varepsilon} + \varepsilon |f_n(s,Y_s^n,Z_s^n)|^2 \\ &\le \frac{|Y_s^n|^2}{\varepsilon} + \varepsilon(\kappa(1+|Y_s^n|+|Z_s^n|))^2. \end{aligned}$$

and

$$|g(s,Y_s^n,Z_s^n)|^2 \quad \le \quad (1+\varepsilon)L|Y_s^n|^2 + (1+\varepsilon)\alpha\, |Z_s^n|^2 + (1+\frac{1}{\varepsilon})\, |g(s,0,0)|^2 .$$

On the other hand, we have from (5.10)

$$K_T^{n+} - K_T^{n-} = Y_0^n - \xi - \int_0^T f_n(s,Y_s^n,Z_s^n)ds - \int_0^T g(s,Y_s^n,Z_s^n)dB_s + \int_0^T Z_s^n dW_s, \tag{5.14}$$

then

$$E(K_T^{n+} - K_T^{n-})^2 \leq C\left(1 + E\int_0^T |Z_s^n|^2 ds\right)$$

we obtain

$$E\int_0^T |Z_s^n|^2 ds \leq C \quad \forall n \geq 1$$

For $n, m \geq N$, Itô's formula gives,

$$\begin{aligned}
E(Y_0^n - Y_0^m)^2 + E\int_0^T |Z_s^n - Z_s^m|^2 ds =& 2E\int_0^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds \\
&+ 2E\int_0^T (Y_s^n - Y_s^m)dK_s^{n+} + 2E\int_0^T (Y_s^m - Y_s^n)dK_s^{m+} \\
&+ 2E\int_t^T (Y_s^m - Y_s^n)dK_s^{n-} \\
&+ 2E\int_t^T (Y_s^n - Y_s^m)dK_s^{m-} \\
&+ E\int_0^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)|^2 .ds.
\end{aligned}$$

But

$$E\int_0^T (Y_s^n - Y_s^m)dK_s^{n+} = E\int_0^T (L_s - Y_s^m)dK_s^{n+} \leq 0.$$

Similarly, we have $E\int_0^T (Y_s^m - Y_s^n)dK_s^{m+} \leq 0$, $E\int_0^T (Y_s^m - Y_s^n)dK_s^{n-} \leq 0$ and $E\int_0^T (Y_s^n - Y_s^m)dK_s^{m-} \leq 0$.

On the other hand we also have,

$$\begin{aligned}
E(Y_0^n - Y_0^m)^2 + E\int_0^T |Z_s^n - Z_s^m|^2 ds \leq& 2E\int_0^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds \\
&+ E\int_0^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)|^2 .ds.
\end{aligned}$$

By Hôlder's inequality and the fact that g is Lipschitz, we get

$$\begin{aligned}
&E\int_0^T |Z_s^n - Z_s^m|^2 ds \\
&\leq \left(E\int_0^T (Y_s^n - Y_s^m)^2 ds\right)^{\frac{1}{2}} \left(E\int_0^T (f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))^2 ds\right)^{\frac{1}{2}} \\
&+ CE\int_0^T |Y_s^n - Y_s^m|^2 ds + \alpha E\int_0^T |Z_s^n - Z_s^m|^2 ds.
\end{aligned}$$

Since $\sup_n E\int_0^T |f_n(s,Y_s^n,Z_s^n)|^2 \leq C$, we obtain,

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq C\left(E\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}}.$$

Hence

$$E\int_0^T |Z_s^n - Z_s^m|^2 ds \longrightarrow 0;\ as\ \ n,m \to \infty.$$

Thus $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}^2(\mathbb{R}^d)$, which end the proof of this Lemma.

□

Proof of Theorem 5.4.1. we can define $Y_t := sup_n Y_t^n$. The arguments used in the proof of the previous Lemma allow us to show that $(Y^n,Z^n) \to (Y,Z)$ in $M^2 \times M^2$. Then, along a subsequence which we still denote (Y^n,Z^n), we get

$$(Y^n,Z^n) \to (Y,Z), \quad dt \otimes d\mathrm{P}\ a.e$$

then, using Lemma 4.4.2, we get $f_n(t,Y_t^n,Z_t^n) \to f(t,Y_t,Z_t) \quad dPdt\ a.e.$
On the other hand, since $Z^n \longrightarrow Z$ in $\mathrm{M}^2(\mathbb{R}^d)$, then there exists $\Lambda \in M^2(\mathbb{R})$ and a subsequence which we still denote Z^n such that $\forall n, |Z^n| \leq \Lambda$, $Z^n \longrightarrow Z,\ dt\otimes dP\ a.e.$
Moreover from **H5.5)**, and Lemma 4.4.2 we have

$$|f_n(t,Y_t^n,Z_t^n)| \leq \kappa(1+\sup_n |Y_t^n| + \Lambda_t) \in \mathrm{L}^2([0,T],\ dt), \quad P-a.s.$$

It follows from the dominated convergence theorem that,

$$E\int_0^T |f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)|^2\, ds \longrightarrow 0, \quad n\to\infty. \tag{5.15}$$

We have,

$$\begin{aligned}
&\mathbb{E}\int_0^T |g(s,Y_s^n,Z_s^n) - g(s,Y_s,Z_s)|^2 ds \\
&\leq C\mathbb{E}\int_0^T |Y_s^n - Y_s|^2 ds + \alpha\mathbb{E}\int_0^T |Z_s^n - Z_s|^2 ds \longrightarrow 0, \quad \text{as } n\to\infty.
\end{aligned}$$

Let

$$\overline{Y}_t = \xi + \int_t^T f\,(s,Y_s,Z_s)\,ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) + \int_t^T g\,(s,Y_s,Z_s)\,dB_s - \int_t^T \overline{Z}_s dW_s, \tag{5.16}$$

$\overline{Z} \in M^2$, $\overline{Y} \in S^2$, $K_T^+ \in L^2$, $U_t \geq \overline{Y}_t \geq L_t$, $\left(K_t^+, K_t^-\right)$ is continuous and nondecreasing, $K_0^+ = K_0^- = 0$ and $\int_0^T \left(\overline{Y}_t - L_t\right) dK_t^+ = \int_0^T (U_t - \overline{Y}_t t) dK_t^- = 0$. By Itô's formula we have

$$\begin{aligned}(Y_t^n - \overline{Y}_t)^2 = 2\int_t^T (Y_s^n - \overline{Y}_s)(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)ds + 2\int_t^T (Y_s^n - \overline{Y}_s)(dK_s^{n+} - dK_s^+) \\ - 2\int_t^T (Y_s^n - \overline{Y}_s)(dK_s^{n-} - dK_s^-) \\ + \int_t^T (Y_s^n - \overline{Y}_s)(g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s))dB_s + 2\int_t^T (Y_s^n - \overline{Y}_s)(Z_s^n - \overline{Z}_s)dW_s \\ + \int_t^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|^2 ds - \int_t^T |Z_s^n - \overline{Z}_s|^2 ds\end{aligned}$$

Passing to expectation and using the fact that $\int_t^T (Y_s^n - \overline{Y}_s)(dK_s^{n+} - dK_s^+) \leq 0$ and $\int_t^T (Y_s^n - \overline{Y}_s)(dK_s^{n-} - dK_s^-) \geq 0$, we get

$$\begin{aligned}E(Y_t^n - \overline{Y}_t)^2 + E\int_t^T |Z_s^n - \overline{Z}_s|^2 ds \leq 2E\int_t^T (Y_s^n - \overline{Y}_s)(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)ds \\ + E\int_t^T |g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|^2 ds\end{aligned}$$

Letting n goes to ∞, we have $\overline{Y}_t = Y_t$ and $\overline{Z}_t = Z_t$ $dP \times dt$ $a.e.$ $\square$

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