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Contents

0.1 Introduction

In this thesis we consider the optimal stochastic control problems, where the systems are dynamic, namely, they evolve over time. Moreover, they are described by Itô's stochastic differential equations, and are sometimes called diffusion models. Since the systems are dynamic, the relevant controls, which are made based on the most updated information available to the controllers, must also change over time. The controllers must select an optimal decision among all possible ones to achieve the best expected result related to their goals. Historically handled with Bellman's and Pontryagin's optimality principles, the research on control theory considerably developed over these last years, inspired in particular by problems emerging from mathematical finance.

The Dynamic Programming Principle. We first consider standard control problem on finite horizon $[0, T]$ as follows: Let (Ω, F, P) be a probability space, let $(F_t)_t$ be a filtration satisfying the usual conditions, and B a d-dimensional Brownian motion defined on the filtered probability space (Ω, F, P), we consider a model in which the time evolution of the system is actively influenced by another stochastic process u_s, called a control process. In this case we suppose that x_s satisfies a stochastic diffurential equation of the form

$$dx_s = b(s, x_s, u_s)\, ds + \sigma(s, x_s, u_s)\, dB_s, \tag{1}$$

with initial data $x_t = x$, the coefficients, $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$, satisfies the usual conditions in order to ensure the existence and unicity of solution to SDE (1), this is typicaly satisfied when b, and σ satisfies a Lipschitz condition on x, uniformly in a, with linear growth condition. The control process u_s is a progressively measurable process valued in the control set $\mathrm{A} \subset \mathbb{R}^n$, satisfies a square integrability condition. We note that, for each constant control v, the state process x_s^v is Markov with infinitesimal generator $\pounds^v$. One must also specify what kind of information is available to the controller of time s, the controller is allowed to know the past history of states x_r, for $r \leq s$, when control u_s is chosen. The Markovian nature of the problem suggests that it should suffice to consider control processes of the form $u(s, x_s)$, such a control u is called a Markov control policy. Formally, we expect that if $u_s = u(s, x_s)$, x_s should be a Markov process with infinitesimal generator acting on function Φ, coincides on $C_b^2(\mathbb{R}^n, \mathbb{R})$ with partial differential operator

$$\pounds^u \Phi = \sum_{i=1}^{n} b_i(t, x, u(t, x)) \Phi_{x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, x, u(t, x)) \Phi_{x_i x_j}, \tag{2}$$

where $a_{ij} = \left(\sigma\sigma^T\right)_{ij}$ denotes the generic term of the matrix $\sigma\sigma^T$. The control problem on a finite time interval $[t, T]$ is to minimize the functional

$$J(t, x, u) = E\left[\int_t^T f(s, x_s, u_s)\, ds + g(x_T)\right], \tag{3}$$

we call f a running cost function and g a terminal cost function. We always assume that f and g are continuous, together with further integrability assumptions needed to insure that J is well defined. If $g(x) = 0$ then the problem is said to be in Lagrange form. If $f(t, x, u) = 0$, the problem is in Mayer form. The starting point for dynamic programming is to regard the infimum of the quantity J, being minimized as a function $V(t, x)$ of the initial data

$$V(t, x) = \inf_{u \in U} J(t, x, u) \tag{4}$$

where $x_t = x$, is the initial state given at time t. the infimum being over the class of controls admitted, V is called the value function. The first step is to obtain Bellman's optimality principle, and also called the dynamic programming principle. This states that for $t \leq t+h \leq T$

$$V(t,x) = \inf_{u \in U} E\left[\int_t^{t+h} f(s, x_s, u_s)\, ds + V(t+h, x_{t+h}) \right], \tag{5}$$

not that, the expression in brackets represents the sum of the runing cost on $[t, t+h]$, withe $(t+h, x_{t+h})$ as initial data. The proof of the dynamic programming principle is technical and, has been studied by different methods, we refer the reader to Krylov [83] , Lions [87], Fleming and Soner [51] , and Yong and Zhou [121] . For using dynamic programming, we are naturally led to vary the probability spaces and so to consider the weak formulation of the stochastic control problem, for which one shows the dynamic programming principle.

Next, by assuming that the value function is $\mathrm{C}^{1,2}([t,T] \times \mathbb{R}^n)$, applying Itô's formula to $V\left(s, X_s^{t,x}\right)$ between t and $t+h$, and then sending h to zero into (5) , The classical HJB equation associated to the stochastic control problem (3) and (4) is given becomes a partial differential equation of second order for V

$$-\frac{\partial V}{\partial t}(t,x) - \sup_{v \in A}\left[\pounds^v V(t,x) + f(t,x,v)\right] = 0, \text{ on } [0,T) \times \mathbb{R}^n, \tag{6}$$

where $\pounds^v$ is the infinitisimal generator associated to the diffusion X with constant control v, given by (2) . Equation (6) is to be considered in $[0,T] \times \mathbb{R}^n$ whith the terminal data $V(T,x) = g(x)$.

The classical verification theorem consists in finding a smooth solution to the HJB equation, and to check, if W is a classical solution of the HJB equation, then W equals the minimum total expected cost among an appropriately defined class of admissible control systems. The proof is quite simple, but the assumption that W is a classical solution is quite restrictive.

In the singular cas the state evolves according to the d-dimensional stochastic differential equation

$$\begin{cases} dX_s & = b\left(s,x_s,u_s\right)ds+\sigma\left(s,x_s,u_s\right)dB_s+G\left(s\right)d\xi\left(s\right), \qquad \text{for } s\in\left[t,T\right], \\ X_t & = x, \end{cases} \tag{7}$$

where b, σ, and G are given deterministic functions, x is the initial state, the control variable is a suitable process (u,ξ) where $u:[0,T]\times\Omega\rightarrow A_1\subset\mathbb{R}^d$, $\xi:[0,T]\times\Omega\rightarrow A_2=([0,\infty))^m$ are $B[0,T]\otimes F$ measurable, (F_t) adapted, and ξ is an increasing process, continuous on the left with limits on the right with $\xi_0=0$. The cost functional has the form

$$J\left(u,\xi\right)=E\left[\int_t^T f\left(s,X_s,u_s\right)dt+\int_t^T k\left(s\right)d\xi\left(s\right)+g\left(X_T\right)\right], \tag{8}$$

As is well known, the Bellman's dynamic programming pinciple is satisfied for the classical stochastic control problem (without the singular control), and under certain regularity conditions the value function it satisfies the HJB equation. This is still the case for singular stochastic control where the HJB equation is a second order variational inequality, we refer the reader also to Fleming and Soner [51] . Haussmann and Suo [66], discusses the dynamic programming principle fo this problem in the case where the coefficients are Lipschitz continuous in the state variable. By the compactification method, it was shown that, the value function is continuous and is the unique viscosity solution of the HJB variational inequality

$$\max\left\{\sup_u H_1\left(t,x,W,\partial_t W,D_x W,D_x^2 W,u\right),H_2\left(t,x,D_x W,u\right),l=1,..,m\right\}=0, \tag{3.3}$$

with H_1, and H_2 are given by

$$H_1\left(t,x,W,\partial_t W,D_x W,D_x^2 W,u\right)=\frac{\partial W}{\partial t}\left(t,x\right)+\pounds^u W\left(t,x\right)+f\left(t,x,u\right),$$
$$H_2\left(t,x,D_x W,u\right)=\sum_{i=1}^n\frac{\partial W}{\partial x_i}\left(t,x\right)G_{il}\left(t\right)+k_l\left(t\right).$$

D_xW and D_x^2W represent respectively, the gradient and the Hessian matrix of W.

Stochastic dynamic programming was introduced into continuous time finance by Merton [91, 92] , who construct explicit solutions of the single agent consumption and portfolio problem. He assumed that the returns of assetin perfect markets satisfy the geometric Brownian motion hypothesis, and he considered utility functions belonging to the hyperbolic absolute risk aversion (HARA) family. Under these assumptions he found explicit formulae for the optimal consumption and portfolio in both the finite and infinite horizon case. A martingale representation technology has been used by Karatzas, Lehoczky and Shreve [74] , to study optimal portfolio and consumption policies in models with general market coefficients. The cas of the Merton problem with a general utilities was analyzed by Kratzas et al. in [73, 74] , who produced the value function in closed form. Models with general utilities and trading constraints were subsequently studied by varios authors, see Karatzas et al. [75] , Zariphopoulou [122] . The notion of recursive utility was first by Duffie and Epstein [39], Duffie and Skiadas [42] have considered the optimization problem when the utility is nonlinear. Using BSDE techniques, El karoui et al. [47] have generalized the characterization of optimality obtained by Duffie and Skiadas [42] . Recall that these BSDE have been introduced by Pardoux and Peng [97] and that theire applications to finance have been developed by El Karoui, Peng and Quenez [48] . Variations of the one-dimensional singular problem have been studied by many authors. It is shown that the value function satisfies a variational inequality which gives rise to a free boundary problem, and the optimal state process is a diffusion reflected at the free boundary. Bather and Chernoff [13] where the first to formulate such a problem. Beněs, Shepp and Witsenhaussen

[14] explicitly solved a one dimensional example by observing that the value function in their example is twice cotinuously differentiable, since this regularity of the value function reduces to a condition at the interface, this regularity property is called the principle of smooth fit. The optimal control can be constructed by using the reflected Brownian motion see Lions and Sznitman [89] for more detail. see also Baldursson and Karatzas [10] for the "social planner's" problem with the associated "small investor's" problem, when the authors considered the capital stock dynamics, corresponding to the cumulative investment process ξ. The stochastic control problems that arise in models with transaction costs are of singular type and their HJB equation becomes a variational Inequality with gradient constraints. This problem was formulated by Magil and Constantinides [90] , who conjectured that the no-transaction region is a cone in the two-dimensional space of position vectors. See., also Constantinided [30, 31] . Note that in these models, because there is a single risky asset, the value function depends only on two state variables, say (x, y) , with x and y special form of the power utility functions. Davis and Norman [36] obtained a closed form expression for the value function employing the homogeneity of the problem. They also showed that the optimal policy confines the investor's portfolio to a certain wedge-shaped region in the wealth plane and they provided an algorithm and numirical computations for the optimal investment rules. The same class of utility functions was later further explored by Shreve and Soner [105] , who relaxed some of the technical assumptions on the market parameters of Davis and Norman and provided further results related to the regularity of the value function. The cas of general utilities was examined through numirical methods by Tourin and Zariohopoulou [110, 111, 112] who built a coherent class of approximation schemes for

investment models with transaction costs. We refer the reader to ∅ksendal and Sulem [96] for the same problem with jump diffusion. For further contributions concerning the singular stochastic control problem and its applications the reader is referred to [12, 24, 28, 65].

As it is well-known, it does not follow directly that the value functionis smooth, and there is not in general a smooth solution of the HJB equation, especially when the diffusion coefficient is degenerate, one is forced to use a notion of weak solution such as viscosity solutionsintroduced by Grandall and Lions [34], in the first order cas and by Lions [87] in the second order case. Lions proved that any viscosity solution is the value function of the related stochastic optimal control problem. Jensen [71] was first to prove uniqueness result for a second orde PDE. Another important step in the development of the second-order problems is Ishii's Lemma [68]. For a general overview of the theory we refer to the "User's Guide...." by Grandall, Ishii and Lions [32], and the book by Fleming and Soner. Viscosity solutions in stochastic control problems arising in mathematical finance were first introduced by Zariphopoulou [122] in the context of optimal investment decisions with trading constraints, see., also Davis, Panas and Zariphopoulou [41], Shreve and Soner [105], Barles and Soner [12], Duffie et al. [40].

The characterization of the value function as the unique viscosity solution is given in [121].

The Stochastic Maximum Principle. An other classical approach for control problem is to derive necessary conditions satisfied by an optimal solution, the argument is to use an appropriate calculus of variations of the cost functional $J(u)$ with respect to the control variable in order to derive a necessary condition of optimality. The maximum principle initiated by Pontryagin, states that an optimal state trajectory must solve a Hamilton system together with a maximum condition of a function called a generalized Hamilton. The Pontryagin's maximum principle was derived for deterministic problems as in calculus of variation.

In stochastic control, the measurability assumptions made on the control variables and the nature of solutions of the underlying SDE, play an essential role in the statement of the maximum principle. The first version of the stochastic maximum principle was established by Kushner [80] (see also Bismut [19], Bensoussan [15] and Haussmann [58]). However, at that time, the results where essentially obtained under the condition that σ independent of control is as follows: assume that b, σ, f and g are bounded, continuously differentiable in the space variable with the first order derivative satisfying the Lipschitz condition, we confine ourselves to (F_t)-atapted controls u_t. The basic idea is to perturb an optimal control and to use some sort of Taylor expansion of the state trajectory around the optimal control, by sending the perturbation to zero, and by martingale representation, the maximum principle is expressed in term of an adjoint process. Let p_t, q_t be processes adapted to the natural filtration of B_t, and satisfying the backward stochastic differential equation

$$\begin{cases} dp_t = -H_x\left(t, x_t, u_t, p_t\right) + q_t dB_t, \\ p_T = -g_x\left(x_T\right), \end{cases} \tag{10}$$

where the Hamiltonian H is defined by

$$H\left(t, x, u, p\right) = p.b\left(t, x, u\right) - f\left(t, x, u\right). \tag{11}$$

The maximum principle then states that, if $(\hat{x}_t, \hat{u}_t)$ is an optimal pair, then one must have

$$\max_{u} H(t, \hat{x}_t, u_t, p_t) = H(t, \hat{x}_t, \hat{u}_t, p_t) \quad a.e.\ t \in [0, T], \ P\text{-a.s.} \tag{12}$$

The first version of the stochastic maximum principle when the diffusion coefficient σ depends explicitly on the control variable and the control domain is not convex, was obtained by Peng [98], in which he studied the second order term in the Taylor expansion of the perturbation method arising from the Itô integral. He then obtained a maximum principle for control-dependent diffusion, which involves in addition to the first-order adjoint process, a second-order adjoint process.

In deterministic control, some efforts have been made to derive optimality necessary conditions with differentiability assumptions on the data weakened or eliminated, many authors have developed optimality necessary conditions, incliding Warga [114] . The most powerful of these results remains the maximum principle developed by Clarke, based on a differential calculus for locally Lipschitz functions.

Recently, in the stochastic case the smoothness conditions on the coefficients have been weakened, in this case the first result has been derived by Mezerdi [93], in the case of a SDE with a non smooth drift, by using Clarke generalized gradients and stable convergence of probability measures. The method performed by Bahlali-Mezerdi-Ouknine in [9] is intimately linked to the Krylov estimate, they proved that (10) and (12) remain true when the coefficients are only Lipschitz but not necessarily differentiable and the diffusion coefficient is uniformly elliptic. However, If b, σ are Lipschitz continuous and f and g are C^1 in space variable, Bahlali-Djehiche-Mezerdi [5] proved a stochastic maximum principle in optimal control of a general class of degenerate diffusion process, this case is treated by using techniques introduced by Bauleau and Hirsch [20, 21], this property (on absolute continuity of probability measures) was the key fact to define a unique linearized version of the stochastic differential equation (1) . The objective of the paper Chighoub, Djehiche and Mezerdi [27] is to extend the results of [7] to the case where f and g are only Lipschitz continuous, how prove the analogue of (10) and (12) holds. The idea is to define a slightly different stochastic differential equation defined on an enlarged probability space, where the initial condition α will be taken as a random elements.

The difficulty to get the stochastic maximum principle for the control problems for systems governed by a forward and bakward SDE for controled diffusion and non-covex control domain is how to use spike variation method for the variational equations with enough higher estimate order and use the duality technique for the adjoint equation. Peng [99] firstly studied one kind of FBSDE control system which had the economic background and could be used to study the recursive utility problem inthe mathematical finance. He obtained the maximum principle for this kind of control system with the control domain being convex . Xu [119] studied the nonconvex control domain case and obtained the corresponding maximum principle. But he assumed that the diffusion coefficient in the forward control system does not contain the control variable, see., also Shi and Wu [118] for the same problem for fully coupled FBSDE. El Karoui, Peng and Quenez [47] consider a portfolio consumption model where the objective is to optimize the recursive utility of consumption and terminal wealth.

The maximum principle for Risk-sensitive control problems have been studied by many authors including Whittle [115, 116] and Hibey [25] . In [17, 18] the maximum principle for bouth full observation and parial observation problems are obtained. In [25] , a measure-valued decomposition and weak control variations are used to obtain a minimum principle for the partial observation problem, see also Lim and Zhou [86] for a new type of the Risk-sensitive maximum principle by using an approach based on the logarithmic transformation and the relationship between the adjoint variables and the value function, for this subject, a kind of portfolio choice problem in certain financial market is given by Wang and Wu [113] .

The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas and Haussmann [22], in which they consider linear dynamics convex cost criterion and convex state constraints. The method used in [22] is based on the known principle of convex analysis, related to the minimization of convex, Gâteaux differentiable functionals defined on a convex closed set. A first order weak maximum principle has been derived by Bahlali and Chala [2], in which convex perturbations are used for both absolutely continuous and singular components. Another result about the second order stochastic maximum principle for nonlinear SDEs with a controlled diffusion matrix were obtained by Bahlali and Mezerdi [8], extending the Peng maximum principle to singular control problems, this result is based on two perturbation of the optimal control, the first is a spike variation, on the absolutely continuous component of the control, and the second one is convex on the singular component. A similar approach has been used by Bahlali et al. in [5] to study the stochastic maximum principle in relaxed-singular optimal control in the case of uncontrolled diffusion. Bahlali et al. in [3] discusses the stochastic maximum principle in singular optimal control to the case where the coefficients are Lipschitz continuous in x, provided that the classical derivatives are replaced by the generalized one.

Both maximum principle and dynamic programming can be regarded as some necessary coditions of optimal controls, under certain conditions they become sufficient ones, the relationship between the maximum principle and dynamic programming is essentially the relationship between the value function, and the solution of the adjoint equation along an optimal state, see e.g. [121] in the classical case. More precisely, the solution of the adjoint process can be expressed in terms of the derivatives of the value function, a vertion of the SMP and DPP still holds true. However, a weaker notion superdifferential and subdifferentiel are needed, see., Yong and Zhou [121].

Chapter 01 and chapter 02: This introductory chapters is intended to give a through description of the maximum principle. Some basic facts, which are widely used throughout the thesis, are also presented.

Chapter 03: The results of this chapter were the subject of the following paper Chighoub, Djehiche, and Mezerdi: The stochastic maximum principle in optimal control of degenerate diffusions with non-smooth coefficients, Random Oper. Stochastic Equations, 17, (2009) 35-53

The objective of this chapter is to derive necessary conditions for optimality in stochastic control problems, where the state process is a solution to a d-dimensional stochastic differential equation, whose coefficients are non smooth. For this model, we use an approximation argument in order to obtain a sequence of control problems with smooth coefficients, and we apply Ekeland's principle in order to establish the necessary conditions satisfied by a near optimal control, to pass to the limit, we will use Egorov and Portmanteau-Alexandrov Teorems, we will use also the notion of extention of the initial filtered probability space, defined by Bouleau and Hirsch.

Chapter 04: The results of this chapter were the subject of the following papers

Bahlali, K., Chighoub, F., Djehiche, B., Mezerdi, B.: *Optimality Necessary conditions in singular stochastic control problems with non smooth data*, J. Math. Anal. Appl., 355, (2009) 479-494

Chighoub, F., Djehiche, B., Mezerdi B.: A stochastic maximum principle in singular control of diffusions with non smooth coefficients (To appear in Australian J. of math. Anal. and Appl.)

Chapter 05: The results of this chapter were the subject of

Bahlali, K., Chighoub, F., Mezerdi, B.: *On the relationship between the SMP and DPP in singular optimal controls and its applications* (Preprint)

Chapter 1

The Dynamic Programming Principle

In this Chapter we present the HJB equation which arise in the optimal control of diffusion processes in $\mathbb{R}^n$. We introduced the standard class of stochastic control problem, the associated dynamic programming principle, and the resuting HJB equation describing the local behavior of the value function of the control problem. Throughout this first introduction to HJB equation the value function is assumed to be as smooth as required. Further , we established the continuity of the value function when the controls take values in a bounded domain, we showed how the HJB equation can be written rigorously in the viscosity sense without any regularity assumption on the value function.

1.1 The Bellman principle

Let (Ω, F, P) be a filtered probability space with filtration $(F_t)_{t\geq 0}$ satisfying the usual conditions. Let $(B_t)_{t\geq 0}$ be a Brownian motion valued in $\mathbb{R}^d$ defined on (Ω, F, F_t, P). We denote by A the set of all progressively mesurable processes $\{u_t\}_{t\geq 0}$ valued in $U \subset \mathbb{R}^k$. The elements of A are called control processes. We consider the state stochastic differential equation, for each control process u_t

$$dx_t = b\left(t, x_t, u_t\right) dt + \sigma\left(t, x_t, u_t\right) dB_t,\ \ t \in [0, T]\,, \tag{1.1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$; $\sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$, be two given functions satisfying, for some constant M

$$\left|b\left(t, x, u\right) - b\left(t, y, u\right)\right| + \left|\sigma\left(t, x, u\right) - \sigma\left(t, y, u\right)\right| \leq M\left|x - y\right|, \tag{1.2}$$

$$\left|b\left(t, x, u\right)\right| + \left|\sigma\left(t, x, u\right)\right| \leq M\left(1 + |x|\right). \tag{1.3}$$

Under (1.2) and (1.3) the above equation has a unique solution x, for a given initial data. We define the cost functional J:$[0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$, by

$$J\left(u\right) = E^{t,x}\left[\int_t^T f\left(t, x_t, u_t\right) dt + g\left(x_T\right)\right], \tag{1.4}$$

where $E^{t,x}$ is the expectation operator conditional on $x_t = x$, and $f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$; $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, we assume that

$$\left|f\left(t, x, u\right)\right| + \left|g\left(x\right)\right| \leq M\left(1 + |x|^2\right), \tag{1.5}$$

for some constant M. The quadratic growth condition (1.5), ensure that J is well defined. The purpose of this Section is to study the minimization problem

$$V\left(t, x\right) = \inf_{u\in U} J\left(t, x, u\right),\ \text{ for }\ (t, x) \in [0, T) \times \mathbb{R}^n, \tag{1.6}$$

which is called the value function of the problem (1.1) and (1.4).

The dynamic programming is a fundamental principle in the theory of stochastic control, we give a version of the stochastic Bellman's principle of optimality. For mathematical treatments of this problem , we refer the reader to Lions [87], Krylov [83], Yong and Zhou [121], Fleming and Soner [51].

Theorem 1.1 Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. Then, for every $h \in (0, T-t)$, we have

$$V(t,x) = \inf_{u \in U} E^{t,x} \left[\int_t^{t+h} f(s, x_s, u_s)\, ds + V(t+h, x_{t+h}) \right]. \tag{1.7}$$

Proof. Suppose that for $h \succ 0$, we given by $\hat{u}_s = \hat{u}(s,x)$ the optimal feedback control for the problem (1.1) and (1.4) over the time interval $[t,T]$ starting at point x_{t+h}. i.e.

$$J(t+h, x_{t+h}, \hat{u}_{t+h}) = V(t+h, x_{t+h}), \quad \text{a.s.} \tag{1.8}$$

Now, we consider

$$\tilde{u} = \begin{cases} u(s,x), & \text{for } s \in [t, t+h], \\ \hat{u}(s,x), & \text{for } s \in [t+h, T]. \end{cases}$$

for some control u. By definition of $V(t,x)$, and using (1.4), we obtain

$$\begin{aligned} V(t,x) &\leq J(t,x,\tilde{u}), \\ &= E^{t,x} \left[\int_t^{t+h} f(s, x_s, u_s)\, ds + \int_{t+h}^{T} f(s, x_s, \hat{u}_s)\, ds + g(x_T) \right], \end{aligned}$$

By the unicity of solution for the SDE (1.1), we have for $s \geq t+h$, $x_s^{t,x} = x_s^{t+h,x_{t+h}^{t,x}}$, then

$$\begin{aligned} J(t,x,\tilde{u}) &= E\left[\int_t^{t+h} f(s,x_s,u_s)\,ds + \int_{t+h}^{T} f\left(s, x_s^{t+h,x_{t+h}^{t,x}}, \hat{u}_s\right) ds + g\left(x_T^{t+h,x_{t+h}^{t,x}}\right)\right], \\ &= E\left[\int_t^{t+h} f(s,x_s,u_s)\,ds + E\left[\int_{t+h}^{T} f(s,x_s,\hat{u}_s)\,ds + g(x_T) \diagup x_{t+h}^{t,x}\right]\right], \\ &= E\left[\int_t^{t+h} f(s,x_s,u_s)\,ds + V\left(t+h, x_{t+h}^{t,x}\right)\right]. \end{aligned}$$

So we get

$$V(t,x) \leq E\left[\int_t^{t+h} f(s,x_s,u_s)\,ds + V\left(t+h, x_{t+h}^{t,x}\right)\right], \tag{1.9}$$

and the equality holds if $\tilde{u} = \hat{u}$, which proves (1.7). ■

1.2 The Hamilton Jacobi Bellman equation

Now, we introduce the HJB equation by deriving it form the dynamic programming principle under smoothness assumptions on the value function. Let $G : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$ into $\mathbb{R}$, be defined by

$$G(t,x,r,p,A) = b(t,x,u)^T p + \frac{1}{2} Tr\left[\sigma\sigma^T(t,x,u)\,A\right] + f(t,x,u), \tag{1.10}$$

we also need to introduce the linear second order operator $\mathcal{L}^u$ associated to the controlled processes x_t, $t \geq 0$, we consider the constant control u

$$\mathcal{L}^u \varphi(t,x) = b(t,x,u)^T D_x\varphi(t,x) + \frac{1}{2} Tr\left[\sigma\sigma^T(t,x,u)\,D_x^2\varphi(t,x)\right]. \tag{1.11}$$

where D_x, D_{xx}^2 denote the gradient and the Hessian operator with respect to the x variable. Assume the value function $V \in C([0,T], \mathbb{R}^n)$, and $f(.,.,u)$ be continuous in (t,x) for all fixed $u \in A$, then we have by Itô's formula

$$V\left(t+h, x_{t+h}\right)=V\left(t, x\right)+\int_{t}^{t+h}\left(\frac{\partial V}{\partial s}+\mathcal{L}^{u} V\right)\left(s, x_{s}^{t, x}\right) ds+\int_{t}^{t+h} DV\left(s, x_{s}^{t, x}\right)^{T} \sigma\left(s, x_{s}^{t, x}, u\right) dB_{s},$$

by taking the expectation, we get

$$E\left[V\left(t+h, x_{t+h}\right)\right]=V\left(t, x\right)+E\left[\int_{t}^{t+h}\left(\frac{\partial V}{\partial s}+\mathcal{L}^{u} V\right)\left(s, x_{s}^{t, x}\right) ds\right],$$

thenm, we have by (1.9)

$$0 \leq E\left[\frac{1}{h} \int_{t}^{t+h}\left(\frac{\partial V}{\partial s}+\mathcal{L}^{u} V\right)\left(s, x_{s}^{t, x}\right)+f\left(s, x_{s}^{t, x}, u\right) ds\right],$$

we now send h to zero, we obtain

$$0 \leq \frac{\partial V}{\partial t}\left(t, x\right)+\mathcal{L}^{u} V\left(t, x\right)+f\left(t, x, u\right),$$

this provides

$$-\frac{\partial V}{\partial t}\left(t, x\right)-\inf_{u \in A}\left[\mathcal{L}^{u} V\left(t, x\right)+f\left(t, x, u\right)\right] \leq 0, \tag{1.12}$$

Now we shall assume that $\hat{u} \in U$, and using the same procedure as above, we conclude that

$$-\frac{\partial V}{\partial t}\left(t, x\right)-\mathcal{L}^{\hat{u}} V\left(t, x\right)-f\left(t, x, u\right)=0, \tag{1.13}$$

by (1.12), then the value function solves the HJB equation

$$-\frac{\partial V}{\partial t}\left(t, x\right)-\inf_{u \in A}\left[\mathcal{L}^{u} V\left(t, x\right)+f\left(t, x, u\right)\right]=0, \quad \forall\left(t, x\right) \in[0, T] \times \mathbb{R}. \tag{1.14}$$

We give sufficient conditions which allow to conclude that the smooth solution of the HJB equation coincides with the value functionm this is the so-called verification result.

Theorem 1.2 *Let W be a $C^{1,2}\left([0, T), \mathbb{R}^{n}\right) \cap C\left([0, T], \mathbb{R}^{n}\right)$ function. Assume that f and g are quadratic growth, i.e. there is a constant M such that*

$$\left|f\left(t, x, u\right)\right|+\left|g\left(x\right)\right| \leq M\left(1+|x|^{2}\right), \text{ for all } \left(t, x, u\right) \in[0, T) \times \mathbb{R}^{n} \times U.$$

(1) *Suppose that* $W(T,.) \leq g$*, and*

$$\frac{\partial W}{\partial t}(t,x) + G(t,x,W(t,x), D_x W(t,x), D_{xx}W(t,x)) \geq 0, \tag{1.15}$$

on $[0,T) \times \mathbb{R}^n$*, then* $W \leq V$ *on* $[0,T) \times \mathbb{R}^n$.

(2) *Assume further that* $W(T,.) = g$*, and there exists a minimizer* $\hat{u}(t,x)$ *of*

$$\mathcal{L}^u V(t,x) + f(t,x,u),$$

such that

$$\begin{aligned} 0 &= \frac{\partial W}{\partial t}(t,x) + G(t,x,W(t,x), D_x W(t,x), D_{xx}W(t,x)), \\ &= \frac{\partial W}{\partial t}(t,x) + \mathcal{L}^{\hat{u}(t,x)} W(t,x) + f(t,x,u), \end{aligned} \tag{1.16}$$

the stochastic differential equation

$$dx_s = b(s, x_s, \hat{u}(s,x))\, ds + \sigma(s, x_s, \hat{u}(s,x))\, dB_s, \tag{1.17}$$

defines a unique solution x_t *for each given initial data* $x_t = x$*, and the process* $\hat{u}(s,x)$ *is a well-defined control process in* U*. Then* $W = V$*, and* $\hat{u}$ *is an optimal Markov control process.*

Proof. The function $W \in C^{1,2}([0,T), \mathbb{R}^n) \cap C([0,T], \mathbb{R}^n)$, then for all $0 \leq t \leq s \leq T$, by Itô's Lemma we get

$$W\left(t, x_r^{t,x}\right) + \int_t^s \left(\frac{\partial W}{\partial t} + \mathcal{L}^{u_r} W\right)\left(r, x_r^{t,x}\right) dr + \int_t^s D_x W\left(r, x_r^{t,x}\right)^T \sigma\left(r, x_r^{t,x}, u_r\right) dB_r,$$

the process $\int_t^s D_x W\left(r, x_r^{t,x}\right)^T \sigma\left(r, x_r^{t,x}, u_r\right)$ is a martingal, then by taking expectation, it follows that

$$E\left[W\left(s, x_s^{t,x}\right)\right] = W(t,x) + E\left[\int_t^s \left(\frac{\partial W}{\partial t} + \mathcal{L}^{u_r} W\right)\left(r, x_r^{t,x}\right) dr\right],$$

by (1.15), we get

$$\frac{\partial W}{\partial t}\left(r,x_r^{t,x}\right)+\mathcal{L}^{u_r}W\left(r,x_r^{t,x}\right)+f\left(r,x_r^{t,x},u_r\right)\geq 0,\ \forall u\in A,$$

then

$$E\left[W\left(s,x_s^{t,x}\right)\right]\geq W\left(t,x\right)-E\left[\int_t^s f\left(r,x_r^{t,x},u_r\right)dr\right],\ \forall u\in A,$$

we now take the limit as $s\rightarrow T$, then by the fact that $W\left(T\right)\leq g$ we obtain

$$E\left[g\left(x_T^{t,x}\right)\right]\geq W\left(t,x\right)-E\left[\int_t^T f\left(r,x_r^{t,x},u_r\right)dr\right],\ \forall u\in A,$$

then $W\left(t,x\right)\leq V\left(t,x\right), \forall\left(t,x\right)\in[0,T]\times\mathbb{R}^n$. Statement (2) is proved by repeating the above argument and observing that the control $\hat{u}$ achieves equality at the crucial step (1.15).

We now state without proof an existence result for the HJB equation (1.14), together with the terminal condition $W\left(T,x\right)=g\left(x\right)$. ■

Theorem.1.3 assume that

$$\exists C\succ\ \ 0\diagup\xi^T\sigma\sigma^T\left(t,x,u\right)\xi\geq C\left|\xi\right|^2,\ \text{for all }\left(t,x,u\right)\in[0,T]\times\mathbb{R}^n\times U.$$

U is compact,

b,σ and f are in $C_b^{1,2}\left([0,T]\times\mathbb{R}^n\right)$,

$g\in C_b^3\left(\mathbb{R}^n\right)$.

Then the HJB equation (1.14), with the terminal data $V\left(T,x\right)=g\left(x\right)$, has a unique solution $V\in C_b^{1,2}\left([0,T]\times\mathbb{R}^n\right)$..

Proof. See Fleming and Rischel [52]. ■

we conclude this section by reviewing briefly the celebrated Merton's optimal management problem.

Example 1 *We consider a market with two securities, a bond whose price solves*

$$dS_t^0 = rS_t^0 dt,\ S_0^0 = s, \tag{1.18}$$

and a stock whose price process satisfies the stochastic differentiel equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{1.19}$$

The market parameters μ *and* σ *are, respectively, the mean rate of return and the volatility, it is assumed that* $\mu \succ r \succ 0$, *and* $\sigma \succ 0$. *The process* B_t *is a standard Brownian motion defined on a probability space* (Ω, F, P). *The wealth process satistfies* $X_s = p_s^0 + p_s$, *with the amountes* u_s^0 *and* u_s *representing the current holdings in the bond and the stock accounts. The state wealth equation is given by*

$$dX_s = rX_s ds + (\mu - r)\, u_s ds + \sigma u_s dB_t. \tag{1.20}$$

The wealth process must satisfy the state costraint

$$X_s \geq 0, a.e.t \leq s \leq T. \tag{1.21}$$

The control u_s, *is admissible if it is* F_s-*progressively measurable, it satisfies* $E \int_t^T u_s^2 ds \prec \infty$, *and it is such that the state costraint* (1.21) *is satisfied. We denote the set of admissible policies by* $\tilde{A}$. *The value function is defined by*

$$V(t, x) = \sup_{\tilde{A}} E\left[\frac{1}{\gamma} X_T^{\gamma} \diagup X_t = x\right]. \tag{1.22}$$

Using stochastic analysis and under appropriate regularity and growth conditions on the *value function, we get that V solves the associated HJB equation, for $x \geq 0$, and $t \in [0,T]$,*

$$\begin{cases} V_t + \max_u \left[\frac{1}{2}\sigma^2 u^2 V_{xx} + (\mu - r) V_x \right] + rxV_x = 0, \\ V(T,x) = \frac{1}{\gamma} x^\gamma, \\ V(t,0) = 0, t \in [0,T]. \end{cases} \tag{1.23}$$

The homogeneity of the utility function and the linearity of the state dynamics with respect to both the wealth and the control portfolio process, suggest that the value function must be of the form

$$V(T,x) = \frac{x^\gamma}{\gamma} f(t), \text{ with } f(T) = 1. \tag{1.24}$$

Using the above form in (1.23), *and after some cancellations, one gets that f must satisfy the first order equation*

$$\begin{cases} f'(t) + \lambda f(t) = 0, \\ f(T) = 1. \end{cases}$$

where

$$\lambda = r\gamma + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2}. \tag{1.25}$$

Therefore,

$$V(t,x) = \frac{x^\gamma}{\gamma} e^{\lambda(T-t)}. \tag{1.26}$$

Once the value function is determined, the optimal policy may be obtained in the so-called feedback form as follows: first, we observe that the maximum of the quadratic term appearing in (1.23) *is achieved at the point*

$$u^*(t,x) = -\frac{(\mu - r) V_x(t,x)}{\sigma^2 V_{xx}(t,x)}, \tag{1.27}$$

or, otherwise,

$$u^*(t,x) = \frac{(\mu - r)}{\sigma^2 (1-\gamma)} x, \tag{1.28}$$

where we used (1.26)*. Next, we recall classical Verification results, which yield that the candidate solution, given in* (1.26) *is indeed the value function and that, moreover, the policy* $u^*(t,x) = -\frac{(\mu-r)}{\sigma^2(1-\gamma)} X_t^*$*, is the optimal investment strategy. In the other words,*

$$V(t,x) = \sup_{\tilde{A}} E\left[\frac{1}{\gamma} X_T^{*\gamma} \diagup X_t^* = x\right].$$

where X_s^* *solves*

$$dX_s^* = \left(r + \frac{(\mu-r)^2}{(1-\gamma)\sigma^2}\right) X_s^* ds + \frac{(\mu-r)}{\sigma(1-\gamma)} X_s^* dB_s. \tag{1.29}$$

The solutionof the optimal state wealth equation is, for $X_t = x$,

$$X_s^* = x \exp\left[\left(r + \frac{(\mu-r)^2}{(1-\gamma)\sigma^2} - \frac{(\mu-r)^2}{2(1-\gamma)^2\sigma^2}\right)(s-t) + \frac{(\mu-r)}{\sigma(1-\gamma)} B_{s-t}\right].$$

The Merton optimal strategy dictates that it is optimal tokeep a fixed proportion, namely $\frac{(\mu-r)}{\sigma^2(1-\gamma)}$*, of the current total wealth invested in the stock account.*

Next, we recall the notion of viscosity solutions for non-linear second order partial differential equation (The HJB equation). For more detail we refer the reader to Crandall, Ishii and Lion[32], and Fleming and Soner [51].

1.3 Viscosity solutions

It is well known that the HJB equation (1.14) does not necessarily admit smooth solutios in general. This makes the applicability of the classical verification theorems very restrictive and is a major deficiency in dynamic programming theory. In recent years, The notion of viscosity solutions was introduced by Crandall and Lions [34] for first-order equations, and by Lions [87] for second-order equations. For a general overview of the theory we refer to the User's Guide by Crandall, Ishii and Lions [32] and the book by Fleming and Soner [51] In this theory all the derivatives involved are replaced by the so-called superdifferentials and subdifferentials, and the solutions in the viscosity sense can be merely continuous functions. The existence and uniqueness of viscosity solutions of the HJB equation can be guaranteed under very mild and reasonable assumptions, which are satisfied in the great majority of cases arising in optimal control problems. For example, the value function turns out to be the unique viscosity solution of the HJB equation (1.14).

Definition 1.5 . A function $V \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity subsolution of (1.14), if $V(T,x) \leq g(x)$, $\forall x \in \mathbb{R}^n$, and for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$, whenever $V - \varphi$ attains a local maximum at $(t,x) \in [0,T) \times \mathbb{R}^n$, we have

$$-\frac{\partial \varphi}{\partial t}(t,x) + \sup_{u \in U} G(t,x,u,-D_x\varphi(t,x), -D_{xx}\varphi(t,x)) \leq 0. \tag{1.30}$$

A function $V \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity supersolution of (1.14), if $V(T,x) \leq g(x)$, $\forall x \in \mathbb{R}^n$, and for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$, whenever $V - \varphi$ attains a local minimum at $(t,x) \in [0,T) \times \mathbb{R}^n$, we have

$$-\frac{\partial \varphi}{\partial t}(t,x) + \sup_{u \in U} G(t,x,u,-D_x\varphi(t,x), -D_{xx}\varphi(t,x)) \geq 0. \tag{1.31}$$

Further, if $V \in C([0,T] \times \mathbb{R}^n)$ is both a viscosity subsolution and viscosity supersolution of (1.14), then it is called a viscosity solution of (1.14).

Theorem. 1.6 Let (1.2) and (1.3) hold, then the value function V is a viscosity solution of (1.14).

Proof. For any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$, let $V - \varphi$ attains a local maximum at $(s,y) \in [0,T) \times \mathbb{R}^n$. Fix a $u \in U$, let x_t be the state trajectory with the control $u_t = u$.

Then by the dynamic programming principle, and Itô's formula, we have for $\hat{s} \succ s$, with $\hat{s} - s \succ 0$ small enough

$$\begin{aligned} 0 &\leq \frac{1}{\hat{s}-s} E\left[V(s,y) - \varphi(s,y) - V(\hat{s}, x_{\hat{s}}) + \varphi(\hat{s}, x_{\hat{s}})\right] \\ &\leq \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, x_t, u)\, dt - \varphi(s,y) + \varphi(\hat{s}, x_{\hat{s}})\right] \\ &\rightarrow -\frac{\partial \varphi}{\partial t}(s,y) - G(s, y, u, -D_x\varphi(t,x), -D_{xx}\varphi(t,x)). \end{aligned}$$

This leads to

$$-\frac{\partial \varphi}{\partial t}(s,y) + G(s, y, u, -D_x\varphi(s,y), -D_{xx}\varphi(s,y)) \leq 0, \forall u \in U.$$

Hence

$$-\frac{\partial \varphi}{\partial t}(s,y) + \sup_{u \in U} G(s, y, u, -D_x\varphi(s,y), -D_{xx}\varphi(s,y)) \leq 0, \forall u \in U. \tag{1.32}$$

On the other hand, if $V - \varphi$ attains a local minimum at $(s,y) \in [0,T) \times \mathbb{R}^n$, then for any $\epsilon \succ 0$, and $\hat{s} \succ s$ with $\hat{s} - s \succ 0$ small enough, we can find a $u_t = u_{\hat{s}}^{\epsilon} \in U$, such that

$$\begin{aligned} 0 &\geq E\left[V(s,y) - \varphi(s,y) - V(\hat{s}, x_{\hat{s}}) + \varphi(\hat{s}, x_{\hat{s}})\right], \\ &\geq -\epsilon(\hat{s}-s) + E\left[\int_s^{\hat{s}} f(t, x_t, u_t)\, dt + \varphi(\hat{s}, x_{\hat{s}}) - \varphi(s,y)\right], \end{aligned}$$

dividing by $(\hat{s} - s)$, and applying Itô's formula to the process $\varphi(t, x_t)$, we get

$$\begin{aligned} -\epsilon &\leq \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} -\frac{\partial \varphi}{\partial t}(t, x_t) + G(t, x_t, u, -D_x\varphi(t,x_t), -D_{xx}\varphi(t,x_t))\, dt\right] \\ &\leq \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} -\frac{\partial \varphi}{\partial t}(t, x_t) + \sup_{u \in U} G(t, x_t, u, -D_x\varphi(t,x_t), -D_{xx}\varphi(t,x_t))\, dt\right] \\ {}_{\hat{s}\rightarrow s} &\rightarrow -\frac{\partial \varphi}{\partial t}(t, x_t) + \sup_{u \in U} G(s, y, u, -D_x\varphi(s,y), -D_{xx}\varphi(s,y)). \end{aligned} \tag{1.33}$$

Combining (1.32), and (1.33), we conclude that V is a viscosity solution of the HJB equation (1.14).

The following Theorem is devoted to a proof of uniqueness of the viscosity solution to the HJB equation ■

Theorem 1.7. Let $V, W \in C_b^{1,2}([0,T]\times\mathbb{R}^n)$. We suppose that V is a supersolution of (1.14), with $V(T,x) \leq W(T,x)$ for all $x \in \mathbb{R}^n$,then $V(t,x) \leq W(t,x) \forall (t,x) \in [0,T] \times \mathbb{R}^n$.

Proof. Let, for $(\alpha, M, N) \in \mathbb{R}_+^* \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, we define

$$G(x,M) = -tr\left[A(x)A(x)^T M\right],$$

then, we obtain

$$\begin{aligned} G(y,N) - G(x,M) &= tr\left[A(x)A(x)^T M\right] - tr\left[A(y)A(y)^T N\right], \\ &= tr\left[A(x)A(x)^T M - A(y)A(y)^T N\right], \\ &\leq 3\alpha\left|A(x) - A(y)\right|^2, \end{aligned}$$

because the matrix

$$C := \begin{pmatrix} A(y)A^T(y) & A(y)A^T(x) \\ A(x)A^T(y) & A(x)A(x)^T \end{pmatrix},$$

is a non negative matrix, we have

$$\begin{aligned} tr\left[A(x)A(x)^T M - A(y)A(y)^T N\right] &= tr\left[C\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix}\right], \\ &\leq 3\alpha tr\left[C\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}\right], \\ &\leq 3\alpha tr\left[(A(x) - A(y))\left(A(x)^T - A(y)^T\right)\right], \\ &\leq 3\alpha\left|A(x) - A(y)\right|^2. \qquad (1.34) \end{aligned}$$

now, we consider the function

$$F:\mathbb{R}^n\times\mathbb{R}^n\rightarrow F\left(x,y\right)=V\left(x\right)-W\left(y\right)-\frac{1}{2\epsilon}\left|x-y\right|^2,$$

with $\epsilon\succ 0$. Suppose that there exists a point $(\bar{x},\bar{y})$ such that F attaints a maximum at $(\bar{x},\bar{y})$, then $x\rightarrow F\left(x,\bar{y}\right)$ attaints a maximum at $\bar{x}$, hence

$$x\rightarrow V\left(x\right)-\frac{1}{2\epsilon}\left|x-\bar{y}\right|^2$$

attaints a maximum at $\bar{x}$. Moreover, $y\rightarrow -F\left(\bar{x},y\right)$ attaints a minimum at $\bar{y}$, then we have

$$y\rightarrow W\left(y\right)-\frac{1}{2\epsilon}\left|\bar{x}-y\right|^2$$

attaints a minimum at $\bar{y}$. By the definition of viscosity subsolution at point $\bar{x}$, we obtain for V with $\varphi\left(x\right)=\frac{1}{2\epsilon}\left|x-\bar{y}\right|^2$ we get

$$\frac{\partial}{\partial t}V\left(\bar{x}\right)+\sup_{u\in A}\left\{-b\left(\bar{x},u\right)\left(\frac{\bar{x}-\bar{y}}{\epsilon}\right)-\frac{1}{2}tr\left(\sigma\sigma^T\left(\bar{x},u\right)\left(\frac{-1}{\epsilon}\right)\right)-f\left(\bar{x},u\right)\right\}\leq 0.$$

by the definition of viscosity subsolution at point $\bar{y}$, we obtain for W with $\varphi\left(y\right)=-\frac{1}{2\epsilon}\left|\bar{x}-y\right|^2$,we get

$$\frac{\partial}{\partial t}W\left(\bar{y}\right)+\sup_{u\in A}\left\{-b\left(\bar{y},u\right)\left(\frac{\bar{x}-\bar{y}}{\epsilon}\right)-\frac{1}{2}tr\left(\sigma\sigma^T\left(\bar{y},u\right)\left(\frac{-1}{\epsilon}\right)\right)-f\left(\bar{y},u\right)\right\}\geq 0.$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t}\left(V\left(\bar{x}\right)-W\left(\bar{y}\right)\right)\leq\\&\sup_{u\in A}\left\{\left|b\left(\bar{x},u\right)-b\left(\bar{y},u\right)\right|\left|\frac{\bar{x}-\bar{y}}{\epsilon}\right|+\left|f\left(\bar{x},u\right)-f\left(\bar{y},u\right)\right|\right.\\&\left.\frac{1}{2}\left|tr\left(\sigma\sigma^T\left(\bar{x},u\right)\left(\frac{-1}{\epsilon}\right)\right)-tr\left(\sigma\sigma^T\left(\bar{y},u\right)\left(\frac{-1}{\epsilon}\right)\right)\right|\right\},\end{aligned}$$

the functions $b, f, \sigma\sigma^T$ are Lipshitz on x uniformly on u then by (1.34), we get

$$\begin{aligned} & \frac{\partial}{\partial t}\left(V\left(\bar{x}\right)-W\left(\bar{y}\right)\right) \\ \leq \quad & c\frac{\left|\bar{x}-\bar{y}\right|^{2}}{\epsilon}+c\left|\bar{x}-\bar{y}\right|+3\grave{\alpha}\left|\bar{x}-\bar{y}\right|^{2}. \end{aligned}$$

on the other hand, $F\left(x,x\right)\leq F\left(\bar{x},\bar{y}\right), \forall x\in\mathbb{R}^{n}$

$$\begin{aligned} V\left(x\right)-W\left(x\right) & \leq V\left(\bar{x}\right)-W\left(\bar{y}\right)-\frac{\left|\bar{x}-\bar{y}\right|^{2}}{2\epsilon} \\ & \leq V\left(\bar{x}\right)-W\left(\bar{y}\right). \end{aligned} \tag{1.35}$$

Because $F\left(\bar{x},\bar{y}\right)\geq F\left(\bar{x},\bar{x}\right)$, we get

$$V\left(\bar{x}\right)-W\left(\bar{y}\right)-\frac{\left|\bar{x}-\bar{y}\right|^{2}}{2\epsilon}\geq V\left(\bar{x}\right)-W\left(\bar{x}\right), \tag{1.36}$$

then

$$W\left(\bar{x}\right)-W\left(\bar{y}\right)-\frac{\left|\bar{x}-\bar{y}\right|^{2}}{2\epsilon}\geq 0, \tag{1.37}$$

Moreover, $F\left(\bar{x},\bar{y}\right)\geq F\left(\bar{y},\bar{y}\right)$, then

$$V\left(\bar{x}\right)-V\left(\bar{y}\right)-\frac{\left|\bar{x}-\bar{y}\right|^{2}}{2\epsilon}\geq 0, \tag{1.38}$$

This proves that

$$\frac{\left|\bar{x}-\bar{y}\right|^{2}}{\epsilon}\leq\left(V+W\right)\left(\bar{x}\right)-\left(V+W\right)\left(\bar{y}\right), \tag{1.39}$$

V, W are bonded, then $\frac{\left|\bar{x}-\bar{y}\right|^{2}}{\epsilon}\leq c$, which means that

$$m\left(\eta\right)=\sup\left\{\left|\left(V+W\right)\left(x\right)-\left(V+W\right)\left(y\right)\right|,\left|x-y\right|\leq\eta\right\}: m\left(\eta\right)\underset{\eta\rightarrow 0}{\rightarrow 0}.$$

by (1.39), we get

$$\frac{\left|\bar{x}-\bar{y}\right|^{2}}{\epsilon}\leq m\left(\left|\bar{x}-\bar{y}\right|\right), \tag{1.40}$$

under (1.38) , on has

$$\frac{1}{2}|\bar{x}-\bar{y}|^2 \leq m\left(c\sqrt{\epsilon}\right). \tag{1.41}$$

combining (1.39) , (1.40) and (1.41) , we obtain

$$\frac{\partial}{\partial t}\left(V\left(\bar{x}\right)-W\left(\bar{y}\right)\right) \leq c\sqrt{\epsilon}+\grave{c}\epsilon+m\left(c\sqrt{\epsilon}\right).$$

finally, by (1.35) on has

$$V\left(x\right)-W\left(y\right) \leq V\left(\bar{x}\right)-W\left(\bar{y}\right) \underset{\epsilon\longrightarrow 0}{\longrightarrow} 0, \text{ for all } x \in \mathbb{R}^n.$$

hence

$$V\left(x\right) \leq W\left(y\right).$$

■

Definition 1.8. Let $V \in C\left([0,T]\times\mathbb{R}^n\right)$, the right superdifferential (resp., subdifferential) of V at $(t,x) \in [0,T) \times \mathbb{R}^n$, denoted by $D_{t,x}^{1,2+}V\left(t,x\right)$ $\left(\text{resp., } D_{t,x}^{1,2-}V\left(t,x\right)\right)$, is a set defined by

$$\begin{aligned} D_{t,x}^{1,2,+}V\left(t,x\right) &= \left\{(p,q,Q) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n} \diagup \lim_{\substack{y\to x, s\to t \\ s\in[0,T)}} \sup \frac{I\left(s,y\right)}{|s-t|+|y-x|^2} \leq 0\right\}, \\ D_{t,x}^{1,2,-}V\left(t,x\right) &= \left\{(p,q,Q) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n} \diagup \lim_{\substack{y\to x, s\to t \\ s\in[0,T)}} \inf \frac{I\left(s,y\right)}{|s-t|+|y-x|^2} \geq 0\right\}, \end{aligned}$$

where

$$I\left(s,y\right)=V\left(s,y\right)-V\left(t,x\right)-q\left(s-t\right)-\langle p,y-x\rangle-\frac{1}{2}\left(y-x\right)^T P\left(y-x\right).$$

Definition 1.9 A function $V \in C\left([0,T]\times\mathbb{R}^n\right)$ is called a viscosity solution of the

HJB equation (1.14) if

$$\begin{aligned} -p+\sup_{u\in U} G(t,x,u,q,Q) &\le 0,\ \forall (p,q,Q)\in D_{t,x}^{1,2,+}V(t,x),\ \forall (t,x)\in[0,T]\times\mathbb{R}^n, \\ -p+\sup_{u\in U} G(t,x,u,q,Q) &\ge 0,\ \forall (p,q,Q)\in D_{t,x}^{1,2,-}V(t,x),\ \forall (t,x)\in[0,T]\times\mathbb{R}^n, \\ V(T,x) &= g(x),\ \forall x\in\mathbb{R}^n. \end{aligned}$$

Lemma. 1.10 The value function V satisfies

$$|V(t,x)-V(s,y)|\le C\left(|t-s|^{\frac{1}{2}}+|x-y|\right).$$

Proof. See Zhou [121]. ■

Corollary.1.11 We have

$$\inf_{(p,q,Q)\in D_{t,x}^{1,2,+}V(t,x)\times U}\left\{[p-G(t,x,u,q,Q)]\ge 0,\forall (t,x)\in[0,T)\times\mathbb{R}^n\right\}. \tag{1.42}$$

Proof. See Zhou [121]. ■

Lemma 1.12. Let $g\in C[0,T]$. Suppose that there is $\rho\in L^1[0,T]$ such that for sufficiently small $h\succ 0$,

$$\frac{g(t+h)-g(t)}{h}\le\rho(t),\ \text{a.e. } t\in[0,T]. \tag{1.43}$$

Then

$$g(t)-g(0)\le\int_0^t \overline{\lim_{h\to 0+}}\frac{g(r+h)-g(r)}{h}dr,\ \forall t\in[0,T]. \tag{1.44}$$

Proof. First fix $t\in[0,T)$, By (1.43) we can apply Fatou's Lemma to get

$$\begin{aligned} \int_0^t \overline{\lim_{h\to 0+}}\frac{g(r+h)-g(r)}{h}dr &\ge \overline{\lim_{h\to 0+}}\int_0^t \frac{g(r+h)-g(r)}{h}dr, \\ &= \overline{\lim_{h\to 0+}}\frac{\int_h^{h+t} g(r)\,dr-\int_0^t g(r)\,dr}{h}, \\ &= \overline{\lim_{h\to 0+}}\frac{\int_h^{h+t} g(r)\,dr-\int_0^h g(r)\,dr}{h}, \\ &= g(t)-g(0). \end{aligned}$$

This proves (1.44) $\forall t \in [0,T]$, finally, the $t=T$ case is obtained by continuity. ■

Theorem $(verification)$**1.13.** Let $W \in C([0,T]\times\mathbb{R}^n)$ be a viscosity solution of the HJB equation (1.14), then

(1) $W(s,y) \leq J(s,y;u)$ for any $(s,y) \in [0,T]\times\mathbb{R}^n$ and any $u \in U$.

(2) Let $(\hat{x},\hat{u})$ be a given admissible pair for the problem $(1.1)-(1.4)$. Suppose that there exists

$$\left(\hat{p},\hat{q},\hat{Q}\right) \in L_F^2(s,T;\mathbb{R}) \times L_F^2(s,T;\mathbb{R}^n) \times L_F^2\left(s,T;\mathbb{R}^{n\times d}\right),$$

such that for $a.e.$ $t \in [s,T]$,

$$\left(\hat{p}(t),\hat{q}(t),\hat{Q}(t)\right) \in D_{t,x}^{1,2,+}W(t,\hat{x}_t),\ \text{P-a.s.}, \tag{1.45}$$

and

$$-\hat{p}(t) + G\left(t,\hat{x}_t,\hat{u}_t,\hat{q}_t,\hat{Q}_t\right) = 0,\ \text{P-a.s.}, \tag{1.46}$$

then $(\hat{x}_t,\hat{u}_t)$ is an optimal pair for the problem $(1.1)-(1.4)$.

Proof. Part (1) is trivial since $W=V$ in view of the uniqueness of the viscosity solutions. We prove only part (2) of the Theorem, set $\varphi(t,\hat{x}_t,\hat{u}_t) = \hat{\varphi}(t)$, for $\varphi = b,\sigma,f$, ect., to simplify the notation. Fix $t \in [0,T)$ such that (1.45) and (1.46) hold. Choose a test function $\phi \in C([t,T]\times\mathbb{R}^n) \cap C^{1,2}((t,T]\times\mathbb{R}^n)$as determined by $\left(\hat{p}(t),\hat{q}(t),\hat{Q}(t)\right) \in D_{t,x}^{1,2,+}W(t,\hat{x}_t)$ and Lemma (1.10). Applying Ito's formula to ϕ, we have for any $h \succ 0$,

$$\begin{aligned} & W(t+h,\hat{x}_{t+h}) - W(t,\hat{x}_t) \leq \phi(t+h,\hat{x}_{t+h}) - \phi(t,\hat{x}_t) \\ = & \int_h^{h+t}\left\{\phi_t(r,\hat{x}_r) + \phi_x(r,\hat{x}_r).\hat{b}(r) + \frac{1}{2}tr\left[\hat{\sigma}^T(r)\,\phi_{xx}(r,\hat{x}_r).\hat{\sigma}(r)\right]\right\}dr. \end{aligned} \tag{1.47}$$

It is well known by the martingale property of stochastic integrals that there are constant

C, indepedent of t,such that

$$E\left|\hat{x}_r-\hat{x}_t\right|^2 \leq C\left|r-t\right| \qquad ,\forall r\geq t, \tag{1.48}$$

$$E\left[\sup_{s\leq r\leq T}\left|\hat{x}_r\right|^{\alpha}\right]\leq C\left(\alpha\right) \qquad ,\forall\alpha\geq T. \tag{1.49}$$

hence, in view of Lemma (1.10), we have

$$\sup_{s\leq r\leq T}\left|\phi_t\left(r,\hat{x}_r\right)\right|^2\leq C^2\sup_{s\leq r\leq T}E\left[1+\frac{\left|\hat{x}_r-\hat{x}_t\right|^2}{r-t}\right]\leq C \tag{1.50}$$

or

$$\sup_{s\leq r\leq T}E\left|\phi_t\left(r,\hat{x}_r\right)\right|\leq\sqrt{C},$$

Moreover, by Lemma 1.12, assumption (1.2) and (1.3), one can show that

$$\sup_{s\leq r\leq T}E\left|\phi_x\left(r,\hat{x}_r\right).\hat{b}\left(r\right)+\frac{1}{2}tr\left[\hat{\sigma}^T\left(r\right)\phi_{xx}\left(r,\hat{x}_r\right).\hat{\sigma}\left(r\right)\right]\right|\leq C.$$

It then follows from (1.48) that for sufficiently small $h\geq 0$,

$$\frac{E\left[W\left(t+h,\hat{x}_{t+h}\right)-W\left(t,\hat{x}_t\right)\right]}{h}\leq C. \tag{1.51}$$

Now we calculate, for any fixed $N\succ 0$,

$$\begin{aligned}
\frac{1}{h}\int_h^{h+t}E\left[\phi_t\left(r,\hat{x}_r\right)-\hat{p}\left(t\right)\right]dr &= \frac{1}{h}\int_h^{h+t}E\left[\left(\phi_t\left(r,\hat{x}_r\right)-\hat{p}\left(t\right)\right)1_{\left|\hat{x}_r-\hat{x}_t\right|\succ N\left|r-t\right|^{\frac{1}{2}}}\right]dr\\
&\quad+\frac{1}{h}\int_h^{h+t}E\left[\left(\phi_t\left(r,\hat{x}_r\right)-\hat{p}\left(t\right)\right)1_{\left|\hat{x}_r-\hat{x}_t\right|\leq N\left|r-t\right|^{\frac{1}{2}}}\right]dr,\\
&= I_1\left(N,h\right)+I_2\left(N,h\right).
\end{aligned}$$

By virtue of (1.49) and (1.51), we have

$$\begin{aligned}
I_1\left(N,h\right) &\leq \frac{1}{h}\int_h^{h+t}E\left[\left|\phi_t\left(r,\hat{x}_r\right)-\hat{p}\left(t\right)\right|^2\right]^{\frac{1}{2}}\left[P\left(\left|\hat{x}_r-\hat{x}_t\right|\succ N\left|r-t\right|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}dr\\
&\leq \frac{C}{N}\rightarrow 0,\ \text{uniformly in } h\succ 0 \text{ as } N\rightarrow\infty.
\end{aligned}$$

On the other hand, for fixed $N \succ 0$, we apply Lemma 1.12 to get

$$\lim_{h\to 0+} \sup_{t\le r\le t+h} \left[\left(\phi_t\left(r,\hat{x}_r\right) - \hat{p}\left(t\right)\right) 1_{\left|\hat{x}_r-\hat{x}_t\right|\le N\left|r-t\right|^{\frac{1}{2}}} \right] \to 0,\ \text{as } h \to 0+,\ \text{P-a.s.}$$

Thus we conclude by the dominated convergence theorem that

$$\lim_{h\to 0+} I_2\left(N,h\right) \to 0,\ \text{as } h \to 0+,\ \text{for each fixed } N.$$

Therefore, we have proved that

$$\lim_{h\to 0+} \frac{1}{h} \int_h^{h+t} E\left[\phi_t\left(r,\hat{x}_r\right)\right] dr \to E\left[\hat{p}\left(t\right)\right]. \tag{1.52}$$

Similarly (in fact, more easily) , we can show that

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h} \int_h^{h+t} E\left[\phi_x\left(r,\hat{x}_r\right).\hat{b}\left(r\right)\right] dr &= E\left[\phi_x\left(t,\hat{x}_t\right).\hat{b}\left(t\right)\right], \\ &= E\left[\hat{q}\left(t\right).\hat{b}\left(t\right)\right], \end{aligned} \tag{1.53}$$

and

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h} \int_h^{h+t} E\left[\frac{1}{2} tr\left[\hat{\sigma}^T\left(r\right)\phi_{xx}\left(r,\hat{x}_r\right).\hat{\sigma}\left(r\right)\right]\right] dr &= E\left[\frac{1}{2} tr\left[\hat{\sigma}^T\left(t\right)\phi_{xx}\left(t,\hat{x}_t\right).\hat{\sigma}\left(t\right)\right]\right] \\ &= E\left[\frac{1}{2} tr\left[\hat{\sigma}^T\left(t\right)\hat{Q}\left(t\right).\hat{\sigma}\left(t\right)\right]\right]. \end{aligned}$$

Consequently (1.48) gives

$$\begin{aligned} \overline{\lim_{h\to 0+}} \frac{E\left[W\left(t+h,\hat{x}_{t+h}\right) - W\left(t,\hat{x}_t\right)\right]}{h} &\le E\left[\hat{p}\left(t\right) + \hat{q}\left(t\right).\hat{b}\left(t\right) + \frac{1}{2} tr\left[\hat{\sigma}^T\left(t\right)\hat{Q}\left(t\right).\hat{\sigma}\left(t\right)\right]\right] \\ &= -E\left[\hat{g}\left(t\right)\right], \end{aligned} \tag{1.54}$$

where the last equality is due to (1.47) . Noting (1.52) and applying Lemma 1.10 to the $g\left(t\right) = E\left[W\left(t,\hat{x}_t\right)\right]$, we arrive at

$$E\left[W\left(T,\hat{x}_T\right) - W\left(s,y\right)\right] \le \int_s^T E\left[\hat{g}\left(t\right)\right] dt, \tag{1.55}$$

which leads to $W(s,y) \geq J(s,y;\hat{u})$. It follows that $(\hat{x},\hat{u})$ is an optimal pair for (1.1) and (1.4). ■

Remark 1.14. In view of Corollary (1.11), the condition (1.47) impllies that $\left(\hat{p}(t), \hat{q}(t), \hat{Q}(t), \hat{u}_t\right)$ achieves the infimum of $p - G(t, \hat{x}_t, u, q, Q)$ over $D_{t,x}^{1,2,+}V(t,\hat{x}_t) \times U$. Meanwhile, it also shows that (1.46) is equivalent to

$$\hat{p}(t) \leq G\left(t, \hat{x}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{Q}(t)\right). \tag{1.56}$$

Remark 1.15. The condition (1.47) implies that

$$\max_{u \in U} G\left(t, \hat{x}_t, u, \hat{p}(t), \hat{q}(t), \hat{Q}(t)\right) = G\left(t, \hat{x}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{Q}(t)\right). \tag{1.57}$$

This easily seen by recalling the fact that V is the viscosity solution of (1.14) ,hence

$$-\hat{p}(t) + \sup_{u \in U} G\left(t, \hat{x}_t, u, \hat{p}(t), \hat{q}(t), \hat{Q}(t)\right) \leq 0,$$

which yields (1.57) under (1.47).

Chapter 2

The Stochastic Maximum Principle

The optimal control problems we are interested in, consists to find an admissible control $\hat{u}$ that minimizes a cost functional subject to an SDE on a finite time horizon. If $\hat{u}$ is some optimal control, we may ask how we can characterize it, in other words, what conditions must $\hat{u}$ necessarily satisfy? These conditions are called the stochastic maximum principle or the necessary conditions for optimality. The original version of Pontryagin's maximum principle was derived for deterministic problems, as in classical calculus of variation. The first version of the stochastic maximum principle was extensively established in the 1970s by Bismut [19], Kushner [80], and Haussmann [62], under the condition that there is no control on the diffusion coefficient. Haussmann [59] developed a powerful form of the stochastic maximum principle for the feedback class of controls by Girsanov's transformation, and applied it to solve some problems in stochastic control.

2.1 The first-order maximum principle

Throghout this section, let us suppose that $\hat{u} \in U$ is an optimal control and denote by $\hat{x}$ the corresponding optimal trajectory, i.e. the solution of the SDE(1.21) controlled by $\hat{u}$. The maximum principale will be proved as follows, first we define a family of perturbed controles u^{ϵ} ,where u^{ϵ} is spike variation of the optimal control $\hat{u}$ on a small time interval, further we use some sort of Taylor expantion of the state trajectory and the cost functional around the optimal control. By sending the perturbation to zero, one obtains some inequality

$$J(u^{\epsilon}) - J(\hat{u}) \geq 0,$$

then by Itô's representation theorem of martingale Brownian, the maximum principle can be expressed in terms of an adjoint process.

We suppose that a d-dimensional Brownian motion B is define on a complete probability space $(\Omega, F, (F_t), P)$. where (F_t) is the p-augmentation of the naturel filtration (F_t^w) defined by $F_t^w = \sigma(B_s : 0 \leq s \leq t)$ $\forall t \in [0, \infty]$. Let us consider the SDE

$$dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t. \tag{2.1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$, are given

The control problem which we will be studying is of the form

$$J(u) = E\left[\int_0^T f(t, x_t, u_t)dt + g(x_T)\right]. \tag{2.2}$$

where $f : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$, and $g : \mathbb{R}^d \to \mathbb{R}$

We might consider strong solution whose existence is given by the work of Itô, under the condition that b, and σ are Lipschitz continuous in x, The following assumptions will be in force throughout this chapter:

$$b, \sigma, f, g \text{ are continuously differentiable with respect to } x, \tag{2.3}$$

$$\text{They and all their derivatives } b_x, \sigma_x, f_x, g_x \text{ are continuous in } (x, u), \tag{2.4}$$

$$\text{The derivatives } b_x, f_x \text{ are bounded uniformly in } u, \tag{2.5}$$

$$\text{The derivatives } \sigma_x, g_x \text{ are bounded,} \tag{2.6}$$

$$b, \sigma \text{ are bounded by } C(1 + |x| + |u|), \tag{2.7}$$

The problem is to minimize the functional $J(u)$ over all $u \in U$, i.e. we seek $\hat{u}$ such that

$$J(\hat{u}) = \inf_{u \in U} J(u), \tag{2.8}$$

such controls $\hat{u}$ are called optimal controls, $\hat{x}$ is the corresponding solution of the SDE (2.1). Under the above hypothesise, the SDE (2.1) has a unique strong solution, such that for any $p > 0$,

$$E\left[\sup_{0 \leq t \leq T} |x_t|^p\right] < \infty. \tag{2.9}$$

and the functional J is a well defined.

The stochastic maximum principle is given by the following theorem.

Theorem 2.1. *Let $\hat{u}$ be an optimal control minimizing the cost J over U, and let $\hat{x}$ be the corresponding optimal trajectory. Then there exists a unique pair of adapted processes*

$$(p, q) \in L^2\left([0, T]; \mathbb{R}^{n \times d}\right) \times L^2\left([0, T]; \mathbb{R}^{n \times d}\right)$$

which is the solution of the BSDE

$$\begin{cases} -dp_t = [f_x(t,\hat{x}_t,\hat{u}_t) + b_x^T(t,\hat{x}_t,\hat{u}_t) + \sigma_x^T(t,\hat{x}_t)q_t]dt - q_t dB_t, \\ p_T = g_x(\hat{x}_t), \end{cases}$$

such that for all $a \in A$,

$$H(t,\hat{x}_t,a,p_t) \leq H(t,\hat{x}_t,\hat{u}_t,p_t), P - a.e,$$

2.1.1 Approximation of trajectories.

To obtain the variational inequality in the stochastic maximum principle, we define the strong perturbation of the control, sometimes called the spike variation

$$u_t^\epsilon = \begin{cases} v & \text{if } t \in [\tau, \tau+\epsilon], \\ \hat{u}_t & \text{otherwise}, \end{cases} \tag{2.11}$$

where $0 \leq \tau < T$ is fixed, $\epsilon > 0$ is sufficiently small, and v is an arbitrary $A-$valued, $F_\tau-$measurable random variable such that $E[|v|^2] < +\infty$. If x_t^ϵdenoted the trajectory associated with u^ϵ, then

$$\begin{aligned} x_t^\epsilon &= x_t,\ t \leq \tau, \\ dx_t^\epsilon &= b\left(t, x_t^\epsilon, v\right) dt + \sigma\left(t, x_t^\epsilon\right) dB_t,\ \tau < t < \tau+\epsilon, \\ dx_t^\epsilon &= b\left(t, x_t^\epsilon, u_t\right) dt + \sigma\left(t, x_t^\epsilon\right) dB_t,\ \tau+\epsilon < t < T. \end{aligned}$$

Lemma 2.2. under the assumption (2.3)-(2.7). *We have*

$$\lim_{\epsilon \to 0} E\left[\sup_{t\in[0,T]} |x_t^\epsilon - \hat{x}_t|^2\right] = 0. \tag{2.12}$$

Proof. By squaring and taking the expectation we get

$$\begin{aligned} E\left[|x_t^\epsilon - \hat{x}_t|^2\right] &\leq 3E\left[\int_0^t |b(s, x_s^\epsilon, u_s^\epsilon) - b(s, \hat{x}_s, u_s^\epsilon)|^2 ds\right] \\ &+3E\left[\left|\int_0^t b(s, \hat{x}_s, u_s^\epsilon) - b(s, \hat{x}_s, \hat{u}_s) ds\right|^2\right] \\ &+3E\left[\int_0^t |\sigma(s, x_s^\epsilon) - \sigma(s, \hat{x}_s)|^2 ds\right]. \end{aligned}$$

by (2.7) we obtain

$$\begin{aligned} E\left[\left|\int_0^t b(s, \hat{x}_s, u_s^\epsilon) - b(s, \hat{x}_s, \hat{u}_s) ds\right|^2\right] &\leq \left|C\left(1 + E\left[\sup_{t\in[0,T]} |\hat{x}_t|\right]\right)\right|^2 \epsilon^2 \\ &\leq C^2 (1+M)^2 \epsilon^2. \end{aligned}$$

Since b, σ are Lipschitz in x, then

$$E\left[|x_t^\epsilon - \hat{x}_t|^2\right] \leq KE\left[\int_0^t |x_s^\epsilon - \hat{x}_s|^2 ds\right] + K\epsilon^2,$$

Finally, by using Burkholder-Davis-Gundy inequality and Gronwall Lemma we conclude, since $\hat{u}$ is optimal, then

$$J(\hat{u}) \leq J(u^\epsilon) = J(\hat{u}) + \epsilon \frac{dJ(u^\epsilon)}{d\epsilon}|_{\epsilon=0} + o(\epsilon),$$

if the indicated derivation exists, thus necessary condition for optimality is that

$$\frac{dJ(u^\epsilon)}{d\epsilon}|_{\epsilon=0} \geq 0.$$

■

The first of this subsection is devoted to the computation of this derivative. Note that since $b(t, x, u)$ and $f(t, x, u)$ are sufficiently integrable, then the following property holds:

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} E[|h(s, x_s, u_s) - h(s, x_s, u_s)|^2] ds = 0, \; ds - a.e, \tag{2.13}$$

where h stands for b or f.

Choose τ such that (2.13) holds. We define y as the solution of the linear SDE

$$\begin{cases} dy_t = b_x(s, x_s, u_s)y_s ds + \sigma_x(s, x_s)y_s dB_s, \ \tau \leq s \leq T, \\ y_\tau = b(\tau, x_\tau, v) - b(\tau, x_\tau, u_\tau). \end{cases} \tag{2.14}$$

and define ς by

$$\begin{cases} d\varsigma_t = f_x(s, x_s, u_s)y_s ds, \ \tau \leq s \leq T, \\ \varsigma_\tau = f(\tau, x_\tau, v) - f(\tau, x_\tau, u_\tau). \end{cases}$$

We are now able to obtain the following differentiation results,

Lemma 2.3. *under the assumption (2.3)-(2.7). We have*

$$\lim_{\epsilon \to 0} E\left[\left|\frac{x_t^\epsilon - \hat{x}_t}{\epsilon} - y_t\right|^2\right] = 0, \tag{2.15}$$

$$\lim_{\epsilon \to 0} E\left[\left|\frac{1}{\epsilon}\int_\tau^T (f(t, \hat{x}_t, u_t^\epsilon) - f(t, \hat{x}_t, \hat{u}_t))dt - \varsigma_T\right|^2\right] = 0 \tag{2.16}$$

Proof. Denote

$$\tilde{x}_t^\epsilon = \frac{x_t^\epsilon - \hat{x}_t}{\epsilon} - y_t$$

then, we have for $t \in [\tau, \tau + \epsilon]$

$$\begin{aligned} d\tilde{x}_t^\epsilon &= \frac{1}{\epsilon}\left[b(t, \hat{x}_t + \epsilon\left(y_t + \tilde{x}_t^\epsilon\right), v) - b(t, \hat{x}_t, \hat{u}_t) - \epsilon b_x(t, \hat{x}_t, \hat{u}_t)y_t\right] dt \\ &\quad + \frac{1}{\epsilon}\left[\sigma(t, \hat{x}_t + \epsilon\left(y_t + \tilde{x}_t^\epsilon\right)) - \sigma(t, \hat{x}_t) - \epsilon\sigma_x(t, \hat{x}_t)y_t\right] dB_t, \\ \tilde{x}_\tau^\epsilon &= -\left[b(\tau, \hat{x}_\tau, v) - b(\tau, \hat{x}_\tau, \hat{u}_\tau)\right], \end{aligned}$$

or also

$$\begin{aligned}
\tilde{x}^{\epsilon}_{\tau+\epsilon} &= \frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\left[b(s,\hat{x}_s+\epsilon\left(y_s+\tilde{x}^{\epsilon}_s\right),v)-b(s,\hat{x}_s,v)\right]ds \\
&+\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\left[b(s,\hat{x}_s,v)-b(\tau,\hat{x}_\tau,v)\right]ds-\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\left[b(s,\hat{x}_s,\hat{u}_s)-b(\tau,\hat{x}_\tau,\hat{u}_\tau)\right]ds \\
&+\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\left[\sigma(s,\hat{x}_s+\epsilon\left(y_s+\tilde{x}^{\epsilon}_s\right))-\sigma(s,\hat{x}_s)\right]dB_t \\
&-\int_{\tau}^{\tau+\epsilon} b_x(s,\hat{x}_s,\hat{u}_s)y_s ds-\int_{\tau}^{\tau+\epsilon}\sigma_x(s,\hat{x}_s)y_s dB_t.
\end{aligned}$$

from which we can deduce

$$\begin{aligned}
E\left|\tilde{x}^{\epsilon}_{\tau+\epsilon}\right|^2 &\leq KE\left[\sup_{t\in[\tau,\tau+\epsilon]}\left|x^{\epsilon}_t-\hat{x}_t\right|^2\right]+KE\left[\sup_{t\in[\tau,\tau+\epsilon]}\left|\hat{x}_t-\hat{x}_\tau\right|^2\right] \\
&+KE\left[\int_{\tau}^{\tau+\epsilon}|y_t|^2dt\right]+\frac{K}{\epsilon}E\left[\int_{\tau}^{\tau+\epsilon}|b(t,\hat{x}_t,\hat{u}_t)-b(\tau,\hat{x}_\tau,\hat{u}_\tau)|^2dt\right].
\end{aligned}$$

by the choice of τ and Lemma (2.2) the last term tendes to 0.

Now, for $\tau+\epsilon\leq t\leq T$,

$$\begin{aligned}
d\tilde{x}^{\epsilon}_t &= \frac{1}{\epsilon}\left[b(t,\hat{x}_t+\epsilon\left(y_t+\tilde{x}^{\epsilon}_t\right),\hat{u}_t)-b(t,\hat{x}_t,\hat{u}_t)-\epsilon b_x(t,\hat{x}_t,\hat{u}_t)y_t\right]dt \\
&+\frac{1}{\epsilon}\left[\sigma(t,\hat{x}_t+\epsilon\left(y_t+\tilde{x}^{\epsilon}_t\right))-\sigma(t,\hat{x}_t)-\epsilon\sigma_x(t,\hat{x}_t)y_t\right]dB_t, \\
&= \int_0^1 b_x(t,\hat{x}_t+\lambda\epsilon\left(y_t+\tilde{x}^{\epsilon}_t\right),\hat{u}_t)\tilde{x}^{\epsilon}_t d\lambda dt+\int_0^1\sigma_x(t,\hat{x}_t+\lambda\epsilon\left(y_t+\tilde{x}^{\epsilon}_t\right))\tilde{x}^{\epsilon}_t d\lambda dB_t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\left|\tilde{x}^{\epsilon}_t\right|^2 &\leq E\left|\tilde{x}^{\epsilon}_{\tau+\epsilon}\right|^2+KE\left[\int_{\tau+\epsilon}^{T}\left|\tilde{x}^{\epsilon}_s\right|^2ds\right] \\
&+E\left[\left\{\int_{\tau+\epsilon}^{T}|y_s|.\left|\int_0^1\left(b_x(s,\hat{x}_s+\lambda\epsilon\left(x^{\epsilon}_s-\hat{x}_s\right),\hat{u}_s)-b_x(s,\hat{x}_s,\hat{u}_s)\right)d\lambda\right|ds\right\}^2\right] \\
&+E\left[\int_{\tau+\epsilon}^{T}|y_s|^2.\left|\int_0^1\left(\sigma_x(s,\hat{x}_s+\lambda\epsilon\left(x^{\epsilon}_s-\hat{x}_s\right))-\sigma_x(s,\hat{x}_s)\right)d\lambda\right|^2ds\right],
\end{aligned}$$

b_x, σ_x are bounded by the Lipschitz constant, we conclude by Lemma (2.2), and Burkholder-Davis-Gundy inequality that

$$\lim_{\epsilon \to 0} \sup_{\tau+\epsilon \leq t \leq T} E\left[|\tilde{x}_t^\epsilon|^2\right] = 0, .$$

(2.16) can be proved by the similar way. ■

2.1.2 Adjoint process and variational inequality

Corollary 2.4. Under the assumption (2.3)-(2.7). We have

$$\frac{dJ(u^\epsilon)}{d\epsilon}|_{\epsilon=0} = E[g_x(x_T).y_T + \varsigma_T] \tag{2.17}$$

.Let us introduce the adjoint process and the first variational inequality from (3.6). Let $\Phi(t,\tau)$ be the solution of the linear SDE

$$\begin{cases} d\Phi(t,\tau) = b_x(t,\hat{x}_t,\hat{u}_t)\Phi(t,\tau)dt + \sigma_x(t,\hat{x}_t)\Phi(t,\tau)dB_t, \ t > \tau, \\ \Phi(\tau,\tau) = I_d. \end{cases} \tag{2.18}$$

This equation is linear with bounded coefficients. Hence it admits a unique strong solution which is invertible, and its inverse Ψ_t is the unique solution of

$$\begin{cases} d\Psi_t = [\sigma_x(t,x_t)\Psi_t\sigma_x^*(t,x_t) - \Psi_t b_x(t,x_t,u_t)]dt - \Psi_t\sigma_x(t,x_t)dB_t \ , t > \tau \ , \\ \Psi(\tau,\tau) = I_d. \end{cases} \tag{2.19}$$

Morever $\Phi(t,\tau)$ satisfies a semigroup property; that is, if $t > s > \tau$ then $\Phi(t,r) = \Phi(t,s).\Phi(t,r)$, which implies in particular that $\Phi(t,\tau) = \Phi_t.\Psi_\tau$, where $\Phi(t) = \Phi(t,0)$ and $\Psi(t) = \Psi(t,0)$.

By applying the Itô's formula to process $\Psi_t y_t$, it is easy to check that

$$y_t = \Phi(t,\tau)(b(\tau,\hat{x}_\tau,v) - b(\tau,\hat{x}_\tau,\hat{u}_\tau))$$

Then replacing y_t with its value in (2.17), it holds that

$$\begin{aligned}\frac{dJ(u^{\epsilon})}{d\epsilon}|_{\epsilon=0} &= E\left[\int_{\tau}^{T} f_x(s,\hat{x}_s,\hat{u}_s)\Phi(s,\tau)(b(\tau,\hat{x}_\tau,v)-b(\tau,\hat{x}_\tau,\hat{u}_\tau))ds\right] \\ &+E\left[g_x(\hat{x}_T).\Phi(T,\tau)(b(\tau,\hat{x}_\tau,v)-b(\tau,\hat{x}_\tau,\hat{u}_\tau))\right] \\ &+E\left[f(\tau,\hat{x}_\tau,v)-f(\tau,\hat{x}_\tau,\hat{u}_\tau)\right].\end{aligned}$$

Now if we define the adjoint process by

$$p_t = E\left[\Phi(T,t)g_x(\hat{x}_T)+\int_t^T \Phi(s,t)f_x(s,\hat{x}_s,\hat{u}_s)ds\diagup F_t\right], \tag{2.20}$$

it follows that

$$\frac{dJ(u^{\epsilon},\xi)}{d\epsilon}|_{\epsilon=0} = E[p_\tau.\left\{b(\tau,\hat{x}_\tau,v)-b(\tau,\hat{x}_\tau,\hat{u}_\tau)\right\}+\left\{f(\tau,\hat{x}_\tau,v)-f(\tau,\hat{x}_\tau,\hat{u}_\tau)\right\}]$$

If define the Hamiltonian H from $[0,T]\times\mathbb{R}^n\times A\times\mathbb{R}^n$ into $\mathbb{R}$ by

$$H(t,x,v,p) = f(t,x,v)+p.b(t,x,v),$$

Then we get from the optimality of $\hat{u}$ the variational inequality

$$0\leq E[H(\tau,x_\tau,v,p_\tau)-H(\tau,x_\tau,u_\tau,p_\tau)],d\tau-a.e. \tag{2.21}$$

We now look for an equation satisfied by the adjoint process (2.20), one has

$$\begin{aligned}& E\left[\Phi_T g_x(\hat{x}_T)+\int_t^T \Phi_s f_x(s,\hat{x}_s,\hat{u}_s)ds\diagup F_t\right] \\ = \;& E\left[\Phi_T g_x(\hat{x}_T)+\int_0^T \Phi_s f_x(s,\hat{x}_s,\hat{u}_s)ds\diagup F_t\right]-\int_0^t \Phi_s f_x(s,\hat{x}_s,\hat{u}_s)ds.\end{aligned}$$

The term

$$E\left[\Phi_T g_x(\hat{x}_T)+\int_0^T \Phi_s f_x(s,\hat{x}_s,\hat{u}_s)ds\diagup F_t\right]$$

is a square integrable F_t-martingale, then by the Itô representation of Browanian martingales, we get

$$\begin{aligned} & E\left[\Phi_T^* g_x(\hat{x}_T) + \int_0^t \Phi_t^* h_x(t, \hat{x}_t, \hat{u}_t) dt \diagup F_t\right] \\ = \; & E\left[\Phi_T^* g_x(\hat{x}_T) + \int_0^t \Phi_t^* h_x(t, \hat{x}_t, \hat{u}_t) dt\right] + \int_0^t Q_s dB_s, \end{aligned}$$

where Q is an adapted process. Next, by applying Itô's formula to

$$p_t = \Psi_t^* . E\left[\Phi_T^* g_x(\hat{x}_T) + \int_t^T \Phi_s^* h_x(s, \hat{x}_s, \hat{u}_s) dt \diagup F_t\right]$$

it is easy to see that p_t satisfies the linear $BSDE$

$$\begin{cases} -dp_t & = \left[f_x(t, \hat{x}_t, \hat{u}_t) + b_x^*(t, \hat{x}_t, \hat{u}_t) p_t + \sigma_x^*\left(t, \hat{x}_t\right) q_t\right] dt - q_t dB_t, \\ p_T & = g_x(\hat{x}_T). \end{cases} \tag{2.22}$$

Where $q_t \in L^2([0.T]\,; \mathbb{R}^{n\times d})$, is given by

$$q_t = \Psi_t^* Q_t - \sigma_t^*(t, \hat{x}_t) p_t.$$

Theorem 1.5. (the stochastic maximum principle).*Let $\hat{u}$ be an optimal control minimizing the cost J over U, and let $\hat{x}$ be the corresponding optimal trjectory. Then there exists a unique pair of adapted processes*

$$(p, q) \in L^2\left([0, T]\,; \mathbb{R}^{n\times d}\right) \times L^2\left([0, T]\,; \mathbb{R}^{n\times d}\right)$$

which is the solution of the $BSDE$ () such that for all $a \in A$,

$$H(t, x_t, a, p_t) - H(t, x_t, a, p_t) \geq 0, \;\; P - a.e,$$

of Theorem 2.1. From (2.21) we get

$$E\left[H\left(t, \hat{x}_t, v, p_t\right) - H\left(t, \hat{x}_t, \hat{u}_t, p_t\right)\right] \geq 0, \;\; dt - a.e,$$

for every bounded A-valued, F_t-measurable random variable v such that $E\left|v\right|^2 < +\infty$. Let $a \in A$ be a deterministic element and F be an arbitrary element of the σ-algebra F_t, and set

$$w_t = a 1_F + \hat{u}_t 1_{\Omega - F}.$$

it is obvius that w is an admissible control Applying (2.21) with w we get

$$E\left[1_F (H\left(t, \hat{x}_t, a, p_t\right) - H\left(t, \hat{x}_t, \hat{u}_t, p_t\right))\right] \geq 0, \ \forall F \in F_t,$$

which implies that

$$E\left[(H\left(t, \hat{x}_t, a, p_t\right) - H\left(t, \hat{x}_t, \hat{u}_t, p_t\right)) \diagup F_t\right] \geq 0.$$

The quantity inside the conditional expectation is F_t-measuranle, and thus the result follows immediately. ■

2.2 The near maximum principle

Near optimal controls is as important as exact optimal controls for both theory and applications, Indeed, optimal controls may not even exist in many situations, while near optimal controls always exist. This section concerns necessary conditions for near optimality or the maximum principle in near optimal controls, for systems governed by the Ito stochastic differential equations with diffusion-independant, and the system are allowed to be degenerate. It is shown that any near optimal control nearly maximizes the Hamiltonian. The proof is based on some stability and continuity of the cost functional and the adjoint process with respect to the control variable, together with the Ekeland's principle.For more detail for this subject we refer the reader to Mezerdi [93],Zhou [123].

Definition 2.5. *For a given* $\epsilon > 0$*, an admissible control* $u_{\cdot}^{\epsilon}$*, is called* $\epsilon -$ *optimal* (or near optimal) *if* $J(u^{\epsilon}) \leq J(u) + \epsilon$. for all $u \in U$

In this section we derive necessary optimality conditions for near optimal controls.

This result is based on Ekeland's variational principale which is given by the following.

Lemma 2.6 (Ekland principle)**.** *Let(S,d) be a metric space and* $\rho : S \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be lower-semicontinuous and bounded from below .For* $\epsilon > 0$, sup*pose* $u^{\epsilon} \in S$ *satisfies* $\rho(u^{\epsilon}) \leq \inf_{u \in S} \rho(u) + \epsilon$.*Then for any* $\lambda > 0$*, there exists* $u^{\lambda} \in S$ *such that*

$$\rho(u^{\lambda}) \leq \rho(u^{\epsilon}).$$

$$d(u^{\lambda}, u^{\epsilon}) \leq \lambda.$$

$$\rho(u^{\lambda}) \leq \rho(u) + \frac{\epsilon}{\lambda}.d(u, u^{\lambda}), \text{ for all } u \in S.$$

define a matric on U by

$$d(u, v) = P\{(t, w) \in [0, T] \times \Omega : u(t, x) \neq v(t, x)\}$$

where P is the product measure of the Lebesgue measure and P. Since A is closed, it can be shown similarly to [50] , that $U[0, T]$ is a complete metric space under d.

This Lemma is mainly devoted to investigating certain continuity of the controlled SDE, the functional J, and the adjoint process (p, q) with respect to the metric d.

Lemma 2.7.

(1) For any $p \geq 0$, there is a constant $C > 0$ such that, for any $u, v \in U$, along with the corresponding trajectories x^u, x^v, it holds that

$$E\left[\sup_{t\in[0,T]} |x_t^u - x_t^v|^{2p}\right] \leq Cd\,(u,v)^{\frac{1}{2}}.$$

(2) There is a constant $C > 0$ such that for any $u,\ v \in U$ along with the corresponding adjoints processes p^u, p^v it holds that

$$E\left[|p_t^u - p_t^v|^2 + \int_0^T |q_t^u - q_t^v|^2\, dt\right] \leq Cd\,(u,v)^{\frac{1}{2}}.$$

(3) The cost functional $J : (U,d) \to \mathbb{R}$ is continuous. More precisely, there is a constant $C_2 > 0$ for any $u,\ v \in U$ such that

$$|J\,(u) - J\,(v)| \leq C_2 d\,(u,v)^{\frac{1}{2}}.$$

Proof. For(1) and (3), see Mezerdi [93], Zhou [123].

(2) Applying Itô's formula to $|p_t^u - p_t^v|^2$, it holds that

$$\begin{aligned}
&|p_t^u - p_t^v|^2 + \int_t^T |q_s^u - q_s^v|^2\, ds = |g_x\,(x_T^u) - g_x\,(x_T^v)|^2 \\
&+2\int_t^T \langle p_s^u - p_s^v, F^u\,(s, x_s^u, p_s^u, q_s^u, u_s) - F^v\,(s, x_s^v, p_s^v, q_s^v, v_s)\rangle\, ds \\
&+\int_t^T \langle p_s^u - p_s^v, q_s^u - q_s^v\rangle\, dW_s,
\end{aligned}$$

where

$$\begin{aligned}
F^u\,(s, x_s^u, p_s^u, q_s^u, u_s) &= b_x\,(s, x_s^u, u_s)\, p_s^u + \sigma_x\,(s, x_s^u)\, q_s^u + f\,(s, x_s^u, u_s)\,, \\
F^v\,(s, x_s^v, p_s^v, q_s^v, v_s) &= b_x\,(s, x_s^v, v_s)\, p_s^v + \sigma_x\,(s, x_s^v)\, q_s^v + f\,(s, x_s^v, v_s)\,,
\end{aligned}$$

by using the Young's inequality and taking expectations in bouth sides, we get

$$E\left[|p_t^u - p_t^v|^2 + \int_t^T |q_s^u - q_s^v|^2 \, ds\right] \leq E\left[|g_x(x_T^u) - g_x(x_T^v)|^2\right]$$
$$+\alpha^2 E\left[\int_t^T |p_s^u - p_s^v|^2 \, ds\right]$$
$$+\frac{2}{\alpha^2} E\left[\int_t^T |F^u(s, x_s^u, p_s^u, q_s^u, u_s) - F^u(s, x_s^v, p_s^v, q_s^v, v_s)|^2 \, ds\right]$$
$$+\frac{2}{\alpha^2} E\left[\int_t^T |F^u(s, x_s^v, p_s^v, q_s^v, v_s) - F^v(s, x_s^v, p_s^v, q_s^v, v_s)|^2 \, ds\right],$$

$$\leq \; E\left[|g_x(x_T^u) - g_x(x_T^v)|^2\right] + \alpha^2 E\left[\int_t^T |p_s^u - p_s^v|^2 \, ds\right]$$
$$+\frac{2}{\alpha^2} E\left[\int_t^T |b_x(s, x_s^u, u_s)|^2 |p_s^u - p_s^v|^2 + |\sigma_x(s, x_s^u)|^2 |q_s^u - q_s^v|^2 \, ds\right]$$
$$+\frac{2}{\alpha^2} E\left[\int_t^T |b_x(s, x_s^u, u_s) - b_x(s, x_s^v, v_s)|^2 |p_s^v|^2 \, ds\right.$$
$$\left. + \int_t^T |\sigma_x(s, x_s^u) - \sigma_x(s, x_s^v)|^2 |q_s^v|^2 \, ds\right]$$
$$+\frac{2}{\alpha^2} E\left[\int_t^T |f_x(s, x_s^u, u_s) - f_x(s, x_s^v, v_s)|^2 \, ds\right].$$

The result follows from (1), the fact that b_x, σ_x, f_x and g_x are Lipschitz continuous in x and Gronwall's Lemma. ■

Define the Hamiltonian H into $\mathbb{R}$ by

$$H(t, x, u, p) = p.f(t, x, u) + q.\sigma(t, x) + f(t, x, u), \tag{2.23}$$

where (p_t, q_t) is the adjoint equation corresponding to (u, x).

Theorem 2.8. *For any $\epsilon > 0$, let (p^ϵ, q^ϵ) the solution to (2.22) corresponding to (u^ϵ, x^ϵ) an $\epsilon-$optimal pair of the problem $(2.1) - (2.2)$, then for any $\gamma \in [0, 1/3)$, there exists a constant $C(\gamma) > 0$, such that*

$$E\left[H(t, x_t^\epsilon, u_t^\epsilon, p_t^\epsilon) - H(t, x_t^\epsilon, v, p_t^\epsilon)\right] \geq C\epsilon, \;\forall v \in A. \tag{2.24}$$

Proof. By lemme 2.7 and the Ekeland's principle with $\lambda = \epsilon^{\frac{2}{3}}$, there is an admissible pair $(\tilde{u}^{\epsilon}, \tilde{x}^{\epsilon})$ such that

$$d\left(u^{\epsilon}, \tilde{u}^{\epsilon}\right) \leq \epsilon^{\frac{2}{3}}, \text{ and } \tilde{J}\left(\tilde{u}^{\epsilon}\right) \leq \tilde{J}\left(u\right) \text{ for any } u \in U,$$

where

$$\tilde{J}\left(u\right) = J\left(u\right) + \epsilon^{\frac{1}{3}} d\left(u, \tilde{u}^{\epsilon}\right),$$

this means that $\tilde{u}^{\epsilon}$ is an optimal control for the system (2.1), with a new cost function $\tilde{J}$. Next, we use the spike variation technique to derive a maximum principle for $\tilde{u}^{\epsilon}$. To this end, let $\tau \in [0, T]$ and $v \in A$ be fixed. For any $\theta > 0$, define $\tilde{u}^{\epsilon,\theta} \in U$, by

$$\tilde{u}_t^{\epsilon,\theta} = \begin{cases} \tilde{u}_t^{\epsilon} & \text{if } t \in [0, T] \diagdown [\tau, \tau + \epsilon], \\ v & \text{if } t \in [\tau, \tau + \epsilon], \end{cases} \tag{2.25}$$

the fact that $\tilde{J}\left(\tilde{u}^{\epsilon}\right) \leq \tilde{J}\left(\tilde{u}^{\epsilon,\theta}\right)$, and $d\left(\tilde{u}^{\epsilon,\theta}, \tilde{u}^{\epsilon}\right) \leq \theta$, imply that

$$J\left(\tilde{u}^{\epsilon,\theta}\right) - J\left(\tilde{u}^{\epsilon}\right) \geq -\epsilon^{\frac{1}{3}} \theta,$$

therefor the map $\theta \to J\left(\tilde{u}^{\epsilon,\theta}\right)$ is differentiable at $\theta = 0$, and that

$$\frac{dJ\left(\tilde{u}^{\epsilon,\theta}\right)}{d\theta} \mid_{\theta=0} = E\left[H\left(t, \tilde{x}_t^{\epsilon}, \tilde{u}_t^{\epsilon}, \tilde{p}_t^{\epsilon}\right)\right] - E\left[H\left(t, \tilde{x}_t^{\epsilon}, v, \tilde{p}_t^{\epsilon}\right)\right] + \epsilon^{\frac{1}{3}} \geq 0,$$

Now, we are derive an estimate for the terme $E\left[H\left(t, \tilde{x}_t^{\epsilon}, \tilde{u}_t^{\epsilon}, \tilde{p}_t^{\epsilon}\right) - H\left(t, \tilde{x}_t^{\epsilon}, v, \tilde{p}_t^{\epsilon}\right)\right]$ with all the $\tilde{x}^{\epsilon}$, $\tilde{u}^{\epsilon}$ and $\tilde{p}^{\epsilon}$ replaced by $x^{\epsilon}, u^{\epsilon}$ and p^{ϵ}. To this end, we first estimate the following difference

$$E\left[\int_0^T \tilde{p}_t^{\epsilon} \cdot \left\{b\left(t, \tilde{x}_t^{\epsilon}, u\right) - b\left(t, \tilde{x}_t^{\epsilon}, \tilde{u}_t^{\epsilon}\right)\right\} dt - \int_0^T p_t^{\epsilon} \cdot \left\{b\left(t, x_t^{\epsilon}, u\right) - b\left(t, x_t^{\epsilon}, u_t^{\epsilon}\right)\right\} dt\right] = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= E\left[\int_0^T \{\tilde{p}_t^\epsilon - p_t^\epsilon\}\{b(t,\tilde{x}_t^\epsilon,u) - b(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon)\}\,dt\right], \\ I_2 &= E\left[\int_0^T p_t^\epsilon\{b(t,\tilde{x}_t^\epsilon,u) - b(t,x_t^\epsilon,u)\}\,dt\right], \\ I_3 &= E\left[\int_0^T p_t^\epsilon\{b(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon) - b(t,x_t^\epsilon,u_t^\epsilon)\}\,dt\right], \end{aligned}$$

so that

$$\begin{aligned} I_1 &\le \left\{E\left[\int_0^T |\tilde{p}_t^\epsilon - p_t^\epsilon|^2\,dt\right]\right\}^{\frac{1}{2}}.\left\{E\left[\int_0^T |b(t,\tilde{x}_t^\epsilon,u) - b(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon)|^2\,dt\right]\right\}^{\frac{1}{2}}, \\ &\le Cd(u^\epsilon,\tilde{u}^\epsilon)^{\frac{1}{4}}.\left\{CE\left[\int_0^T (1+|\tilde{x}_t^\epsilon|)^2\,dt\right]\right\}^{\frac{1}{2}} \le C\epsilon^\gamma. \end{aligned}$$

Next, by the Schwartz inequality, one has

$$\begin{aligned} I_1 &\le E\left[\int_0^T |p_t^\epsilon|^2\,dt\right]^{\frac{1}{2}}.E\left[\int_0^T |b(t,\tilde{x}_t^\epsilon,u) - b(t,x_t^\epsilon,u)|^2\,dt\right]^{\frac{1}{2}}, \\ &\le CE\left[\int_0^T |\tilde{x}_t^\epsilon - x_t^\epsilon|^2\,dt\right]^{\frac{1}{2}}, \\ &\le Cd(u^\epsilon,\tilde{u}^\epsilon)^{\frac{1}{4}} = C\epsilon^\gamma \end{aligned}$$

Further,

$$\begin{aligned} I_3 &= E\left[\int_0^T p_t^\epsilon\{b(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon) - b(t,\tilde{x}_t^\epsilon,u_t^\epsilon)\}\,dt\right] \\ &\quad + E\left[\int_0^T p_t^\epsilon\{b(t,\tilde{x}_t^\epsilon,u_t^\epsilon) - b(t,x_t^\epsilon,u_t^\epsilon)\}\,dt\right], \\ &\le E\left[\int_0^T |p_t^\epsilon|^2\,dt\right]^{\frac{1}{2}} E\left[\int_0^T |b(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon) - b(t,\tilde{x}_t^\epsilon,u_t^\epsilon)|^2\,\chi_{\tilde{u}_t^\epsilon \neq u_t^\epsilon} dt\right]^{\frac{1}{2}} + C\epsilon^\gamma, \\ &\le Cd(u^\epsilon,\tilde{u}^\epsilon)^{\frac{1}{4}} + C\epsilon^\gamma, \\ &\le C\epsilon^\gamma + C\epsilon^\gamma. \end{aligned}$$

Similarly,

$$E\left[\int_0^T \{(f(t,\tilde{x}_t^\epsilon,u) - f(t,\tilde{x}_t^\epsilon,\tilde{u}_t^\epsilon)) - (f(t,x_t^\epsilon,u) - f(t,x_t^\epsilon,u_t^\epsilon))\}\,dt\right] \le C\epsilon^\gamma.$$

■

2.3 Peng's maximum principle

The main difficulty when facing a general controlled diffusion is that the Itô integral term is not of the same order as the Lebesgue term and thus the first-order variation method fails. This difficulty was overcomed by Peng [98] , who studied the second-order term in the Taylor expansion of the perturbation method arising from the Itô integral. He then obtained a maximum principle for possibly degenerate and control-dependent diffusion, which involves in addition to the first-order adjoint process, a second-order adjoint process.

In this Section we consider the stochastic maximum principle in stochastic control problems of systems governed by a SDE with controlled diffusion coefficient, Let (Ω, F, F_t, P) be a probability space with filtration. Let B_t be an $\mathbb{R}^n$-valued standard Wiener process. We assume that $F_t = \sigma\{B_s : 0 \leq s \leq t\}$. Consider the following stochastic control system:

$$\begin{cases} dx_t = b(t, x_t, u_t)\, dt + \sigma(t, x_t, u_t)\, dB_t, & \text{for } t \in [0, T], \\ x_0 = \alpha, \end{cases} \tag{2.26}$$

where, $b : [0, T] \times \mathbb{R}^d \times U \times \Omega \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$. An admissible control $v.$ is an F_t-adapted process with values in U such that

$$\sup_{0\leq t\leq T} E\,|v_t|^m \prec \infty,\ m \geq 1.$$

where U is a nonempty subset of $\mathbb{R}^k$. We denote the set of all admissible controls by U_{ad}.

Our optimal control problem is to minimize the following cost functional over U_{ad}

$$J(u) = E\left[\int_0^T f(t, x_{t,} u_t)\,dt + g(x_T)\right]. \tag{2.27}$$

where $f : [0, T] \times \mathbb{R}^d \times U \times \Omega \to \mathbb{R}$, $g : \mathbb{R}^d \times \Omega \to \mathbb{R}$, satisfy the following

(H) b, σ, f, g are twice continuously differentiable with respect to x. They and all their derivatives b_x, b_{xx}, σ_x, σ_{xx}, f_x, f_{xx}, g_x, g_{xx}, are continuous in (x, v). b_x, b_{xx}, σ_x, $\sigma_{xx}, f_x, f_{xx}, g_x, g_{xx}$ are bounded, and b,σ, f_x, g_x are bounded by $C(1 + |x| + |v|)$.

2.3.1 Second-order expansion

The second order maximum principle is studied by Peng [98] for a general case, where the diffusion coefficient can contain the control variable and the domain can be non convex. In this section we treat this problem by the second order expantion method based on a kind of variational equation and variational inequality, becausethe usual first order expansion method does not work here. Let $(\hat{x}, \hat{u})$ be an optimal solution of the problem. It is classical to construct a perturbated admissible control in the following way (spike variation)

$$u_t^\epsilon = \begin{cases} v \text{ if } t \in [\tau, \tau + \epsilon], \\ \hat{u}_t \text{ otherwise}, \end{cases}$$

Where $0 \leq \tau < T$ is fixed, $\epsilon > 0$ is sufficiently small, and v is an arbitrary -measurable random variable with values in U, such that $\sup_{\omega \in \Omega} |v(\omega)|$. Let x^ϵ be the trajectory of the control system (2.26) corresponding to the control u^ϵ. We would like to derive the variational inequality from the fact that

$$J(u^\epsilon) - J(\hat{u}) \geq 0.$$

To this end, we need the following estimation.

Lemma.2.9 We suppose (H). Then the following estimate holds

$$E\left[\sup_{0\leq t\leq T}\left|\hat{x}_t - x_t^{\epsilon} - \acute{x}_t - \H{x}_t\right|^2\right] \leq C\epsilon^2. \tag{2.28}$$

where $\acute{x}_t$, $\H{x}_t$ are solutions of

$$\begin{aligned}
d\acute{x}_t &= \left\{b\left(\hat{x}_t, u_t^{\epsilon}\right) - b\left(\hat{x}_t, \hat{u}_t\right) + b_x\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t\right\}dt \qquad (2.29)\\
&\quad + \left\{\sigma\left(\hat{x}_t, u_t^{\epsilon}\right) - \sigma\left(\hat{x}_t, \hat{u}_t\right) + \sigma_x\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t\right\}dB_t,\\
d\H{x}_t &= \left\{b_x\left(\hat{x}_t, \hat{u}_t\right)\H{x}_t + \frac{1}{2}b_{xx}\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t\acute{x}_t\right\}dt\\
&\quad + \left\{\sigma_x\left(\hat{x}_t, \hat{u}_t\right)\H{x}_t + \frac{1}{2}\sigma_{xx}\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t\acute{x}_t\right\}dB_t\\
&\quad + \left\{b_x\left(\hat{x}_t, u_t^{\epsilon}\right) + b_x\left(\hat{x}_t, \hat{u}_t\right)\right\}\acute{x}_t dt\\
&\quad + \left\{\sigma_x\left(\hat{x}_t, u_t^{\epsilon}\right) + \sigma_x\left(\hat{x}_t, \hat{u}_t\right)\right\}\acute{x}_t dB_t, \qquad (2.30)
\end{aligned}$$

Proof. See Peng [98]. ■

Lemma.2.10 Under the assumption (H), we have

$$\begin{aligned}
\circ\left(\epsilon\right) \leq\ & E\left[\int_0^T\left\{f_x\left(\hat{x}_t, \hat{u}_t\right)\left(\acute{x}_t + \H{x}_t\right) + \frac{1}{2}f_{xx}\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t\acute{x}_t\right\}dt\right]\\
& + E\left[g_x\left(\hat{x}_T\right)\left(\acute{x}_T + \H{x}_T\right) + \frac{1}{2}g_{xx}\left(\hat{x}_T\right)\acute{x}_T\acute{x}_T\right]\\
& + E\left[\int_0^T f\left(\hat{x}_t, u_t^{\epsilon}\right) - f\left(\hat{x}_t, \hat{u}_t\right)dt\right]. \qquad (2.31)
\end{aligned}$$

Proof. See Peng [98]. ■

Remark.2.11 In the case where σ does not contain the control variable v, the relation (2.31) can be reduced to

$$\begin{aligned}
O\left(\epsilon\right) \leq\ & E\left[\int_0^T\left\{f_x\left(\hat{x}_t, \hat{u}_t\right)\acute{x}_t +\right\}dt g_x\left(\hat{x}_T\right)\acute{x}_T\right]\\
& + E\left[\int_0^T f\left(\hat{x}_t, u_t^{\epsilon}\right) - f\left(\hat{x}_t, \hat{u}_t\right)dt\right]. \qquad (2.32)
\end{aligned}$$

Thus we need only the first-order variational equation (2.29).

2.3.2 Adjoint processes and variational inequality

Equation (2.29) and (2.30) are called the first order and the second order variational equations. We introduce the first-order and second-order adjoint processes for (2.29) and (2.30). With these processes, we can easily derive the variational inequality from (2.31). The linear terms in the inequality (2.31) can be treated in the following way For simplicity, we let

$$\varsigma_x(t) = \varsigma_x(t, \hat{x}_t, \hat{u}_t), \text{ and } \varsigma_{xx}(t) = \varsigma_{xx}(t, \hat{x}_t, \hat{u}_t), \text{ for } \varsigma = b, \sigma, f, g.$$

Consider a linear stochastic system

$$dz_t = (b_x(t) z_t - \phi(t)) dt + (\sigma_x(t) z_t + \psi(t)) dt, z_0 = 0, \tag{2.33}$$
$$(\phi(.), \psi(.)) \in L^2_F(0, T; \mathbb{R}^n) \times \left(L^2_F(0, T; \mathbb{R}^n)\right)^d,$$

where $\psi(.) = (\psi_1(.), ..., \psi_d(.))$, and $L^2_F(0, T; \mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued adapted processes such that

$$E \int_0^T |\phi(t)|^2 dt \prec \infty.$$

We can construct a linear functional on the Hilbert space $L^2(0, T; \mathbb{R}^n) \times (L^2(0, T; \mathbb{R}^n))^d$ as follows:

$$I(\phi(.), \psi(.)) = E\left[\int_0^T f_x(t) z_t dt + g_x(T) z_T\right].$$

It is easy to verify that $I(.,.)$ is continuous. Then by the Riesz Representation Theorem, there is a unique

$$(p(.), q(.)) \in L^2(0, T; \mathbb{R}^n) \times (L^2(0, T; \mathbb{R}^n))^d, q = (q_1, ..., q_d),$$

such that

$$E\left[\left\{\int_0^T (p(t),\phi(t)) + \sum_{j=1}^{d} (q_i(t),\psi_i(t))\right\} dt\right] = I(\phi(.),\psi(.)), \tag{2.34}$$

$\forall \phi(.), \psi(.) \in L_F^2(0,T;\mathbb{R}^n) \times (L_F^2(0,T;\mathbb{R}^n))^d$. With (2.29) and (2.30), we can apply this result to some of the terms of (2.31)

$$\begin{aligned} E\left[\int_0^T f_x(t)\acute{x}_t dt + g_x(T)\acute{x}_T\right] &= E\int_0^T (p_t,(b(\hat{x}_t,u_t^\epsilon) - b(\hat{x}_t,\hat{u}_t)))\,dt \\ &\quad + E\int_0^T tr\left(q_t^T.(\sigma(\hat{x}_t,u_t^\epsilon) - \sigma(\hat{x}_t,\hat{u}_t))\right) dt. \end{aligned}$$

$$\begin{aligned} &E\left[\int_0^T f_x(\hat{x}_t,\hat{u}_t)\,\ddot{x}_t dt + g_x(T)\,\ddot{x}_T\right] = E\int_0^T p_t^T\left(b_x(\hat{x}_t,u_t^\epsilon) - b_x(\hat{x}_t,\hat{u}_t)\right)\acute{x}_t dt \\ &+E\int_0^T \sum_{j=1}^{d} q_t^{jT}\left(\sigma_x^j(\hat{x}_t,u_t^\epsilon) - \sigma_x^j(\hat{x}_t,\hat{u}_t)\right)\acute{x}_t dt \\ &+E\int_0^T \frac{1}{2}\left[\left(p_t b_{xx}(\hat{x}_t,\hat{u}_t) + \sum_{j=1}^{d} q_t^j\sigma_{xx}^j(\hat{x}_t,\hat{u}_t)\right)\acute{x}_t\acute{x}_t\right] dt, \end{aligned}$$

Then we can rewrite (2.31) as

$$\begin{aligned} o(\epsilon) \leq\; & E\int_0^T (H(\hat{x}_t,u_t^\epsilon,p_t,q_t) - H(\hat{x}_t,\hat{u}_t,p_t,q_t))\,dt \\ & +\frac{1}{2}E\int_0^T \acute{x}_t^T H_{xx}(\hat{x}_t,\hat{u}_t,p_t,q_t)\,\acute{x}_t dt \\ & +\frac{1}{2}E\left[\acute{x}_T^T g_{xx}(\hat{x}_T)\,\acute{x}_t\right], \end{aligned} \tag{2.35}$$

where the Hamiltonian H is defined by

$$H(x,u,p_t,q_t) = f(x,u) + (p,b(x,u)) + \sum_{j=1}^{d}\left(q_j,\sigma^j(x,u)\right).$$

The interesting thing is that the quadratic terms of (2.35) can still be treated by applying the Riesz Representation Theorem. Indeed, applying Ito's formula to the matrix-valued processes $X_t = \acute{x}_t \acute{x}_t^T$

$$\begin{aligned} dX_t &= \left\{ X_t b_x^T(\hat{x}_t, \hat{u}_t) + b_x(\hat{x}_t, \hat{u}_t) X_t^T + \sum_{j=1}^{d} \sigma_x^j X_t \sigma_x^{jT}(\hat{x}_t, \hat{u}_t) \right\} dt \\ &\quad + \left\{ X_t^T \sigma_x(\hat{x}_t, \hat{u}_t) + \sigma_x(\hat{x}_t, \hat{u}_t) X_t^T + \psi^\epsilon(t) \right\} dB_t + \phi^\epsilon(t) dt, \end{aligned} \tag{2.36}$$

where ϕ^ϵ and ψ^ϵ are adapted processes given by

$$\begin{aligned} \phi^\epsilon(t) &= \acute{x}_t \left(b(\hat{x}_t, u_t^\epsilon) - b(\hat{x}_t, \hat{u}_t) \right)^T + \left(b(\hat{x}_t, u_t^\epsilon) - b(\hat{x}_t, \hat{u}_t) \right) \acute{x}_t^T \\ &\quad + \sigma_x(\hat{x}_t, \hat{u}_t) \acute{x}_t \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right)^T \\ &\quad + \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right) \acute{x}_t^T \sigma_x^T(\hat{x}_t, \hat{u}_t) \\ &\quad + \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right) \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right)^T . \\ \psi^\epsilon(t) &= \acute{x}_t \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right)^T + \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right) \acute{x}_t^T, \end{aligned}$$

$$\begin{aligned} E\int_0^T \phi^\epsilon(t)\, dt &\leq E\int_0^T \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right) \left(\sigma(\hat{x}_t, u_t^\epsilon) - \sigma(\hat{x}_t, \hat{u}_t) \right)^T dt + o(\epsilon), \\ E\int_0^T \psi^\epsilon(t)\, dt &\leq o(\epsilon). \end{aligned}$$

Consider the following symmetric matrix-valued linear stochastic differential equations:

$$\begin{aligned} & dZ_t = \left\{ Z_t b_x^T(\hat{x}_t, \hat{u}_t) + b_x(\hat{x}_t, \hat{u}_t) Z_t^T + \sum_{j=1}^{d} \sigma_x^j Z_t \sigma_x^{jT}(\hat{x}_t, \hat{u}_t) \right\} dt \\ & + \left\{ Z_t^T \sigma_x(\hat{x}_t, \hat{u}_t) + \sigma_x(\hat{x}_t, \hat{u}_t) Z_t^T + \psi(t) \right\} dB_t + \phi(t) dt, \\ & Z_0 = 0. \\ & (\phi(.), \psi(.)) \in L_F^2(0,T;\mathbb{R}^{n\times n}) \times (L_F^2(0,T;\mathbb{R}^{n\times n}))^d, \psi = (\psi_1, ..., \psi_d). \end{aligned}$$

where $\mathbb{R}^{n\times n}$ is the space of all nxn real symmetric matrices with the following scalar product:

$$(A_1, A_2)_* = tr(A_1, A_2), \ \forall A_1, A_2 \in \mathbb{R}^{n\times n}.$$

Now, let us construct a linear functional via (2.36):

$$M\left(\phi\left(.\right),\psi\left(.\right)\right)=E\int_{0}^{T}\left(Z_{t},H_{xx}\left(t\right)\right)_{*}dt+E\left[\left(Z_{T},g_{xx}\left(\hat{x}_{T}\right)\right)_{*}\right]$$

Obviously, $M((.),(.))$ is a linear continuous functional on

$$L_{F}^{2}(0,T;\mathbb{R}^{n\times n})\times(L_{F}^{2}(0,T;\mathbb{R}^{n\times n}))^{d}$$

thus there exists a unique $(P,Q)\in L_{F}^{2}(0,T;\mathbb{R}^{n\times n})\times(L_{F}^{2}(0,T;\mathbb{R}^{n\times n}))^{d}$, such that

$$M\left(\phi\left(.\right),\psi\left(.\right)\right)=E\int_{0}^{T}\left[\left(P_{t},\phi\left(t\right)\right)_{*}+\sum_{j=1}^{d}\left(Q_{t}^{j},\psi^{j}\left(t\right)\right)_{*}\right]dt. \tag{2.37}$$

Since for all $x\in\mathbb{R}^{n},A\in\mathbb{R}^{n\times n}$, $\left(xx^{*},A\right)_{*}=tr\left[\left(xx^{*}\right)A\right]=x^{*}Ax$, from (2.36) and (2.37) we can rewrite (2.35) as

$$\begin{aligned}o\left(\epsilon\right)\leq{}&E\int_{0}^{T}\left(H\left(\hat{x}_{t},u_{t}^{\epsilon},p_{t},q_{t}\right)-H\left(\hat{x}_{t},\hat{u}_{t},p_{t},q_{t}\right)\right)dt\\&+\frac{1}{2}E\int_{0}^{T}\left[\left(P_{t},\phi^{\epsilon}\left(t\right)\right)_{*}+\sum_{j=1}^{d}\left(Q_{t}^{j},\psi^{\epsilon j}\left(t\right)\right)_{*}\right]dt.\end{aligned} \tag{2.38}$$

From the definition of,ϕ^{ϵ} and ψ^{ϵ}, we obtain

$$\begin{aligned}o\left(\epsilon\right)\leq{}&E\int_{0}^{T}\left(H\left(\hat{x}_{t},u_{t}^{\epsilon},p_{t},q_{t}\right)-H\left(\hat{x}_{t},\hat{u}_{t},p_{t},q_{t}\right)\right)dt\\&+\frac{1}{2}E\int_{0}^{T}tr\left[\left(\sigma\left(\hat{x}_{t},u_{t}^{\epsilon}\right)-\sigma\left(\hat{x}_{t},\hat{u}_{t}\right)\right)^{T}P_{t}\left(\sigma\left(\hat{x}_{t},u_{t}^{\epsilon}\right)-\sigma\left(\hat{x}_{t},\hat{u}_{t}\right)\right)\right]dt,\end{aligned}$$

Finally, we have

$$\begin{aligned}0\leq{}&H\left(\hat{x}_{\tau},v,p_{\tau},q_{\tau}\right)-H\left(\hat{x}_{\tau},\hat{u}_{\tau},p_{\tau},q_{\tau}\right)\\&+\frac{1}{2}tr\left\{\left(\sigma\left(\hat{x}_{\tau},v\right)-\sigma\left(\hat{x}_{\tau},\hat{u}_{\tau}\right)\right)^{T}P_{\tau}\left(\sigma\left(\hat{x}_{\tau},v\right)-\sigma\left(\hat{x}_{\tau},\hat{u}_{\tau}\right)\right)\right\},\end{aligned} \tag{2.39}$$

for all $v \in U$, $a.e., a.s.$, or, equivalently

$$\begin{aligned} 0 &\leq H\left(\hat{x}_\tau, v, p_\tau, q_\tau - P_\tau \sigma\left(\hat{x}_\tau, \hat{u}_\tau\right)\right) + \frac{1}{2} tr\left(\sigma\sigma^T\left(\hat{x}_\tau, v\right) P_\tau\right) \\ &\quad - H\left(\hat{x}_\tau, \hat{u}_\tau, p_\tau, q_\tau - P_\tau \sigma\left(\hat{x}_\tau, \hat{u}_\tau\right)\right) + \frac{1}{2} tr\left(\sigma\sigma^T\left(\hat{x}_\tau, \hat{u}_\tau\right) P_\tau\right), \\ \forall v &\in U, a.e., a.s. \end{aligned}$$

Where the first-order adjoint process (p, q) is the unique solution of the folowing BSDE

$$\begin{cases} -dp_t = \left(b_x\left(\hat{x}_t, \hat{u}_t\right)^T p_t + \sum\limits_{j=1}^{d} \sigma_x^j\left(\hat{x}_t, \hat{u}_t\right)^T q_t^j - f_x\left(\hat{x}_t, \hat{u}_t\right)\right) dt \\ +q_t dB_t, \\ p_T = -g_x\left(\hat{x}_T\right). \end{cases} \tag{2.40}$$

The second-order adjoint process (P, Q) is the unique solution of the folowing BSDE

$$\begin{cases} -dP_t = b_x\left(\hat{x}_t, \hat{u}_t\right)^T P_t + P_t b_x\left(\hat{x}_t, \hat{u}_t\right) + \sum\limits_{j=1}^{d} \sigma_x^j\left(\hat{x}_t, \hat{u}_t\right)^T P_t \sigma_x^j\left(\hat{x}_t, \hat{u}_t\right) \\ + \sum\limits_{j=1}^{d} \sigma_x^j\left(\hat{x}_t, \hat{u}_t\right)^T Q_t^j + \sum\limits_{j=1}^{d} Q_t^j \sigma_x^j\left(\hat{x}_t, \hat{u}_t\right)^T + H_{xx}\left(\hat{x}_t, \hat{u}_t, p_t, q_t\right) dt \\ + \sum\limits_{j=1}^{d} Q_t^j dB_t^j, \\ p_T = -g_{xx}\left(\hat{x}_T\right). \end{cases} \tag{2.41}$$

Now we are ready to state The stochastic maximum principle

Theorem 2.12. Let (H) hold. If $(\hat{x}_t, \hat{u}_t)$ is a solution of the optimal control problem $(2.26), (2.27)$, then there exists a first order (resp a second order) adjoint process

$$\begin{aligned} (p(.), q(.)) &\in L^2(0, T; \mathbb{R}^n) \times (L^2(0, T; \mathbb{R}^n))^d, \\ (P(.), Q(.)) &\in L_F^2(0, T; \mathbb{R}^{n\times n}) \times (L_F^2(0, T; \mathbb{R}^{n\times n}))^d, \end{aligned}$$

which are, respectively, solutions of (2.40), (resp(2.41)) such that the variational inequality (2.39) holds.

2.4 Connection to dynamic programming principle

It is known that is a relation between the maximum principle and dynamic programming this relation is essentially connection between the value function and the solution of the adjoint equation in the optimal state. In this section we treat two cases, the first when the value function $V \in C^{1,2}([0,T] \times \mathbb{R}^n)$, we prove that

$$(p_t, q_t) = (D_x V(t, \hat{x}_t), D_x^2 V(t, \hat{x}_t)\sigma(t, \hat{x}_t, \hat{u}_t)).$$

But the value function is not in general $C^{1,2}([0,T] \times \mathbb{R}^n)$, then by the viscosity solutions notion can replace the casical derivatives of the value function by superdifferentials.

Let us recall the stochastic optimal control problem formulated in Chapter 1. Consider the stochastic controlled system

$$\begin{cases} dx_t = b\left(t, x_t, u_t\right) dt + \sigma\left(t, x_t, u_t\right) dB_t, \qquad \text{for } t \in \left[0, T\right], \\ x_s = y, \end{cases} \tag{2.45}$$

along with the cost functional

$$J\left(u\right) = E\left[\int_0^T f\left(t, x_{t,} u_t\right) dt + g\left(x_T\right)\right]. \tag{2.46}$$

We denote by $U\left\{[s,T]\right\}$ the set of all (Ω, F, F_t, P), satisfying: (Ω, F, P) is a complete probability space, $\left\{B_t\right\}_{t \geq s}$ is a d-dimensional standard Brownian motion defined on (Ω, F, P) over $\left[s, T\right]$, with $B_0 = 0$ a.s., and $F_t = \sigma\left\{x_r; s \leq r \leq t\right\}$.

The optimal control problem can be stated as follows: For given $(s,y) \in (0,T] \times \mathbb{R}^n$ minimize (2.46) subject to (2.45) over $U\left\{[s,T]\right\}$. The value function is defined as

$$V\left(s, y\right) = \inf_{u \in U} J\left(s, y, u\right),$$

Recall that the HJB equation associated with the optimal control problem (2.45) and (2.46) is as follows

$$-\frac{\partial V}{\partial t}(t,x)+\sup_{u\in U} G\left(t,x,u,-D_xV(t,x),-D_{xx}V(t,x)\right)=0, \tag{2.47a}$$

with the terminal condition

$$V(T,x)=g(x),\forall x\in\mathbb{R}^n. \tag{2.47b}$$

where the Hamiltonian G is defined by

$$G(t,x,u,p,P)=\frac{1}{2}tr\left(\sigma(t,x,u)^T P\sigma(t,x,u)\right)+p.b(t,x,u)-f(t,x,u).$$

On the other hand, let (p,q) be a solution of the BSDE (2.40) associated with the optimal pair $(\hat{u},\hat{x})$. We suppose in this Section all conditions of the Chapter 1 are satisfied.

2.4.1 The smooth case.

We first study the case where the value function V is sufficiently smooth.

Theorem 2.5. *Let $(t,x)\in[0,T]\times\mathbb{R}^n$ be fixed, let $(\hat{u},\hat{x})$ be an optimal solution of the problem $(2.45)-(2.46)$, and W be a classical solution of the HJB $(2.47a)-(2.47b)$, suppose that $W\in C^{1,3}([0,T]\times\mathbb{R}^n;\mathbb{R})$. Then the solution of the BSDE (2.40) is given by*

$$(p_t,q_t)=\left(D_xW(t,\hat{x}_t),D_x^2W(t,\hat{x}_t)\sigma(t,\hat{x}_t,\hat{u}_t)\right). \text{ P-a.s.}$$

Proof. Since $W\in C^{1,3}([0,T]\times O;\mathbb{R})$, we may apply the Itô's rule to $\frac{\partial W}{\partial x_k}(t,\hat{x}_t)$, we obtain

$$\frac{\partial W}{\partial x_k}(T,\hat{x}_t)=\frac{\partial W}{\partial x_k}(t,\hat{x}_t)+\int_t^T\frac{\partial^2 W}{\partial s\partial x_k}(s,\hat{x}_s)\,ds+\int_t^T\sum_{i=1}^n\frac{\partial^2 W}{\partial x_k\partial x_i}(s,\hat{x}_s)\,d\hat{x}_i(s)$$
$$+\frac{1}{2}\int_t^T\sum_{i,j=1}^n a_{ij}(s,\hat{x}_s,\hat{u}_s)\frac{\partial^3 W}{\partial x_k\partial x_i\partial x_j}(s,\hat{x}_s)\,ds$$

Therefore

$$\frac{\partial W}{\partial x_k}\left(T,\hat{x}_t\right)=\frac{\partial W}{\partial x_k}\left(t,\hat{x}_t\right)+\int_t^T\left\{\frac{\partial^2 W}{\partial s\partial x_k}\left(s,\hat{x}_s\right)+\sum_{i=1}^n b_i\left(s,\hat{x}_s,\hat{u}_s\right)\frac{\partial^2 W}{\partial x_k\partial x_i}\left(s,\hat{x}_s\right)\right.$$

$$\left.+\frac{1}{2}\sum_{i,j=1}^n a_{ij}\left(s,\hat{x}_s,\hat{u}_s\right)\frac{\partial^3 W}{\partial x_k\partial x_i\partial x_j}\left(s,\hat{x}_s,\hat{u}_s\right)\right\}ds+\int_t^T\sum_{i=1}^n\frac{\partial^2 W}{\partial x_k\partial x_i}\left(s,\hat{x}_s\right)\sigma\left(s,\hat{x}_s,\hat{u}_s\right)dB_s \tag{2.48}$$

On the other hand, define

$$\begin{aligned} A\left(t,x,u\right) &= \frac{\partial W}{\partial t}\left(t,x\right)+\sum_{i=1}^n b_i\left(t,x,u\right)\frac{\partial W}{\partial x_i}\left(t,x\right) \\ &\quad +\frac{1}{2}\sum_{i,j=1}^n a_{ij}\left(t,x,u\right)\frac{\partial^2 W}{\partial x_i\partial x_j}\left(t,x\right)+f\left(t,x,u\right). \end{aligned}$$

If we differentiate $A\left(t,x,u\right)$ with respect to x_k, and evaluate the result at $\left(x,u\right)=\left(\hat{x}_t,\hat{u}_t\right)$ we get

$$\begin{aligned} &\frac{\partial^2 W}{\partial t\partial x_k}\left(t,\hat{x}_t\right)+\sum_{i=1}^n b_i\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial^2 W}{\partial x_k\partial x_i}\left(t,\hat{x}_t\right)+\frac{1}{2}\sum_{i,j=1}^n a_{ij}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial^3 W}{\partial x_k\partial x_i\partial x_j}\left(t,\hat{x}_t\right) \\ =\; & -\sum_{i=1}^n\frac{\partial b_i}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial W}{\partial x_i}\left(t,\hat{x}_t\right)-\frac{1}{2}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial^2 W}{\partial x_i\partial x_j}\left(t,\hat{x}_t\right) \\ & -\frac{\partial f}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right). \end{aligned} \tag{2.49}$$

Finally, substituting (2.49) into (2.48) which simplifies to

$$\begin{aligned} d\left(\frac{\partial W}{\partial x_k}\left(t,\hat{x}_t\right)\right) &= -\left\{\sum_{i=1}^n\frac{\partial b_i}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial W}{\partial x_i}\left(t,\hat{x}_t\right)\right. \\ &\quad \left.+\frac{1}{2}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial^2 W}{\partial x_i\partial x_j}\left(t,\hat{x}_t\right)+\frac{\partial f}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\right\}dt \\ &\quad +\sum_{i=1}^n\frac{\partial^2 W}{\partial x_k\partial x_i}\left(t,\hat{x}_t\right)\sigma\left(t,\hat{x}_t,\hat{u}_t\right)dB_t. \end{aligned} \tag{2.50}$$

For each $k=1,..,n$. Clearly,

$$\sum_{i=1}^n\frac{\partial b_i}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)\frac{\partial W}{\partial x_i}\left(t,\hat{x}_t\right)=\frac{\partial b^T}{\partial x_k}\left(t,\hat{x}_t,\hat{u}_t\right)D_x W\left(t,\hat{x}_t\right),$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_k}(t,\hat{x}_t,\hat{u}_t)\frac{\partial^2 W}{\partial x_i \partial x_j}(t,\hat{x}_t) = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_k}\left(\sum_{h=1}^{d}\sigma_{ih}\sigma_{jh}\right)(t,\hat{x}_t,\hat{u}_t)\frac{\partial^2 W}{\partial x_i \partial x_j}(t,\hat{x}_t)$$
$$= tr\left(\frac{\partial \sigma^T}{\partial x_k}(t,\hat{x}_t,\hat{u}_t) D_x^2 W(t,\hat{x}_t)\sigma(t,\hat{x}_t,\hat{u}_t)\right).$$

Then (2.50) given by the form

$$d\left(\frac{\partial W}{\partial x_k}(t,\hat{x}_t)\right) = -\left\{\frac{\partial b^T}{\partial x_k}(t,\hat{x}_t,\hat{u}_t) D_x W(t,\hat{x}_t)\right.$$
$$\left. + tr\left(\frac{\partial \sigma^T}{\partial x_k}(t,\hat{x}_t,\hat{u}_t) D_x^2 W(t,\hat{x}_t)\sigma(t,\hat{x}_t,\hat{u}_t)\right) + \frac{\partial f}{\partial x_k}(t,\hat{x}_t,\hat{u}_t)\right\} dt$$
$$+\sum_{i=1}^{n}\frac{\partial W}{\partial x_k \partial x_i}(t,\hat{x}_t)\sigma_i(t,\hat{x}_t,\hat{u}_t) dB_t. \tag{2.51}$$

Now, by definition of the Hamiltonian H we get

$$\frac{\partial H}{\partial x_k}(t,x,u,p,q) = \frac{\partial b^T}{\partial x_k}(t,x,u)p + tr\left(\frac{\partial \sigma^T}{\partial x_k}(t,x,u)q\right) + \frac{\partial f}{\partial x_k}(t,x,u),$$

and define p_t^k the kth coordinate of the column vector p_t by

$$\begin{cases} dp_t^k &= -\dfrac{\partial H}{\partial x_k}(t,\hat{x}_t,\hat{u}_t,p_t,q_t) dt + q_t^k dB_t, \quad \text{for } t\in[0,T], \\ p_T &= \dfrac{\partial g}{\partial x_k}(\hat{x}_T), \end{cases}$$

with $q_t^k dB_t = \sum_{1\leq h\leq d} q_t^{kh} dB_t^h$, for $k = 1,..,n$. Hence, by the uniqueness of the solution to (2.40) and (2.51), we obtain

$$p_t^k = \frac{\partial W}{\partial x_k}(t,\hat{x}_t),$$

and

$$q_t^{kh} = \sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_k \partial x_i}(t,\hat{x}_t)\sigma_{ih}(t,\hat{x}_t,\hat{u}_t)$$

q_t^{kh} the $khth$ element of q_t for $k = 1,..,n$, and $h = 1,..,d$. In particular, note that (p_t, q_t) represents

$$\left(D_x W(t,\hat{x}_t), D_x^2 W(t,\hat{x}_t)\sigma(t,\hat{x}_t,\hat{u}_t)\right)$$

where $\hat{x}_t$ is the optimal solution of the controlled SDE (2.45). ■

Exemple 2.16. Consider the following control problem $U = [-1, 1], n = m = 1.$

$$\begin{cases} dX_t & = 2u_t dt + \sqrt{2} dW_t, \quad \text{for } t \in [0, T], \\ X_s & = y, \end{cases} \tag{2.52}$$

the cost functional is given by

$$J(s, y, u) = E\left[\int_s^T \left(u_t^2 + 1\right) dt - \log ch(X_T)\right], \tag{2.53}$$

for any fixed (s, y, u) applying Itô's formula to the process $\log ch(X_t)$,

$$d(\log ch(X_t)) = th(X_t) dX_t + \frac{1}{2} ch^{-2}(X_t) d\langle X, X \rangle_t$$

then

$$\begin{aligned} \log ch(X_t) &= \log ch(y) + \int_s^T th(X_t)\left[2u_t dt + \sqrt{2} dW_t\right] + \int_s^T ch^{-2}(X_t) dt \\ &= \log ch(y) + \int_s^T \left\{2u_t th(X_t) + ch^{-2}(X_t)\right\} dt + \int_s^T \sqrt{2} th(X_t) dW_t, \end{aligned}$$

combining with (2.53), we get

$$\begin{aligned} J(s, y, u) + \log ch(y) &= E\int_s^T \left[\left(u_t^2 + 1\right) - 2u_t th(X_t) - ch^{-2}(X_t)\right] dt \\ &= E\int_s^T (u_t - th(X_t))^2 dt \geq 0, \end{aligned}$$

because

$$1 - ch^{-2}(x) = th(x),$$

and

$$E\int_s^T (u_t - th(X_t))^2 dt \geq 0$$

hence

$$\begin{aligned} V(s,y) &= -\log ch(y) \\ &= -\log\left(\frac{1}{2}\left(e^{y}+e^{-y}\right)\right), \end{aligned}$$

with $u_t^* = th(X_t^*)$ is an optimal control, and X_t^* satisfies

$$\begin{cases} dX_t^* &= 2th(X_t^*)\,dt + \sqrt{2}dW_t, \quad \text{for } t \in [0,T], \\ X_0^* &= 0, \end{cases} \tag{2.54}$$

applying Itô's formula to the process $th(X_t^*)$

$$\begin{aligned} d(th(X_t^*)) &= \frac{1}{ch^2(X_t^*)}dX_t^* - \frac{th(X_t^*)}{ch^2(X_t^*)}d\langle X^*, X^*\rangle_t, \\ &= \frac{1}{ch^2(X_t^*)}\left(2th(X_t^*)\,dt + \sqrt{2}dW_t\right) - \frac{2th(X_t^*)}{ch^2(X_t^*)}dt, \\ &= \frac{\sqrt{2}}{ch^2(X_t^*)}dW_t. \end{aligned}$$

then

$$d(th(X_t^*)) = \sqrt{2}\,[ch(X_t^*)]^{-2}\,dW_t, \quad \text{for } t \in [0,T], \tag{2.55}$$

The uniqueness of the adapted solution (p,q) to the adjoint process (2.40) yields

$$\begin{cases} p_t &= th(X_t^*)\,dt, \quad \text{for } t \in [0,T], \\ q_t &= \sqrt{2}\,[ch(X_t^*)]^{-2}, \quad \text{for } t \in [0,T]. \end{cases} \tag{2.56}$$

2.4.2 The non smooth case.

Next, we drop the differentiability condition on the value function. It is clear that the method used earlier will no longer be valid. Now, the idea is based on the viscosity solution theory for second order PDE$_s$ to study the relationship between the maximum principle and dynamic programming.principle

1. Differentials in the spatial variable. Let us first recall the partial superdifferentials and subdifferentials of the value function in the the spatial variable x. Therefore, we need the following notations

$$D_x^{2,+}V(t,x) = \left\{(p,P)\in\mathbb{R}^n\times\mathbb{R}^{n\times n}\diagup \lim_{y\to x}\sup\frac{I_1(s,y)}{|s-t|+|y-x|^2}\leq 0\right\},$$
$$D_x^{2,-}V(t,x) = \left\{(p,P)\in\mathbb{R}^n\times\mathbb{R}^{n\times n}\diagup \lim_{y\to x}\inf\frac{I_1(s,y)}{|s-t|+|y-x|^2}\geq 0\right\},$$

where

$$I_1(s,y) = V(s,y) - V(t,x) - \langle p, y-x\rangle - \frac{1}{2}(y-x)^T P(y-x).$$

Theorem.2.17 Let $(t,x)\in[0,T)\times\mathbb{R}^n$ be fixed, let $(\hat{u},\hat{x})$ be an optimal solution of the problem $(2.45)-(2.46)$, and W be a viscosity solution of the HJB equation $(2.47a)-(2.47b)$. Then the solution of the BSDE (2.40) along an optimal trajectory, satisfied

$$\{-p_t\}\times[-P_t,+\infty)\subseteq D_x^{2,+}V(t,\hat{x}),\ \forall t\in[s,T],\ \text{P-a.s.},$$

$$D_x^{2,-}V(t,\hat{x})\subseteq\{-p_t\}\times[-\infty,-P_t),\ \forall t\in[s,T],\ \text{P-a.s.}$$

Proof. See Zhou [121]. ∎

2. Differentials in the time variable. Now, Let us recall the partial superdifferentials and subdifferentials of the value function in the time variable t

$$D_{t+}^{1,+}V(t,x) = \left\{q\in\mathbb{R}\diagup \lim_{\substack{s\downarrow t\\ s\in[0,T)}}\sup\frac{I(s,y)}{|s-t|}\leq 0\right\},$$
$$D_{t+}^{1-}V(t,x) = \left\{q\in\mathbb{R}\diagup \lim_{\substack{s\downarrow t\\ s\in[0,T)}}\inf\frac{I(s,y)}{|s-t|}\geq 0\right\},$$

where

$$I_2(s,y) = V(s,y) - V(t,x) - (q, s-t).$$

Define the function H by

$$H(t,x,u)=G(t,x,u,p,P)-tr\left(\sigma(t,x,u)\left[q-P\sigma(t,\hat{x},\hat{u})\right]\right),$$

where $p,q,$ and P are the solution to (2.40), and (2.41), associated with the optimal pair $(\hat{x},\hat{u})$. Therefore, the following result appears rather natural.

Theorem.2.18 Under the assumptions of Theorem (2.14), we have

$$H(t,\hat{x}_t,\hat{u}_t)\in D_{t+}^{1,+}V(t,\hat{x}),\ \text{a.e. } t\in[s,T],\ \text{P-a.s.}$$

Proof. See Zhou [121.] . ∎

Now, let us combine Theorem 2.17, and Theorem 2.18, to get the following result.

Theorem. 2.19 Under the assumptions of Theorem (2.17), we have

$$[H(t,\hat{x}_t,\hat{u}_t),\infty)\times\{-p_t\}\times[-P_t,+\infty)\subseteq D_{t,x}^{1,2,+}V(t,\hat{x}_t),\ \text{a.e. } t\in[s,T],\ \text{P-a.s.}$$

and

$$D_{t,x}^{1,2,-}V(t,\hat{x}_t)\subseteq(-\infty,H(t,\hat{x}_t,\hat{u}_t)]\times\{-p_t\}\times(-\infty,-P_t],\ \text{a.e. } t\in[s,T],\ \text{P-a.s.}$$

2.5 The SMP in singular optimal controls

In this section we consider the maximum principle in stochastic control problems of systems governed by a SDE with uncontrolled diffusion coefficient (see Bahlali et. al [5]), where the control variable has two components, the first being abselutely continuous and the second singular, we suppose that an optimal control exists. The expected cost to be minimized over the class of admissible controls is given by (2.58). For this subject, the first version of the stochastic maximum principle that covers singular control was obtained by

Cadenillas and Haussmann in [22] , the first order weak maximum principle has ben derived by Bahlali and Chala [2] , see, also Bahlali and Mezerdi [8] for the general maximum principle in singular controls problem. Our objectif in this section is to establish the optimality necessary conditions of this kind of problems. First, we formulate the control problem and describe the assumptions of the model.

Let (Ω, F, F_t, P) be a filtered probability space satisfying the usual conditions, on which a d-dimensional Brownian motion (B_t) is defined with the filtration (F_t). Let T be a strictly positive real number, A_1 is a non empty subset of $\mathbb{R}^n$ and $A_2 = ([0,\infty))^m$. U_1 is the class of measurable, adapted processes $u : [0,T] \times \Omega \to A_1$, and U_2 is the class of measurable, adapted processes $\xi : [0,T] \times \Omega \to A_2$.

Definition 2.20. An admissible control is a pair (u,ξ) of $A_1 \times A_2$-valued, measurable F_t-adapted processes, such that

Definition 2 *1. u is absolutely continuous, and ξ is of bounded variation, non decreasing left-continuous with right limits and $\xi_0 = 0$,*

2. $E\left[\sup_{t\in[0,T]} |u_t|^2 + |\xi_T|^2\right] < \infty.$

We denote by $U = U_1 \times U_2$ the set of all admissible controls.

For $(u,\xi) \in U$, suppose the state $x_t = x_t^{(u,\xi)} \in \mathbb{R}^n$ is described by the equation

$$\begin{cases} dx_t &= b\left(t, x_t, u_t\right) dt + \sigma\left(t, x_t\right) dB_t + G\left(t\right) d\xi\left(t\right), \qquad \text{for } t \in [0,T], \\ x_0 &= x, \end{cases} \tag{2.57}$$

Where, $b : [0,T] \times \mathbb{R}^n \times A_1 \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d, G : [0,T] \to \mathbb{R}^d \otimes \mathbb{R}^m$, are given. Suppose we are given a cost functional $J(u,\xi)$ of the form

$$J(u,\xi)=E\left[\int_0^T f(t,x_t,u_t)\,dt+\int_0^T k(t)\,d\xi(t)+g(x_T)\right], \tag{2.58}$$

where, $f:[0,T]\times\mathbb{R}^d\times A_1\to\mathbb{R}$, $g:\mathbb{R}^d\to\mathbb{R}$, and $k:[0,T]\to([0,\infty))^m$, with $k(t)\,d\xi(t)=\sum_{l=1}^{m}k_l(t)\,d\xi_l(t)$. The following assumptions will be in force throughout this section:

$$b,\sigma,f,g \text{ are continuously differentiable with respect to } x, \tag{2.59}$$

$$\text{They and all their derivatives } b_x,\sigma_x,f_x,g_x \text{ are continuous in } (x,u), \tag{2.60}$$

$$\text{The derivatives } b_x,f_x \text{ are bounded uniformly in } u, \text{ and } \sigma_x,g_x \text{ are bounded}, \tag{2.61}$$

$$b,\sigma \text{ are bounded by } C(1+|x|+|u|), \tag{2.62}$$

$$G,\ k \text{ are continuous and } G \text{ is bounded.} \tag{2.63}$$

The problem is to minimize the functional $J(u,\xi)$ over all $(u,\xi)\in U$, i.e., we seek $(\hat{u},\hat{\xi})\in U$ such that

$$J\left(\hat{u},\hat{\xi}\right)=\sup_{(u,\xi)\in U}J(u,\xi), \tag{2.64}$$

such controls $(\hat{u},\hat{\xi})$ are called optimal controls, $x^{(\hat{u},\hat{\xi})}$ is the corresponding optimal solution of the SDE (2.57). Under the above hypothesise, the SDE (2.57) has a unique strong solution, such that for any $p>0$,

$$E\left[\sup_{0\le t\le T}|x_t|^p\right]<\infty, \tag{2.65}$$

and the functional J is a well defined.

The maximum principle will be proved in two steps. First we define a family of perturbed controls $(u^\epsilon,\hat{\xi})$, where u^ϵ is a spike variation of the absolutely continuous part $\hat{u}$ on a small time interval $[\tau,\tau+\epsilon]$. The first variational inequality is derived from the fact that

$$J(u^{\epsilon},\hat{\xi})-J(\hat{u},\hat{\xi})\geq 0.$$

The second step is to introduce another family of perturbed controls $(\hat{u},\xi^{\epsilon})$, where ξ^{ϵ} is a convex perturbation of $\hat{\xi}$. The second variational inequality is then obtained from the inequality

$$J(\hat{u},\xi^{\epsilon})-J(\hat{u},\hat{\xi})\geq 0.$$

The stochastic maximum principle in its integral form is given by the following Theorem.

Theorem 2.21 (The strict stochastic maximum principle in integral form). Let $(\hat{u},\hat{\xi})$ be a strict optimal control minimizing the cost J over U, and let $\hat{x}$ be the corresponding optimal trjectory. Then there exists a unique pair of adapted processes

$$(p,q)\in L^{2}\left([0,T];\mathbb{R}^{n}\right)\times L^{2}\left([0,T];\mathbb{R}^{n\times d}\right)$$

which is the solution of the BSDE (3.18), such that for all $a\in A_1$, and $\eta\in U_2$,

$$\begin{aligned}&H(t,x_t,a,p_t)-H(t,x_t,\hat{u}_t,p_t)\geq 0,\ \ P-a.e.,\\&E\int_0^T(k_t+G_t^{*}p_t)d(\eta-\hat{\xi})_t\geq 0.\end{aligned}$$

2.5.1 The first variational inequality

To obtain the first variational inequality in the stochastic maximum principle, we define the strong perturbation of the absolutely continuous parts of the control, sometimes called the spike variation

$$(u_t^\epsilon, \hat{\xi}_t) = \begin{cases} (v, \hat{\xi}_t) & \text{if } t \in [\tau, \tau+\epsilon], \\ (\hat{u}_t, \hat{\xi}_t) & \text{otherwise,} \end{cases} \tag{2.66}$$

where $0 \leq \tau < T$ is fixed, $\epsilon > 0$ is sufficiently small, and v is an arbitrary A_1-valued, $F_\tau-$measurable random variable such that $E|v|^2 < +\infty$. Note that the singular part is not affected by the perturbation. If $x_t^{(u^\epsilon,\hat{\xi})}$denoted the trajectory associated with $(u^\epsilon, \hat{\xi})$, then

$$\begin{cases} x_t^{(u^\epsilon,\hat{\xi})} = x_t, \ t \leq \tau, \\ dx_t^{(u^\epsilon,\hat{\xi})} = b\left(t, x_t^{(u^\epsilon,\hat{\xi})}, v\right) dt + \sigma\left(t, x_t^{(u^\epsilon,\hat{\xi})}\right) dB_t + G_t d\hat{\xi}_t, \ \tau < t < \tau+\epsilon, \\ dx_t^{(u^\epsilon,\hat{\xi})} = b\left(t, x_t^{(u^\epsilon,\hat{\xi})}, \hat{u}_t\right) dt + \sigma\left(t, x_t^{(u^\epsilon,\hat{\xi})}\right) dB_t + G_t d\hat{\xi}_t, \ \tau+\epsilon < t < T. \end{cases}$$

It is easy to check by standard arguments that

$$\lim_{\epsilon \longrightarrow 0} E\left(\sup_{t\in[0,T]} |x_t^{(u^\epsilon,\hat{\xi})} - \hat{x}_t|^2\right) = 0, \tag{2.67}$$

Arguing as in the section 1, we define y as the solution of the linear SDE

$$\begin{cases} dy_t = b_x(s, \hat{x}_s, \hat{u}_s) y_s ds + \sigma_x(s, \hat{x}_s) y_s B_s, \ \tau \leq s \leq T, \\ y_\tau = b(\tau, \hat{x}_\tau, v) - b(\tau, \hat{x}_\tau, \hat{u}_\tau). \end{cases} \tag{2.68}$$

Let ς be defined by

$$\begin{cases} d\varsigma_t = f_x(s, \hat{x}_s, \hat{u}_s) y_s ds, \ \tau \leq s \leq T, \\ \varsigma_\tau = f(\tau, \hat{x}_\tau, v) - f(\tau, \hat{x}_\tau, \hat{u}_\tau). \end{cases}$$

We can prove the following approximation result

Lamma 2.22. Under the assumptions $(2.59) - (2.63)$, *we have*

$$\lim_{\epsilon \longrightarrow 0} E\left[\left|\frac{x_T^{(u^\epsilon,\hat{\xi})} - \hat{x}_T}{\epsilon} - y_T\right|^2\right] = 0,$$

$$\lim_{\epsilon \longrightarrow 0} E\left[\left|\frac{1}{\epsilon}\int_\tau^T (f(t, \hat{x}_t, u_t^\epsilon) - f(t, \hat{x}_t, \hat{u}_t))dt - \varsigma_T\right|^2\right] = 0.$$

Proof. Since $x_T^{(u^\epsilon,\hat{\xi})} - \hat{x}_T$ does not depend on the singular part, the proof follows that of Lemma 2.2. ■

Corollary 2.23 Under the assumptions $(2.59) - (2.63)$, *we have*

$$\frac{dJ(u^\epsilon,\hat{\xi})}{d\epsilon}|_{\epsilon=0} = E[g_x(\hat{x}_T).y_T + \varsigma_T].$$

Proof. By using the estimate (2.67), the result follows by mimicking the same proof as in corollary (2.4). ■

Let us introduce the adjoint process and the first variational inequality from corollary 2.22. We proced as in section 01. Let $\Phi(t,\tau)$ be the solution of the linear equation

$$\begin{cases} d\Phi(t,\tau) = b_x(t,\hat{x}_t,\hat{u}_t)\Phi(t,\tau)dt + \sigma_x(t,\hat{x}_t)\Phi(t,\tau)dB_t, \ t > \tau, \\ \Phi(\tau,\tau) = I_d. \end{cases} \tag{2.69}$$

By the uniqueness property, it is easy to check that

$$y(t) = \Phi(t,\tau)(b(\tau,\hat{x}_\tau,v) - b(\tau,\hat{x}_\tau,\hat{u}_\tau)),$$

if we define the adjoint process by

$$p_t = E\left[\Psi_t^*\Phi_T^* g_x(\hat{x}_T) + \Psi_t^*\int_{ts}^T \Phi^* f_x(s,\hat{x}_s,\hat{u}_s)ds / F_t\right], \tag{2.70}$$

then we get from the optimality of $(\hat{u},\hat{\xi})$ the first variational inequality

$$0 \leq E[H(\tau,\hat{x}_\tau,v,p_\tau) - H(\tau,\hat{x}_\tau,\hat{u}_\tau,p_\tau)], d\tau - a.e.,$$

where the Hamiltonian H is given from $[0,T]\times\mathbb{R}^n\times A_1\times\mathbb{R}^n$ into $\mathbb{R}$ by

$$H(t,x,v,p) = f(t,x,v) + p.b(t,x,v). \tag{2.71}$$

2.5.2 The second variational inequality

To obtain the second variational inequality of the stochastic maximum principle, we introduce the convex perturbation applied on the singular part of the control process

$$(\hat{u}_t, \xi_t^\epsilon) = (\hat{u}_t, \hat{\xi}_t + \epsilon(\eta_t - \hat{\xi}_t)), \tag{2.72}$$

where $\theta > 0$ and η is an arbitrary element of U_2. Note that the first part of the control is not affected by the perturbation . Since $(\hat{u}, \hat{\xi})$ is an optimal control, we'll derive the second variational inequality from the fact that

$$0 \leq J(\hat{u}, \xi^\epsilon) - J(\hat{u}, \hat{\xi}).$$

Lemma 2.24. *Let $x_t^{(\hat{u},\xi^\epsilon)}$ be the trajectory associated with $(\hat{u}, \xi^\epsilon)$. then the following estimation holds*:

$$\lim_{\epsilon \to 0} E\left[\sup_{t\in[0,T]} \left|x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t\right|^2\right] = 0.$$

Proof. From assumption (2.60)-(2.61) and by using the Burkholder–Davis–Gundy inequality for the martingale part, we get

$$\begin{aligned} E\left[\sup_{t\in[0,T]} \left|x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t\right|^2\right] \quad &\leq \quad 6KE\left[\int_0^t \sup_{\alpha\in[0,s]} \left|x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t\right|^2 ds\right] \\ &\quad +3M\epsilon^2 E\left[\left|\eta_T - \hat{\xi}_T\right|^2\right]. \end{aligned}$$

From Definition 2.20 and using Gronwall's inequality, the result follows immediately by letting ϵ go to zero. ■

Lemma 2.25. *Under assumption (2.60) and (2.61), the following estimation holds*:

$$\lim_{\epsilon \to 0} E\left[\left|\frac{x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t}{\epsilon} - z_t\right|^2\right] = 0,$$

where z the solution of the integral equation

$$z_t = \int_0^t b_x(s, \hat{x}_s, \hat{u}_s) z_s d_s + \int_0^t \sigma_x(s, \hat{x}_s) z_s dB_s + \int_0^t G_s d(\eta - \xi)_s. \tag{2.73}$$

Proof. From Definition 2.20 and assumption (2.60)-(2.61), it is easy to verify by Gronwall's inequality that

$$E\left[\sup_{t\in[0,T]} |z_t|^2\right] < \infty. \tag{2.74}$$

Let

$$\gamma_t^\epsilon = \frac{x_t^{(u,\xi^\epsilon)} - \hat{x}_t}{\epsilon} - z_t.$$

It is easy to see that

$$\begin{aligned} E\left|\gamma_t^\epsilon\right|^2 \leq\ & 3\int_0^t \left|\int_0^1 b_x\left(s, x_s^{(\hat{u},\xi^\epsilon)} + \lambda\left[x_s^{(\hat{u},\xi^\epsilon)} - \hat{x}_s\right], \hat{u}_s\right)\gamma_s^\epsilon d\lambda\right|^2 ds \\ & + 3\int_0^t \left|\int_0^1 \sigma_x\left(s, x_s^{(\hat{u},\xi^\epsilon)} + \lambda\left[x_s^{(\hat{u},\xi^\epsilon)} - \hat{x}_s\right], \hat{u}_s\right)\gamma_s^\epsilon d\lambda\right|^2 ds \\ & + 3E\left|\rho_t^\epsilon\right|^2, \end{aligned}$$

where ρ_t^ϵ is given by

$$\begin{aligned} \rho_t^\epsilon =\ & \int_0^t\int_0^1 z_s\left[b_x\left(s, x_s^{(\hat{u},\xi^\epsilon)} + \lambda\left[x_s^{(\hat{u},\xi^\epsilon)} - \hat{x}_s\right], \hat{u}_s\right) - b_x(s, \hat{x}_s, \hat{u}_s)\right] d\lambda ds \\ & + \int_0^t\int_0^1 z_s\left[\sigma_x\left(s, x_s^{(\hat{u},\xi^\epsilon)} + \lambda\left[x_s^{(\hat{u},\xi^\epsilon)} - \hat{x}_s\right]\right) - \sigma_x(s, \hat{x}_s)\right] d\lambda dB_s. \end{aligned}$$

Since b_x, σ_x are bounded, it holds that

$$E\left|\gamma_t^\epsilon\right|^2 \leq 6C\int_0^t E\left|\gamma_s^\epsilon\right|^2 dt + 3E\left|\rho_t^\epsilon\right|^2.$$

By using Lemma 2.24 and (2.74),together with the Dominated Convergence theorem, we obtain

$$\lim_{\epsilon\to 0} E\left|\rho_t^\epsilon\right|^2 = 0.$$

We conclude by applying Gronwall's lemma and letting ϵ go to zero. ■

Lemma 2.26 *The following inequality holds*:

$$0 \leq E\left[g_x(\hat{x}_T)z_T\right] + E\int_0^T h_x(t,\hat{x}_t,\hat{u}_t)z_t dt + E\int_0^T k_t d(\eta - \hat{\xi})_t. \tag{2.75}$$

Proof. From the second variational inequality, we have

$$\begin{aligned} 0 &\leq \frac{1}{\epsilon}E\left[g\left(x_T^{(\hat{u},\xi^\epsilon)}\right) - g(\hat{x}_T)\right] + \frac{1}{\epsilon}E\int_0^T \left(f\left(t, x_t^{(\hat{u},\xi^\epsilon)}, \hat{u}_t\right) - f(t,\hat{x}_t,\hat{u}_t)\right)dt \\ &\quad + E\int_0^T k_t d(\eta_t - \hat{\xi}_t), \\ &= E\int_0^1 \left(\frac{x_T^{(\hat{u},\xi^\epsilon)} - \hat{x}_T}{\epsilon}\right) g_x\left(\hat{x}_T + \lambda\left(x_T^{(\hat{u},\xi^\epsilon)} - \hat{x}_T\right)\right) d\lambda \\ &\quad + E\int_0^T\int_0^1 \left(\frac{x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t}{\epsilon}\right) f_x\left(t, \hat{x}_t + \lambda\left(x_t^{(\hat{u},\xi^\epsilon)} - \hat{x}_t\right), \hat{u}_t\right) d\lambda dt \\ &\quad + E\int_0^T k_t d(\eta - \hat{\xi})_t. \end{aligned}$$

Since the derivatives g_x and f_x are continuous and bounded, by letting ϵ go to 0, we see that the result follows from Lemma 2.24 and Lemma 2.25. By the same method as in the last subsection, we are able to derive the second variational inequality from (2.75). If $\Phi(t,s)$ denotes the solution of (2.69), it is easy to check that z_t is given explicitly by

$$z_t = \int_0^t \Phi(t,s)G_s d(\eta - \hat{\xi})_s.$$

Replacing z_t with its value, we obtain the second variational inequality

$$0 \leq E\int_0^T (k_t + G_t^* p_t) d(\eta - \hat{\xi})_t,$$

where p_t is the adjoint process defined in the last subsection by (2.70). ■

2.5.3 The adjoint equation and the stochastic maximum principe

Applying Itô's formula to p_t given by (2.70), it is easy to see that p_t satisfies the linear backward SDE

$$\begin{cases} -dp_t &= \left\{ f_x(t, \hat{x}_t, \hat{u}_t) + b_x^T(t, \hat{x}_t, \hat{u}_t)p_t + \sigma_x^T(t, \hat{x}_t)\, q_t \right\} dt - q_t dB_t, \\ p_T &= g_x(\hat{x}_T). \end{cases} \tag{2.76}$$

where $q_t \in L^2\left([0.T]\,; \mathbb{R}^{n\times d}\right)$, is given by

$$q_t = \Psi_t^T Q_t - \sigma_t^T(t, x_t)p_t,$$

and Q_t is given by the Itô representation Theorem of Browanian martingales

$$\begin{aligned} \int_0^t Q_s dB_s &= E\left[\Phi_T^T g_x(\hat{x}_T) + \int_0^t \Phi_t^T f_x(t, \hat{x}_t, \hat{u}_t)dt \diagup F_t\right] \\ &\quad - E\left[\Phi_T^T g_x(\hat{x}_T) + \int_0^t \Phi_t^T f_x(t, \hat{x}_t, \hat{u}_t)dt\right], \end{aligned}$$

The stochastic maximum principle in its integral form is given by the following Theorem.

Theorem 3 ***2.27 (The strict stochastic maximum principle)*** *Let $(\hat{u}, \hat{\xi})$ be an optimal control minimizing the cost J over U, and let $\hat{x}$ be the corresponding optimal trajectory. Then there exists a unique pair of adapted processes*

$$(p, q) \in L^2([0, T]\,; \mathbb{R}^n) \times L^2([0, T]\,; \mathbb{R}^{n\times d}),$$

which is the solution of the BSDE (3.18), such that

$$H\left(t, \hat{x}_t, \hat{u}_t, p_t\right) = \min_{a\in A_1} H\left(t, \hat{x}_t, a, p_t\right), dt - a.e.,\ \ P - a.s., \tag{2.77}$$

$$P\left\{\forall t \in [0, T], \forall i; k_i\left(t\right) + G_i^*\left(t\right)p_t \geq 0\right\} = 1, \tag{2.78}$$

$$P\left\{\sum_{i=1}^m 1_{\left\{k_i(t)+G_i^*(t)p_t\geq 0\right\}d\xi_t^i} = 0\right\} = 1. \tag{2.79}$$

Proof. To prove (2.78) and (2.79) we follow [5]. Since $(\hat{u}, \hat{\xi})$ is optimal, the inequality

$$E\int_0^T (k_t + G_t^* p_t) d(\eta - \hat{\xi})_t \geq 0,$$

holds for every $\eta \in U_2$. In particular, let $\eta \in U_2$ be defined by

$$d\eta_t^i = \begin{cases} 0 & \text{if } k_i(t) + G_i^*(t)\, p_t > 0, \\ d\hat{\xi}_i^t & \text{otherwise.} \end{cases}$$

then

$$\begin{aligned} E\int_0^T (k_t + G_t^* p_t) d(\eta - \hat{\xi})_t &= E\left[\sum_{i=1}^m \int_0^T (k_i(t) + G_i^*(t)\, p_t)\, 1_{\{k_i(t)+G_i^*(t)p_t>0\}} d\left(-\hat{\xi}_t^i\right)\right] \\ &= 0, \end{aligned}$$

and relation (2.79) follows immediately.

Let us prove (2.78). For each $i \in \{1, 2, ..., m\}$, let

$$\begin{aligned} A_t^i &= \{\omega \in \Omega : k_i(t) + G_i^*(t)\, p_t < 0\}, \\ A^i &= \{(t, \omega) \in [0, T] \times \Omega : k_i(t) + G_i^*(t)\, p_t < 0\}, \end{aligned}$$

and define

$$\eta_t^i = \hat{\xi}_t^i + \int_0^t 1_{A^i}(s, \omega) ds,$$

It is easy to see that $\eta_t = \left(\eta_t^1, \eta_t^2, ..., \eta_t^m\right)$ is in U_2. Moreover

$$E\int_0^T (k_t + G_t^* p_t) d(\eta - \hat{\xi})_t = E\left[\sum_{i=1}^m \int_0^T (k_i(t) + G_i^*(t)\, p_t)\, 1_{A^i} dt\right] < 0,$$

which contradicts (2.78), unless for every $i = 1, 2, ..., m$, $dt \otimes P(A^i) = 0$. This proves the desired result since k, G, and p are continuous. ■

Chapter 3

The SMP For Degenerate Diffusions With Non Smooth Coefficients

For a controlled stochastic differential equation with a finite horizon cost functional, a necessary conditions for optimal control of degenerate diffusions with non smooth coefficients is derived in this chapter. The main idea is to show that the SDE_S admit a unique linearized version interpreted as its distributional derivative with respect to the initial condition, defined on an enlarged probabilty space, where the initial condition α will be taken as a random element, we use technique of Bouleau-Hirsch on absolute contunuity of probability measures in order to define the adjoint process on an extension of the initial probability space.

3.1 Assumptions and main result

In this section we will make some preliminaries. First of all, besides the Euclidean space $\mathbb{R}^d$, for any $x, y \in \mathbb{R}^d$, we use $x.y$ to denote the inner product of these two vectors. We put $\partial_x = \left(\frac{\partial}{\partial x_j}\right)_{j=1,\ldots,d}$, and note that if $\psi : \mathbb{R}^d \to \mathbb{R}^d$ then $\partial_x \psi \triangleq \left(\partial_{x_j} \psi^i\right)_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$.

From now on, we let $\Omega = C_0\left(\mathbb{R}_+, \mathbb{R}^d\right)$ be the space of continuous functions w such that $w(0) = 0$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. F is the Borel σ-field over Ω, P is the Wiener measure on (Ω, F), $(F_t)_{t\geq 0}$ is the filtration of coordinates augmented with P-null sets of F. We define the canonical process $B_t(w) = w(t)$, for all $t \geq 0$. Thus, $\left(\Omega, F, (F_t)_{t\geq 0}, P, B_t\right)$ is a Brownian motion. Let T be a fixed strictly positive real number, we consider stochastic optimal control problems by the set of admissible controls U we mean the colluction of $\left(\Omega, F, (F_t)_{t\geq 0}, P, B_t\right)$ and A-valued F_t-adapted measurable process $u_. = \{u_t : 0 \leq t \leq T\}$. A is a given closed set in some Euclidean space $\mathbb{R}^d$, we denote $(\Omega, F, P, B, u) \in U$ the set of all admissible controls, but occasionall we will write only $u \in U$ if no ambiguity arises. Now, for each $u \in U$ let x_t be the solution of the controlled stochastic differential equation

$$\begin{cases} dx_t = b(t, x_t, u_t)\, dt + \sigma(t, x_t)\, dB_t, & \text{for } t \in [0, T], \\ x_0 = \alpha, \end{cases} \tag{3.1}$$

and the objective is to minimize over controls $u \in U$ the cost functional

$$J(u) = E\left[\int_0^T f(t, x_{t,} u_t)\, dt + g(x_T)\right]. \tag{3.2}$$

We introduce the standing assumptions:

Maps $b : [0,T]\times\mathbb{R}^d\times U\times\Omega \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d\times\Omega \to \mathbb{R}^{d\times d}$, $f : [0,T]\times\mathbb{R}^d\times U\times\Omega \to \mathbb{R}$, $g : \mathbb{R}^d \times \Omega \to \mathbb{R}$, satisfy the following: b, f are $B\left([0,T] \times \mathbb{R}^d \times U\right) \otimes F_T$-measurable,

σ is $B\left([0,T]\times\mathbb{R}^d\right)\otimes F_T$-measurable, and g is $B\left(\mathbb{R}^d\right)\otimes F_T$-measurable, where $B(G)$ is the Borel σ-field of the metric space G. There exist $M>0$, such that for all (t,x,y,a) in $\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d\times A$

$$|b(t,x,a)-b(t,y,a)|+|\sigma(t,x)-\sigma(t,y)|\leq M|x-y|, \tag{3.3}$$

$$|f(t,x,a)-f(t,y,a)|+|g(x)-g(y)|\leq M|x-y|, \tag{3.4}$$

$$|b(t,x,a)|+|\sigma(t,x)|\leq M(1+|x|), \tag{3.5}$$

$$|f(t,x,a)|+|g(x)|\leq M(1+|x|), \tag{3.6}$$

and

$$b(t,x,a) \text{ and } f(t,x,a) \text{ are continuous in } a \text{ uniformly in } (t,x). \tag{3.7}$$

Assumptions (3.3) and (3.5) guarantee the existence and uniqueness of strong solution for (3.1), such that for any $p>0$,

$$E\left[\sup_{0\leq t\leq T}|x_t|^p\right]<+\infty.$$

Since b, σ^j (the jth column of the matrix σ), f and g are Lipschitz continuous functions in the state variable they are differentiable almost everywhere in the sense of Lebesgue measure (Rademacher Theorem). Let us denote by b_x, σ_x, f_x and g_x any Borel measurable functions such that

$$\begin{aligned}
\partial_x b(t,x,a) &= b_x(t,x,a) \quad dx\text{-a.e.},\\
\partial_x f(t,x,a) &= f_x(t,x,a) \quad dx\text{-a.e.},\\
\partial_x \sigma(t,x) &= \sigma_x(t,x) \quad dx\text{-a.e.},\\
\partial_x g(x) &= g_x(x) \quad dx\text{-a.e.}
\end{aligned}$$

It is clear that these almost everywhere derivatives are bounded by the Lipschitz constante M. Finally, assume that $b_x(t,x,a)$, $f_x(t,x,a)$ are continuous in a uniformly in (t,x). We assume throughout this paper that an optimal control $\hat{u}$ of the control problem assosiated with (3.1) and (3.2) exists. That is

$$J(\hat{u}) = \inf_{u \in U} J(u).$$

Let h be a continuous positive function on $\mathbb{R}^d$ such that $\int h(x)\,dx = 1$ and $\int |x|^2 h(x)\,dx < \infty$. We set $D = \left\{ f \in L^2(hdx),\ \text{such that } \frac{\partial f}{\partial x_j} \in L^2(hdx) \right\}$, where $\frac{\partial f}{\partial x_j}$ denotes the derivative in the distribution sense. Equipped with the norm

$$\|f\|_D = \left[\int_{\mathbb{R}^d} f^2 hdx + \sum_{1 \le j \le d} \int_{\mathbb{R}^d} \left(\frac{\partial f}{\partial x_j} \right)^2 hdx \right]^{\frac{1}{2}},$$

D is a Hilbert space, which is a classical Dirichlet space (see [21]). Moreover D is a subset of the Sobolev space $H^1_{loc}(\mathbb{R}^d)$.

Let $\widetilde{\Omega} = \mathbb{R}^d \times \Omega$, and $\widetilde{F}$ the Borel σ-field over $\widetilde{\Omega}$ and $\widetilde{P} = hdx \otimes P$. Let $\widetilde{B}_t(x,w) = B_t(w)$ and $\widetilde{F}_t$ the natural filtration of $\widetilde{B}_t$ augmented with $\widetilde{P}$-negligible sets of $\widetilde{F}$. It is clear that $\left(\widetilde{\Omega}, \widetilde{F}, \left(\widetilde{F}_t\right)_{t \ge 0}, \widetilde{P}, \widetilde{B}_t\right)$ is a Brownian motion. We introduce the process $\tilde{x}_t$ defined on the enlarged space $\left(\widetilde{\Omega}, \widetilde{F}, \left(\widetilde{F}_t\right)_{t \ge 0}, \widetilde{P}, \widetilde{B}_t\right)$ solution of the stochastic differential equation

$$\begin{cases} d\tilde{x}_t = b(t, \tilde{x}_t, \tilde{u}_t)\,dt + \sigma(t, \tilde{x}_t)\,d\widetilde{B}_t, \\ \tilde{x}_0 = \alpha, \end{cases} \tag{3.8}$$

associated to the control $\tilde{u}_t(x,w) = u_t(w)$. Since the coefficients b and σ are Lipschitz continuous and grow at most linearly, equations (3.8) has a unique $\widetilde{F}_t$-adapted solution with continuous trajectories. Equations (3.1) and (3.8) are almost the same except that uniqueness of the solution of (3.8) is slightly weaker, one can easily prove that the uniqueness implies that for each $t \geq 0$, $\tilde{x}_t = x_t$, $\widetilde{P}$-a.s.

The main result of this paper is stated in the following Theorem.

Theorem 3.1. *(Stochastic maximum principle) Let* $(\hat{u}, \hat{x})$ *be an optimal pair for the controlled system* (3.1) *and* (3.2), *then there exist an* F*-adapted process (the adjoint process) satisfying*

$$p_t := -\widetilde{E}\left[\int_t^T \Phi^*(s,t).f_x(s,\hat{x}_s,\hat{u}_s)\,ds + \Phi^*(T,t).g_x(\hat{x}_T)\diagup\widetilde{F}_t\right], \tag{3.9}$$

for which the following stochastic maximum principle holds:

$$H(t,\hat{x}_t,\hat{u}_t,p_t) = \max_{a\in A} H(t,\hat{x}_t,a,p_t) \quad dt\text{-a.e}, \ \widetilde{P}\text{-a.s.}, \tag{3.10}$$

where $\Phi(s,t)$, $(s \geq t)$ *is the fundamental solution of the linear equation*

$$\begin{cases} d\Phi(s,t) = b_x(s,\hat{x}_s,\hat{u}_s).\Phi(s,t)\,ds + \sum\limits_{1\leq j\leq d} \sigma_x^j(s,\hat{x}_s).\Phi(s,t)\,d\widetilde{B}_s^j, \\ \Phi(t,t) = Id. \end{cases} \tag{3.11}$$

where the Hamiltonian H *is defined by*

$$H(t,x,u,p) = p.b(t,x,u) - f(t,x,u). \tag{3.12}$$

Here Φ^* *denotes the transpose of the matrix* Φ.

3.2 Proof of the main result

Let us recall some preliminaries and notation on the Bouleau-Hirsch method which will be applied in this paper to establish the stochastic maximum principle of the controlled system (3.1), (3.2).

Theorem 3.2. *(The Bouleau-Hirsch flow property) For $\widetilde{P}$-almost every w*

(1) For all $t \geq 0$, $\tilde{x}_t$ is in D^d.

(2) There exists a $\widetilde{F}_t$-adapted $GL_d(\mathbb{R})$-valued continuous process $\left(\widetilde{\Phi}_t\right)_{t\geq 0}$ such that for every $t \geq 0$

$$\frac{\partial}{\partial x}\left(x_t^{\alpha}(w)\right) = \widetilde{\Phi}_t(\alpha, w) \quad dx\text{-}a.e.$$

where $\dfrac{\partial}{\partial x}$ denotes the derivative in the ditribution sense.

(3) The distributional derivative $\widetilde{\Phi}_t$ is the unique fundamental solution of the linear stochastic differential equation

$$\begin{cases} d\widetilde{\Phi}(s,t) = b_x(s, \tilde{x}_s, \tilde{u}_s) . \widetilde{\Phi}(s,t)\, ds + \sum\limits_{1 \leq j \leq d} \sigma_x^j(s, \tilde{x}_s) . \widetilde{\Phi}(s,t)\, d\widetilde{B}_s^j, \ s \geq t, \\ \widetilde{\Phi}(t,t) = Id, \end{cases} \tag{3.13}$$

where b_x and σ_x^j are versions of the almost everywhere derivatives of b and σ^j.

Remark 3.3. *It is proved in* [21] *that the image measure of $\widetilde{P}$ by the map $\tilde{x}_t$ is absolutely continuous with respect to the Lebesgue measure.*

From now on, let us assume that the initial time $s = 0$ and initial state α of the system are fixed. Define a metric on the space U of admissible controls

$$d(u_., v_.) = \tilde{P}\left\{(t,w) \in [0,T] \times \Omega : u_t(w) \neq v_t(w)\right\}, \tag{3.14}$$

where $\tilde{P}$ is the product measure of the Lebesgue measure and P. Since A is closed, it can be shown similarly to $[50]$, that $U[0,T]$ is a complete metric space under d.

3.2.1 The maximum principle for a Family of perturbed control problems

Now, let φ be a non negative smooth function defined on $\mathbb{R}^d$, with support in the unit ball such that $\int_{\mathbb{R}^d} \varphi(y)\, dy = 1$. For $n \in \mathbb{N}^T$ define the following smooth functions by convolution

$$
\begin{aligned}
b^n(t,x,a) &= n^d \int_{\mathbb{R}^d} b(t,x-y,a)\,\varphi(ny)\,dy,\\
f^n(t,x,a) &= n^d \int_{\mathbb{R}^d} f(t,x-y,a)\,\varphi(ny)\,dy,\\
\sigma^{j,n}(t,x) &= n^d \int_{\mathbb{R}^d} \sigma^j(t,x-y)\,\varphi(ny)\,dy,\\
g^n(x) &= n^d \int_{\mathbb{R}^d} g(x-y)\,\varphi(ny)\,dy.
\end{aligned}
$$

In the next Lemma we list some properties satisfied by these functions.

Lemma 3.4. *(1) The functions* $b^n(t,x,a)$, $\sigma^{j,n}(t,x)$, $f^n(t,x,a)$, *and* $g^n(x)$ *are Borel measurable bounded functions and Lipschitz continuous with constant* K *in* x.

(2) There exists a constant C *positive independent of* t, x *and* n *such that for every* t *in* $[0,T]$

$$
\begin{aligned}
&|b^n(t,x,a) - b(t,x,a)| + \left|\sigma^{j,n}(t,x) - \sigma^j(t,x)\right| \le \frac{C}{n},\\
&|f^n(t,x,a) - f(t,x,a)| + |g^n(x) - g(x)| \le \frac{C}{n}.
\end{aligned}
$$

(3) The functions $b^n(t,x,a)$, $f^n(t,x,a)$, $\sigma^{j,n}(t,x)$ *and* $g^n(x)$ *are* C^∞*-functions in* x, *and for all* t *in* $[0,T]$, *we have*

$$
\begin{aligned}
&\lim_{n\to+\infty} b^n_x(t,x,a) = b_x(t,x,a) \quad dx\text{-a.e.},\\
&\lim_{n\to+\infty} f^n_x(t,x,a) = f_x(t,x,a) \quad dx\text{-a.e.},\\
&\lim_{n\to+\infty} \sigma^{j,n}_x(t,x) = \sigma^j_x(t,x) \quad dx\text{-a.e.},\\
&\lim_{n\to+\infty} g^n_x(x) = g_x(x) \qquad dx\text{-a.e.}
\end{aligned}
$$

(4) For every $p \geq 1$ *and* $R > 0$

$$\lim_{n\to\infty} \iint\limits_{B(0,R)\times[0,T]} \sup_{a\in A} |b_x^n(t,x,a) - b_x(t,x,a)|^p \, dxdt = 0,$$

$$\lim_{n\to\infty} \iint\limits_{B(0,R)\times[0,T]} \sup_{a\in A} |f_x^n(t,x,a) - f_x(t,x,a)|^p \, dxdt = 0.$$

where $B(0,R)$ *denotes a ball in* $\mathbb{R}^d$ *of radius* R.

Proof. Statements (1), (2) and (3) are classical facts (see [53] for the proof).

(4) is proved as in [7]. ■

Now, consider the process y_t, $t \geq 0$, solution of the stochastic differential equation, defined on the enlarged probability space $\left(\widetilde{\Omega}, \widetilde{F}, \left(\widetilde{F}_t\right)_{t\geq 0}, \widetilde{P}, \widetilde{B}_t\right)$ by

$$\begin{cases} dy_t = b^n(t, y_t, u_t)\, dt + \sigma^n(t, y_t)\, d\widetilde{B}_t, \\ y_0 = \alpha, \end{cases} \tag{3.15}$$

and define the cost functional

$$J^n(u_t) = \widetilde{E}\left[\int_0^T f^n(t, y_t, u_t)\, dt + g^n(y_T)\right], \tag{3.16}$$

where b^n, σ^n, f^n and g^n be the regularized functions of b, σ, f and g.

The following result gives the estimates which relate the original control problem with the perturbed ones.

Lemma 3.5. *Let* (x_t) *and* (y_t) *the solutions of* (3.1) *and* (3.15) *respectively, corresponding to an admissible control* u. *Then there exists positive constants* M_1 *and* M_2 *such that:*

(1) $\widetilde{E}\left[\sup\limits_{0\leq t\leq T} |x_t^u - y_t^u|^2\right] \leq M_1.(\epsilon_n)^2$.

(2) $|J^n(u_t) - J(u_t)| \leq M_2.\epsilon_n$, *where* $\epsilon_n = \dfrac{C}{n}$.

Proof. This lemma follows from standard arguments from stochastic calculus and lemma 3.4. ■

Let $\hat{u}$ be an optimal for the original control problem (3.1) and (3.2). Note that $\hat{u}$ is not necessarily optimal for the perturbed control problem (3.15) and (3.16). However, by Lemma 3.5 we obtain the existence of $(\delta_n) \equiv (2M_2.\epsilon_n)$, a sequence of positive real numbers converging to 0 such that:

$$J^n(\hat{u}) \leq \inf_{u \in U} J^n(u) + \delta_n.$$

That is $\hat{u}$ is δ_n-optimal for the perturbed control problem. According to Lemma 3.5 it is easy to see that $J^n(.)$ is continuous on U endowed with the metric d defined by (3.14). By the Ekeland principle for $\hat{u}$ with $\lambda_n = \delta_n^{\frac{2}{3}}$, there is an admissible control u^n such that

$$d(\hat{u}, u^n) \leq \delta_n^{\frac{2}{3}},$$

and

$$J_\delta^n(u^n) \leq J_\delta^n(u), \text{ for any } u \in U,$$

where

$$J_\delta^n(u) = J^n(u) + \delta_n^{\frac{1}{3}} d(u, u^n).$$

This means that u^n is an optimal for the perturbed system (3.15) with a new cost function J_δ^n. Denote by x^n the unique solution of (3.15) corresponding to u^n, and let $\Phi^n(s,t)$ $(s \geq t)$, be the fundamental solution of the linear equation

$$\begin{cases} d\Phi^n(s,t) = b_x^n(s, x_s^n, u_s^n).\Phi^n(s,t)\,dt + \sum_{1 \leq j \leq d} \sigma_x^{j,n}(s, x_s^n).\Phi^n(s,t)\,d\widetilde{B}_s^j, \\ \Phi^n(t,t) = Id. \end{cases} \tag{3.17}$$

Remark 3.6. *Since u^n is optimal for J_δ^n, and the functions b^n, σ^n, f^n and g^n are smooth, we can use the spike variation technique to derive a maximum principle for u^n.*

Proposition 3.7. *For each integer n, there exists an admissible control u^n and a $\left(\widetilde{F}_t\right)$-adapted process q_t^n given by*

$$q_t^n = -\widetilde{E}\left[\int_t^T \Phi^{n,T}(s,t).f_x^n(s,x_s^n,u_s^n)\,ds + \Phi^{n,T}(T,t).g_x^n(x_T^n)\diagup\widetilde{F}_t\right], \tag{3.18}$$

and a Lebesgue null set N such that for $t \in N^c$

$$\widetilde{E}\left[H^n(t,x_t^n,u_t^n,q_t^n)\right] \geq \widetilde{E}\left[H^n(t,x_t^n,v,q_t^n)\right] - \delta_n^{\frac{1}{3}},$$

for every A-valued F_t-measurable random variable v, where the Hamiltonian H^n is defined by

$$H^n(t,x,u,p) = p.b^n(t,x,u) - f^n(t,x,u). \tag{3.19}$$

Where $\Phi^{n,*}$ denotes the transpose of the matrix Φ^n.

Proof. Let $t_0 \in [0,T]$ and v a A-valued F_t-measurable random variable. For any $\varepsilon \geq 0$, define $u_\varepsilon^n \in U$ by

$$u_\varepsilon^n = \begin{cases} v & t \in [t_0, t_0+\varepsilon], \\ u_t^n & t \in [0,T] \diagdown [t_0, t_0+\varepsilon]. \end{cases}$$

The fact that

$$J_\delta^n(u^n) \leq J_\delta^n(u_\varepsilon^n),$$

and

$$d(u_\varepsilon^n, u^n) \leq \varepsilon,$$

imlpy that

$$J^n(u_\varepsilon^n) - J^n(u^n) \geq -\delta_n^{\frac{1}{3}}.\varepsilon.$$

However, according to Lemma 3.5 the data defining the perturbed control problem (3.15), (3.16) are differentiable, therefore the map $\varepsilon \to J^n(u_\varepsilon^n)$ is differentiable at $\varepsilon = 0$, and that

$$\frac{dJ^n(u_\varepsilon^n)}{d\varepsilon}\mid_{\varepsilon=0} = \widetilde{E}\left[H^n\left(t, x_t^n, u_t^n, q_t^n\right)\right] - \widetilde{E}\left[H^n\left(t, x_t^n, v, q_t^n\right)\right] + \delta_n^{\frac{1}{3}} \geq 0,$$

for every A-valued F_t-measurable random variable v. ■

Remark 3.8. *This inequality can be proved for every near optimal control u_δ, using the stability of the state equation and adjoint process with respect to the control variable (see Zhou* [123]*).*

Let $\Phi^n(s,t)$ $(s \geq t)$ the $d \times d$-matrix valued process, satisfying the following linear equation

$$\begin{cases} d\Phi^n(s,t) = b_x^n\left(s, \hat{x}_s^n, \hat{u}_s\right).\Phi^n(s,t)\,dt + \sum\limits_{1\leq j\leq d} \sigma_x^{j,n}\left(s, \hat{x}_s^n\right).\Phi^n(s,t)\,d\widetilde{B}_s^j, \\ \Phi^n(t,t) = Id, \end{cases} \tag{3.20}$$

where $\hat{x}_t^n$ is the unique solution of (3.15) *corresponding to the optimal control $\hat{u}$*

$$\begin{cases} d\hat{x}_t^n = b^n\left(t, \hat{x}_t^n, \hat{u}_t\right)dt + \sigma^n\left(t, \hat{x}_t^n\right)d\widetilde{B}_t, \\ \hat{x}_0^n = \alpha. \end{cases} \tag{3.21}$$

Corollary 3.9. *there exists an $\left(\widetilde{F}_t\right)$-adapted process satisfying*

$$p_t^n := -\widetilde{E}\left[\int_t^T \Phi^{n,T}(s,t).f_x\left(s, \hat{x}_s^n, \hat{u}_s\right)ds + \Phi^{n,T}(T,t).g_x\left(\hat{x}_T\right) \diagup \widetilde{F}_t\right], \tag{3.22}$$

and a Lebesgue null set N such that, for $t \in N^c$,

$$\widetilde{E}\left[H^n\left(t, \hat{x}_t^n, \hat{u}_t, p_t^n\right)\right] \geq \widetilde{E}\left[H^n\left(t, \hat{x}_t^n, v, p_t^n\right)\right] - \delta_n^{\frac{1}{3}}, \tag{3.23}$$

for every A-valued F_t-measurable random variable v.

3.2.2 Passing to the Limit

Our aim is now to give a maximum principle of diffusion processes with Lipschitz coefficients (problem (3.9) and (3.10)). To pass to the Limit in (3.24) and (3.25). We will use Egorov and Portmanteau-Alexandrov Theorems, we will use also the notion of extension of the initial filtered probability space, defined by Bouleau and Hirsch.

Lemma 3.10. *We have*

$$\lim_{n\to+\infty} \widetilde{E}\left[\sup_{s\le t\le T} \left|\Phi^n(s,t)-\Phi(s,t)\right|^2\right]=0, \tag{3.24}$$

$$\lim_{n\to+\infty} \widetilde{E}\left[\sup_{0\le t\le T} \left|p_t^n-p_t\right|^2\right]=0, \tag{3.25}$$

$$\lim_{n\to+\infty} \widetilde{E}\left[\left|H^n(t,\hat{x}_t^n,\hat{u}_t,p_t^n)-H(t,\hat{x}_t,\hat{u}_t,p_t)\right|\right]=0, \tag{3.26}$$

where Φ_t, p_t and H are determined by the fundamental solution (3.11), the adjoint process (3.9) and the associated Hamiltonian (3.12), corresponding to the optimal pair $(\hat{x},\hat{u})$. Φ_t^n, p_t^n and H^n are determined by the fundamental solution (3.20), the adjoint process (3.22) and the associated Hamiltonian (3.19), corresponding to the approximating sequence $\hat{x}_t^n$, given by (3.21).

Proof. In view of the Burkholder, Schwartz inequalities and the Gronwall Lemma, we have

$$\widetilde{E}\left[\sup_{t\le s\le T}\left|\Phi^n(s,t)-\Phi(s,t)\right|^2\right]\le$$
$$M\widetilde{E}\left[\sup_{t\le s\le T}\left|\Phi^n(s,t)\right|^4\right]^{\frac{1}{2}}.\left\{\widetilde{E}\left[\int_0^T\left|b_x^n(t,\hat{x}_t^n,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)\right|^4dt\right]^{\frac{1}{2}}\right.$$
$$\left.+\sum_{1\le j\le d}\widetilde{E}\left[\int_0^T\left|\sigma_x^{j,n}(t,\hat{x}_t^n)-\sigma_x^j(t,\hat{x}_t)\right|^4dt\right]^{\frac{1}{2}}\right\},$$

since the coefficients in the linear stochastic differential equation (3.21) are bounded, it is easy to see that $\widetilde{E}\left[\sup_{t\leq s\leq T}|\Phi^n(s,t)|^4\right]<+\infty$. To derive (3.24), it is sufficient to prove the following

$$\widetilde{E}\left[\int_0^T |b_x^n(t,\hat{x}_t^n,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)|^4\,dt\right]\to 0 \qquad \text{as } n\to+\infty,$$

and

$$\widetilde{E}\left[\int_0^T \left|\sigma_x^{j,n}(t,\hat{x}_t^n)-\sigma_x^j(t,\hat{x}_t)\right|^4 dt\right]\to 0 \qquad \text{as } n\to+\infty,\ j=1,2,.....,d.$$

Let us prove the first Limit. We have

$$\widetilde{E}\left[\int_0^T |b_x^n(t,\hat{x}_t^n,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)|^4\,dt\right]\leq M\left(I_1^n+I_2^n\right),$$

where

$$I_1^n=\widetilde{E}\left[\int_0^T\sup_{a\in A}|b_x^n(t,\hat{x}_t^n,a)-b_x(t,\hat{x}_t^n,a)|^4\,dt\right],$$

and

$$I_2^n=\widetilde{E}\left[\int_0^T\sup_{a\in A}|b_x(t,\hat{x}_t^n,a)-b_x(t,\hat{x}_t,a)|^4\,dt\right],$$

Since the law of $\hat{x}_t^n$ is absolutely continuous with respect to the Lebesgue measure, let $\rho_t^n(y)$ its density. Then

$$I_1^n=\int_0^T\int_{\mathbb{R}^d}\sup_{a\in A}|b_x^n(t,y,a)-b_x(t,y,a)|^4\,\rho_t^n(y)\,dydt.$$

Let us show that, for all $t\in[0,T]$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\sup_{a\in A}|b_x^n(t,y,a)-b_x(t,y,a)|^4\,\rho_t^n(y)\,dy=0.$$

For each $p>0$, $\widetilde{E}\left[\sup_{0\leq t\leq T}|\hat{x}_t^n|^p\right]<\infty$. Thus, $\lim_{R\to+\infty}\widetilde{P}\left(\sup_{0\leq t\leq T}|\hat{x}_t^n|>R\right)=0$, then it is enough to show that for every $R>0$,

$$\lim_{n\to+\infty}\int_{B(0,R)}\sup_{a\in A}|b_x^n(t,y,a)-b_x(t,y,a)|^4\,\rho_t^n(y)\,dy=0.$$

According to Lemma 3.4

$$\sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|^4 \to 0 \quad dy\text{-}a.e,$$

at least for a subsequence. Then by Egorov's Theorem, for every $\delta > 0$, there exists a measurable set F with $\lambda(F) < \delta$, such that $\sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|$ converges uniformly to 0 on the set F^c. Note that, since the Lebesgue measure is regular, F may be chosen closed. This implies that

$$\begin{aligned} & \lim_n \int_{F^c} \sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ \leq\ & \lim_n \left(\sup_{y\in F^c}\sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|^4\right) = 0. \end{aligned}$$

Now, by using the boundness of the derivatives b_x^n, b_x by the Lipschitz constant M, we have

$$\begin{aligned} & \int_F \sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ =\ & \widetilde{E}\left[\sup_{a\in A}\left|b_x^n(t,\hat{x}_t^n,a)-b_x(t,\hat{x}_t^n,a)\right|^4 \chi_{\{\hat{x}_t^n\in F\}}\right] \\ \leq\ & 2M^4\widetilde{P}\left(\hat{x}_t^n \in F\right). \end{aligned}$$

Since $(\hat{x}_t^n)$ converges to $\hat{x}_t$ in probability, then in distribution. Applying the Portmanteau-Alexandrov Theorem, we obtain

$$\begin{aligned} \lim_{n\to+\infty}\int_F \sup_{a\in A}\left|b_x^n(t,y,a)-b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy &\leq 2M^4 \limsup \widetilde{P}\left(\hat{x}_t^n \in F\right) \\ &\leq 2M^4\widetilde{P}\left(\hat{x}_t \in F\right) \\ &= 2M^4\int_F \rho_t(y)\,dy < \varepsilon. \end{aligned}$$

where $\rho_t(y)$ denotes the density of $\hat{x}_t$ with respect to Lebesgue measure.

Now, since

$$\begin{aligned} &\int_{B(0,R)} \sup_{a\in A} \left|b_x^n(t,y,a) - b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ = &\int_{F} \sup_{a\in A} \left|b_x^n(t,y,a) - b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ &+ \int_{F^c} \sup_{a\in A} \left|b_x^n(t,y,a) - b_x(t,y,a)\right|^4 \rho_t^n(y)\,dy, \end{aligned}$$

we get $\lim\limits_{n\to+\infty} I_1^n = 0.$

Let $k \geq 0$ be a fixed integer, then it holds that $I_2^n \leq C\left(J_1^k + J_2^k + J_3^k\right)$, where

$$\begin{aligned} J_1^k &= \widetilde{E}\left[\int_0^T \left|b_x(t,\hat{x}_t^n,\hat{u}_t) - b_x^k(t,\hat{x}_t^n,\hat{u}_t)\right|^4 dt\right], \\ J_2^k &= \widetilde{E}\left[\int_0^T \left|b_x^k(t,\hat{x}_t^n,\hat{u}_t) - b_x^k(t,\hat{x}_t,\hat{u}_t)\right|^4 dt\right], \\ J_3^k &= \widetilde{E}\left[\int_0^T \left|b_x^k(t,\hat{x}_t,\hat{u}_t) - b_x(t,\hat{x}_t,\hat{u}_t)\right|^4 dt\right]. \end{aligned}$$

Applying the same argements used in the first limit (Egorov and Portmanteau-Alexandrov Theorems), we obtain that $\lim\limits_{n\to+\infty} J_1^k = 0$. We use the continuity of b_x^k in x and the convergence in probability of $\hat{x}_T^n$ to $\hat{x}_T$ to deduce that $b_x^k(t,\hat{x}_t^n,\hat{u}_t)$ converges to $b_x^k(t,\hat{x}_t,\hat{u}_t)$ in probability as $n\to\infty$, and to infer by using the Dominated convergence Theorem that $\lim\limits_{n\to+\infty} J_2^k = 0$. Since b_x^k, b_x are bounded by the Lipschitz constant and by using the absolute continuity of the law of $\hat{x}_t$ with respect to the Lebesgue measure, the convergence of b_x^k to b_x, and the Dominated convergence Theorem, we get $\lim\limits_{n\to+\infty} J_3^k = 0$.

Next, let use prove the limit (3.25). Clearly

$$\widetilde{E}\left[\left|p_t^n - p_t\right|^2\right] \leq C_1\left(\alpha_1^n + \alpha_2^n\right),$$

where

$$\alpha_1^n = \widetilde{E}\left[\int\limits_t^T \left|\left(\Phi^{n,T}(s,t).f_x^n(s,\hat{x}_s^n,\hat{u}_s) - \Phi^T(s,t).f_x(s,\hat{x}_s,\hat{u}_s)\right)\right|^2 ds\right]$$

and

$$\alpha_2^n = \widetilde{E}\left[\left|\Phi^{n,T}(T,t).g_x^n(\hat{x}_T^n) - \Phi^T(T,t).g_x(\hat{x}_T)\right|^2\right].$$

Since f_x is bounded by the Lipschitz constant M, and applying the Schwartz inequality, we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}\alpha_1^n &\leq C\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^{n,T}(s,t)\,ds\right|^4\right]^{\frac{1}{2}}.\widetilde{E}\left[\int_0^T \left|f_x^n(s,\hat{x}_s^n,\hat{u}_s) - f_x(s,\hat{x}_s,\hat{u}_s)\right|^4 ds\right]^{\frac{1}{2}} \\ &\quad + CM.\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^{n,T}(s,t) - \Phi^*(s,t)\right|^2\right]\end{aligned}$$

It is easy to see that $\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^{n,T}(s,t)\,ds\right|^4\right] < +\infty$. Applying the same arguments used in the first Limit (Egorov and Portmanteau - Alexandrov Theorems) it holds that

$$\lim_{n\to+\infty}\widetilde{E}\left[\int_0^T \left|f_x^n(s,\hat{x}_s^n,\hat{u}_s) - f_x(s,\hat{x}_s,\hat{u}_s)\right|^4 ds\right]^{\frac{1}{2}} = 0.$$

On the other hand, since g_x is bounded by the Lipschitz constant, and applying the Schwartz inequality we get

$$\begin{aligned}\alpha_2^n &\leq C\left\{\widetilde{E}\left[\left|\Phi^{n,T}(T,t)\right|^4\right]\right\}^{\frac{1}{2}}.\left\{\widetilde{E}\left[\left|g_x^n(\hat{x}_T^n) - g_x(\hat{x}_T)\right|^4\right]\right\}^{\frac{1}{2}} \\ &\quad + CM.\widetilde{E}\left[\left|\Phi^{n,T}(T,t) - \Phi^T(T,t)\right|^2\right],\end{aligned}$$

where M is a positive constant.

Let $k \geq 0$ be a fixed integer, then it holds that

$$\begin{aligned}\widetilde{E}\left[\left|g_x(\hat{x}_T^n) - g_x(\hat{x}_T)\right|^4\right] &\leq \widetilde{E}\left[\left|g_x^n(\hat{x}_T^n) - g_x^k(\hat{x}_T^n)\right|^4\right] + \widetilde{E}\left[\left|g_x^k(\hat{x}_T^n) - g_x^k(\hat{x}_T)\right|^4\right] \\ &\quad + \widetilde{E}\left[\left|g_x^k(\hat{x}_T) - g_x(\hat{x}_T)\right|^4\right].\end{aligned}$$

The law of $\hat{x}_T^n$ is absolutely continuous with respect to the Lebesgue measure, let $\rho_T^n(y)$ it is density, and by the same fashion (by applying Egorov and Portmanteau - Alexandrov Theorems), we get

$$\lim_{n\to+\infty} \widetilde{E}\left[\left|g_x^n(\hat{x}_T^n) - g_x^k(\hat{x}_T^n)\right|^4\right] = \lim_{n\to+\infty}\int\limits_{\mathbb{R}^d} \left|g_x^n(y) - g_x^k(y)\right|^4 \rho_T^n(y)\,dy = 0.$$

We use the continuity of g_x^k in x and the convergence in probability of $\hat{x}_T^n$ to $\hat{x}_T$ to deduce that $g_x^k(\hat{x}_T^n)$ converges to $g_x^k(\hat{x}_T)$ in probability as $n \to \infty$, and to infer by using the Dominated convergence Theorem that

$$\lim_{n\to+\infty} \widetilde{E}\left[\left|g_x^k(\hat{x}_T^n) - g_x^k(\hat{x}_T)\right|^4\right] = 0.$$

Since

$$\widetilde{E}\left[\left|g_x^k(\hat{x}_T) - g_x(\hat{x}_T)\right|^4\right] = \int_{\mathbb{R}^d} \left|g_x^k(y) - g_x(y)\right|^4 \rho_T(y)\,dy,$$

g_x^k, g_x are bounded by the Lipschitz constant, and g_x^k converges to g_x dx-$a.e$, we conclude by the Dominated convergence Theorem that

$$\lim_{n\to+\infty} \widetilde{E}\left[\left|g_x^k(\hat{x}_T) - g_x(\hat{x}_T)\right|^4\right] = 0.$$

Finally, by using Burkholder-Davis-Gundy inequality, we obtain (3.25).

Now, let use prove that

$$\lim_{n\to+\infty} \widetilde{E}\left[\left|H^n(t,\hat{x}_t^n,\hat{u}_t,p_t^n) - H(t,\hat{x}_t,\hat{u}_t,p_t)\right|\right] = 0.$$

Applying the Schwartz inequality we get

$$\widetilde{E}\left[\left|H^n(t,\hat{x}_t^n,\hat{u}_t,p_t^n) - H(t,\hat{x}_t,\hat{u}_t,p_t)\right|\right] \le \left\{\widetilde{E}\left|p_t^n - p_t\right|^2\right\}^{\frac{1}{2}} . \left\{\widetilde{E}\left|b^n(t,\hat{x}_t^n,\hat{u}_t)\right|^2\right\}^{\frac{1}{2}}$$
$$+ \left\{\widetilde{E}\left|b^n(t,\hat{x}_t^n,\hat{u}_t) - b(t,\hat{x}_t,\hat{u}_t)\right|^2\right\}^{\frac{1}{2}} . \left\{\widetilde{E}\left|p_t\right|^2\right\}^{\frac{1}{2}} + \widetilde{E}\left|f^n(t,\hat{x}_t^n,\hat{u}_t) - f(t,\hat{x}_t,\hat{u}_t)\right|.$$

Lemma 3.4 and (3.25) imply that the first expression in the right hand side converges to 0 as $n \to \infty$. Since

$$\begin{aligned} \widetilde{E}\left|f^n\left(t, \hat{x}_t^n, \hat{u}_t\right) - f\left(t, \hat{x}_t, \hat{u}_t\right)\right| &\leq \widetilde{E}\left|f^n\left(t, \hat{x}_t^n, \hat{u}_t\right) - f^n\left(t, \hat{x}_t, \hat{u}_t\right)\right| \\ &\quad + \widetilde{E}\left|f^n\left(t, \hat{x}_t, \hat{u}_t\right) - f\left(t, \hat{x}_t, \hat{u}_t\right)\right|, \end{aligned}$$

f^n being continuous and bounded, $\hat{x}_t^n$ converges uniformly in probability to $\hat{x}_t$, we conclude by the Dominated convergence Theorem that

$$\lim_{n \to +\infty} \widetilde{E}\left|f^n\left(t, \hat{x}_t^n, \hat{u}_t\right) - f^n\left(t, \hat{x}_t, \hat{u}_t\right)\right| = 0.$$

Using Lemma 3.5 and the Dominated convergence Theorem to conclude that

$$\lim_{n \to +\infty} \widetilde{E}\left|f^n\left(t, \hat{x}_t, \hat{u}_t\right) - f\left(t, \hat{x}_t, \hat{u}_t\right)\right| = 0.$$

The convergence of the second term in the right hand side can be performed in a similar way. ■

Proof of Theorem 3.1.. Use the Corollary 3.9 and the Lamma 3.10. ■

Chapter 4

The SMP for singular optimal control of diffusions with non smooth coefficients

Our aim in this Chapter is to extend the stochastic maximum principle in singular optimal control to the case where the coefficients b, σ, f and g are Lipschitz continuous in x, we prove that the analogue of the section 6 of chapter 2 holds, provided that the classical derivatives are replaced by the generalized one. We approximate the initial control problem by smooth ones, and we apply Ekeland's principle to derive the associated adjoint processes and use Krylov's inequality to prove the convergence in the uniformly elliptic case. In the degenerate case, we use techniques of Bouleau-Hirsch on the differentiability of the solution of an SDE with Lipschitz coefficients with respect to initial data, in the distribution sense, and we use Egorov and Portmanteau-Alexandrov Theorems to prove the convergence of the derivatives.

4.1 Assumption

Let (Ω, F, F_t, P) be a filtered probability space satisfying the usual conditions, on which a d-dimensional Brownian motion (B_t) is defined with the filtration (F_t). Let T be a strictly positive real number, A_1 is a non empty subset of $\mathbb{R}^n$ and $A_2 = ([0, \infty))^m$. U_1 is the class of measurable, adapted processes $u : [0, T] \times \Omega \to A_1$, and U_2 is the class of measurable, adapted processes $\xi : [0, T] \times \Omega \to A_2$.

Definition 4.1. *An admissible control is a pair (u, ξ) of $A_1 \times A_2$-valued, measurable F_t-adapted processes, such that u is absolutely continuous, and ξ is of bounded variation, non decreasing left-continuous with right limits and $\xi_0 = 0$.*

We denote by $U = U_1 \times U_2$ the set of all admissible controls. For $(u, \xi) \in U$, suppose the state $x_t = x_t^{(u,\xi)} \in \mathbb{R}^d$ is described by the equation

$$\begin{cases} dx_t &= b(t, x_t, u_t)\, dt + \sigma(t, x_t)\, dB_t + G_t d\xi_t, \qquad \text{for } t \in [0, T], \\ x_0 &= \alpha, \end{cases} \tag{4.1}$$

Since $d\xi_t$ may be singular with respect to Lebesgue measure dt, we call ξ our singular control. and the process u is our absolutely continuous control. Suppose we are given a cost functional $J(u, \xi)$ of the form

$$J(u, \xi) = E\left[\int_0^T f(t, x_t, u_t)\, dt + \int_0^T k_t d\xi_t + g(x_T)\right], \tag{4.2}$$

Where $b : [0, T] \times \mathbb{R}^d \times A_1 \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f : [0, T] \times \mathbb{R}^d \times A_1 \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$, $G : [0, T] \to \mathbb{R}^d \otimes \mathbb{R}^m$, and $k : [0, T] \to ([0, \infty))^m$. Satisfy the following: b, σ,

f and g are Borel measurable and bounded functions and there exist $M > 0$, such that for all (t, x, y, a) in $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times A_1$

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x) - \sigma(t, y)| \leq M |x - y|, \tag{4.3}$$

$$|f(t, x, a) - f(t, y, a)| + |g(x) - g(y)| \leq M |x - y|, \tag{4.4}$$

$$b(t, x, a) \text{ and } f(t, x, a) \text{ are continuous in } a \text{ uniformly in } (t, x), \tag{4.5}$$

$$\exists c > 0, \forall \zeta \in \mathbb{R}^d, \forall (t, x) \in [0, T] \times \mathbb{R}^d, \zeta^* \sigma(t, x) \sigma^*(t, x) \zeta \geq c |\zeta|^2, \tag{4.6}$$

and

$$G,\ k \text{ are continuous and bounded.} \tag{4.7}$$

Find $(\hat{u}, \hat{\xi}) \in U$ such that

$$J(\hat{u}, \hat{\xi}) = \min_{(u,\xi) \in U} J(u, \xi),$$

any $(\hat{u}, \hat{\xi})$ satisfying the above is called an optimal control of problem (4.1), (4.2), the corresponding state process $\hat{x}$ is called an optimal state process, and $\hat{u}$ (resp $\hat{\xi}$) is called absolutely continuous (resp singular) optimal control.

Under the above hypothesise, the SDE (4.1) has a unique strong solution x_t and the cost is a well defined from U into $\mathbb{R}$. Such that for any $p > 0$,

$$E\left[\sup_{0 \leq t \leq T} |x_t|^p\right] < +\infty.$$

Since b, σ^j (the jth column of the matrix σ), f and g are Lipschitz continuous functions in the state variable they are differentiable almost everywhere in the sense of Lebesgue measure (Rademacher Theorem). Let us denote by b_x, σ_x, f_x and g_x any Borel measurable functions such that

$$\begin{aligned} \partial_x b(t,x,a) &= b_x(t,x,a) \quad dx\text{-}a.e., \\ \partial_x f(t,x,a) &= f_x(t,x,a) \quad dx\text{-}a.e., \\ \partial_x \sigma(t,x) &= \sigma_x(t,x) \quad dx\text{-}a.e., \\ \partial_x g(x) &= g_x(x) \quad dx\text{-}a.e. \end{aligned}$$

It is clear that these almost everywhere derivatives are bounded by the Lipschitz constant M. Finally, assume that $b_x(t,x,a)$ and $f_x(t,x,a)$ are continuous in a uniformly in (t,x)

4.2 The non degenerate case

4.2.1 The main result

The main result of this section is stated in the following Theorem.

Theorem 4.2. (Stochastic maximum principle) Let $(\hat{u},\hat{\xi})$ be an optimal control for the controlled system (4.1), (4.2) and let $\hat{x}$ be the corresponding optimal trajectory. Then there exists a measurable F_t-adapted process p_t satisfying

$$p_t := E\left[\int_t^T \Phi^T(s,t).f_x(s,\hat{x}_s,\hat{u}_s)\,ds + \Phi^T(T,t).g_x(\hat{x}_T)\diagup F_t\right], \tag{4.8}$$

such that for all $a \in A_1$ and $\eta \in U_2$

$$0 \le H(t,\hat{x}_t,a,p_t) - H(t,\hat{x}_t,\hat{u}_t,p_t) \quad dt\text{-a.e},\ \ P\text{-a.s.}, \tag{4.9}$$

and

$$0 \le E\int_0^T \left(k_t + G_t^T p_t\right) d\left(\eta - \hat{\xi}\right)_t \tag{4.10}$$

where the Hamiltonian H associated to the control problem is

$$H(t,x,u,p) = p.b(t,x,u) + f(t,x,u), \tag{4.11}$$

and $\Phi(s,t)$, $(s \geq t)$ is the fundamental solution of the linear equation

$$\begin{cases} d\Phi(s,t) = b_x(s,\hat{x}_s,\hat{u}_s).\Phi(s,t)\,ds + \sum\limits_{1\leq j\leq d} \sigma_x^j(s,\hat{x}_s).\Phi(s,t)\,dB_s^j, \\ \Phi(t,t) = Id. \end{cases} \tag{4.12}$$

Theorem 4 *Here* ^{T} *denotes the transpose.*

4.2.2 Proof of the main result

Let us recall Krylov's inequality for diffusion processes which will be used in the sequel.

Theorem 4.3. (Krylov [17]) Let (Ω, F, F_t, P) be a filtered probability space, $(B_t)_{t\geq 0}$ a d-dimensional Brownian motion, $b : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d$, $\sigma : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d\otimes\mathbb{R}^d$ bounded adapted processes such that: $\exists c > 0$, $\forall\zeta \in \mathbb{R}^d$, $\forall(t,x) \in [0,T]\times\mathbb{R}^d$, $\zeta^*\sigma\sigma^*\zeta \geq c|\zeta|^2$. Let

$$x_t = x + \int_0^t b(t,w)\,dt + \int_0^t \sigma(t,w)\,dB_t,$$

be an Itô process. Then for every Borel function $f : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}$ with support in $[0,T]\times B(0,M)$, the following inequality holds

$$E\left[\int_0^T |f(t,x_t)|\,dt\right] \leq K\left[\int_0^T \int_{B(0,M)} |f(t,x)|^{d+1}\,dt dx\right]^{\frac{1}{d+1}},$$

where K is a constant and $B(0,M)$ is the ball of center 0 and radius M.

To apply Ekeland's variational principle in the non degenerate case, we have to endow the set of controls with an appropriate metric. For any $(u,\xi)\,,\,(\upsilon,\eta)\in U$, we set

$$d_1\left(u,v\right)=P\otimes dt\left\{(w,t)\in\Omega\times[0,T]\,,v\left(w,t\right)\neq u\left(w,t\right)\right\}, \tag{4.13}$$

$$d_2\left(\xi,\eta\right)=\left(E\left[\sup_{0\leq t\leq T}\left|\xi_t-\eta_t\right|^2\right]\right)^{\frac{1}{2}}, \tag{4.14}$$

$$d\left(\left(u,\xi\right),\left(\upsilon,\eta\right)\right)=d_1\left(u,v\right)+d_2\left(\xi,\eta\right). \tag{4.15}$$

where $P\otimes dt$ is the product measure of P with the Lebesgue measure dt.

Lemma 4.4.

Lemma 5 *(1)* (U,d) *is a complete metric space.*

(2) The cost functional J *is continuous from* U *into* $\mathbb{R}$.

Proof. (1) It is clear that (U_2,d_2) is a complete metric space. Moreover, it was shown in [50] that (U_1,d_1) is a complete metric space. Hence (U,d) is a complete metric space.

Item (2) is proved as in [93][123]. ■

Necessary conditions for a family of perturbed control problems

For $n\in\mathbb{N}^*$, let us consider the sequence of perturbed control problems obtained by replacing b, σ, f and g by b^n, σ^n, f^n and g^n. Let us denote y the solution of the controlled stochastic differential equation.

$$\begin{cases} dy_t=b^n\left(t,y_t,u_t\right)dt+\sigma^n\left(t,y_t\right)dB_t+G_t d\xi_t, \\ y_0=\alpha, \end{cases} \tag{4.16}$$

The corresponding cost is given by

$$J^n(u,\xi) = E\left[\int_0^T f^n(t, y_t, u_t)\, dt + \int_0^T k_t d\xi_t + g^n(y_T)\right], \tag{4.17}$$

Lemma 4.5. Let $(u,\xi) \in U$, x_t and y_t the solutions of (4.1) and (4.16) respectively corresponding to the control (u,ξ), then we have

Lemma 6 *(1)* $E\left[\sup_{0\le t\le T} |x_t - y_t|^2\right] \le M_1.(\epsilon_n)^2$, *where* $\epsilon_n = \dfrac{C}{n}$.

(2) $|J^n(u,\xi) - J(u,\xi)| \le M_2.\epsilon_n$.

Proof. Since $x_t - y_t$ and $J^n(u,\xi) - J(u,\xi)$ does not depend on the singular part, then This lemma follows from standard arguments from stochastic calculus and lemma 3.4.

■

Let us suppose that $\left(\hat{u},\hat{\xi}\right) \in U$ is an optimal control for the initial control problem (4.1) and (4.2). Note that $\left(\hat{u},\hat{\xi}\right)$ is not necessarily optimal for the perturbed control problem (4.16) and (4.17). However, by Lemma 4.5 we obtain the existence of $(\delta_n) \equiv (2M_2.\epsilon_n)$ a sequence of positive real numbers converging to 0, such that

$$J^n\left(\hat{u},\hat{\xi}\right) \le \inf_{(\upsilon,\eta)\in U} J^n(\upsilon,\eta) + \delta_n.$$

The control $\left(\hat{u},\hat{\xi}\right)$ will be δ_n-optimal for the perturbed control problem. According to Lemma 4.4, it is easy to see that $J^n(.,.)$ is continuous on $U = U_1 \times U_2$ endowed with the metric $d = d_1 + d_2$ defined by (4.15). By the Ekeland principle (lemma 3.4) for $\left(\hat{u},\hat{\xi}\right)$ with $\lambda_n = \delta_n^{\frac{2}{3}}$. There is an admissible control (u^n,ξ^n) such that

$$d\left(\left(\hat{u},\hat{\xi}\right),(u^n,\xi^n)\right) \le \delta_n^{\frac{2}{3}},$$

and

$$J_{\delta}^{n}\left(u^{n},\xi^{n}\right)\leq J_{\delta}^{n}\left(\upsilon,\eta\right),\quad \text{for a general control }\ (\upsilon,\eta)\in U,$$

where

$$J_{\delta}^{n}\left(\upsilon,\eta\right)=J^{n}\left(\upsilon,\eta\right)+\delta_{n}^{\frac{1}{3}}.d\left(\left(\upsilon,\eta\right),\left(u^{n},\xi^{n}\right)\right).$$

This means that (u^n,ξ^n) is an optimal control for the perturbed system (4.16) with a new cost function J_{δ}^{n}. The controlled process x^n is then defined as the unique solution to the stochastic differential equation,

$$\begin{cases} dx_t^n=b^n\left(t,x_t^n,u_t^n\right)dt+\sigma^n\left(t,x_t^n\right)dB_t+G_t d\xi_t^n, \\ y_0=\alpha, \end{cases} \tag{4.18}$$

We consider $\Phi^n\left(s,t\right)\ \left(s\geq t\right)$, the fundamental solution of the linear stochastic differential equation

$$\begin{cases} d\Phi^n\left(s,t\right)=b_x^n\left(s,x_s^n,u_s^n\right).\Phi^n\left(s,t\right)ds+\sum\limits_{1\leq j\leq d}\sigma_x^{j,n}\left(s,x_s^n\right).\Phi^n\left(s,t\right)dB_s^j, \\ \Phi^n\left(t,t\right)=Id. \end{cases} \tag{4.19}$$

Note that b_x^n, $\sigma_x^{n,j}$ $(j=1,..,d)$ are respectively the matrices of first order partial derivatives of b^n, $\sigma^{n,j}$ $(j=1,..,d)$ with respect to x.

Proposition 4.6. For each integer n, there exists an admissible control (u^n,ξ^n) and a (F_t)-adapted process p_t^n given by

$$p_t^n=E\left[\int_t^T\Phi^{n,T}\left(s,t\right).f_x^n\left(s,x_s^n,u_s^n\right)ds+\Phi^{n,T}\left(T,t\right).g_x^n\left(x_T^n\right)\diagup F_t\right], \tag{4.20}$$

and a Lebesgue null set N such that for $t\in N^c$

$$E\left[H^n\left(t,x_t^n,\upsilon,p_t^n\right)-H^n\left(t,x_t^n,u_t^n,p_t^n\right)\right]\geq-\delta_{n}^{\frac{1}{3}}.M_1, \tag{4.21}$$

and

$$E\int_0^T \left(k_t + G_t^T p_t^n\right) d\left(\eta - \xi^n\right)_t \geq -\delta_n^{\frac{1}{3}}.M_2. \tag{4.22}$$

for all $\upsilon \in A_1$, and $\eta \in U_2$, where the Hamiltonian H^n is defined by

$$H^n\left(t,x,u,p\right) = p.b^n\left(t,x,u\right) + f^n\left(t,x,u\right). \tag{4.23}$$

Here ^{T} denotes the transpose.

Proof. According to the optimality of (u^n, ξ^n) for the perturbed system with the cost function J_δ^n, we can use the spike variation method to derive a maximum principle for (u^n, ξ^n). Let $t_0 \in [0,T]$, $\upsilon \in A_1$ and $\eta \in U_2$, for any $\varepsilon > 0$, define the two perturbations $(u_t^{n,\varepsilon}, \xi_t^n)$ and $(u_t^n, \xi_t^{n,\varepsilon})$ by

$$\left(u_t^{n,\varepsilon}, \xi_t^n\right) = \begin{cases} \left(\upsilon, \xi_t^n\right) & t \in [t_0, t_0+\varepsilon], \\ \left(u_t^n, \xi_t^n\right) & t \in [0,T] \diagdown [t_0, t_0+\varepsilon]. \end{cases}$$

and

$$\left(u_t^n, \xi_t^{n,\varepsilon}\right) = \left(u_t^n, \xi_t^n + \varepsilon\left(\eta_t - \xi_t^n\right)\right)$$

Since (u_t^n, ξ_t^n) is optimal for the cost J_δ^n, then

$$0 \leq J_\delta^n\left(u_t^{n,\varepsilon}, \xi_t^n\right) - J_\delta^n\left(u_t^n, \xi_t^n\right)$$

and

$$0 \leq J_\delta^n\left(u_t^n, \xi_t^{n,\varepsilon}\right) - J_\delta^n\left(u_t^n, \xi_t^n\right)$$

this imply that

$$0 \leq J^n\left(u_t^{n,\varepsilon}, \xi_t^n\right) - J^n\left(u_t^n, \xi_t^n\right) + \delta_n^{\frac{1}{3}}.d_1\left(u_t^n, u_t^{n,\varepsilon}\right),$$

and

$$0 \leq J^n\left(u_t^n, \xi_t^{n,\varepsilon}\right) - J^n\left(u_t^n, \xi_t^n\right) + \delta_n^{\frac{1}{3}}.d_2\left(\xi_t^n, \xi_t^{n,\varepsilon}\right),$$

using the definitions of d_1 and d_2 it holds that

$$0 \leq J^n\left(u_t^{n,\varepsilon}, \xi_t^n\right) - J^n\left(u_t^n, \xi_t^n\right) + \delta_n^{\frac{1}{3}}.M_1\varepsilon, \tag{4.24}$$

and

$$0 \leq J^n\left(u_t^n, \xi_t^{n,\varepsilon}\right) - J^n\left(u_t^n, \xi_t^n\right) + \delta_n^{\frac{1}{3}}.M_2\varepsilon. \tag{4.25}$$

Where M_i $(i = 1, 2)$ is a positive constant. From inequalities (4.24) and (4.25) respectively we use the same method as in section 5 in chapter 2 to obtain respectively (4.21) and (4.22). ■

We use a transformation that makes it possible to apply Krylov's estimate for diffusion processes. Define the dynamics $\overline{b} : [0,T]\times\mathbb{R}^d\times A_1 \to \mathbb{R}^d$, $\overline{b}^n : [0,T]\times\mathbb{R}^d\times A_1 \to \mathbb{R}^d$, $\overline{\sigma} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, and $\overline{\sigma}^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, by

$$\begin{aligned}
&\overline{b}\left(t,x,a\right) = b\left(t, x + \int_0^t G_s d\xi_s, a\right), \\
&\overline{b}^n\left(t,x,a\right) = b^n\left(t, x + \int_0^t G_s d\xi_s, a\right), \\
&\overline{\sigma}\left(t,x\right) = \sigma\left(t, x + \int_0^t G_s d\xi_s\right), \\
&\overline{\sigma}^n\left(t,x\right) = \sigma^n\left(t, x + \int_0^t G_s d\xi_s\right).
\end{aligned}$$

Let z the unique solution of

$$\begin{cases} dz_t = \overline{b}\left(t, z_t, u_t\right)dt + \overline{\sigma}\left(t, z_t\right)dB_t, \\ z_0 = \alpha. \end{cases} \tag{4.26}$$

This implies that $x_t = z_t + \int_0^t G_s d\xi_s$ solves the SDE (4.1) with data (b, σ).

Similary, let z^n the unique solution of

$$\begin{cases} dz_t^n = \overline{b}^n\left(t, z_t^n, u_t\right)dt + \overline{\sigma}^n\left(t, z_t^n\right)dB_t, \\ z_0^n = \alpha. \end{cases} \tag{4.27}$$

Then $x_t^n = z_t^n + \int_0^t G_s d\xi_s$ solves the SDE (4.16) with data (b^n, σ^n).

Note that, $\overline{b}, \overline{b}^n, \overline{\sigma}^j$, and $\overline{\sigma}^{j,n}$ $(j = 1, ..., d)$ are measurable bounded functions and Lipschitz continuous with constant M in x, we conclude that the generalized derivatives (in the distribution sense) $\overline{b}_x, \overline{b}_x^n, \overline{\sigma}_x^j$, and $\overline{\sigma}_x^{j,n}$ $(j = 1, ..., d)$ are well defined.

Lemma 4.7. We have

$$\lim_{n \to +\infty} E\left[\sup_{0 \le t \le T} \left|x_t^n - \hat{x}_t\right|^2\right] = 0 \tag{4.28}$$

$$\lim_{n \to +\infty} E\left[\sup_{t \le s \le T} \left|\Phi^n(s,t) - \Phi(s,t)\right|^2\right] = 0 \tag{4.29}$$

$$\lim_{n \to +\infty} E\left[\sup_{0 \le t \le T} \left|p_t^n - p_t\right|^2\right] = 0 \tag{4.30}$$

$$\lim_{n \to +\infty} E\left[\left|H^n\left(t, x_t^n, u_t^n, p_t^n\right) - H\left(t, \hat{x}_t, \hat{u}_t, p_t\right)\right|\right] = 0. \tag{4.31}$$

Where Φ_t, p_t and H are determined by the fundamental solution (4.12), the adjoint process (4.8) and the associated Hamiltonian (4.11), corresponding to the optimal state process $\hat{x}_t$. Φ_t^n, p_t^n and H^n are determined by the fundamental solution (4.19), the adjoint process (4.20) and the associated Hamiltonian (4.23), corresponding to the approximating sequence x_t^n, given by (4.18).

In what follows, C represents a generic constant, which can be different from line to line.

Proof. By squaring and take expectation, we get

$$E\left[\left|x_t^n - \hat{x}_t\right|^2\right] \le C\left(A_1^n + A_2^n + A_3^n + M.\left(d_2\left(\xi^n, \hat{\xi}\right)\right)^2\right),$$

where M is a positive constant, and

$$A_1^n = E\left[\int_0^t \left|b^n\left(s,x_s^n,u_s^n\right)-b^n\left(s,x_s^n,\hat{u}_s\right)\right|^2 \chi_{\{u^n\neq\hat{u}\}}\left(s\right)ds\right],$$

$$A_2^n = E\left[\int_0^t \left|b^n\left(s,x_s^n,\hat{u}_s\right)-b^n\left(s,\hat{x}_s,\hat{u}_s\right)\right|^2+\left|\sigma^n\left(s,x_s^n\right)-\sigma^n\left(s,\hat{x}_s\right)\right|^2 ds\right],$$

$$A_3^n = E\left[\int_0^t \left|b^n\left(s,\hat{x}_s,\hat{u}_s\right)-b\left(s,\hat{x}_s,\hat{u}_s\right)\right|^2+\left|\sigma^n\left(s,\hat{x}_s\right)-\sigma\left(s,\hat{x}_s\right)\right|^2 ds\right].$$

By using the boundness of the coefficient b^n and the fact that $d_1\left(u^n,\hat{u}\right)\to 0$ as $n\to+\infty$, we have $\lim\limits_{n\to\infty} A_1^n = 0$. Since b^n and σ^n are Lipschitz in the state variable, then

$$A_2^n \leq CE\left[\int_0^t \left|x_s^n-\hat{x}_s\right|^2 ds\right].$$

Finally, we conclude from the Lemma 3.2 that $\lim\limits_{n\to+\infty} A_3^n = 0$. Then by using Burkholder-Davis-Gundy inequality and the Gronwall Lemma, we obtain (4.28).

Again, using standard arguments based on Burkholder-Davis-Gundy, Schwartz inequalities and the Gronwall Lemma, we easily check that

$$E\left[\sup_{t\leq s\leq T}\left|\Phi^n\left(s,t\right)-\Phi\left(s,t\right)\right|^2\right] \leq$$
$$CE\left[\sup_{t\leq s\leq T}\left|\Phi^n\left(s,t\right)\right|^4\right]^{\frac{1}{2}}\left\{E\left[\int_0^T \left|b_x^n\left(t,x_t^n,u_t^n\right)-b_x\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right]^{\frac{1}{2}}\right.$$
$$\left.+\sum_{1\leq j\leq d} E\left[\int_0^T \left|\sigma_x^{j,n}\left(t,x_t^n\right)-\sigma_x^j\left(t,\hat{x}_t\right)\right|^4 dt\right]^{\frac{1}{2}}\right\},$$

since the coefficients in the linear stochastic differential equation (4.19) are bounded it is easy to see that $E\left[\sup\limits_{s\leq t\leq T}\left|\Phi^n\left(s,t\right)\right|^4\right] < +\infty$. To obtain the desired result it is sufficient to prove that

$$\lim_{n\to+\infty} E\left[\int_0^T \left|b_x^n\left(t,x_t^n,u_t^n\right)-b_x\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right] = 0,$$

$$\lim_{n\to+\infty} E\left[\int_0^T \left|\sigma_x^{j,n}\left(t,x_t^n\right)-\sigma_x^j\left(t,\hat{x}_t\right)\right|^4 dt\right] = 0, \text{ for } j=1,..,d,$$

we have, $E\left[\int_0^T \left|b_x^n\left(t,x_t^n,u_t^n\right)-b_x\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right]\leq C\left(I_1^n+I_2^n\right)$, where

$$\begin{aligned} I_1^n &= E\left[\int_0^T \left|b_x^n\left(t,x_t^n,u_t^n\right)-b_x^n\left(t,x_t^n,\hat{u}_t\right)\right|^4 \chi_{\{u^n\neq\hat{u}\}}(t)\,dt\right], \\ I_2^n &= E\left[\int_0^T \left|b_x^n\left(t,x_t^n,\hat{u}_t\right)-b_x^n\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right]. \end{aligned}$$

First, in view of the boundness of the derivative b_x^n by the Lipschitz constant and the fact that $d_1\left(u^n,\hat{u}\right)\to 0$ as $n\to+\infty$, we obtain $\lim_{n\to+\infty} I_1^n=0$. Next, Let $k\geq 1$ be a fixed integer, we then get

$$\lim_{n\to+\infty} I_2^n \leq \overline{\lim_n} C.\left\{J_1^n+J_2^n+J_3^n\right\},$$

where

$$\begin{aligned} J_1^n &= E\left[\int_0^T \left|b_x^n\left(t,x_t^n,\hat{u}_t\right)-b_x^k\left(t,x_t^n,\hat{u}_t\right)\right|^4 dt\right], \\ J_2^n &= E\left[\int_0^T \left|b_x^k\left(t,x_t^n,\hat{u}_t\right)-b_x^k\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right], \\ J_3^n &= E\left[\int_0^T \left|b_x^k\left(t,\hat{x}_t,\hat{u}_t\right)-b_x\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right]. \end{aligned}$$

Now, let $\hat{z}$ (resp z^n) denotes the unique solution of the SDE (4.26) (resp (4.27)) corresponding to $\left(\hat{u},\hat{\xi}\right)$ (resp $\left(u^n,\xi^n\right)$), then it holds that

$$J_1^n=E\left[\int_0^T \left|\overline{b}_x^n\left(t,z_t^n,\hat{u}_t\right)-\overline{b}_x^k\left(t,z_t^n,\hat{u}_t\right)\right|^4 dt\right],$$

and

$$J_3^n=E\left[\int_0^T \left|\overline{b}_x^k\left(t,\hat{z}_t,\hat{u}_t\right)-\overline{b}_x\left(t,\hat{z}_t,\hat{u}_t\right)\right|^4 dt\right],$$

The following argument is taken as in [83] page 87, let $w(t,x)$ be a continuous function such that $w(t,x)=0$ if $t^2+x^2\geq 1$, and $w(0,0)=1$. Then for $M>0$, we have

$$\overline{\lim_n} J_1^n \leq CE\left[\int_0^T \left(1-w\left(\frac{t}{M},\frac{\hat{z}_t}{M}\right)\right)dt\right]$$
$$+C\overline{\lim_n} E\left[\int_0^T w\left(\frac{t}{M},\frac{\hat{z}_t}{M}\right).\left|\overline{b}_x^n\left(t,z_t^n,\hat{u}_t\right)-\overline{b}_x^k\left(t,z_t^n,\hat{u}_t\right)\right|^4 dt\right].$$

Therefore without loss of generality, we may suppose that for all $n\in\mathbb{N}^*$, the functions $\overline{b}_x,\overline{\sigma}_x$ $\overline{b}_x^n$, and $\overline{\sigma}_x^n$ have compact support in $[0,T]\times B(0,M)$. Since the diffusion matrix $\overline{\sigma}^n$ satisfies the non degeneracy condition with the same constant as σ, then by applying Krylov's inequality, we obtain

$$\begin{aligned}\overline{\lim_n} J_1^n &\leq CE\left[\int_0^T \left(1-w\left(\frac{t}{M},\frac{\hat{z}_t}{M}\right)\right)dt\right]\\ &\quad +C\overline{\lim_n}\left\|\sup_{a\in A_1}\left|\overline{b}_x^n(t,x,a)-\overline{b}_x^k(t,x,a)\right|^4\right\|_{d+1,M}.\end{aligned}$$

Since b_x^n converges to b_x dx-*a.e.*, it is simple to see that $\overline{b}_x^n$ converges to $\overline{b}_x$ dx-*a.e.* and

$$\overline{\lim_n}\left\|\sup_{a\in A_1}\left|\overline{b}_x^n(t,x,a)-\overline{b}_x^k(t,x,a)\right|^4\right\|_{d+1,M}=0.$$

Next, let M goes to $+\infty$, then from the properties of the function $w(t,x)$ we have $\overline{\lim_n} J_1^n=0$. Istimating J_3^n similarily, it holds that $\overline{\lim_n} J_3^n=0$. We use the continuity of b_x^k in x. From (4.28), and by using the Dominated convergence theorem we deduce that $\overline{\lim_n} J_2^n=0$. Hence $\lim_{n\to+\infty} I_1^n=0$. Using the same technique, we prove that

$$\lim_{n\to+\infty} E\left[\int_0^T \left|\sigma_x^{j,n}\left(t,x_t^n\right)-\sigma_x^j\left(t,\hat{x}_t\right)\right|^4 dt\right]=0,\ \text{for } j=1,...,d.$$

Now, let us prove that $\lim_{n\to+\infty} E\left[\sup_{0\leq t\leq T}\left|p_t^n-p_t\right|^2\right]=0$. Clearly,

$$E\left[\left|p_t^n-p_t\right|^2\right]\leq C\left(\alpha_1^n+\alpha_2^n\right), \tag{4.32}$$

where

$$\alpha_1^n = E\left[\int_t^T \left|\Phi^{n,T}(s,t).f_x^n(s,x_s^n,u_s^n) - \Phi^T(s,t).f_x(s,\hat{x}_s,\hat{u}_s)\right|^2 ds\right],$$

and

$$\alpha_2^n = E\left[\left|\Phi^{n,T}(T,t).g_x^n(x_T^n) - \Phi^T(T,t).g_x(\hat{x}_T)\right|^2\right]$$

Since f_x is bounded by the Lipschitz constant M, and applying the Schwartz inequality, we get

$$\begin{aligned}\alpha_1^n &\leq CE\left[\sup_{t\leq s\leq T}\left|\Phi^{n,T}(s,t)\right|^4\right]^{\frac{1}{2}}.E\left[\int_0^T \left|f_x^n(s,x_s^n,u_s^n) - f_x(s,\hat{x}_s,\hat{u}_s)\right|^4 ds\right]^{\frac{1}{2}} \\ &+CM.E\left[\sup_{t\leq s\leq T}\left|\Phi^{n,T}(s,t) - \Phi^T(s,t)\right|^2\right].\end{aligned}$$

Hence, by the continuity and the boundness of derivatives f_x^n, f_x, relations (4.28), (4.29) and the fact that $d_1(u^n,\hat{u}) \to 0$ as $n \to \infty$, together with the Krylov's inequality and the Dominated convergence theorem, for the term involving $f_x^n(s,x_s^n,u_s^n) - f_x(s,\hat{x}_s,\hat{u}_s)$, we get by sending n to infinity $\lim_{n\to+\infty}\alpha_1^n = 0$.

On the other hand, since g_x is bounded by the Lipschitz constant, and applying the Schwartz inequality we get

$$\begin{aligned}\alpha_2^n &\leq C\left\{E\left[\left|\Phi^{n,T}(T,t)\right|^4\right]\right\}^{\frac{1}{2}}.\left\{E\left[\left|g_x^n(x_T^n) - g_x(\hat{x}_T)\right|^4\right]\right\}^{\frac{1}{2}} \\ &+CM.E\left[\left|\Phi^{n,T}(T,t) - \Phi^T(T,t)\right|^2\right],\end{aligned}$$

Since, g_x^n and g_x are bounded by the Lipschitz constant and g_x^n converges to g_x, we conclude by (4.28) and the dominated convergence theorem that

$$\lim_{n\to+\infty} E\left[\left|g_x^n(x_T^n) - g_x(\hat{x}_T)\right|^4\right] = 0.$$

From (4.32), then by using Burkholder-Davis-Gundy inequality, we obtain (4.30).

The Schwartz inequality, gives

$$E\left[\left|H^{n}\left(t,x_{t}^{n},u_{t}^{n},p_{t}^{n}\right)-H\left(t,\hat{x}_{t},\hat{u}_{t},p_{t}\right)\right|\right]\leq\left\{E\left|p_{t}^{n}-p_{t}\right|^{2}\right\}^{\frac{1}{2}}\left\{E\left|b^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)\right|^{2}\right\}^{\frac{1}{2}}$$

$$+\left\{E\left|b^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)-b\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|^{2}\right\}^{\frac{1}{2}}\left\{E\left|p_{t}\right|^{2}\right\}^{\frac{1}{2}}+E\left|f^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)-f\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|.$$

Lemma 4.5 and (4.30) imply that the first expression in the right hand side converges to 0 as $n\to+\infty$.

Next,

$$E\left|b^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)-b\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|^{2}\leq C\left(\beta_{1}^{n}+\beta_{2}^{n}+\beta_{3}^{n}\right),$$

where

$$\begin{aligned}
\beta_{1}^{n} &= E\left[\left|b^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)-b^{n}\left(t,x_{t}^{n},\hat{u}_{t}\right)\right|^{2}\chi_{\{u^{n}\neq\hat{u}\}}\left(t\right)\right],\\
\beta_{2}^{n} &= E\left[\left|b^{n}\left(t,x_{t}^{n},\hat{u}_{t}\right)-b^{n}\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|^{2}\right],\\
\beta_{3}^{n} &= E\left[\left|b^{n}\left(t,\hat{x}_{t},\hat{u}_{t}\right)-b\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|^{2}\right].
\end{aligned}$$

The boundness of b^{n} and the fact that $d_{1}\left(u^{n},\hat{u}\right)\underset{n\to\infty}{\to}0$, guarantee the convergence of β_{1}^{n} to 0 as $n\to+\infty$. By virtue of (3.21), and the dominated convergence theorem we get, $\lim\limits_{n\to+\infty}\beta_{2}^{n}=0$. In view of the Lemma 3.2, we have $\lim\limits_{n\to+\infty}\beta_{3}^{n}=0$.

The term $E\left|f^{n}\left(t,x_{t}^{n},u_{t}^{n}\right)-f\left(t,\hat{x}_{t},\hat{u}_{t}\right)\right|$ can be treated by the same technique. ■

Proof of Theorem 3.1. Let n goes to $+\infty$, then from Proposition 3.7 and Lemma 3.8, we get

$$\begin{aligned}
E\left[H\left(t,\hat{x}_{t},v,p_{t}\right)-H\left(t,\hat{x}_{t},\hat{u}_{t},p_{t}\right)\right] &\geq 0, \quad dt\text{-a.e., } P\text{-a.s.},\\
E\int_{0}^{T}\left(k_{t}+G_{t}^{*}p_{t}\right)d\left(\eta-\hat{\xi}\right)_{t} &\geq 0,
\end{aligned}$$

for every A_{1}-valued F_{t}-measurable random variable v, and $\eta\in U_{2}$.

Let $a \in A_1$, then for every $A_t \in F_t$

$$E\left[\left(H\left(t,\hat{x}_t,a,p_t\right)-H\left(t,\hat{x}_t,\hat{u}_t,p_t\right)\right)\chi_{A_t}\right]\geq 0,\quad dt\text{-a.e., } P\text{-a.s.,}$$

which implies that

$$E\left[\left(H\left(t,\hat{x}_t,a,p_t\right)-H\left(t,\hat{x}_t,\hat{u}_t,p_t\right)\right)\diagup F_t\right]\geq 0$$

Since $H\left(t,\hat{x}_t,a,p_t\right)-H\left(t,\hat{x}_t,\hat{u}_t,p_t\right)$ is F_t-measurable, then the first variational inequality without expectation follows immediately.

4.3 The Degenerate case

In this section we drop the uniform ellipticity condition on the diffusion matrix. It is clear that the method used earlier will no longer be valid. Now, the idea is based on a result by Bouleau and Hirsch [8] on absolute continuity of probability measures, and the differentiability of the solution of an SDE with Lipschitz coefficients with respect to initial data in the sense of distributions on an extension of the initial probability space.

Let $\widetilde{\Omega}=\mathbb{R}^d\times\Omega$, and $\widetilde{F}$ the Borel σ-field over $\widetilde{\Omega}$ and $\widetilde{P}=hdx\otimes P$. Let $\widetilde{B}_t\left(x,w\right)=B_t\left(w\right)$ and $\widetilde{F}_t$ the natural filtration of $\widetilde{B}_t$ augmented with $\widetilde{P}$-negligible sets of $\widetilde{F}$. It is clear that $\left(\widetilde{\Omega},\widetilde{F},\left(\widetilde{F}_t\right)_{t\geq 0},\widetilde{P},\widetilde{B}_t\right)$ is a Brownian motion. We introduce the process $\tilde{x}_t$ defined on the enlarged space $\left(\widetilde{\Omega},\widetilde{F},\left(\widetilde{F}_t\right)_{t\geq 0},\widetilde{P},\widetilde{B}_t\right)$ solution of the stochastic differential equation

$$\begin{cases} d\tilde{x}_t=b\left(t,\tilde{x}_t,\tilde{u}_t\right)dt+\sigma\left(t,\tilde{x}_t\right)d\widetilde{B}_t+G_t d\tilde{\xi}_t, \text{ for } t\in[0,T], \\ \tilde{x}_0=\alpha, \end{cases} \tag{4.33}$$

associated to the control $\left(\tilde{u}_t,\tilde{\xi}_t\right)(x,w)=\left(u_t,\xi_t\right)(w)$. Since the coefficients are Lipschitz continuous and bounded, equations (4.1) has a unique $\widetilde{F}_t$-adapted solution. Equations (2.1), and (4.1) are almost the same except that uniqueness of the solution of (4.1) is slightly weaker, one can easily prove that the uniqueness implies that for each $t \geq 0$, $\tilde{x}_t = x_t$, $\widetilde{P}$-a.s.

4.3.1 The main result

The main result of this section is stated in the following Theorem.

Theorem 4.8. (Stochastic maximum principle) Let $(\hat{u}, \hat{\xi})$ be an optimal control for the controlled system (2.1), (2.2) and let $\hat{x}$ be the corresponding optimal trajectory. Then there exists a measurable F_t-adapted process p_t satisfying

$$p_t := \widetilde{E}\left[\int_t^T \Phi^*(s,t).f_x(s,\hat{x}_s,\hat{u}_s)\,ds + \Phi^*(T,t).g_x(\hat{x}_T) \diagup \widetilde{F}_t\right], \tag{4.34}$$

such that for all $a \in A_1$ and $\eta \in U_2$

$$0 \leq H(t,\hat{x}_t,a,p_t) - H(t,\hat{x}_t,\hat{u}_t,p_t) \quad dt\text{-a.e},\ \widetilde{P}\text{-a.s.}, \tag{4.35}$$

and

$$0 \leq \widetilde{E}\int_0^T (k_t + G_t^* p_t)\, d\left(\eta - \hat{\xi}\right)_t \tag{4.36}$$

where the Hamiltonian H is defined by

$$H(t,x,u,p) = p.b(t,x,u) + f(t,x,u), \tag{4.37}$$

and $\Phi(s,t)$, $(s \geq t)$ is the fundamental solution of the linear equation

$$\begin{cases} d\Phi_s = b_x(s,\hat{x}_s,\hat{u}_s).\Phi(s,t)\,ds + \sum\limits_{1\leq j\leq d} \sigma_x^j(s,\hat{x}_s).\Phi(s,t)\,d\widetilde{B}_s^j, \\ \Phi(t,t) = Id. \end{cases} \tag{4.38}$$

Theorem 7 *Here * denotes the transpose.*

4.3.2 Proof of the main result

Let $\tilde{z}_t = \tilde{x}_t - \int_0^t G_s d\xi_s$ the unique solution of the SDE

$$\begin{cases} d\tilde{z}_t = \overline{b}\left(t, \tilde{z}_t, u_t\right) dt + \overline{\sigma}\left(t, \tilde{z}_t\right) d\widetilde{B}_t, \\ \tilde{z}_0 = \alpha. \end{cases} \tag{4.39}$$

on the enlarged space $\left(\widetilde{\Omega}, \widetilde{F}, \left(\widetilde{F}_t\right)_{t\geq 0}, \widetilde{P}, \widetilde{B}_t\right)$, where $\overline{b}$ and $\overline{\sigma}$ are defined in sebsection 3.2.

Theorem 4.9. (The Bouleau-Hirsch flow property) For $\widetilde{P}$-almost every w

(1) For all $t \geq 0$, $\tilde{z}_t$ is in D^d.

(2) There exists a $\widetilde{F}_t$-adapted $\mathrm{GL}_d\left(\mathbb{R}\right)$-valued continuous process $\left(\widetilde{\Phi}_t\right)_{t\geq 0}$ such that for every $t \geq 0$

$$\frac{\partial}{\partial x}\left(z_t^{\alpha}\left(w\right)\right) = \widetilde{\Phi}_t\left(\alpha, w\right) \quad dx\text{-}a.e.,$$

where $\frac{\partial}{\partial x}$ denotes the derivative in the ditribution sense.

(3) The distributional derivative $\widetilde{\Phi}_t$ is the unique fundamental solution of the linear stochastic differential equation

$$\begin{cases} d\widetilde{\Phi}\left(s,t\right) = \overline{b}_x\left(s, \tilde{z}_s, \tilde{u}_s\right).\widetilde{\Phi}\left(s,t\right) ds + \sum\limits_{1\leq j\leq d} \overline{\sigma}_x^j\left(s, \tilde{z}_s\right).\widetilde{\Phi}\left(s,t\right) d\widetilde{B}_s^j, \ s \geq t, \\ \widetilde{\Phi}\left(t,t\right) = Id, \end{cases} \tag{4.40}$$

where $\overline{b}_x$ and $\overline{\sigma}_x^j$ are versions of the almost everywhere derivatives of $\overline{b}$ and $\overline{\sigma}^j$.

(4) The image measure of $\widetilde{P}$ by the map $\tilde{z}_t$ is absolutely continuous with respect to the Lebesgue measure.

The maximum principle for a Family of perturbed control problems

Now, consider the process y_t, $t \geq 0$, solution of the system valued in $\mathbb{R}^d$, defined on the enlarged probability space $\left(\widetilde{\Omega}, \widetilde{F}, \left(\widetilde{F}_t\right)_{t\geq 0}, \widetilde{P}, \widetilde{B}_t\right)$ by

$$\begin{cases} dy_t = b^n\left(t, y_t, u_t\right)dt + \sigma^n\left(t, y_t\right)d\widetilde{B}_t + G_t d\xi_t, \\ y_0 = \alpha, \end{cases} \tag{4.41}$$

and define the cost functional

$$J^n\left(u_t\right) = \widetilde{E}\left[\int_0^T f^n\left(t, y_t, u_t\right)dt + \int_0^T k_t d\xi_t + g^n\left(y_T\right)\right], \tag{4.42}$$

where b^n, σ^n, f^n and g^n be the regularized functions of b, σ, f and g.

The following result gives the estimates which relate the original control problem with the perturbed ones.

Lemma 4.10. Let (x_t) and (y_t) the solutions of (2.1) and (4.9) respectively, corresponding to an admissible control (u, ξ). Then

(1) $\widetilde{E}\left[\sup\limits_{0\leq t\leq T}\left|x_t - y_t\right|^2\right] \leq M_1.\left(\epsilon_n\right)^2.$

(2) $\left|J^n\left(u, \xi\right) - J\left(u, \xi\right)\right| \leq M_2.\epsilon_n, \quad$ where $\epsilon_n = \dfrac{C}{n}.$

Where M_1 and M_2 are positive constants.

Let $\left(\hat{u}, \hat{\xi}\right)$ be an optimal control for the initial problem (2.1) and (2.2). Note that $\left(\hat{u}, \hat{\xi}\right)$ is not necessarily optimal for the perturbed control problem (4.9) and (4.10), however, according to Lemma 4.3, there exists $(\delta_n) \equiv (2M_2.\epsilon_n)$ a sequence of positive real numbers converging to 0, such that

$$J^n(\hat{u}, \hat{\xi}) \leq \inf_{(\upsilon,\eta)\in U} J^n\left(\upsilon, \eta\right) + \delta_n.$$

The functional J^n defined by (4.10) being continuous on $U = U_1 \times U_2$ with respect to the topology induced by the metric $d'\left(\left(u,\xi\right),\left(\upsilon,\eta\right)\right) = d_1'\left(u,\upsilon\right) + d_2'\left(\xi,\eta\right)$, for all $\left(u,\xi\right)$, $\left(\upsilon,\eta\right) \in U$, where

$$d_1'\left(u,v\right) = \widetilde{P} \otimes dt \left\{ \left(w,t\right) \in \widetilde{\Omega} \times \left[0,T\right], v\left(w,t\right) \neq u\left(w,t\right) \right\},$$
$$d_2'\left(\xi,\eta\right) = \left(\widetilde{E}\left[\sup_{0 \le t \le T} \left|\xi_t - \eta_t\right|^2 \right] \right)^{\frac{1}{2}},$$

Then by applying Ekeland principle to J^n for $\left(\hat{u},\hat{\xi}\right)$ with $\lambda_n = \delta_n^{\frac{2}{3}}$, there exists an admissible control $\left(u^n,\xi^n\right)$ such that

$$d'\left((\hat{u},\hat{\xi}), \left(u^n,\xi^n\right) \right) \le \delta_n^{\frac{2}{3}},$$

$$J_\delta^n\left(u^n,\xi^n\right) \le J_\delta^n\left(\upsilon,\eta\right), \quad \text{for any } \left(\upsilon,\eta\right) \in U,$$

and $\left(u^n,\xi^n\right)$ is an optimal control for the perturbed system (4.9) with a new cost function

$$J_\delta^n\left(\upsilon,\eta\right) = J^n\left(\upsilon,\eta\right) + \delta_n^{\frac{1}{3}}.d'\left(\left(\upsilon,\eta\right),\left(u^n,\xi^n\right)\right).$$

Denote by x^n the unique solution of (4.9) corresponding to $\left(u^n,\xi^n\right)$

$$\begin{cases} dx_t^n = b^n\left(t,x_t^n,u_t^n\right)dt + \sigma^n\left(t,x_t^n\right)d\widetilde{B}_t + G_t d\xi_t^n, \\ x_0^n = \alpha, \end{cases} \tag{4.43}$$

The controlled process $z_t^n = x_t^n - G_t d\xi_t^n$ is then defined as the solution to the stochastic differential equation

$$\begin{cases} dz_t^n = \overline{b}^n\left(t,z_t^n,u_t^n\right)dt + \overline{\sigma}^n\left(t,z_t^n\right)d\widetilde{B}_t, \\ z_0^n = \alpha. \end{cases} \tag{4.44}$$

where $\overline{b}^n$ and $\overline{\sigma}^n$ are defined in sebsection 3.2. Let $\Phi^n\left(s,t\right)$ $\left(s \ge t\right)$, be the fundamental solution of the linear equation

$$\begin{cases} d\Phi^n(s,t) = b_x^n(s,x_s^n,u_s^n).\Phi^n(s,t)\,ds + \sum\limits_{1\le j\le d} \sigma_x^{j,n}(s,x_s^n).\Phi^n(s,t)\,d\widetilde{B}_s^j, \\ \\ \Phi^n(t,t) = Id. \end{cases} \tag{4.45}$$

Proposition 4.11. For each integer n, there exists an admissible control (u^n, ξ^n) and a $\left(\widetilde{F}_t\right)$-adapted process p_t^n given by

$$p_t^n = \widetilde{E}\left[\int_t^T \Phi^{n,*}(s,t).f_x^n(s,x_s^n,u_s^n)\,ds + \Phi^{n,*}(T,t).g_x^n(x_T^n) \diagup \widetilde{F}_t\right], \tag{4.46}$$

and a Lebesgue null set N such that for $t \in N^c$

$$\widetilde{E}\left[H^n(t,x_t^n,v,p_t^n) - H^n(t,x_t^n,u_t^n,p_t^n)\right] \ge -\delta_n^{\frac{1}{3}}.M_1, \tag{4.47}$$

and

$$\widetilde{E}\int_t^T (k_t + G_t^* p_t^n)\,d(\eta - \xi^n)_t \ge -\delta_n^{\frac{1}{3}}.M_2, \tag{4.48}$$

for all $v \in A_1$, and $\eta \in U_2$, where the Hamiltonian H^n is defined by

$$H^n(t,x,u,p) = p.b^n(t,x,u) + f^n(t,x,u). \tag{4.49}$$

Here * denotes the transpose.

By the same method as in the sebsection 3.2, we are able to derive the proof.

Lemma 4.12. We have

$$\lim_{n\to+\infty} \widetilde{E}\left[\sup_{0\le t\le T} |x_t^n - \hat{x}_t|^2\right] = 0, \tag{4.50}$$

$$\lim_{n\to+\infty} \widetilde{E}\left[\sup_{s\le t\le T} |\Phi^n(s,t) - \Phi(s,t)|^2\right] = 0, \tag{4.51}$$

$$\lim_{n\to+\infty} \widetilde{E}\left[\sup_{0\le t\le T} |p_t^n - p_t|^2\right] = 0, \tag{4.52}$$

$$\lim_{n\to+\infty} \widetilde{E}\left[|H^n(t,x_t^n,u_t^n,p_t^n) - H(t,\hat{x}_t,\hat{u}_t,p_t)|\right] = 0. \tag{4.53}$$

Where Φ_t, p_t and H are determined by (4.6), (4.2), and (4.5), corresponding to the optimal solution $\hat{x}_t$. Φ_t^n, p_t^n and H^n are determined by (4.13), (4.14) and (4.17), corresponding to the approximating sequence x_t^n, given by (4.11).

In the sequel, we denote by C a positive constant which may vary from line to line.

Proof. The limit (4.18) is proved by the same fashion as the limit (3.21).

In view of the Burkholder, Schwartz inequalities and the Gronwall Lemma, we obtain

$$\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^n(s,t)-\Phi(s,t)\right|^2\right]\leq$$
$$C\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^n(s,t)\right|^4\right]^{\frac{1}{2}}\left\{\widetilde{E}\left[\int_0^T\left|b_x^n(t,x_t^n,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)\right|^4dt\right]^{\frac{1}{2}}\right.$$
$$\left.+\sum_{1\leq j\leq d}\widetilde{E}\left[\int_0^T\left|\sigma_x^{j,n}(t,x_t^n)-\sigma_x^j(t,\hat{x}_t)\right|^4dt\right]^{\frac{1}{2}}\right\},$$

since the coefficients in the linear stochastic differential equation (4.13) are bounded, it is easy to see that $\widetilde{E}\left[\sup_{t\leq s\leq T}\left|\Phi^n(s,t)\right|^4\right]<+\infty$. To derive (4.19), it is sufficient to prove the following two assertions

$$\widetilde{E}\left[\int_0^T\left|b_x^n(t,x_t^n,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)\right|^4dt\right]\to 0\qquad\text{as n}\to+\infty,$$

and

$$\widetilde{E}\left[\int_0^T\left|\sigma_x^{j,n}(t,x_t^n)-\sigma_x^j(t,\hat{x}_t)\right|^4dt\right]\to 0\qquad\text{as n}\to+\infty,\ \text{for j=1,2,.....,d.}$$

Let us prove the first Limit. We have

$$\widetilde{E}\left[\int_0^T\left|b_x^n(t,x_t^n,u_t^n)-b_x(t,\hat{x}_t,\hat{u}_t)\right|^4dt\right]\leq C\left(I_1^n+I_2^n+I_3^n\right),$$

where

$$\begin{aligned}
I_1^n &= \widetilde{E}\left[\int_0^T \left|b_x^n\left(t,x_t^n,u_t^n\right)-b_x^n\left(t,x_t^n,\hat{u}_t\right)\right|^4 \chi_{\{u^n\neq\hat{u}\}}(t)\,dt\right],\\
I_2^n &= \widetilde{E}\left[\int_0^T \left|b_x^n\left(t,x_t^n,\hat{u}_t\right)-b_x\left(t,x_t^n,\hat{u}_t\right)\right|^4 dt\right],\\
I_3^n &= \widetilde{E}\left[\int_0^T \left|b_x\left(t,x_t^n,\hat{u}_t\right)-b_x\left(t,\hat{x}_t,\hat{u}_t\right)\right|^4 dt\right],
\end{aligned}$$

In view of the boundness of the derivative b_x^n by the Lipschitz constant and the fact that $d_1'\left(u^n,\hat{u}\right)\to 0$ as $n\to+\infty$, we obtain $\lim\limits_{n\to+\infty} I_1^n = 0.$

Indeed, we have

$$\begin{aligned}
I_2^n &\leq \widetilde{E}\left[\int_0^T \sup_{a\in A_1}\left|\overline{b}_x^n\left(t,z_t^n,a\right)-\overline{b}_x\left(t,z_t^n,a\right)\right|^4 dt\right],\\
&= \int_0^T\int_{\mathbb{R}^d}\sup_{a\in A_1}\left|\overline{b}_x^n\left(t,y,a\right)-\overline{b}_x\left(t,y,a\right)\right|^4 \rho_t^n(y)\,dydt,
\end{aligned}$$

where z_t^n denotes the unique solution of the SDE (3.20), corresponding to (u^n,ξ^n), and $\rho_t^n(y)$ its density with respect to the Lebesgue measure. Let us show

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\sup_{a\in A_1}\left|\overline{b}_x^n\left(t,y,a\right)-\overline{b}_x\left(t,y,a\right)\right|^4 \rho_t^n(y)\,dydt = 0.$$

For each $p>0$, $\widetilde{E}\left[\sup\limits_{0\leq t\leq T}\left|z_t^n\right|^p\right]<+\infty$. Thus, $\lim\limits_{R\to\infty}\widetilde{P}\left(\sup\limits_{0\leq t\leq T}\left|z_t^n\right|>R\right)=0$, then it is enough to show that for every $R>0$,

$$\lim_{n\to+\infty}\int_{B(0,R)}\sup_{a\in A_1}\left|\overline{b}_x^n\left(t,y,a\right)-\overline{b}_x\left(t,y,a\right)\right|^4 \rho_t^n(y)\,dy = 0.$$

According to Lemma 3.2, it is easy to see that

$$\begin{aligned}
&\sup_{a\in A_1}\left|\overline{b}_x^n\left(t,y,a\right)-\overline{b}_x\left(t,y,a\right)\right|^4\\
=\;&\sup_{a\in A_1}\left|b_x^n\left(t,y+\int_0^T G_t d\xi_t^n,a\right)-b_x\left(t,y+\int_0^T G_t d\xi_t^n,a\right)\right|^4 \to 0 \quad dy\text{-}a.e,
\end{aligned}$$

at least for a subsequence. Then by Egorov's Theorem, for every $\delta > 0$, there exists a measurable set F with $\lambda(F) < \delta$, such that $\sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a) - \overline{b}_x(t,y,a)\right|$ converges uniformly to 0 on the set F^c. Note that, since the Lebesgue measure is regular, F may be chosen closed. This implies that

$$\begin{aligned} & \lim_{n\to+\infty} \int_{F^c} \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a) - \overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ \leq \; & \lim_{n\to+\infty} \left(\sup_{y\in F^c} \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a) - \overline{b}_x(t,y,a)\right|^4 \right) = 0. \end{aligned}$$

Now, by using the boundness of the derivatives $\overline{b}_x^n$, $\overline{b}_x$ we have

$$\begin{aligned} & \int_F \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a) - \overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\ = \; & \widetilde{E}\left[\sup_{a\in A_1} \left|\overline{b}_x^n(t,\hat{z}_t^n,a) - \overline{b}_x(t,\hat{z}_t^n,a)\right|^4 \chi_{\{\hat{z}_t^n\in F\}} \right] \\ \leq \; & 2M^4 \widetilde{P}(\hat{z}_t^n \in F). \end{aligned}$$

In view of the relation (4.18), it is easy to see that $z_t^n = x_t^n - \int_0^t G_s d\xi_s^n$ converges to $\hat{z}_t = \hat{x}_t - \int_0^t G_s d\hat{\xi}_s$ in probability, then in distribution. Applying the Portmanteau-Alexandrov Theorem, we obtain

$$\begin{aligned} \lim_n \int_F \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a) - \overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \; & \leq \; 2M^4 \limsup \widetilde{P}(z_t^n \in F) \\ & \leq \; 2M^4 \widetilde{P}(\hat{z}_t \in F) \\ & = \; 2M^4 \int_F \rho_t(y)\,dy < \varepsilon. \end{aligned}$$

where $\rho_t(y)$ denotes the density of $\hat{z}_t$ with respect to Lebesgue measure.

Now, since

$$\begin{aligned}
&\int\limits_{B(0,R)} \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a)-\overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\
&= \int\limits_{F} \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a)-\overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy \\
&\quad + \int\limits_{F^c} \sup_{a\in A_1} \left|\overline{b}_x^n(t,y,a)-\overline{b}_x(t,y,a)\right|^4 \rho_t^n(y)\,dy,
\end{aligned}$$

we get $\lim\limits_{n\to+\infty} I_2^n = 0$.

Let $k \geq 0$ be a fixed integer, then it holds that $I_3^n \leq C\left(J_1^k + J_2^k + J_3^k\right)$, where

$$\begin{aligned}
J_1^k &= \widetilde{E}\left[\int_0^T \left|b_x(t,x_t^n,\hat{u}_t)-b_x^k(t,x_t^n,\hat{u}_t)\right|^4 dt\right], \\
J_2^k &= \widetilde{E}\left[\int_0^T \left|b_x^k(t,x_t^n,\hat{u}_t)-b_x^k(t,\hat{x}_t,\hat{u}_t)\right|^4 dt\right], \\
J_3^k &= \widetilde{E}\left[\int_0^T \left|b_x^k(t,\hat{x}_t,\hat{u}_t)-b_x(t,\hat{x}_t,\hat{u}_t)\right|^4 dt\right].
\end{aligned}$$

Applying the same arguments used in the first limit (Egorov and Portmanteau-Alexandrov Theorems), we obtain that $\lim\limits_{n\to+\infty} J_1^k = 0$. We use the continuity of b_x^k in x and the convergence in probability of x_T^n to $\hat{x}_T$ to deduce that $b_x^k(t,x_t^n,\hat{u}_t)$ converges to $b_x^k(t,\hat{x}_t,\hat{u}_t)$ in probability as $n \to +\infty$, and to infer by using the Dominated convergence Theorem that $\lim\limits_{n\to+\infty} J_2^k = 0$.

$$\begin{aligned}
J_3^k &= \widetilde{E}\left[\int_0^T \sup_{a\in A_1}\left|\overline{b}_x^k(t,\hat{z}_t,a)-\overline{b}_x(t,\hat{z}_t,a)\right|^4 dt\right] \\
&= \int_0^T \int\limits_{\mathbb{R}^d} \sup_{a\in A_1}\left|\overline{b}_x^k(t,y,a)-\overline{b}_x(t,y,a)\right|^4 \rho_t(y)\,dydt
\end{aligned}$$

$\overline{b}_x^k$, $\overline{b}_x$ are bounded, by using the convergence of $\overline{b}_x^k$ to $\overline{b}_x$, and by using the Dominated convergence Theorem, we get $\lim\limits_{n\to+\infty} J_3^k = 0$. ■

Proof of Theorem 4.1. Use the Corollary 4.5 and the Lamma 4.6.

Chapter 5

On the relationship between the SMP and DPP in singular optimal controls and its applications

This chapter investigates the relationship between the maximum principle and dynamic programming for stochastic optimal control problems where the state X_t at time t of the system is governed by a stochastic differential equation with nonlinear coefficients and a nonconvex state domain, allowing both regular control and singular control. We prove that under appropriate differentiability assumptions on the coefficients of the state equation and the gain functional , the solution of the adjoint equation of such problems coincides with the derivatives of the value function. We study the case where the value function is sufficiently smooth, generalizing the classical cases. For this situation a verification theorem is proved and an example is given.

Lemma 5.1. (The sufficients conditions of optimality) Let (u^*, ξ^*) be an admissible control, we denote X^* the associated controlled state process. Suppose there exists a solution (p, q) to the corresponding $BSDE$ (2.11). If we assume that $(x, u) \to H(t, x, u, p_t, q_t)$, and $x \to g(x)$ are concave functions, for all $t \in [0, T]$ for all $v \in A_1$, and $\xi \in U_2$

$$H(t, X_t^*, u_t^*, p_t, q_t) = \sup_{v \in A_1} H(t, X_t^*, v, p_t, q_t), \text{ dt-a.e., P-a.s.,} \tag{5.1}$$

$$E\int_0^T \left\{k(t) + G^T(t) p_t\right\} d(\xi - \xi^*)_t \leq 0. \tag{5.2}$$

then (u^*, ξ^*) is an optimal control.

Let (u, ξ) be an arbitrary admissible pair, and consider

$$\begin{aligned} J(u^*, \xi^*) - J(u, \xi) &= E\left[\int_0^T f(t, X_t^*, u_t^*) - f(t, X_t, u_t)\, dt\right] \\ &+ E\left[\int_0^T k(t)\, d(\xi^* - \xi)(t)\right] + E\left[g(X_T^*) - g(X_T)\right]. \end{aligned} \tag{5.3}$$

Since g is concave, we get

$$\begin{aligned} E\left[g(X_T^*) - g(X_T)\right] &\geq E\left[(X_T^* - X_T)^T \nabla g(X_T^*)\right] \\ &= E\left[(X_T^* - X_T)^T p_T\right], \\ &= E\left[\int_0^T (X_t^* - X_t)^T dp_t\right] + E\left[\int_0^T p_t d(X_t^* - X_t)\right] \\ &+ E\left[\int_0^T tr\left\{(\sigma(t, X_t^*) - \sigma(t, X_t))^T q_t\right\} dt\right]. \end{aligned}$$

With

$$\begin{aligned} E\left[\int_0^T (X_t^* - X_t)^T dp_t\right] &= E\left[\int_0^T (X_t^* - X_t)^T (-\nabla H_x(t, X_t^*, u_t^*, p_t, q_t))\, dt\right] \\ &+ E\left[\int_0^T (X_t^* - X_t)^T q_t dB_t\right], \end{aligned}$$

and

$$
\begin{aligned}
E\left[\int_0^T p_t d\left(X_t^* - X_t\right)\right] &= E\left[\int_0^T p_t \left(b\left(t, X_t^*, u_t^*\right) - b\left(t, X_t, u_t\right)\right)^T dt\right] \\
&+ E\left[\int_0^T p_t \left(\sigma\left(t, X_t^*\right) - \sigma\left(t, X_t\right)\right)^T dB_t\right] \\
&+ E\left[\int_0^T G^T(t)\, p_t d\left(\xi - \xi^*\right)_t\right].
\end{aligned}
$$

On the other hand, the process

$$
\int_0^t \left\{p_s \left(\sigma\left(s, X_s^*\right) - \sigma\left(s, X_s\right)\right)^T + \left(X_s^* - X_s\right)^T q_s\right\} dB_s
$$

is a continuous local martingale for all $0 < t \leq T$, According to (2.9), and the fact that $(p, q) \in L^2\left([0,T]; \mathbb{R}^n\right) \times L^2\left([0,T]; \mathbb{R}^{n\times d}\right)$, together with the Burkholder-Davis-Gundy inequality, we deduce that

$$
E\left[\int_0^t \sup_{0 \leq r \leq s} \left| p_r \left(\sigma\left(r, X_r^*\right) - \sigma\left(r, X_r\right)\right)^T + \left(X_r^* - X_r\right)^T q_r \right| ds\right] < \infty.
$$

Thus, the process

$$
\int_0^t \left\{p_s \left(\sigma\left(s, X_s^*\right) - \sigma\left(s, X_s\right)\right)^T + \left(X_s^* - X_s\right)^T q_s\right\} dB_s
$$

inded a martingale have zero expectation. By the concavity of the Hamiltonian H, we get

$$
\begin{aligned}
E\left[g\left(X_T^*\right) - g\left(X_T\right)\right] &\geq -E\left[\int_0^T \left(H\left(t, X_t^*, u_t^*, p_t, q_t\right) - H\left(t, X_t, u_t, p_t, q_t\right)\right) dt\right] \\
&+ E\left[\int_0^T p_t \left(b\left(t, X_t^*, u_t^*\right) - b\left(t, X_t, u_t\right)\right)^T dt\right] \\
&+ E\left[\int_0^T tr\left\{\left(\sigma\left(t, X_t^*\right) - \sigma\left(t, X_t\right)\right)^T q_t\right\} dt\right] \\
&+ E\left[\int_0^T G^T(t)\, p_t d\left(\xi - \xi^*\right)_t\right].
\end{aligned}
$$

By the definition of the Hamiltonian H and (2.15), we obtain

$$J(u^*,\xi^*) - J(u,\xi) \geq 0,$$

then (u^*,ξ^*) is an optimal control for the problem (2.8).

5.1 Relation to dynamic programming

The other major approach for studying singular stochastic control problems is the Bellman dynamic programming principle, a result about this approach can be found in [17], whoe considered the n-dimensional cas. By the compactification method, it was shown that, the value function is continuous and is the unique viscosity solution of the HJB variational inequality (3.2). An advantage of this approach is that it does not require any regularity of the value function, and thus needs only very mild hypothesis on the data. Let $X_s^{t,x}$ be the solution of the controlled SDE (2.1) for $s \geq t$, with initial value $X_t = x$, and we define the gain function

$$J(u,\xi) = E\left[\int_t^\tau f(s, X_s, u_s)\, ds + \int_t^\tau k(s)\, d\xi(s) + g(X_\tau)\right], \tag{5.4}$$

We have to impose differentiabilty conditions on the coefficients b, σ, f, and g, as in section 2. Now, since our objective is to maximize this gain function, the value function of our singular control problem is defined as

$$V(t,x) = \sup_{(u,\xi)\in U} J(u,\xi). \tag{5.5}$$

for an initial state (t,x), we say that (u^*,ξ^*) is an optimal control if $V(t,x) = J(u^*,\xi^*)$.

If we do not apply any singular control, then the infinitesimal generator A^u, associated with (2.1), acting on functions φ, coincides on $C_b^2(\mathbb{R}^n;\mathbb{R})$ with partial differential operator A^u given by

$$A^u(t,x) = \sum_{i=1}^{n} b_i(t,x,u)\frac{\partial\varphi}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(t,x),$$

where $a_{ij}(t,x) = \left(\sigma\sigma^T\right)_{ij}(t,x)$ denotes the generic term of the symmetric matrix $\sigma\sigma^T(t,x)$. The variational inequality associated to the singular control problem is

$$\max\left\{\sup_u H_1\left(t,x,W,\partial_t W, D_x W, D_x^2 W, u\right), H_2(t,x,D_x W,u), l = 1,..,m\right\} = 0, \qquad (5.6)$$

for $(t,x) \in S$, with H_1, and H_2 are given by

$$\begin{aligned} H_1\left(t,x,W,\partial_t W, D_x W, D_x^2 W, u\right) &= \frac{\partial W}{\partial t}(t,x) + A^u W(t,x) + f(t,x,u), \\ H_2(t,x,D_x W,u) &= \sum_{i=1}^{n}\frac{\partial W}{\partial x_i}(t,x)\, G_{il}(t) + k_l(t). \end{aligned}$$

$D_x W$ and $D_x^2 W$ represent respectively, the gradient and the Hessian matrix of W. The bondary data satisfying

$$W(\tau,x) = g(x), \quad (\tau,x) \in \partial S. \qquad (5.7)$$

We start with the definition of classical solutions of the variational inequality (3.2).

Definition 5.2. Let us consider a function $W \in C^{1,2}(S) \cap C\left(\overline{S}\right)$, and define

$$C(W) = \left\{(t,x) \in S : \sum_{i=1}^{n}\sum_{l=1}^{m}\left\{\frac{\partial W}{\partial x_i}(t,x)\, G_{il}(t) + k_l(t)\right\} < 0\right\}$$

We say that W is a classical solution of (3.2) if

$$\frac{\partial W}{\partial t}(t,x) + \sup_{u\in U}\left\{A^u W(t,x) + f(t,x,u)\right\} = 0, \forall (t,x) \in C(W), \qquad (5.8)$$

$$\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}(t,x)\, G_{il}(t) + k_l(t) \leq 0, \text{ for all } (t,x) \in S, l = 1,..,m, \qquad (5.9)$$

$$\frac{\partial W}{\partial t}(t,x) + A^u W(t,x) + f(t,x,u) \leq 0, \text{ for every } (t,x,u) \in S \times U. \qquad (5.10)$$

The following verification Theorem is very similar to Theorem 6.1. in [8] . We drop here the convexity condition for the state domain, and we show that the classical solution to the variational inequality (3.2) with the boundary condition (3.3) coincides with the value function. To this end we first show that $W(t,x)$ majorizes the gain functional $J(u,\xi)$ for any control (u,ξ), and if (u^*,ξ^*) is an optimal control then $W(t,x) = J(u^*,\xi^*)$. Let us denote for $l = 1,..,m$,

$$C_l = \left\{(t,x) \in S : \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(t,x)\, G_{il}(t) + k_l(t) < 0\right\}, \tag{5.11}$$

$$D_l = \left\{(t,x) \in S : \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(t,x)\, G_{il}(t) + k_l(t) = 0\right\}. \tag{5.12}$$

Theorem 5.3. Let W be a classical solution of (3.2) , such that for some constants $k \geq 1,\ M \in (0,\infty)$, $|W(t,x)| \leq M\left(1+|x|^k\right)$. Then, for all $(t,x) \in S$, and $(u,\xi) \in U$

$$W(t,x) \geq J(u,\xi).$$

Furthermore, if there exists $(u^*,\xi^*) \in A$ such that with probability 1

$$(t, X_t^*) \in C(W),\ \text{Lebesgue almost every } t \leq T, \tag{5.13}$$

$$u_t^* \in \arg\max_{u \in A_1} \left\{A^u W(t,X_t^*) + f(t,X_t^*,u)\right\}, \tag{5.14}$$

$$\sum_{l=1}^{m} \left\{\sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(t,X_t^*)\, G_{il}(t) + k_l(t)\right\} d\xi_l^*(t) = 0, \tag{5.15}$$

$$W\left(t,X_{t+}^*\right) - W(t,X_t^*) = -\sum_{l=1}^{m} k_l(t)\, \Delta\xi_l^*(t), \tag{5.16}$$

For all jumping times t of $\xi^*(t)$., then

$$W(t,x) = J(u^*,\xi^*).$$

Let us define $\forall (u,\xi) \in U$, $(t,x) \in S$, and $R > 0$

$$\tau_R = \tau_R^{(u,\xi)} = \tau \curlywedge R \curlywedge \inf \left\{ s > t : \sup_s |X_s| > R \right\},$$

$\lim\limits_{R\to\infty} \tau_R = \tau$. Furthermore, $\int\limits_t^{\tau_R} \sum\limits_{i=1}^n \frac{\partial W}{\partial x_i}(s, X_s)\,\sigma(s, X_s)\,dB_s$ is an Itô integral with finite quadratic variation, so its expected value is zero. Thus, applying Itô formula and taking expectation, we get

$$\begin{aligned}
E\left[W\left(\tau_R, X_{\tau_R}\right)\right] &= W(t,x) + E\left[\int\limits_t^{\tau_R} \left\{ \frac{\partial W}{\partial s}(s, X_s) + A^u W(s, X_s) \right\} ds \right] \\
&\quad + E\left[\int\limits_t^{\tau_R} \sum_{l=1}^m \sum_{i=1}^n \left\{ \frac{\partial W}{\partial x_i}(s, X_s)\, G_{il}(s) \right\} d\xi_l^c(s) \right] \\
&\quad + E\left[\sum_{t \le s \le \tau_R} \left\{ W(s, X_{s+}) - W(s, X_s) \right\} \right].
\end{aligned}$$

By (3.6), and (3.7) we get

$$\begin{aligned}
W(t,x) &\geq E\left[\int\limits_t^{\tau_R} f(s, X_s, u_s) + \int\limits_t^{\tau_R} k(s)\, d\xi^c(s)\right] \\
&\quad - E\left[\sum_{t \le s \le \tau_R} \left\{ W(s, X_{s+}) - W(s, X_s) \right\}\right] + E\left[W\left(\tau_R, X_{\tau_R}\right)\right],
\end{aligned}$$

by the mean value theorem and (3.6), we have

$$W(s, X_{s+}) - W(s, X_s) = \sum_{l=1}^m \sum_{i=1}^n \frac{\partial W}{\partial x_i}(s, x(s))\, G_{il}(s)\, \Delta\xi_l(s) \le -k(s)\, \Delta\xi(s),$$

where $x(s)$ is some point on the straight line between X_s and X_{s+}, hence

$$W(t,x) \geq E\left[\int\limits_t^{\tau_R} f(s, X_s, u_s) + \int\limits_t^{\tau_R} k(s)\, d\xi(s) + W\left(\tau_R, X_{\tau_R}\right)\right],$$

from the dominated convergence Theorem, we so that

$$E\left[\int\limits_t^{\tau_R} f(s, X_s, u_s) + \int\limits_t^{\tau_R} k(s)\, d\xi(s)\right] \underset{R\to\infty}{\longrightarrow} E\left[\int\limits_t^{\tau} f(s, X_s, u_s) + \int\limits_t^{\tau} k(s)\, d\xi(s)\right]$$

by the left continuity of X and the continuity of W, we get

$$\lim_{R\to\infty} W\left(\tau_R, X_{\tau_R}\right) = W\left(\tau, X_\tau\right) = g\left(X_\tau\right),$$

$W\left(\tau_R, X_{\tau_R}\right)$ is uniformly integrable and the dominated convergence Theorem implies

$$\lim_{R\to\infty} E\left[W\left(\tau_R, X_{\tau_R}\right)\right] = E\left[g\left(X_\tau\right)\right],$$

hence $W\left(t,x\right) \geq J\left(u,\xi\right).$

Now, apply the above argument to $\left(u^*,\xi^*\right) \in U$, with $\tau_R^* = \tau_R^{\left(u^*,\xi^*\right)}$. Hence if $\left(3.10\right) - \left(3.13\right)$ hold, then by $\left(3.4\right)$ and $\left(3.5\right)$, we get

$$W\left(t,x\right) = E\left[\int_t^{\tau_R^*} f\left(s, X_s^*, u_s^*\right) + \int_t^{\tau_R^*} k\left(s\right) d\xi^*\left(s\right) + W\left(\tau_R^*, X_{\tau_R^*}^*\right)\right].$$

Finally, using the same limiting procedure as above, we conclude that

$$W\left(t,x\right) = J\left(u^*,\xi^*\right).$$

The following result is a generalization to the classical case, [see, e.g., Theorem 4.1.in Chapter 5 of [24]], and a generalization to linear dynamics, convex cost criterion and convex state constraints of Theorem 6.2. in [9]. Comparing with the stochastic maximum principle, one would expect the solution $\left(p,q\right)$ of the BSDE (2.11) to correspond to the derivatives of the classical solution of $\left(3.2\right) - \left(3.3\right).$

Theorem 5.4. Let W be a classical solution of (3.2), with the boundary condition $\left(3.3\right)$, suppose that $W \in C^{1,3}\left(S\right)$, with all derivatives are continuous on $\overline{S}$, and there exists $\left(u^*,\xi^*\right) \in U$ such that the conditions $\left(3.10\right) - \left(3.13\right)$ are satisfied. Then the solution of the BSDE (2.11) is given by

$$\left(p_t, q_t\right) = \left(D_x W\left(t, X_t^*\right), D_x^2 W\left(t, X_t^*\right)\sigma\left(t, X_t^*\right)\right),$$

with the termial condition is given at the time $T=\tau$ by $p_\tau = D_x g\left(X_\tau^*\right)$.

By the above conditions, we may apply the Itô's rule for semimartingals to $\dfrac{\partial W}{\partial x_k}\left(t, X_t^*\right)$, we obtain

$$\frac{\partial W}{\partial x_k}\left(T, X_{\tau_R}^*\right) = \frac{\partial W}{\partial x_k}\left(t, X_t^*\right) + \int_t^{\tau_R^*} \frac{\partial^2 W}{\partial s \partial x_k}\left(s, X_s^*\right) ds + \int_t^{\tau_R^*} \sum_{i=1}^n \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) dX_i^*\left(s\right)$$
$$+\frac{1}{2}\int_t^{\tau_R^*} \sum_{i,j=1}^n a_{ij}\left(s, X_s^*\right) \frac{\partial^3 W}{\partial x_k \partial x_i \partial x_j}\left(s, X_s^*\right) ds$$
$$+\sum_{t\le s\le \tau_R^*} \left\{ \frac{\partial W}{\partial x_k}\left(s, X_{s+}^*\right) - \frac{\partial W}{\partial x_k}\left(s, X_s^*\right) - \sum_{i=1}^n \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) \Delta X_i^*\left(s\right) \right\},$$

where the sum is taken over all jumping times $s \in (t, \tau_R^*]$ of ξ^*, and

$$\begin{aligned} \Delta X_i^*\left(s\right) &= X_i^*\left(s+\right) - X_i^*\left(s\right), \\ &= \sum_{l=1}^m G_{il}\left(s\right) \Delta\xi_l^*\left(s\right), \text{ for } i = 1, .., n. \end{aligned}$$

where $\Delta\xi_l^*\left(s\right) = \xi_l^*\left(s+\right) - \xi_l^*\left(s\right)$. Therefore

$$\frac{\partial W}{\partial x_k}\left(\tau_R^*, X_{\tau_R^*}^*\right) = \frac{\partial W}{\partial x_k}\left(t, X_t^*\right) + \int_t^{\tau_R^*} \left\{ \frac{\partial^2 W}{\partial s \partial x_k}\left(s, X_s^*\right) + \sum_{i=1}^n b_i\left(s, X_s^*, u_s^*\right) \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) \right.$$
$$\left. + \frac{1}{2}\sum_{i,j=1}^n a_{ij}\left(s, X_s^*\right) \frac{\partial^3 W}{\partial x_k \partial x_i \partial x_j}\left(s, X_s^*\right) \right\} ds \tag{5.17}$$
$$+ \int_t^{\tau_R^*} \sum_{i=1}^n \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) \sigma\left(s, X_s^*\right) dB_s + \int_t^{\tau_R^*} \sum_{i=1}^n \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) \sum_{l=1}^m G_{il}\left(s\right) d\xi_l^*\left(s\right)$$
$$+ \sum_{t\le s\le \tau_R^*} \left\{ \frac{\partial W}{\partial x_k}\left(s, X_{s+}^*\right) - \frac{\partial W}{\partial x_k}\left(s, X_s^*\right) - \sum_{i=1}^n \frac{\partial^2 W}{\partial x_k \partial x_i}\left(s, X_s^*\right) \sum_{l=1}^m G_{il}\left(s\right) \Delta\xi_l^*\left(s\right) \right\},$$

Now, let $\xi^{*c}\left(s\right)$ denote the continuous part of $\xi^*\left(s\right)$, i.e.

$$\xi^{*c}\left(s\right) = \xi^*\left(s\right) - \sum_{t\le s\le \tau_R^*} \Delta\xi_l^*\left(s\right),$$

it holds that

$$\begin{aligned}
\int_t^{\tau_R^*} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) d\xi_l^{*c}(s) &= \int_t^{\tau_R^*} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) d\xi_l^{*}(s) \\
&\quad - \sum_{i=1}^{n} \sum_{t \le s \le T} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) \Delta\xi_l^{*}(s), \\
&= \int_t^{\tau_R^*} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) 1_{\{(s,X_s^*) \in D_l\}} d\xi_l^{*c}(s) \\
&\quad + \int_t^{\tau_R^*} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) 1_{\{(s,X_s^*) \in C_l\}} d\xi_l^{*c}(s),
\end{aligned}$$

For every $(t, x) \in D_l$, we have by (3.8)

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i} (t, x) G_{il}(t) = \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} (t, x) G_{il}(t) + k_l(t) \right\} = 0, \text{ for } l = 1, .., m.$$

hence

$$\int_t^{\tau_R^*} \sum_{i=1}^{n} \sum_{l=1}^{m} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) 1_{\{(s,X_s^*) \in D_l\}} d\xi_l^{*c}(s) = 0. \tag{5.18}$$

Furthermore, for every $(t, x) \in C_l$, and $l = 1, .., m$, we have $\sum_{i=1}^{n} \frac{\partial W}{\partial x_k \partial x_i} (t, x) G_{il}(t) <$ 0, but the equation (3.12) implies that

$$\sum_{l=1}^{m} 1_{\{(s,X_s^*) \in C_l\}} d\xi_l^{*c}(s) = 0,$$

hence

$$\int_t^{\tau_R^*} \sum_{i=1}^{n} \sum_{l=1}^{m} \frac{\partial^2 W}{\partial x_k \partial x_i} (s, X_s^*) G_{il}(s) 1_{\{(s,X_s^*) \in C_l\}} d\xi_l^{*c}(s) = 0, \tag{5.19}$$

By the mean value theorem we have

$$\frac{\partial W}{\partial x_k} \left(s, X_{s+}^*\right) - \frac{\partial W}{\partial x_k} (s, X_s^*) = D_x \left(\frac{\partial W}{\partial x_k} \right)^T (s, x(s)) \Delta X_s^*,$$

where $x(s)$ is some point on the straight line between X_s^* and X_{s+}^*. To prove that the right hand side above vanishes, it is enough to check that, if $\Delta\xi_l^*(s) > 0$ then

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i}(s, x(s))\, G_{il}(s) = 0, \text{ for } l = 1, .., m.$$

It is clear that

$$\begin{aligned} & W\left(s, X_{s+}^*\right) - W\left(s, X_s^*\right) + \sum_{l=1}^{m} k_l(s)\,\Delta\xi_l^*(s) \\ = & \sum_{l=1}^{m} \left\{ \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(s, x(s))\, G_{il}(s) + k_l(s) \right\} \Delta\xi_l^*(s), \end{aligned}$$

by (3.13) the laste term vanishes. $\Delta\xi_l^*(s) > 0$ then $(s, x(s)) \in D_l$, for $l = 1, .., m$. According to (3.9), we obtain

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i}(s, x(s))\, G_{il}(s) = \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} \frac{\partial W}{\partial x_i}(s, x(s))\, G_{il}(s) + k_l(s) \right\} = 0,$$

hence

$$\sum_{t \le s \le \tau_R^*} \left\{ \frac{\partial W}{\partial x_k}\left(s, X_{s+}^*\right) - \frac{\partial W}{\partial x_k}\left(s, X_s^*\right) \right\} = 0. \tag{5.20}$$

On the other hand, define

$$\begin{aligned} A(t, x, u) &= \frac{\partial W}{\partial t}(t, x) + \sum_{i=1}^{n} b_i(t, x, u) \frac{\partial W}{\partial x_i}(t, x) \\ & + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2 W}{\partial x_i \partial x_j}(t, x) + f(t, x, u). \end{aligned}$$

If we differentiate $A(t, x, u)$ with respect to x_k, and evaluate the result at $(x, u) = (X_t^*, u_t^*)$ we get by (3.4), (3.10), and (3.11)

$$\begin{aligned} & \frac{\partial^2 W}{\partial t \partial x_k}(t, X_t^*) + \sum_{i=1}^{n} b_i(t, X_t^*, u_t^*) \frac{\partial^2 W}{\partial x_k \partial x_i}(t, X_t^*) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, X_t^*) \frac{\partial^3 W}{\partial x_k \partial x_i \partial x_j}(t, X_t^*) \\ = & -\sum_{i=1}^{n} \frac{\partial b_i}{\partial x_k}(t, X_t^*, u_t^*) \frac{\partial W}{\partial x_i}(t, X_t^*) - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k}(t, X_t^*) \frac{\partial^2 W}{\partial x_i \partial x_j}(t, X_t^*) \\ & - \frac{\partial f}{\partial x_k}(t, X_t^*, u_t^*). \end{aligned} \tag{5.21}$$

Finally, substituting $(3.15), (3.16), (3.17)$ and (3.18) into (3.14) which simplifies to

$$d\left(\frac{\partial W}{\partial x_k}(t, X_t^*)\right) = -\left\{\sum_{i=1}^{n} \frac{\partial b_i}{\partial x_k}(t, X_t^*, u_t^*)\frac{\partial W}{\partial x_i}(t, X_t^*)\right.$$
$$\left. + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k}(t, X_t^*)\frac{\partial^2 W}{\partial x_i \partial x_j}(t, X_t^*) + \frac{\partial f}{\partial x_k}(t, X_t^*, u_t^*)\right\} dt$$
$$+\sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_k \partial x_i}(t, X_t^*)\,\sigma(t, X_t^*)\,dB_t. \qquad (5.22)$$

By the continuity of $\frac{\partial W}{\partial x_k}$ on $\overline{S}$, and by the left continuity of X, we get

$$\frac{\partial W}{\partial x_k}(\tau, X_\tau^*) = \lim_{R\to\infty} \frac{\partial W}{\partial x_k}\left(\tau_R^*, X_{\tau_R^*}^*\right) = \frac{\partial g}{\partial x_k}(X_\tau^*).$$

for each $k = 1, .., n$. Clearly,

$$\sum_{i=1}^{n} \frac{\partial b_i}{\partial x_k}(t, X_t^*, u_t^*)\frac{\partial W}{\partial x_i}(t, X_t^*) = \frac{\partial b^T}{\partial x_k}(t, X_t^*, u_t^*)\, D_x W(t, X_t^*),$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_k}(t, X_t^*)\frac{\partial^2 W}{\partial x_i \partial x_j}(t, X_t^*) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_k}\left(\sum_{h=1}^{d} \sigma_{ih}\sigma_{jh}\right)(t, X_t^*)\frac{\partial^2 W}{\partial x_i \partial x_j}(t, X_t^*)$$
$$= tr\left(\frac{\partial \sigma^T}{\partial x_k}(t, X_t^*)\, D_x^2 W(t, X_t^*)\,\sigma(t, X_t^*)\right).$$

Then (3.19) given by the form

$$d\left(\frac{\partial W}{\partial x_k}(t, X_t^*)\right) = -\left\{\frac{\partial b^T}{\partial x_k}(t, X_t^*, u_t^*)\, D_x W(t, X_t^*)\right.$$
$$\left. + tr\left(\frac{\partial \sigma^T}{\partial x_k}(t, X_t^*)\, D_x^2 W(t, X_t^*)\,\sigma(t, X_t^*)\right) + \frac{\partial f}{\partial x_k}(t, X_t^*, u_t^*)\right\} dt$$
$$+\sum_{i=1}^{n} \frac{\partial W}{\partial x_k \partial x_i}(t, X_t^*)\,\sigma_i(t, X_t^*)\,dB_t, \qquad (5.23)$$

with the terminal condition

$$\frac{\partial W}{\partial x_k}(\tau, X_\tau^*) = \frac{\partial g}{\partial x_k}(X_\tau^*).$$

Now, from (3.8) we note that

$$\frac{\partial H}{\partial x_k}(t,x,u,p,q)=\frac{\partial b^T}{\partial x_k}(t,x,u)\,p+tr\left(\frac{\partial \sigma^T}{\partial x_k}(t,x)\,q\right)+\frac{\partial f}{\partial x_k}(t,x,u),$$

and define p_t^k the kth coordinate of the column vector p_t by

$$\begin{cases} dp_t^k &= -\dfrac{\partial H}{\partial x_k}(t,X_t^*,u_t^*,p_t,q_t)\,dt+q_t^k dB_t, \qquad \text{for } t\in[0,T], \\ p_T &= \dfrac{\partial g}{\partial x_k}(X_T^*), \end{cases}$$

with $q_t^k dB_t = \sum\limits_{1\le h\le d} q_t^{kh} dB_t^h$, for $k=1,..,n$. Hence, by the uniqueness of the solution to (2.9) and (3.20), we obtain

$$p_t^k=\frac{\partial W}{\partial x_k}(t,X_t^*),$$

and

$$q_t^{kh}=\sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_k \partial x_i}(t,X_t^*)\,\sigma_{ih}(t,X_t^*)$$

q_t^{kh} the $khth$ element of q_t for $k=1,..,n$, and $h=1,..,d$. In particular, note that (p_t,q_t) represents

$$\left(D_x W(t,X_t^*),D_x^2 W(t,X_t^*)\,\sigma(t,X_t^*)\right)$$

where X_t^* is the optimal solution of the controlled SDE (2.1).

5.2 Application to finance

Suppose the wealth X_t at time t corresponding to initial capital $x>0$ is governed by the linear stochastic differential equation

$$\begin{cases} dX_t &= \mu X_t dt+\sigma X_t dB_t-d\xi(t), \qquad \text{for } t\in[0,T], \\ X_0 &= x, \end{cases} \tag{5.24}$$

This problem can be regarded as a special case of the portfolio selection with transaction costs problem's in the case of a single push direction [see, e.g., Davis and Norman]. We concider here, the situation where an investor only invests in a risky stock of constants rate of return μ and volatility σ and he may consume continuously and costlessly from the wealth. The objective of the investor is to maximize the functional

$$J(\xi) = E\left[\int_0^\tau e^{-\delta t} X_t^\gamma dt + \int_0^\tau e^{-\delta t} d\xi(t)\right], \tag{5.25}$$

with $\gamma \in (0,1)$, $\mu, \sigma, \delta, \theta > 0$ are given constants. $(1-\gamma)$ is the relative risk aversion of the consumer, $\xi(t)$ is an increasing adapted cadlag process satisfying $P\left\{|\xi(T)|^2 < \infty\right\} = 1$ with $\xi_0 = 0$, representing the total transaction taken out up to time t. ξ is called admissible strategy for given initial capital x if the solution of (4.1) satisfies $X_0 = x$, we denote by $\Pi(x)$ the class of such pairs for x. We want to find the optimal strategy $\xi^*(.) \in \Pi(x)$ which maximizes the expected total discounted utility of the . This is an example of a singular stochastic control problem. It is called singular because the investment control measure $d\xi(t)$ is allowed to be singular with respect to Lebesgue measure dt. Other applications of the singular control problems with jump diffusions in finance are developed in the recent textbook [22].

We illustrate a verification result for the maximum principle, in this case the Hamiltonian gets the form

$$H(t, X, c, p, q) = \mu X_t p_t + \sigma X_t q_t + e^{-\delta t} X_t^\gamma. \tag{5.26}$$

Let $\xi^* \in U$ be a candidate for an optimal control, and let X^* be the corresponding wealth process with corresponding solution (p^*, q^*) of the adjoint equation

$$\begin{cases} dp_t^* = -\left(\mu p_t^* + \sigma q_t^* + e^{-\delta t}\gamma X_t^{\gamma-1}\right) dt + q_t^* dB_t, \quad \text{for } t \in [0, \tau), \\ p_\tau^* \text{ at time } \tau, \end{cases} \tag{5.27}$$

with the transversality condition

$$E[p_\tau^*.(X_\tau^* - X_\tau)] \leq 0. \tag{5.28}$$

Here, the conditions (2.14) and () , gets the form, with probability 1

$$-p_t^* + e^{-\delta t} < 0, \text{ for all } t \in [0, \tau], \tag{5.29}$$

$$1_{\left\{-p_t^* + e^{-\delta t} < 0\right\}} d\xi_t^* = 0. \tag{5.30}$$

Explicit solution of the adjoint equation (4.4) satisfies the conditions () , () , and () , is a difficult problem, then we use the relation between the value function and the solutions (p^*, q^*) of the adjoint equation given on the optimal state to solve the problem. Further, for any ξ define

$$\phi(t, x) = \sup_{\xi \in \Pi(t,x)} J(\xi).$$

Note that, the definition of $\Pi(t, x)$ is similar to $\Pi(x)$, except that the starting time is t, and the wealth at t is x.

The generator of time-space process if $\xi = 0$ is

$$A\Phi(t, x) = \frac{\partial \Phi}{\partial t}(t, x) + \mu x \frac{\partial \Phi}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}(t, x),$$

the non-intervention region is described by

$$C = \left\{(t, x) : -\frac{\partial \Phi}{\partial x}(t, x) + e^{-\delta t} < 0\right\}. \tag{5.31}$$

If we guess C has the form $C = \{(t,x) : 0 < x < b\}$ for some barrier point $b > 0$, then by (3.10) – (3.12) the equation (3.4) gets the form,

$$\frac{\partial \Phi}{\partial t}(t,x) + \mu x \frac{\partial \Phi}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}(t,x) + e^{-\delta t} x^\gamma = 0, \text{ for } 0 < x < b, \tag{5.32}$$

and

$$-\frac{\partial \Phi}{\partial x}(t,x) + e^{-\delta t} = 0, \text{for } x \geq b. \tag{5.33}$$

The linearity of the wealth dynamics with respect to the state and control processes, together with the form of the utility function enables us to represent the solution Φ in a separable form, to this end, We try a solution Φ of the form $\Phi(t,x) = e^{-\delta t}\Psi(x)$, then $A\Phi(t,x) = e^{-\delta t} A^0 \Psi(x)$ where Ψ remains to be determined. In terms of Ψ the equation (4.7) has following form

$$-\delta \Psi(x) + \mu x \Psi'(x) + \frac{1}{2}\sigma^2 x^2 \Psi''(x) + x^\gamma = 0.$$

We now choose $\Psi(x) = C_1 x^{r_1} + C_2 x^{r_2} + K x^\gamma$, where C_1, C_2 are arbitrary constants, and $r_1 < 0 < r_2$, are the solution of the equation

$$\frac{1}{2}\sigma^2 r^2 + \mu r - \delta = 0,$$

are given by

$$r_i = \frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2 - \mu \pm \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\delta\sigma^2}\right),$$

and the constant $K \in \mathbb{R}_+^*$, is given by

$$K = -\frac{1}{\frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - \delta}.$$

Outside C we require that $-\Psi'(x) + 1 = 0$, or $\Psi(x) = x + M$, M is a constant to be determined. Hence we put

$$\Phi(t,x) = \{$$

$$\begin{cases} e^{-\delta t}\left(C_1x^{r_1}+C_2x^{r_2}+Kx^{\gamma}\right) & \text{for } 0<x<b, \\ e^{-\delta t}\left(x+M\right) & \text{for } x\geq b. \end{cases} \quad (5.1)$$

we put $C_1 = 0$, assuming smooth fit's principle at point b, we obtain a system of equations for unknowns C_2, M and b. Φ continuous at $x = b$ then

$$C_2b^{r_2} + Kb^{\gamma} = b + M, \tag{5.35}$$

Φ continuously differentiable at $x = b$, then

$$C_2r_2b^{r_2-1} + K\gamma b^{\gamma-1} = 1, \tag{5.36}$$

Φ twice continuously differentiable at $x = b$, then

$$C_2r_2\left(r_2-1\right)b^{r_2-2} + K\gamma\left(\gamma-1\right)b^{\gamma-2} = 0.$$

Then, we get

$$M = C_2b^{r_2} + Kb^{\gamma} - b, \tag{5.37}$$

the barrier point is given by

$$b = \left(\frac{K\gamma\left(1-\gamma\right)}{C_2r_2\left(r_2-1\right)}\right)^{\frac{1}{r_2-\gamma}}, \tag{5.38}$$

and

$$C_2 = \frac{1-K\gamma b^{\gamma-1}}{r_2b^{r_2-1}},$$

$\gamma < 1$, and $r_2 > 1$, then $b > 0$. Next, we look into the conditions (3.6) and (3.7). Accordingly we will study two different cases, the first when $x \geq b$, denote by $F(x)$ the function given by

$$\begin{aligned} F(x) &= A^0\Psi(x), \\ &= -\delta\left(x+M\right) + \mu x + x^{\gamma}, \end{aligned}$$

which is a decreasing function in x on $[b,+\infty[$, if $\delta > \mu$. So we need only to check that $F(b) \le 0$, but this follows from the fact that $A^0\Psi(x) \le 0$ for all $x < b$, and $\Psi \in C^2$. The second case when $0 < x < b$, then $\Psi(x) = Kx^\gamma + C_2x^{r_2}$ and the condition (3.6) gets the form

$$-\left(C_2r_2x^{r_2-1} + K\gamma x^{\gamma-1}\right) + 1 \le 0.$$

Put $G(x) = -\left(C_2r_2x^{r_2-1} + K\gamma x^{\gamma-1}\right) + 1$, from (4.13), we get

$$G(b) = -\left(C_2r_2b^{r_2-1} + K\gamma b^{\gamma-1}\right) + 1 = 0,$$

and

$$G'(b) = -\left(C_2r_2(r_2-1)b^{r_2-1} + K\gamma(\gamma-1)b^{\gamma-2}\right) = 0,$$

since $\gamma \in (0,1)$, then $-K\gamma(\gamma-1)$ and $C_2r_2(r_2-1)$ are a positive constants. Thus $G'(x) \ge G'(b) = 0$, for all $x \in]0,b]$, then G is a increasing function, Thus we have established that $G(x) \le G(b) = 0$ on $]0,b]$.

For construction of the optimal control $\xi^*(.)$, let us consider the stochastic integral equation

$$X_t^* = x_0 + \int_0^t \mu X_s^* ds + \int_0^t \sigma X_s^* dB_s - \xi^*(t), \tag{5.39}$$

$$X_t^* \le b, \qquad t \in [0,T], \tag{5.40}$$

$$\int_0^t 1_{\{X_s^* < b\}} d\xi^*(s) = 0, \quad t \in [0,T]. \tag{5.41}$$

Here b is given by (4.15). $(4.23) - (4.25)$ define so-called Skorohod problem, whose solution is a pair $(X_t^*, \xi^*(t))$, where X_t^* is a diffusion process reflected at b. The conditions $(3.10) - (3.13)$ claim the existence of an increasing process $\xi^*(t)$ such that X_t^* stays in $\overline{C}$ for all times t. If the initial size $x \le b$, $\xi^*(t)$ increases only when X_t^* is at the point b so as to ensure $X_t^* \leq b$, on the other hand, if the initial size $x > b$ then $\xi^*(0+) = x - b$, that is X_0^* jumps to point b immediately and such that $X_{0+}^* = b$ then evolves on as the case of X_t^* whith the initial point b. Such a singular control is called a local time at b. The existence and uniqueness of such a local time is proved in [12]. The existence and uniqueness to the solution of the Skorohod problem $(4.23) - (4.25)$ is established in [21].

Note that, by construction of ξ^*, or by construction of Φ all the conditions of the section 3 are satisfied and the value function is $\phi(t,x) = \Phi(t,x)$.

Now we want to solve the constrainted adjoint equation (4.4), but in order to apply Theorem 3.2., we first prove that, the solution of the adjoint equation is given by

$$(p_t^*, q_t^*) = \left(e^{-\delta t}\left(C_2 r_2 X_t^{*r_2-1} + K\gamma X_t^{*\gamma-1}\right), \sigma e^{-\delta t}\left(C_2 r_2 (r_2-1) X_t^{*r_2-1} + K\gamma(\gamma-1) X_t^{*\gamma-1}\right)\right), \tag{5.42}$$

to this end, we differentiate the process

$$A(t, X_t) = e^{-\delta t}\left(C_2 r_2 X_t^{*r_2-1} + K\gamma X_t^{*\gamma-1}\right), \text{ for } t \in [0,\tau], \tag{5.43}$$

using Itô's rule for semimartingals, we get

$$\begin{aligned} & K\gamma e^{-\delta t} X_t^{\gamma-1} = K\gamma X_0^{\gamma-1} \\ & + \int_0^t K\gamma e^{-\delta s}\left\{-\delta X_s^{\gamma-1} + \mu(\gamma-1) X_s^{\gamma-1} + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2 X_s^{\gamma-1}\right\} ds \\ & + \int_0^t K\gamma(\gamma-1)\sigma e^{-\delta s} X_s^{\gamma-1} dB_t - \int_0^t K\gamma(\gamma-1) e^{-\delta s} X_s^{\gamma-2} d\xi^c(s) \\ & + \sum_{0\leq s\leq t}\left\{K\gamma e^{-\delta s}\left(X_{s+}^{\gamma-1} - X_s^{\gamma-1}\right)\right\}. \end{aligned} \tag{5.44}$$

where $\xi^c(s) = \xi(s) - \sum_{0\leq s\leq t} \Delta\xi(s)$ denote the continuous part of $\xi(s)$. Next, consider the case when $X_s = X_s^* = X_s^{\xi^*(s)}$, for all times s between 0 and at those $t = t^*$, for which $X_{t^*}^* = b$. We merely note that t^* is fixed because b is deterministic, in this case $A(s, X_s) = K\gamma e^{-\delta s} X_s^{*\gamma-1}$ for $s \in [0, t^*]$, then we obtain

$$K\gamma e^{-\delta t^*} X_{t^*}^{*\gamma-1} = K\gamma X_0^{*\gamma-1}$$
$$+ \int_0^{t^*} K\gamma e^{-\delta s} X_s^{*\gamma-1} \left\{-\delta + \mu(\gamma-1) + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2\right\} ds$$
$$+ \int_0^{t^*} K\gamma(\gamma-1)\sigma e^{-\delta s} X_s^{*\gamma-1} dB_t - \int_0^{t^*} K\gamma(\gamma-1) e^{-\delta s} X_s^{*\gamma-2} d\xi^c(s)$$
$$+ \sum_{0 \le s \le t^*} \left\{K\gamma e^{-\delta s}\left(X_{s+}^{*\gamma-1} - X_s^{*\gamma-1}\right)\right\}.$$

replacing by the value of the constant δ we get, the integrant with respect to ds on the last equality is given by

$$\begin{aligned} & K\gamma e^{-\delta s} X_s^{*\gamma-1} \left\{-\delta + (\gamma-1)\mu + \frac{1}{2}\sigma^2(\gamma-1)(\gamma-2)\right\}, \\ = \quad & -K\gamma e^{-\delta s} X_s^{*\gamma-1} \left\{\mu + (\gamma-1)\sigma^2\right\}. \end{aligned} \tag{5.45}$$

Next, by the mean value theorem, we get

$$K\gamma e^{-\delta s}\left(X_{s+}^{*\gamma-1} - X_s^{*\gamma-1}\right) = K\gamma(\gamma-1) e^{-\delta s} y_s^{\gamma-2}.\triangle X_s^*,$$

if $\Delta X_s^* \neq 0$, it is necessary that $y_s \geq b$, then $y_s^{\gamma-2} \leq b^{\gamma-2}$, since $\gamma \in (0,1)$, then $K\gamma(\gamma-1)$ is a negative constant, hence by () we obtain

$$K\gamma(\gamma-1) y_s^{\gamma-2} \geq K\gamma(\gamma-1) b^{\gamma-2} = 0, \tag{5.46}$$

on the other hand $y_s^{\gamma-2} \geq 0$, then

$$K\gamma(\gamma-1) y_s^{\gamma-2} \leq 0, \tag{5.47}$$

hence by () and (), we obtain

$$K\gamma(\gamma-1) y_s^{\gamma-2} = 0.$$

Then

$$K\gamma e^{-\delta s}\left(X_{s+}^{*\gamma-1}-X_{s}^{*\gamma-1}\right)=0 \tag{5.48}$$

Now, the fact that $1_{\{X_s^*<b\}}d\xi^{*c}(s)=0$, and by the same argument as above, we conclude that

$$\begin{aligned}\int_0^{t^*} K\gamma(\gamma-1)\sigma e^{-\delta s}X_s^{*\gamma-2}d\xi^{*c}(s) &= \int_0^{t^*} K\gamma(\gamma-1)e^{-\delta s}X_s^{*\gamma-2}1_{\{X_t^*\geq b\}}d\xi^{*c}(s) \\ &\quad+\int_0^{t^*} K\gamma(\gamma-1)e^{-\delta s}X_s^{*\gamma-2}1_{\{X_t^*<b\}}d\xi^{*c}(s). \\ &= 0. \end{aligned} \tag{5.49}$$

By () it is possible to take a terminal condition for the adjoint equation (4.4) at time $\tau=t^*$, by $p_{t^*}^*=e^{-\delta t^*}$.

Further, by substituting $(4.19), (4.20), (4.21)$ and (4.22) into (4.18), together with (). Then the uniqueness of adapted solutions (p^*,q^*) of the adjoint equation (4.4) implies (4.16).

Note thate, by the regularity smooth fit condition, we can extend p_t, for t$\in[t^*,T]$ by $p_t=e^{-\delta t^*}$. and with this choise all conditions of theorem are satisfied and the conditions () and () are coincide with () and ().

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