République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

UNIVERSITE MOHAMED KHIDER, BISKRA

FACULTÉ des SCIENCES EXACTES et des SCIENCES de la NATURE et de la VIE

DEPARTEMENT DE MATHEMATIQUES

Thèse présentée en vue de l'obtention du Diplôme de

Doctorat en Mathématiques

Option: **Probabilités et Statistiques**

Par

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Titre

Contrôle optimal des équations différentielles stochastiques de type champ moyen

Soutenue le: **23 Avril 2015**

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Dedicace

Je dédie ce modeste travail:

- à mes chers Parents pour tout ce qu'ils m'ont donné,
- à mon mari Hicham et mon fils Younès,
- à mes frères Mohamed Amine et Youcef et à ma soeur Ibtissam.

Un grand merci à tous ceux qui m'ont aidé à concrétiser ce travail.

REMERCIEMENTS

Je tiens à remercier Monsieur Khaled Bahlali, Maître de conférence à l'université du Sud-Toulon-Var, qui m'a proposé cette thèse, rentrant le cadre de l'accord Tassili 13 MDU 887, entre la laboratoire de mathématiques appliquées, université de Biskra et le laboratoire IMATH de l'université de Toulon. Il a été toujours à mon écoute et m'a orienté de façon toujours judicieuse dans mes recherches. Il m'a accueilli au laboratoire IMATH à plusieurs reprises, pendant la préparation de cette thèse et s'est occupé de tous les aspects, de mes multiples séjours avec beaucoup d'attention. Ses encouragements et ses conseils m'ont beaucoup aidé à la concrétisation de ce modeste travail. Qu'il trouve ici l'expression de ma haute considération et de mon respect.
C'est avec un énorme plaisir, que je remercie le Professeur Abdelhakim Necir, qui a accepté de présider mon jury de thèse et pour l'intérêt qu'il a toujours manifesté pour mon travail et ma progression. Mes remerciements vont également à mes professeurs Messieurs Djamel Meraghni et Boubakeur Labed, qui ont accepté de faire partie de mon jury. J'ai beaucoup appris des cours, qu'ils m'ont assuré tant au niveau de la Licence qu'au niveau du Master. Je tiens aussi à remercier le professeur Salah Eddine Rebiai, de l'université de Batna, pour l'intérêt qu'il a manifesté à ce modeste travail en acceptant de se joindre au jury.
Je tiens aussi à adresser mes vifs remerciements à Monsieur Youssef Ouknine, professeur à l'université de Marrakech et membre de l'académie Hassan II des sciences et technologie, pour l'accueil chaleureux qu'il m'a réservé lors de ma visite à Marrakech et pour les facilités, qu'il m'a accordé pour la participation à l'école organisée à Kelaa Mgouna sur le contrôle déterministe et stochastique. Ecole organisée dans le cadre du projet européen FP 07: "*Deterministic and Stochastic Control*".
Aussi, je voudrais remercier chaleureusement le Professeur Ernst Eberlein, pour l'accueil chaleureux et le support financier, dans le cadre du projet DAAD, lors de mon séjour à l'université de Freibourg (Allemagne). Il m'a invité à participer au congrès international "ARS CONJECTANDI", fêtant le passage de trois siècle depuis la parution du premier livre de probabilités, par Johann Bernouilli. Ce fût pour moi une exprérience fabuleuse. Qu'il trouve ici l'expression de ma haute considération.
Que tous ceux qui ont contribué de près ou de loin à la préparation de cette thèse, particulièrement mes enseignants, trouvent ici l'expression de ma gratitude et de mon respect.

Abstract

In this thesis we are interested by optimal control of systems driven by stochastic differential equations of the mean-field type. In these equations, the coefficients depend not only on the state but also on the distribution of the state process, via the expectation of some function of the state. We establish existence of relaxed and strict optimal controls for this type of problems in the case of controlled drift as well as wher both the drift and diffusion coefficient are controlled. Moreover we derive necessary optimality conditions in the form of a stochastic maximum principle for an optimal relaxed control.

Resume

Dans cette thèse, nous nous sommes intéressés au contrôle optimal des équations différentielles stochastiques de type champ moyen. Dans ces équations les coefficients dépendent non seulement de l'état du sytème, mais aussi de la loi de l'état par l'intermédiaire de l'espérance d'une certaine fonction de l'état. On établit l'existence de contrôles optimaux relaxés et stricts dans les cas ou le drift est controlé, aussi bien que dans le cas ou le drift et le coefficient de diffusion son contrôlés. Aussi, on démontre des conditions nécessaires d'optimalité, sous la forme d'un principe du maximum, vérifiées par un contrôle optimal relaxé.

Contents

Introduction

The main goal of this thesis is to investigate the problem of existence of an optimal control as well as the necessary conditions for optimality, for a system governed by a stochastic differential equation of the mean-field type, (MFSDE in short), taking the form:

$$\begin{cases} dX_t = b(t, X_t, E\left(\psi(X_t)\right), u_t)dt + \sigma(t, X_t, E\left(\Phi(X_t)\right), u_t)dW_t \\ X_0 = x. \end{cases} \tag{1}$$

$(W_t, t \geq 0)$ is a d-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, x is the initial state and u_t stands for the control variable. σ, b , ψ, ϕ are deterministic maps.

The expected cost on the time interval $[0, T]$ is of mean field type also and is given by

$$J(u) \quad = \quad E\left(\int_0^T h(t, X_t, E\varphi(X_t), u_t)dt + g(X_T, E\lambda(X_T))\right). \tag{2}$$

In the state equation and the cost functional, the functions depend not only on the state of the system, but also on the distribution of the state process, via the expectation of some function of the state. MFSDEs are obtained as mean square limits of interacting particle systems of the form:

$$dX_t^{i,n} = b(t, X_t^{i,n}, 1/n\sum_{j=1}^{n}\psi(X_t^{i,n}), u_t)dt + \sigma(t, X_t^{i,n}, 1/n\sum_{j=1}^{n}\Phi(X_t^{i,n}), u_t)dW_t \tag{3}$$

When n goes to infinity, it is proved in [43] , in the linear case, that $X_t^{i,n}$ converges to $\overline{X}_t^i$, where all the processes $\overline{X}_t^i$ $(i = 1, ...)$, are independent copies of the same process, called the non linear process or the McKean-Vlasov process, which is the unique solution of the MFSDE (2.1). We refer to [31], to the general case of a non linear dependence of the coefficients upon the process and its distribution and the driving process is a general Lévy process.

Motivated by a recent interest in differential games, control problems where the state process is a MFSDE, where the coefficients depend on the marginal probability law of the solution, have been studied in [1] and provide interesting models in applications, in particular to game problems [15, 34]. A typical example is the continuous-time Markowitz's mean-variance portfolio selection problem, where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term, see [2, 19, 47, 51]. The main drawback, when dealing with mean field stochastic control problems, is that the state process is not a Markov process and as a consequence, the dynamic programming principle is no longer valid. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see [2, 11, 17, 35, 38, 46, 47]. The SMP gives necessary optimality conditions in terms of the maximization of some hamiltonian and an adjoint process which is the solution of a backward SDE of mean field type, see [12, 14].

In this work, we are interested by the existence of an optimal control, where the state equation, as well as the cost function are of mean field type. This kind of result is interesting in itself and particularly when one deals with the stochastic maximum principle. So, it is interesting to know if an optimal control exists and to try to characterize it, by deriving necessary conditions. A control u^* is called optimal if it satifies $J(u^*) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$, where $\mathcal{U}_{ad}$ is the space of admissible controls, that is measurable, adapted processes with values in some action space A. If moreover, u^* is in $\mathcal{U}_{ad}$, it is called strict.

Th first chapter is an introduction to stochastic calculus. We recall the necessary tools such as Brownian motion, stochastic differential equations, backward stochastic differential equations, which will be used in the sequel.

In the second chapter, we establish two main results. We first show the existence of an optimal relaxed control, for control problems driven by non linear MFSDEs. The proof is based on tightness properties of the underlying processes and Skorokhod selection theorem. Our results extend in particular those in [21, 26, 3], for mean field SDEs. Moreover, due to the compactness of the action space, we show that the relaxed control could be choosen among the so-called sliding controls, which are convex combinations of Dirac measures. As a consequence and under some Fillipov convexity condition, the relaxed control is shown to be strict.The second main result is an existence result for control problems driven by linear MFSDEs. For this particular class of problems, we prove the existence of a strict strong optimal control, that is a control process in $\mathcal{U}_{ad}$, which is adapted to the initial filtration. This means that the admissible controls in this case are adapted to a fixed filtration. The method of proof is based essentially on weak convergence techniques on the space $L^2_{\mathcal{F}}$ of square integrable processes and Mazur's theorem on the equality of the strong and weak closure of a convex set. It should be noted that for this case, there is no need to use tightness techniques and to change the initial probability space. The reason is that roughly speaking, in the linear case, the problem reduces to finite dimensional techniques. In particular, our result extends [49], Theorem 5.2, to mean-field control problems.

In the third chapter, we deal with mean-field stochastic conbrol problems where both the drift and the diffusion coefficient are controlled. As it will be shown, the stochastic equation associated with the relaxed generator will be governed by a continuous orthogonal martingale measure, rather than a Brownian motion. For this model, we prove that the strict and relaxed control problems have the same value function and that an optimal relaxed control exists. Our result extends in particular [3, 21, 39] to mean field controls. The proof is based on tightness properties of the underlying processes and Skorokhod selection theorem. Moreover, due to the compactness of the action space, we show that the relaxed control could be choosen among the so-called sliding controls, which are convex combinations of Dirac measures. As a consequence and under the so-called Fillipov convexity condition, the optimal relaxed control is shown to be strict.

In the fourth chapter, we establish necessary conditions for optimality in the form of a relaxed stochastic maximum principle, obtained via the first and second order adjoint processes, extending Peng's maximum principle [41] to mean field control problems and [11] to relaxed controls. The advantage of our result is that the maximum pinciple applies to a natural class of controls, which is the closure of the class of strict controls, for which we know that an optimal control exists. The proof of the main result is based on the approximation of the relaxed control problem by a sequence of strict control problems. Then Ekeland's variational principle is applied to get necessary conditions of near-optimality for the sequence of nearly optimal strict controls. The result is obtained by a passage to the limit in the state equation as well as in the adjoint processes. The resulting first and second order adjoint processes are solutions of linear backward SDEs driven by a Brownian motion and an orthogonal square integrable martingale. The advantage of our result is that it is given via an approximation procedure, so that it could be convenient for numerical computation.

Chapter 1

An introduction to stochastic calculus

1.1 Introduction

In this chapter, we'll introduce the necessary tools, which will be used in the sequel. In particular we'll present briefly the theory of stochastic differential equations driven by a Brownian motion. These equation play a great role in this work. For this end, we present a brief history and some properties of the Brownian motion and the Ito existence and uniqueness theorem of solutions of stochastic differential equations, in the case of globally Lipschitz coefficients. Moreover, due to their importance in control theory, we present a brief account of backward stochastic differential equations (BSDE). In particular we give without proof the Pardoux-Peng theorem on existence and uniqueness of adapted square integrables solutions of BSDEs. It should be mentioned that the notion of BSDE has been introduced by J.M.Bismut, in connection with stochastic control. More precisely the adjoint process arizing in stochastic control satisfies a linear BSDE.

1.2 Some properties of stochastic processes

1.2.1 Notion of stochastic process

To represent a random phenomenon dependent of the time, suitable mathematical model is given by a space of probability $(\Omega, \mathcal{F}, P)$ and a function :

$$X : \Omega \times \mathbb{R}_+ \longrightarrow E$$
$$(\omega, t) \longrightarrow X(\omega, t)$$

For a fixed t, the state of the system is a random variable $X(\omega, t)$
If fixed $\omega \in \Omega$,the states are represented by the function $t \longrightarrow X(\omega, t)$ called path.

Definition 1.2.1 *Let T a set of indices ($\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$...). We Call a stochastic process defined on T with values in a measurable set$(E, \mathcal{E})$, a family of random variables $(X_t)_{t\in T}$ with values in $(E, \mathcal{E})$.*

Let T be any set, we denote E^T the set of applications of T in E .
We denote

$$\pi_t : E^T \longrightarrow E$$
$$f \longrightarrow \pi_t(f) = f(t)$$

π_t :is called coordinate of index t.
If E is equipped with the $\sigma-$algebra $\mathcal{E}$, then the space E^T can be endowed with the product $\sigma-$algebra defined as $\sigma-$algebra generated by the sets of the form $\pi_{t_1}^{-1}(A_1) \cap \pi_{t_2}^{-1}(A_2) \cap ... \cap \pi_{t_n}^{-1}(A_n)$, where $A_1, A_2, ..., A_n$ belong to E and $(t_1, t_2, ..., t_n) \subset T$.
This $\sigma-$algebra can be defined as the smallest $\sigma-$algebra making measurable coordinate applications π_t for all $t \in T$.
The process (π_t) is called the canonical processassociated to the process (X_t).
The essential tool for the construction of process is the theorem of Kolmogorov. It is valid, particularly when the measurable space $(E, \mathcal{E})$ is$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, or E is a complete metric space with a countable basis of open sets, equipped with his Borel $\sigma-$algebra.

Theoreme 1.2.1 ***(Kolmogorov)****Let T be a set of indices and $(E, \mathcal{E})$ a space of the previous form. Let for each finite subset S of T let a probability measure μ_S defined over $(E^S, \mathcal{E}^{\otimes S})$, so that these measures form a compatible system, ie- for each S and S' such that $S \subset S'$ we are $\pi_S(\mu_{S'}) = \mu_S$. Then there exist only one probability measure μ over $(E^T, \mathcal{E}^{\otimes T})$ such that $\pi_S(\mu) = \mu_S$.*

The Kolmogorov theorem is used to construct a probability measure on the canonical space (which is generally an infinite dimensional space) equipped with infinite product $\sigma-$algebra. So it is useful to prove the existence of a stochastic process.

1.2.2 Equivalence of Processes

Equivalents Processes

Definition 1.2.2 *we say that two processes are equivalent if they have the same marginal distributions.*

Let (X_t) defined on $(\Omega_1, \mathcal{F}_1, P_1)$ whith values in $(E, \mathcal{E})$ and (Y_t) defined on $(\Omega_2, \mathcal{F}_2, P_2)$ with values in $(E, \mathcal{E})$.
(X_t) and (Y_t) are equivalent processes if for each $(t_1, t_2, ..., t_n)$, the random vectors $(X_{t_1}, X_{t_2}, ..., X_{t_n})$and $(Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$ have the same probability distribution. This probability distribution is defined on $(E^n, \mathcal{E}^{\otimes n})$.
Two processes are equivalent if they have the same canonical process.

Modification of a process

Definition 1.2.3 *It is said that two processes (X_t) and (Y_t) defined on the same probability space $(\Omega, \mathcal{F}, P)$ with values in the same state space $(E, \mathcal{E})$, are modifications of one another if:*

$$\forall t \in T : X_t = Y_t, \; P - p.s$$

Remark 1.2.1 *it is obvious that if (X_t) and (Y_t) are two modifications processes from one another, then, they are equivalent.*

Indistinguishable processes

Definition 1.2.4 *It is said that two processes (X_t) and (Y_t) defined on the same probability space $(\Omega, \mathcal{F}, P)$ with values in the same state space $(E, \mathcal{E})$, are indistinguishables if:*

$$P(X_t = Y_t, \, \forall t \in T) = 1$$

Remark 1.2.2 *It is clear that if two processes are indistinguishable then they are modifications of one another. The converse is false in general.*

1.2.3 Measurability of processes

Measurable process

Assume that the set of indices is $\mathbb{R}_+$.

Definition 1.2.5 *Let $(\Omega, \mathcal{F})$ a measurable space. A filtration on $(\Omega, \mathcal{F})$ is any increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of sub-$\sigma-$algebras of $\mathcal{F}$. ie: for each $s, t \in \mathbb{R}_+$ such that $s \leq t$ we have: $\mathcal{F}_s \subset \mathcal{F}_t$.*

Let us define $\mathcal{F}_{t^+} = \bigcap\limits_{s>t} \mathcal{F}_s$, we say that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right continuous if $\mathcal{F}_{t^+} = \mathcal{F}_t$.

Remark 1.2.3 *the filtration $(\mathcal{F}_{t^+})$ is always right continuous.*

Definition 1.2.6 *Let $(X_t)_{t\in\mathbb{R}_+}$ a process defined on $(\Omega, \mathcal{F})$ and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the filtration of $\mathcal{F}$. We will say that $(X_t)_{t\in\mathbb{R}_+}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ or $\mathcal{F}_t-$adapted if for each $t \in \mathbb{R}_+$, the random variable X_t is $\mathcal{F}_t-$measurable.*

Remark 1.2.4 *It is clear that any process $(X_t)_{t\in\mathbb{R}_+}$ is adapted to his natural filtration defined as:*

$$\mathcal{F}_t = \sigma\left(X_s,\ s \leq t\right)$$

Definition 1.2.7 *We say that a process $(X_t)_{t\in\mathbb{R}_+}$ is measurable if the map:*

$$\begin{aligned} X : (\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)) &\longrightarrow (E, \mathcal{E}) \\ (\omega, t) &\longrightarrow X_t(\omega) \end{aligned}$$

is measurable.

Definition 1.2.8 *We say that $(X_t)_{t\in\mathbb{R}_+}$ is progressively measurable if for each $t \in \mathbb{R}_+$ the map:*

$$\begin{aligned} X : (\Omega \times [0, t]\,, \mathcal{F}_t \otimes \mathcal{B}(\,[0, t]\,)) &\longrightarrow (E, \mathcal{E}) \\ (\omega, s) &\longrightarrow X_s(\omega) \end{aligned}$$

is measurable.

Theoreme 1.2.2 *If the state space E is a metric space and the processus $(X_t)_{t\in\mathbb{R}_+}$ is adapted and whose paths are continuous from the right, then the process $(X_t)_{t\in\mathbb{R}_+}$ is progressively measurable.*

Predictable and optional processes

Definition 1.2.9 *Let $(\Omega, \mathcal{F}, P)$ probability space equipped whith the filtration $(\mathcal{F}_t)$. We say that the filtration is complete if the space $(\Omega, \mathcal{F}, P)$ is complete and that $\mathcal{F}_0$ contains all subsets negligible compared to P.*

We say that the filtration $(\mathcal{F}_t)$ satisfies the usual conditions if it is complete and continuous from the right.

Definition 1.2.10 *Let $(\mathcal{F}_t)$ the filtration on $(\Omega, \mathcal{F}, P)$. we call stopping time compared to $(\mathcal{F}_t)$ or $\mathcal{F}_t-$stopping time any random variable $T : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that for all $t \in \mathbb{R}_+$, the set $\{T \leq t\} \in \mathcal{F}_t$.*

Definition 1.2.11 *(Predictable process) we call the predictable $\sigma-$algebra on $\Omega \times \mathbb{R}_+$, the smallest $\sigma-$algebra on $\Omega \times \mathbb{R}_+$ which makes measurable the processes which are continuous from the left. A process is predictable if it is measurable with respect to the predictable $\sigma\_$ algebra.*

Definition 1.2.12 *(optional processes)We call the optional $\sigma-$algebra on $\Omega \times \mathbb{R}_+$, the smallest $\sigma-$algebra on $\Omega \times \mathbb{R}_+$making measurable the processes which are continuous from the right and limited from the left (càdlàg). A process is optional if it is measurable with respect to the optional $\sigma-$algebra.*

1.3 Brownian motion

1.3.1 Brief history

The discovery of Brownian motion is attributed to the Scottish botanist, Robert Brown in 1827, after a study of the irregular motion of pollen grains suspended in water. Now,It is known that this movement is the result of the cumulative effects of the shock of water molecules with the particle in question. Also, in the early twentieth century A. Einstein met the Brownian movement, when determining the diameters of some molecules. Bachelier also studied some properties of Brownian motion, in relation with the modeling of price changes of certain shares on the stock exchange, in particular the Markov property.

He discovered the link of Brownian motion with the heat equation. The rigorous mathematical definition and existence of this process, was established by Norbert Wiener in 1923. In particular he constructed a probability measure on the space of trajectories under which the process of coordinates is the Wiener process or Brownian motion.
In1933, Paul Lévy proved that the Brownian motion is a continuous martingale whose quadratic variation is equal to t. It gives a useful characterization in practice in terms of martingales.
Brownian motion is the most popular process and is of very deep interest in many branches of mathematics: the theory of Markov processes, martingale theory, potential theory, partial differential equations, complex analysis etc...
Today Brownian motion is very used by practitioners of finance, engineering sciences, biology and economics.

1.3.2 Definitions

Definition 1.3.1 *A Brownian motion B defined on a probability space equipped with a filtration $(\mathcal{F}_t)$ is a continous $\mathcal{F}_t-$adapted process satisfying:*
1)$B_0 = 0$,P.p.s
2)For all $0 \leq s \leq t$, the random variable $B_t - B_s$ is independent from $\mathcal{F}_s$.
3)For all $0 \leq s \leq t$, $B_t - B_s$ is a gaussian random variable such that:
$E(B_t - B_s) = 0$ and $var(B_t - B_s) = t - s$

Definition 1.3.2 ***(Generalization)*** *X is called a generalized Brownian motion or a Brownien motion with drift μ if $X_t = x + \mu_t + \sigma B_t$ where B is a Brownin motion. The rando variable X_t is gaussian with mean $x + \mu_t$ and variance $\sigma^2 t$.*

1.3.3 Properties of the Brownian motion

Proposition 1.3.1 *1) The Brownian motion is **a** homogeneous Markov process with gaussian transition probabilities:*
$P_t(x, dy) = \frac{1}{\sqrt{2\pi t}} \exp -(\frac{(x-y)^2}{2t})dy$
2)The symetry property:$\{-B_t\}_{t\geq 0}$ is a Brownian motion.
3)The scaling property: pour tout $c > 0, \{cB_{t/c^2}\}_{t\geq 0}$ is a Brownian motion.

Theoreme 1.3.1 ***(Regularity)***

Theoreme 1.3.2 *1)The Brownian motion is of unbounded variation in every interval.*
2) The Brownian motion is nowhere differentiable (Paley,Wiener,Zygmud, 1933).
3) The trajectories of the Brownian motion are locally Hölder continuous of order α*,with* $\alpha < 1/2$*, and are not locally Hôlder for:*$\alpha \geq 1/2$.

Brownian motion and martingales

Theoreme 1.3.3 ***(Paul Levy)*** *Let* $(X_t)_{t\geq 0}$ *a continuous process, then* $(X_t)_{t\geq 0}$ *is a Brownian motion if and only if:*
i)(X_t) *is a martingale.*
ii)$(X_t^2 - t)$ *is a martingale.*

The following inequality is very useful in the study of stochastic differential equations and to prove boundness of martingales.

Theoreme 1.3.4 ***Burkholder–Davis–Gundy inequality***
Let $p \in]0,\infty[$*,there exist two constants* c_p *et* C_p *suach that, for every continuous local martingale* X*, nul at* 0*, we have:*

$$c_p E\left[\langle X,X\rangle_\infty^{p/2}\right] \leq E\left[\sup_{t\geq 0}|X_t|^p\right] \leq C_p E\left[\langle X,X\rangle_\infty^{p/2}\right]$$

For every $T>0$

$$c_p E\left[\langle X,X\rangle_T^{p/2}\right] \leq E\left[\sup_{0\leq t\leq T}|X_t|^p\right] \leq C_p E\left[\langle X,X\rangle_T^{p/2}\right]$$

Ito's representation theorem for Brownian martingales

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ be a probability space satisfying thge usual conditions, B a Brownian motionon this space. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of B that is $\mathcal{F}_t = \sigma\left(B_s; s\leq t\right)$.

Theoreme 1.3.5 ***(Itô)*** *Let* $M \in \mathcal{M}^2\left[0,T\right] \left(resp.\mathcal{M}^{2loc}\left[0,T\right]\right)$*, then there exist a unique* *process* $Z \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)(resp.L^{2loc}_{\mathcal{F}}(0,T;\mathbb{R}^m))$ *such that:*

$$M(t) = \int_0^t \langle Z(s).dB(s) \rangle , \qquad \forall t \in [0,T], \; P - a.s.$$

Proof. See [29] Page(80-83)] ■

1.4 Stochastic differential equations

The stochastic differential equations are extensions of ordinary differential equations taking into account the random perturbations. This theory has been introduced by K. Itô, to give a pathwise representation of diffusion processes. It allows one to study random trajectories and to treat certain problems coming the theory of partial differential equations, both theoritically and numerically.
A stochastic differential equation (SDE) is defined as follows:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB_t$$

This equation has a sense, if we write it in integral form, taking into account that the first integral is a Lebesgue integral, while the second integral is an Itô stochastic integral, with respect to a Brownian motion (B_t):

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s,)dB_s$$

We can allow the coefficients b and σ to depend also on the time t. Then we have the following SDE

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{1.1}$$

1.4.1 The Itô existence and uniqueness theorem

Definition 1.4.1 ***Theoreme 1.4.1*** *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ *a filtered probability space satisfynig the usual conditions and let* σ *and* b *such that:*

$$b : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$
$$\sigma : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow M_{d\times m}(\mathbb{R})$$

are measurable functions such that:
1)There exist a constant k *such that for all* $t \in [0,T]$,*and* $x, y \in \mathbb{R}^d$

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq k\,|x-y|$$

Theoreme 1.4.2

$$|b(t,x)|^2 - \|\sigma(t,y)\|^2 \leq k\left|1+|x|^2\right| \qquad (1.5)$$

2)The initial condition X_0 *is independent of* $(B_t, t\geq 0)$ *and square inetgrable.*
Then the SDE 1.1 has a unique solution (X_t) *such that* $E(\sup\limits_{0\leq t\leq T} |X_t|^2) < \infty$

Theoreme 1.4.3 *Inégalités de Burkholder–Davis–Gundy*
Soit $p \in]0,\infty[$,*Il existe deux constantes* c_p *et* C_p *telles que, pour toute martingale locale continue* X, *nulle en zéro,*

$$c_p E\left[\langle X, X\rangle_\infty^{p/2}\right] \leq \sup_{t\geq 0} E\left[|X_t|^p\right] \leq C_p E\left[\langle X, X\rangle_\infty^{p/2}\right]$$

Pour $T > 0$

$$c_p E\left[\langle X, X\rangle_T^{p/2}\right] \leq \sup_{0\leq t\leq T} E\left[|X_t|^p\right] \leq C_p E\left[\langle X, X\rangle_T^{p/2}\right]$$

1.4.2 Strong and weak solutions of SDEs

Let us denote the SDE 1.1 by $E_x(\sigma, b)$

Strong solution

Definition 1.4.2 *A strong solution of SDE* $E_x(\sigma, b)$ *on a probability space* $(\Omega, \mathcal{F}, P)$ *with a Brownian motion* B ,*a process* X *defined on* $(\Omega, \mathcal{F}, P)$ *satisfying:*

i)X_t *is adapted to* $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$ *the natural filtration of* B.
ii)X *satisfies the SDE* $E_x(\sigma, b)$.
iii)X *est continous in* t *and* $P(\int_0^T (\sigma^2(s, X_s)ds < \infty) = 1, P(\int_0^T (b(s, X_s)ds < \infty) = 1$.

Weak solution

Definition 1.4.3 *A weak solution of* $E_x(\sigma, b)$,*is a collection* $((\Omega, \mathcal{F}, P), (\mathcal{F}_t), B, X)$ *such that:*
i)X *is adapted to* $\mathcal{F}_t$.
ii)B *is a Brownian motion adapted to* $\mathcal{F}_t$.
iii)$P(\int_0^T (\sigma^2(s, X_s)ds < \infty) = 1, P(\int_0^T (b(s, X_s)ds < \infty) = 1$.
iv)(B, X) *satisfies SDE* $E_x(\sigma, b)$.

Remark 1.4.1 *1)The weak solution are not necessarily adapted to* $\mathcal{F}_t^B$. *This is the fundamental difference with the strong solutions*
2)A stron g solution is a weak solution.

1.4.3 Strong and weak uniqueness

Strong or pathwise uniqueness

Definition 1.4.4 *Let* X ,X' *two solutions of* $E_x(\sigma, b, B), E_{x'}(\sigma, b, B')$ *respectively.We have strong uniqueness if:*
i)X ,X', B, B' *are defined on the same probability space.*
ii)if $X_0 = X'_0$ *P.a.s,* $B = B'$ *P.a.s,the :* $P(X_t = X'_t, t \in [0, T]) = 1$.
iii)X *et* X'*are indistinguishable.*

Weak uniqueness

Definition 1.4.5 *Let* (X, B) ,(X', B') *two solutions of* $E_x(\sigma, b)$, $E_{x'}(\sigma, b)$ *respectively. We say that we have weak uniqueness or uniqueness in law if:*

$$law(X, B) = law(X', B')$$

Remark 1.4.2 ***Theoreme 1.4.4*** *(Yamada-Watanabe) i)The pathwise uniqueness implies the uniqueness in law.*
ii)The weak existence and pathwise uniqueness imply strong existence and uniqueness.

Remark 1.4.3 *The Yamada-Watanabe is very useful in practice to prove strong existence and uniqueness. It is sufficient to prove that a solution exists in law and prove then the pathwise uniqueness.*

1.5 Linear stochastic differential equations and and backward stochastic differential equations

Due to their importance in stochastic control, we recall some of the main properties of linear SDEs and BSDEs.

1.5.1 Linear stochastic differential equations

The one domensional case

Consider the linear SDE

$$\begin{cases} dX_t = [A(t)X_t + b(t)]\,dt + [C(t)X_t + \sigma(t)]\,dB_t \\ X(0) = x \end{cases} \tag{1.2}$$

where:
1)B_t is a Brownian motion of dimension 1 .
2)$A(.), C(.) \in L^\infty [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$
3)$b(.), \sigma(.) \in L^\infty [0,T] \times \mathbb{R}^n$.
Using Itô's theorem, and under assumptions it is clear that equation 1.2 has a unique strong solution of the form:

$$X_t = \varphi_t x + \varphi_t \int_0^t \varphi_s^{-1} [b(s) - C(s)\sigma(s)]\, ds + \varphi_t \int_0^t \varphi_s^{-1} \sigma(s) dB_s; t \in [0,T] \tag{1.3}$$

Let φ_t be the solution of:

$$\begin{cases} d\varphi_t = A(t)\varphi_t dt + C(t)\varphi_t dB_s \\ \varphi_t(0) = I \end{cases} \tag{1.4}$$

then $\varphi_t(.)$ admits an inverse $\varphi_t^{-1} = \Psi_t$ satisfying:

$$\begin{cases} d\Psi_t = \Psi_t \left[-A(t)dt + C(t)^2\right] dt - \Psi_t C(t) dB_t \\ \Psi_t(0) = I \end{cases} \tag{1.5}$$

By applying Ito's formula to $\varphi(t)\Psi(t)$ we get $d[\varphi(t)\Psi(t)] = 0$, then $\varphi(t)\Psi(t) = I$.Therefore $\Psi(t) = \varphi^{-1}(t)$.

By applying again Ito's formula to $\Psi(t)X(t)$ where $X(t)$ is the solution of 1.2 we get:

$$\begin{aligned} d[\varphi(t)\Psi(t)] &= \Psi(t)dX(t) + X(t)d(t) + d\left\langle \Psi., X.\right\rangle_t \\ &= \Psi(t)\left[A(t)X(t) + b(t)\right] dt + \left[C(t)X(t) + \sigma(t)\right] dB(t) + X(t)\Psi(t)\left[-A(t)dt + C(t)^2\right] dt \\ &- \Psi(t)C(t)dB(t) - C^2(t)\Psi(t)X(t)dt + \sigma(t)C(t)\Psi(t)dt \\ &= \Psi(t)(b(t) - C(t)\sigma(t))dt + \Psi(t)\sigma(t)dB(t).X(t) \end{aligned}$$

Then the explicit formula 1.3 holds by using :$\Psi(t) = \varphi^{-1}(t)$.

The case of a multidimensional Brownian motion

Let X_t be the solution of the linear SDE

$$\begin{cases} dX_t = \left[A(t)X_t + b(t)\right] dt + \sum_{j=1}^{m} \left[C^j(t)X_t + \sigma^j(t)\right] dB_t^j \\ X(0) = x \end{cases} \tag{1.13}$$

Let φ_t be the solution of the matrix SDE

$$\begin{cases} d\varphi_t = A(t)\varphi_t dt + \sum_{j=1}^{m} C^j(t)\varphi_t dB_s^i \\ \varphi_t(0) = I \end{cases}$$

One can prove that the inverse φ_t^{-1} satisfies

$$\begin{cases} d(\varphi_t^{-1}) = \varphi_t^{-1}\left[-A(t)dt + \sum_{j=1}^{m} C^j(t)^2\right] dt - \sum_{j=1}^{m} \varphi_t^{-1} C^j(t) dB_t^i \\ \varphi_t^{-1}(0) = I \end{cases}$$

By using multidomensional Ito's formula we get

$$X_t = \varphi_t x + \varphi_t \int_0^t \varphi_s^{-1}\left[b(s) - \sum_{j=1}^{m} C^j(s)\sigma^j(s)\right] ds + \sum_{j=1}^{m} \varphi_t \int_0^t \varphi_s^{-1}\sigma^j(s) dB_s^j; t \in [0,T]$$

Remark 1.5.1 *The explicit formula has been obtained by J.M.Bismut.*

1.5.2 Backward stochastic differential equations

Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a filtered probability space and let ζ be a square integrable $\mathcal{F}_T-$measurable random variable.

Consider the differential equation

$$\begin{cases} -dY_t = f(Y_t), t \in [0,T] \\ Y_T = \zeta \end{cases}$$

where the process Y_t is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Suppose that $f \equiv 0$ then:

$$\frac{-dY_t}{dt} = 0$$

It is clear that the solution is given by:

$$Y_T = \zeta \Rightarrow Y_t = \zeta$$

But Y_t is not $\mathcal{F}_t-$measurable.

If we seach for an adapted solution we set:

$$Y_t = E(\zeta/\mathcal{F}_t) \text{ which } \mathcal{F}_t - \text{adapted}$$

From Ito's representation theorem ,there exist an adapted square integrable process Z such that:

$$Y_t = E(\zeta/\mathcal{F}_t) = E(\zeta) + \int_0^t Z_s dB_s \text{ , avec } Y_T = \zeta$$

We see that :$Y_T = \zeta = E(\zeta) + \int_0^T Z_s dB_s$.

By computing the difference $Y_t - Y_T$, we have:

$$Y_t - Y_T = \int_0^t Z_s dB_s - \int_0^T Z_s dB_s = -\int_t^T Z_s dB_s.$$

$$\Rightarrow Y_t = Y_T - \int_t^T Z_s dB_s.$$

Therefore

$$\begin{cases} dY_t = Z_t dB_t \\ Y_T = \zeta \end{cases}$$

In differntial form we have

$$\begin{cases} -dY_t = -Z_t dB_t \\ Y_T = \zeta \end{cases} \quad \text{i.e. } Y_t = \zeta - \int_t^T Z_s dB_s$$

Now let f be a function depending also on Z , such that:

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t$$
$$Y_T = \zeta$$

or in integral form:

$$\begin{cases} Y_t = \zeta + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dB_r \ , \ \ 0 \leq t \leq T \\ Y_T = \zeta \end{cases} \tag{1.6}$$

where f is called the driver of the backward SDE ,and ζ is the terminal value.

Notation and definition of the solution

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a probability space with a filtration, on which a Brownian motion B is defined.

We suppose that $\mathcal{F}_t$ is the natural filtration of B.

We denote $\mathcal{S}^2(\mathbb{R}^m)$ the space of processes Y progressively measurable, continuous and with values in $\mathbb{R}^m$ such that:

$$\|Y\|^2_{\mathcal{S}^2} = E\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty$$

Let $\mathcal{M}^2(\mathbb{R}^{m\times d})$ the space of processes Y progressively measurable, with values in $\mathbb{R}^{m\times d}$ such that:

$$\|Z\|^2_{\mathcal{M}^2} = E\left[\int_0^T \|Z_t\|^2\right] < +\infty$$

We denote $\mathcal{B}^2$ the Banach space :$\mathcal{S}^2_c(\mathbb{R}^m) \times \mathcal{M}^2(\mathbb{R}^{m\times d})$

We search for a couple $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ solution of equation 1.6.

Definition 1.5.1 *A solution of the BSDE 1.6 is a couple* $(Y, Z) = \{(Y_t, Z_t)\}_{t\in[0,T]}$ *of processes in* $\mathcal{S}^2_c(\mathbb{R}^m) \times \mathcal{M}^2(\mathbb{R}^{m\times d})$*,satisfying:*

1)$P-a.s. \int_0^T \left\{|f(r, Y_r, Z_r)| + \|Z_r\|^2\right\} dr < \infty.$

2)$P-a.s.$*:*

$$Y_t = \zeta + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dw_r \ , \ \ 0 \leq t \leq T$$

The Pardoux Peng theorem

Let ζ be a random variable,$\mathcal{F}_t-$adapted with values in $\mathbb{R}^m$,and let f be a driver defined on $[0,T]\times\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$,with values in $\mathbb{R}^m$,such that for any $(y,z)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$,the process f is progressively measurable .

Theoreme 1.5.1 *(Pardoux-Peng).Under the following assumptions,:*
1) f is uniformly Lipschitz in y et z ,i.e for all t,y,y',z,z' :

$$|f(t,y,z)-f(t,y',z')|\leq k\left(|y-y'|+\|z-z'\|\right)$$

2)

$$E\left[|\zeta|^2+|f(t,0,0)|^2\,dt\right]<+\infty$$

the BSDE 1.6 admits a unique solution $(Y,Z)\in\mathcal{S}_c^2(\mathbb{R}^m)\times\mathcal{M}^2(\mathbb{R}^{m\times d})$.

Chapter 2

Existence of optimal controls for mean-field stochastic differential equations: the case where only the drift is controlled

Abstract

In this paper we study the existence of an optimal control for systems, governed by stochastic differential equations of mean-field type. In these equations, the drift and the diffusion coefficient depend not only on the state of the system, but also on the expectation of some function of the state. These equations are obtained as limits of some interacting particle systems and are important in game theory with a large number of small players. For non linear systems, we prove the existence of an optimal relaxed control, by using tightness techniques and Skorokhod selection theorem. The optimal control is a measure valued process defined on another probability space. In the case where the coefficients are linear maps and the cost functions are convex, we prove by using weak convergence techniques, the existence of an optimal strict control, adapted to the initial filtration.

Keys words. Mean-field, stochastic differential equation, relaxed control, existence, tightness,

weak convergence.

MSC 2010 subject classifications. 93E20, 60H30.

2.1 Introduction

The purpose of this paper is to study the problem of existence of an optimal control, for a system governed by a stochastic differential equation of the mean-field type, (MFSDE in short), taking the form:

$$\begin{cases} dX_t = b(t, X_t, E(\psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t)), u_t)dW_t \\ X_0 = x. \end{cases} \tag{2.1}$$

$(W_t, t \geq 0)$ is a d-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, x is the initial state and u_t stands for the control variable. σ, b , ψ, ϕ are deterministic maps.

The expected cost on the time interval $[0, T]$ is of mean field type also and is given by

$$J(u) \quad = \quad E\left(\int_0^T h(t, X_t, E\varphi(X_t), u_t)dt + g(X_T, E\lambda(X_T))\right). \tag{2.2}$$

In the state equation and the cost functional, the functions depend not only on the state of the system, but also on the distribution of the state process, via the expectation of some function of the state. MFSDEs are obtained as mean square limits of interacting particle systems of the form:

$$dX_t^{i,n} = b(t, X_t^{i,n}, 1/n\sum_{j=1}^n \psi(X_t^{i,n}), u_t)dt + \sigma(t, X_t^{i,n}, 1/n\sum_{j=1}^n \Phi(X_t^{i,n}), u_t)dW_t \tag{2.3}$$

When n goes to infinity, it is proved in [43] , in the linear case, that $X_t^{i,n}$ converges to $\overline{X}_t^i$, where all the processes $\overline{X}_t^i$ $(i = 1, ...)$, are independent copies of the same process, called the non linear process or the McKean-Vlasov process, which is the unique solution of the MFSDE (2.1). We refer to [31], to the general case of a non linear dependence of the coefficients upon the process and its distribution and the driving process is a general Lévy process.

Motivated by a recent interest in differential games, control problems where the state process is a MFSDE, where the coefficients depend on the marginal probability law of the solution, have been studied in [1] and provide interesting models in applications, in particular to game problems [15, 34]. A typical example is the continuous-time Markowitz's mean-variance portfolio selection problem, where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term, see [2, 19, 47, 51]. The main drawback, when dealing with mean field stochastic control problems, is that the state process is not a Markov process and as a consequence, the dynamic programming principle is no longer valid. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see [2, 11, 17, 35, 38, 46, 47]. The SMP gives necessary optimality conditions in terms of the maximization of some hamiltonian and an adjoint process which is the solution of a backward SDE of mean field type, see [12, 14].

In this paper, we are interested by the existence of an optimal control, where the state equation, as well as the cost function are of mean field type. This kind of result is interesting in itself and particularly when one deals with the stochastic maximum principle. So, it is interesting to know if an optimal control exists and to try to characterize it, by deriving necessary conditions. A control u^* is called optimal if it satifies $J(u^*) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$, where $\mathcal{U}_{ad}$ is the space of admissible controls, that is measurable, adapted processes with values in some action space A. If moreover, u^* is in $\mathcal{U}_{ad}$, it is called strict.

For classical control problems, driven by classical SDEs without the mean field part, existence of such a strict optimal control follows from the Filipov-type convexity condition. In the absence of this condition, a strict optimal control may fail to exist. The idea is then to introduce the class of relaxed controls, in which the controller chooses at time t, a probability measure $\mu_t(da)$ on the action space A, rather than an element $u_t \in A$. The set of relaxed controls, when equipped with stable convergence, is a compact separable metrizable space. Note that the class of strict controls could be seen as a subset of the space of relaxed controls, by identifying a strict control (u_t) with the Dirac measure $\delta_{u_t}(da)$. The first existence of an optimal relaxed control has been proved in [23], for classical Ito SDEs, where only the drift is controlled. The case of an SDE where the diffusion coefficient depends explicitly on the control variable has been solved in [21, 25], where the optimal relaxed control is shown to be Markovian, see also [26, ?, 3]. Existence results for systems driven by backward and forward-backward SDEs have been investigated in [4, 5, 13].

We establish two main results. We first show the existence of an optimal relaxed control, for control problems driven by non linear MFSDEs. The proof is based on tightness properties of the underlying processes and Skorokhod selection theorem. Our results extend in particular those in [21, 26, 3], for mean field SDEs. Moreover, due to the compactness of the action space, we show that the relaxed control could be choosen among the so-called sliding controls, which are convex combinations of Dirac measures. As a consequence and under some Fillipov convexity condition, the relaxed control is shown to be strict.The second main result is an existence result for control problems driven by linear MFSDEs. For this particular class of problems, we prove the existence of a strict strong optimal control, that is a control process in $\mathcal{U}_{ad}$, which is adapted to the initial filtration. This means that the admissible controls in this case are adapted to a fixed filtration. The method of proof is based essentially on weak convergence techniques on the space $L^2_{\mathcal{F}}$ of square integrable processes and Mazur's theorem on the equality of the strong and weak closure of a convex set. It should be noted that for this case, there is no need to use tightness techniques and to change the initial probability space. The reason is that roughly speaking, in the linear case, the problem reduces to finite dimensional techniques. In particular, our result extends [49], Theorem 5.2, to mean-field control problems.

2.2 Existence of optimal relaxed controls for systems driven by non linear MFSDEs

2.2.1 Controlled mean field stochastic differential equations

Let (W_t) is a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Let A be some compact subset of $\mathbb{R}^k$ called the action space or the control set.

We study the existence of optimal controls for systems driven non linear mean field SDEs of the form

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t))dW_t \\ X_0 = x \end{cases} \tag{2.4}$$

and the cost functional over the time interval $[0, T]$ is given by

$$J(U) = E\left(\int_0^T h(t, X_t, E(\varphi(X_t), u_t\right) dt + g(X_T, E\lambda(X_T)) \tag{2.5}$$

where b, σ, l, h, g and ψ are given functions. The control variable u_t, is a measurable, $\mathcal{F}_t-$ adapted process with values in the action space A.

Let us assume the following conditions:

$(\mathbf{H_1})$ Assume that

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}^d \\ \sigma &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\ \Psi &: \mathbb{R}^d \longrightarrow \mathbb{R}^d, \Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d \end{aligned} \tag{2.6}$$

are bounded continuous functions and there exists $K > 0$ such that for any pairs (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned} &|b(t, x_1, y_1, u) - b(t, x_2, y_2, u)| \leq K(|x_1 - x_2| + |y_1 - y_2|) \\ &|\sigma(t, x_1, y_1, u) - \sigma(t, x_2, y_2, u)| \leq K(|x_1 - x_2| + |y_1 - y_2|) \\ &|\Psi(x_1) - \Psi(x_2)| \leq K(|x_1 - x_2|) \\ &|\Phi(x_1) - \Phi(x_2)| \leq K(|x_1 - x_2|) \end{aligned} \tag{2.7}$$

$(\mathbf{H_2})$ Assume that

$$\begin{aligned} &h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times A \longrightarrow \mathbb{R} \\ &g : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ &\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d \\ &\lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d \end{aligned}$$

are bounded continuous functions and h is K-Lipschiz continuous in the variables (x, y), that is there exists $K > 0$ such that for any pairs (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^d \times \mathbb{R}^d$:

$$|h(t, x_1, y_1, u) - h(t, x_2, y_2, u)| \leq K(|x_1 - x_2| + |y_1 - y_2|) \tag{2.8}$$

Proposition 2.2.1 *Under assumption* $(\mathbf{H_1})$ *the MFSDE (2.4) has a unique strong solution. Moreover for each* $p > 0$ *we have* $E(|X_t|^p) < +\infty$.

Proof. Let us define $\overline{b}(t,x,\mu,a)$ on $[0,T]\times\mathbb{R}^d\times\mathbb{M}_1(\mathbb{R}^d)\times\mathbb{R}^k$ and $\overline{\sigma}(t,x,\mu,a)$ on $[0,T]\times\mathbb{R}^d\times\mathbb{M}_1(\mathbb{R}^d)$ by

$$\begin{aligned}\overline{b}(t,x,\mu,a) &= b(.,.,\int\Psi(x)d\mu(x),.)\\ \overline{\sigma}(t,x,\mu) &= \sigma(t,x,\int\Phi(x)d\mu(x))\end{aligned} \tag{2.9}$$

where $\mathbb{M}_1(\mathbb{R}^d)$ denotes the space of probability measures in $\mathbb{R}^d$.
According to Proposition 1.2 in [31] it is sufficient to check that $\overline{b}$ and $\overline{\sigma}$ are Lipschitz in (x,μ). Indeed since the coefficients b and σ are Lipschitz continuous in x, then $\overline{b}$ and $\overline{\sigma}$ are also Lipschitz in x. Moreover one can verify easily that $\overline{b}$ and $\overline{\sigma}$ are also Lipshitz continuous in μ, with respect to the Wasserstein metric

$$\begin{aligned}&d(\mu,\nu)=\inf\left\{\left(E^Q\left|X-Y\right|^2\right)^{1/2};Q\in\mathbb{M}_1(\mathbb{R}^d\times\mathbb{R}^d),\text{ with marginals }\mu,\nu\right\}\\ &=\sup\left\{\int hd\left(\mu-\nu\right);\ \ |h(x)-h(y)|\leq|x-y|\right\},\end{aligned} \tag{2.10}$$

where $\mathbb{M}_1(\mathbb{R}^d\times\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d\times\mathbb{R}^d$. Note that the second equality is given by the Kantorovich-Rubinstein theorem [33]. Since the mappings b and Ψ in the the MFSDE are Lipschitz continuous in x we have

$$\begin{aligned}&\left|b(.,.,\int\Psi(x)d\mu(x),.)-b(.,.,\int\Psi(x)d\nu(x),.)\right|\\ &\leq K\left|\int\Psi(x)d(\mu(x)-\nu(x))\right|\\ &\leq K'.d\left(\mu,\nu\right)\end{aligned} \tag{2.11}$$

Similar arguments can be used for σ. Using similar techniques as in Proposition 1.2 in [31], it holds that for each $p>0$, $E(|X_t|^p)<+\infty$. ■

2.2.2 Relaxed controls

Our objective is to minimize the cost function, over the class $\mathcal{U}_{ad}$ of admissible controls, that is, adapted processes with values in the set A, called the action space. A control $\widehat{u}$ is called optimal if it satisfies $J\left(\widehat{u}\right)=\inf\left\{J\left(u\right),u\in\mathcal{U}_{ad}\right\}$.
If we do not assume convexity conditions, an optimal control may fail to exist in the

set $\mathcal{U}_{ad}$ of strict controls even in deterministic control. It should be noted that the set $\mathcal{U}_{ad}$ is not equipped with a compact topology. The idea is then to embedd the set of strict controls into a wider class of controls, in which the controller chooses at time t, a probability measure $\mu_t(du)$ on the control set A, rather than an element $u_t \in A$. These measure valued controls are called relaxed controls. It turns out that this class of controls enjoys good topological properties. If $\mu(du) = \delta_{u_t}(du)$ is a Dirac measure charging u_t for each t, then we get a strict control as a special case. Thus the set of strict controls may be identified as a subset of of relaxed controls.

Let us consider a simple deterministic example.

The problem is to minimize the following cost function: $J(u) = \int_0^T (X^u(t))^2 \, dt$ over the set $\mathcal{U}_{ad}$ of open loop controls, that is, measurable functions $u : [0,T] \to \{-1,1\}$, where $X^u(t)$ denotes the solution of $dX^u(t) = u(t)dt$, $X(0) = 0$. We have $\inf_{u\in\mathcal{U}_{ad}} J(u) = 0$. Indeed, consider the following sequence of controls:

$$u_n(t) = (-1)^k \text{ if } \tfrac{kT}{n} \leq t \leq \tfrac{(k+1)T}{n},\ 0 \leq k \leq n-1.$$

Then clearly $|X^{u_n}(t)| \leq 1/n$ and $|J(u_n)| \leq T/n^2$ which implies that $\inf_{u\in\mathcal{U}_{ad}} J(u) = 0$. There is however no control $\widehat{u}$ such that $J(\widehat{u}) = 0$. If this would have been the case, then for every t, $X^{\widehat{u}}(t) = 0$. This in turn would imply that $u_t = 0$, which is impossible.

The problem is that the sequence (u_n) has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality. If we identify $u_n(t)$ with the Dirac measure $\delta_{u_n(t)}(du)$, then $(\mu_t^n(du))_n$ converges weakly to $(T/2)\cdot[\delta_{-1} + \delta_1](du)$. This suggests that the set of strict controls is too narrow and should be embedded into a wider class with a reacher topological structure for which the control problem becomes solvable. The idea of relaxed control is to replace the A-valued process (u_t) with a $\mathbb{M}_1(A)$-valued process (μ_t), where $\mathbb{M}_1(A)$ is the space of probability measures equipped with the topology of weak convergence.

In the relaxed form of our control problem we replace in the state equation the process u_t by μ_t which a process with values on the space of proability measures on the control set A. Then the state process will satisfy, instead of (2.4), the following equation

$$\begin{cases} X_t = x + \int_0^t \int_A b\left(s, X_s, E(\Psi(X_s), a\right) \mu_s(da).ds + \int_0^t \sigma\left(s, X_s, E(\Phi(X_s)\right) dW_s, \\ X_0 = x \end{cases} \tag{2.12}$$

μ_t is called a relaxed control applied at time t. If μ_t is a Dirac measure concentrated at a single point u_t then we get a strict control as a particular case of a relaxed control.

The canonical space of the set of relaxed controls

Let $\mathbb{M}_1(A)$ be the space of probability measures on the control set A. Let $\mathbb{V}$ be the space of measurable tranformations $\mu : [0,T] \longrightarrow \mathbb{M}_1(A)$, then μ can be identified as a nonnegative measure on the product $[0,T] \times A$, by putting for $C \in \mathcal{B}([0,T])$ and $D \in \mathcal{B}(A)$

$$\overline{\mu}(C \times D) = \int_C \mu_t(da)dt$$

$\overline{\mu}$ can be extended uniquely to an element of $\mathbb{M}_+([0,T] \times A)$ the space of Radon measures on $[0,T] \times A$, equipped with the topology of stable convergence. This topology is the weakest topology such that the mapping

$$\overline{\mu} \longrightarrow \int_0^T \int_A \phi(t,a).\overline{\mu}(dt,da)$$

is continuous for all bounded measurable functions ϕ which are continuous in a.

Equipped with this topology, $\mathbb{M}_+([0,T] \times A)$ is a compact separable metrizable space. Therefore $\mathbb{V}$ as a closed subspace of $\mathbb{M}_+([0,T] \times A)$ is also compact (see El Karoui, Hauss-Lepeltier, [30]) for more details.

Notice that $\mathbb{V}$ can be identified as the space of positive Radon measures on $[0,T] \times A$, whose projections on $[0,T]$ coincide with Lebesgue measure.

Let us define the Borel $\sigma-$field $\overline{\mathbb{V}}$ as the smallest $\sigma-$field such that the mappings

$$\int_0^T \int_A \phi(t,u).\mu_t(du)dt$$

are measurable, where ϕ is a bounded measurable function which is continuous in a.

Let us also introduce the filtration $\left(\overline{\mathbb{V}}_t\right)$ on $\mathbb{V}$, where $\overline{\mathbb{V}}_t$ is generated by $\left\{1_{[0,t]}\mu,\ \mu \in \mathbb{V}\right\}$.

Definition 2.2.1 *A measure-valued control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable μ with values in $\mathbb{V}$ such that $\mu(\omega, t, da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.*

Now let us introduce the precise definitions of a strict control and relaxed control.

Definition 2.2.2 *A strict control is a term* $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, u_t, W_t, X_t)$ *such that*
(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is a probability space equipped with a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfying the usual conditions.*
(2) u_t *is a A-valued process, progressively measurable with respect to* $(\mathcal{F}_t)$*.*
(3) W_t *is a* $(\mathcal{F}_t, P)$*- Brownian motion and* (W_t, X_t) *satisfies MFSDE (2.4).*
We denote by $\mathcal{U}_{ad}$ *the space of strict controls.*

The controls as defined in the last definition are called weak controls, because of the possible change of the probability space and the Brownian motion with u_t.

Definition 2.2.3 *A relaxed control is a term* $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, \mu_t, W_t, X_t)$ *such that*
(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is a probability space equipped with a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfying the usual conditions.*
(2) μ *is a measure-valued control on* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$*.*
(3) W_t *is a* $(\mathcal{F}_t, P)$*- Brownian motion and* (W_t, X_t) *satisfies the following MFSDE (2.12).*
We denote by $\mathcal{R}$ *the space of relaxed controls.*

Accordingly, the relaxed cost functional will be given by

$$J(\mu) = E\left(\int_0^T \int_A h(t, X_t, E(\varphi(X_t), a)\mu_t(da)dt + g(X_T, E\lambda(X_T)\right). \tag{2.13}$$

By putting $\widetilde{b}\,(t, X_t, E(\Psi(X_t), \mu_t) = \int_A b\,(t, X_t, E(\Psi(X_t), a)\,\mu_t(da)$, it follows that the new drift $\widetilde{b}$ satisfies the same Lipschitz assumptions $(\mathbf{H}_1)$ as b. Therefore Equation 2.12 has a unique solution such that for each $p > 0$ we have $E(|X_t|^p) < +\infty$.

Approximation of the relaxed model

By defining the relaxed control problem, a natural question arises on the relation between the strict control problem and the relaxed one. Thanks to the so-called chattering lemma and the continuity of the state process with respect to the control variable, one can prove that the two problems are equivalent. That is the value functions for the two problems are the same. In other words, the infimum of the cost function among strict controls is equal to the infimum of the cost function taken among relaxed controls.

Lemma 2.2.1 *(Chattering lemma)*

i) Let (μ_t) be a relaxed control. Then there exists a sequence of adapted processes $(u^n(t))$ with values in A, such that the sequence of random measures $\left(\delta_{u_t^n}(da)\, dt\right)$ converges in $\mathbb{V}$ to $\mu_t(da)\, dt$, $P-a.s.$

ii) For any g continuous in $[0,T]\times\mathbb{M}_1(A)$ such that $g(t,.)$ is linear, we have $P-a.s$

$$\lim_{n\to+\infty}\int_0^t g(s,\delta_{u_s^n})ds=\int_0^t g(s,\mu_s)ds \text{ uniformly in } t\in[0,T]\,. \tag{2.14}$$

Proof. See [21] and [23] Lemma 1 page 152. ■

Proposition 2.2.2 *1) Let X_t, X_t^n be the solutions of state equation (2.12) corresponding to μ and u^n, where μ and u^n are defined as in the last lemma. Then*

$$\lim_{n\to\infty}E\left[\sup_{0\le t\le T}\left|X_t^n-X_t\right|^2\right]=0. \tag{2.15}$$

2) Let $J(u^n)$ and $J(\mu)$ the expected costs corresponding respectively to u^n and μ. Then there exists a subsequence (u^{n_k}) of (u^n) such that $J(u^{n_k})$ converges to $J(\mu)$.

Proof. 1) Let μ a relaxed control and $\left(\delta_{u_t^n}(da)\right)$ the sequence of atomic measures associated to the sequence of strict controls (u^n), as in the last Lemma. Let X_t, X_t^n the corresponding state processes. Then

$$\begin{aligned}
|X_t-X_t^n| &\le \left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(du).ds-\int_0^t\int_A b\left(s,X_s^n,E(\Psi(X_s^n),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\left|\int_0^t\sigma\left(s,X_s,E(\Phi(X_t)\right)ds-\int_0^t\sigma\left(s,X_s^n,E(\Phi(X_s^n)\right)ds\right| \\
&\le \left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(du).ds-\int_0^t\int_A b\left(s,X_s,E(\Psi(X_s),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\delta_{u_s^n}(da).ds-\int_0^t\int_A b\left(s,X_s^n,E(\Psi(X_s^n),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\sup_{s\le t}\left|\int_0^s\sigma\left(v,X_v,E(\Phi(X_v)\right)dW_v-\int_0^t\sigma\left(v,X_v^n,E(\Phi(X_v^n)\right)dW_v\right|
\end{aligned}$$

Then by using Burkholder-Davis-Gundy inequality for the martingale part and the fact that all the functions in equation (2.12) are Lipschitz continuous, it holds that

$$E\left(\sup_{0\le t\le T}\left|X_t-X_t^n\right|^2\right)\le K\left[\int_0^T E\left(\sup_{0\le s\le t}\left|X_s-X_s^n\right|^2\right)dt+\varepsilon_n\right] \tag{2.16}$$

where K is a nonnegative constant and

$$\varepsilon_n = E\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(du).ds-\int_0^t\int_A b\left(s,X_s,E(\Psi(X_s),u\right)\delta_{u_s^n}(da)ds\right|\right) \tag{2.17}$$

By using Lemma 2.5 ii) and the dominated convergence theorem it holds that $\lim\limits_{n\to+\infty}\varepsilon_n = 0$. We conclude by using Gronwall lemma.

2) Property 1) implies that the sequence (X_t^n) converges to X_t in probability uniformly in t, then there exists a subsequence $(X_t^{n_k})$ which converges to X_t, P -$a.s$ uniformly in t. We have

$$\begin{aligned}|J(u^{n_k})-J(\mu)| &\leq E\left[\int_0^T\int_A|h(t,X_t^{n_k},E(\varphi(X_t^n),a)-\ h(t,X_t,E(\varphi(X_t),a)|\,\delta_{u_t^{n_k}}(da)\,dt\right]\\ &+E\left[\left|\int_0^T\int_A h(t,X_t,E(\varphi(X_t),a)\delta_{u_t^{n_k}}(da)\,dt-\int_0^T\int_A h(t,X_t,E(\varphi(X_t),a)\mu_t(da)\,dt\right|\right]\\ &+E\left[|g(X_T^{n_k},E(\lambda(X_T^{n_k}))-g(X_T,E(\lambda(X_T))|\right]\end{aligned}$$

It follows from the continuity and boundness of the functions h, g, φ and λ with respect to x and y, that the first and third terms in the right hand side converge to 0 . The second term in the right hand side tends to 0 by the weak convergence of the sequence μ^n to μ, the continuity and the boundness of h in the variable a. We use the dominated convergence theorem to conclude. ■

Remark 2.2.1 *As a consequence of Proposition 2.6, it holds that the value functions for the strict and relaxed control problems are the same.*

Notation

In the sequel we denote by:

$\mathcal{C}([0,T];\mathbb{R}^d)$: the space of continuous functions from $[0,T]$ into $\mathbb{R}^d$, equipped with the topology of uniform convergence.

2.2.3 The main result

The main result of this section is given by the following theorem. Note that this result extends [23, 21, 24] to systems driven by mean field SDEs with uncontrolled diffusion coefficient.

Theoreme 2.2.1 *Under assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *the relaxed control problem has an optimal solution.*

The proof is based on some auxiliary results related to the tightness of the processes under consideration and the identification of their limits.
Let $(\mu^n)_{n\geq 0}$ be a minimizing sequence, that is $\lim\limits_{n\to\infty} J(\mu^n) = \inf\limits_{q\in\mathcal{R}} J(\mu)$ and let (W^n, X^n) be the unique solution of our MFSDE:

$$\begin{cases} X_t^n = x + \int_0^t \int_A b\left(s, X_s^n, E(\Psi(X_s^n), u\right) \mu_s^n(du).ds + \int_0^t \sigma\left(s, X_s^n, E(\Phi(X_s^n)\right) dW_s^n, \\ X_0^n = x. \end{cases} \tag{2.18}$$

The proof of the main result consists in proving that the sequence of distributions of the processes (μ^n, W^n, X^n) is tight for a certain topology on the state space and then show that we can extract a subsequence which converges in law to a process $(\widehat{q}, \widehat{W}, \widehat{X})$, which satisfies the same MFSDE. To achieve the proof we show that under some regularity conditions the sequence of cost functionals $(J(\mu^n))_n$ converges to $J(\widehat{\mu})$ which is equal to $\inf\limits_{\mu\in\mathcal{R}} J(\mu)$ and then $(\widehat{q}, \widehat{W}, \widehat{X})$ is optimal.

Lemma 2.2.2 *The sequence of distributions of the relaxed controls* $(\mu^n)_n$ *is relatively compact in* $\mathbb{V}$.

Proof. The relaxed controls μ^n are random variables on the space $\mathbb{V}$ which is compact. Then by applying Prohorov's theorem yields that the family of distributions associated to $(\mu^n)_{n\geq 0}$ is tight then it is relatively compact. ■

Lemma 2.2.3 *Let* (W_t^n, X_t^n) *be the solution of the MFSDE (2.18), then the sequence* $P_{(W^n,X^n)}$ *of distributions of* (W^n, X^n) *is relatively compact on the space* $\mathcal{C}\left([0,T], \mathbb{R}^d\right) \times \mathcal{C}\left([0,T], \mathbb{R}^d\right)$, *where* $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$ *is endowed with the topology of uniform convergence.*

Proof. To prove that the sequence $\left(P_{(W^n,X^n)}\right)$ is relatively compact in $\mathcal{C}\left([0,T], \mathbb{R}^d\right) \times \mathcal{C}\left([0,T], \mathbb{R}^d\right)$ it is sufficient to prove that (P_{W^n}) and (P_{X^n}) are relatively compact in $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$. According to Kolmogorov's theorem [29] page 18, we need to verify that

a) $\lim_{A\to+\infty}\inf_n P^n\left(\|x(0)\|\leq A\right)=0$

b) $\lim_{\delta\to 0}\limsup P^n\left(\sup\limits_{\substack{0\leq s\leq t\leq T\\ t-s<\delta}}\|x(t)-x(s)\|\geq\gamma\right)=0$

Condition a) is an immediate consequence of the fact that $W^n(0)=0$ and $X^n(0)=x$.

To prove b) it is sufficient to chech that

$E(\|W^n(t)-W^n(s)\|^4)\leq C\,|t-s|^2$

$E(\|X^n(t)-X^n(s)\|^4)\leq C\,|t-s|^2$

for some constants C_1 and C_2 independant from n.

The first inequality is obvious. Let us verify the second one. We have

$E\left(\|X_t^n-X_s^n\|^4\right)\leq M.E\left(\left\|\int_s^t\int_A b\left(u,X_u^n,E(\Psi(X_u^n),a\right)\mu_s^n(da).ds\right\|^4+\left\|\int_s^t\sigma\left(u,X_u^n,E(\Phi(X_u^n)\right)dW_s^n\right\|^4\right)$ v

M is some positive constant. Using Burkholder-Davis-Gundy inequality to the martingale part and the fact that b and σ are bounded functions yield the desired result. ■

Proof. of Theorem 2.8

By using Lemmas 2.9 and 2.10, it holds that the sequence of processes (μ^n,W^n,X^n) is tight on the space $\mathbb{V}\times\mathcal{C}\left([0,T],\mathbb{R}^d\right)^2$. Then by the Skorokhod representation theorem, there exists a probability space $\left(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}}\right)$, a sequence $\overline{\gamma}^n=\left(\overline{\mu}^n,\overline{W}^n,\overline{X}^n\right)$ and $\overline{\gamma}=\left(\overline{q},\overline{W},\overline{X}\right)$ defined on this space such that:

(i) for each $n\in\mathbb{N}$, $\text{law}(\gamma^n)=\text{law}(\overline{\gamma}^n)$,

(ii) there exists a subsequence $(\overline{\gamma}^{n_k})$ of $(\overline{\gamma}^n)$, still denoted $(\overline{\gamma}^n)$, which converges to $\overline{\gamma}$, $\overline{\mathbb{P}}$-a.s. on the space Γ.

This means in particular that the sequence of relaxed controls $(\overline{\mu}^n)$ converges in the stable topology to $\overline{\mu}$, $\overline{\mathbb{P}}-a.s.$ and $\left(\overline{W}^n,\overline{X}^n\right)$ converges uniformly to $\left(\overline{W},\overline{X}\right)$, $\overline{\mathbb{P}}-a.s.$

According to property (i), we get

$$\begin{cases}\overline{X}_t^n=x+\int_0^t\int_A b\left(s,\overline{X}_s^n,E(\Psi(\overline{X}_s^n),u\right)\overline{\mu}_s^n(du)ds+\int_0^t\sigma\left(s,\overline{X}_s^n,E(\Phi(\overline{X}_s^n)\right)d\overline{W}_s^n,\\ \overline{X}_0^n=x.\end{cases}\tag{2.19}$$

The coefficients b, σ, Ψ and Φ being Lipschitz continuous in (x,y),then according to property (ii) and using similar arguments as in [42] page 32, it holds that

$\int_0^t\int_A b\left(s,\overline{X}_s^n,E(\Psi(\overline{X}_s^n),u\right)\overline{\mu}_s^n(du)ds$ converges in probability to $\int_0^t\int_K b\left(s,\overline{X}_s,E(\Psi(\overline{X}_s),u\right)\overline{\mu}_s(du)ds$

and

$$\int_0^t \sigma\left(s, \overline{X}_s^n, E(\Phi(\overline{X}_s^n)\right) d\overline{W}_s^n \text{ converges in probability to } \int_0^t \sigma\left(s, \overline{X}_s, E(\Phi(\overline{X}_s)\right) d\overline{W}_s.$$

Therefore $\overline{X}$ satisfies the MFSDE

$$\begin{cases} \overline{X}_t = x + \int_0^t \int_K b\left(s, \overline{X}_s, E(\Psi(\overline{X}_s), u\right) \overline{\mu}_s(du)ds + \int_0^t \sigma\left(s, \overline{X}_s, E(\Phi(\overline{X}_s)\right) d\overline{W}_s, \\ \overline{X}_0^n = x. \end{cases} \tag{2.20}$$

To finish the proof of Theorem 2.8, it remains to verify that $\overline{\mu}$ is an optimal control. According to above properties (i)-(ii) and assumption $(\mathbf{H}_2)$, we have

$$\begin{aligned} \inf_{\mu \in \mathcal{R}} J(\mu) &= \lim_{n\to\infty} J(\mu^n), \\ &= \lim_{n\to\infty} E\left[\int_0^T \int_A h(t, X_t^n, E(\varphi(X_t^n), a)\mu_t^n(da)dt + g(X_T^n, E\lambda(X_T^n))\right] \\ &= \lim_{n\to\infty} \overline{E}\left[\int_0^T \int_A h(t, \overline{X}_t^n, E(\varphi(\overline{X}_t^n), a)\overline{\mu}_t^n(da)dt + g(\overline{X}_T^n, E\lambda(\overline{X}_T^n))\right] \\ &= \overline{E}\left[\int_0^T \int_A h(t, \overline{X}_t, E(\varphi(\overline{X}_t), a)\overline{\mu}_t(da)dt + g(\overline{X}_T, E\lambda(\overline{X}_T))\right]. \end{aligned}$$

Hence $\overline{\mu}$ is an optimal control. ■

The action space A being compact, we prove in the next proposition that the investigation for an optimal relaxed control can be reduced to the so called sliding controls also known as chattering controls. A sliding control is a relaxed control of the form

$$q_t = \sum_{i=1}^{p} \alpha_i(t)\delta_{u_i(t)}(da), u_i(t) \in A, \alpha_i(t) \geq 0 and \sum_{i=1}^{p} \alpha_i(t) = 1. \tag{2.21}$$

Proposition 2.2.3 *Let μ be a relaxed control and X the corresponding state process. Then one can choose a sliding control*

$$\nu_t = \sum_{i=1}^{p} \alpha_i(t)\delta_{u_i(t)}(da),\ u_i(t) \in A,\ \alpha_i(t) \geq 0 \textit{ and } \sum_{i=1}^{p} \alpha_i(t) = 1 \tag{2.22}$$

such that

1) X is a solution of the controlled MFSDE

$$\begin{cases} dX_t &= \displaystyle\sum_{i=1}^{p}\alpha_i(t)b(t,X_t,E(\Psi(X_t)),u_i(t))dt+\sigma(t,X_t,E(\Phi(X_t))dW_t \\ X_0 &= x \end{cases} \tag{2.23}$$

2) $J(\mu)=J(\nu)$.

Proof. Let Λ denote the $d+1$-dimensional simplex $\Lambda=\left\{\lambda=(\lambda_0,\lambda_1,...,\lambda_{d+1})\,;\ \lambda_i\geq 0;\ \sum_{i=0}^{d+1}\lambda_i=1\right\}$ and W the $(d+2)$-cartesian product of the set A

$W=\{w=(u_0,u_1,...,u_{d+1})\,;\ u_i\in A\}$

Define the function

$g(t,\lambda,w)=\sum_{i=0}^{d+1}\lambda_i\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)-\int_A\widetilde{b}\,(t,X_t,E(\Psi(X_t),u)\,\mu_t(du)$

where $t\in[0,T]\,,\lambda\in\Lambda,w\in W$ and $\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)=\begin{pmatrix}b(t,X_t,E(\Psi(X_t),u_i)\\ h(t,x_t,E(\Psi(X_t),u_i)\end{pmatrix}$

Let $\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)$, $i=0,1,...,d+1$ be the subset of $(d+1)$ arbitrary points in $P(t,X_t)$ where

$P(t,X_t)=\{(b(t,X_t,E(\Psi(X_t),a),h(t,X_t,E(\Psi(X_t),a))\,;\ a\in A\}\subset\mathbb{R}^{d+1}$

Then the convex hull of this set is the collection of all points of the form

$\sum_{i=0}^{d+1}\lambda_i\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)$

If μ is a relaxed control, then $\int_A\widetilde{b}\,(t,X_t,E(\Psi(X_t),a)\,\mu_t(da)\in Conv\,(P(t,X_t))$, the convex hull of $P(t,X_t)$. Therefore it follows from Carathéodory's Lemma (which says that the convex hull of a d-dimensional set M coincides with the union of the convex hulls of $d+1$ points of M), that for each $(w,t)\in\Omega\times[0,T]$ the equation $g(t,\lambda,w)=0$ admits at least one solution. Moreover the set

$\left\{(\omega,\lambda,w)\in\Omega\times\Lambda\times W:\sum_{i=0}^{d+1}\lambda_i\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)=\int_A\widetilde{b}\,(t,x_t,E(\Psi(x_t),a)\,\mu_t(da)\right\}$

is measurable with respect to $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^{d+1})\otimes\mathcal{B}(A^{d+1})$ with non empty $\omega-$sections for each ω.

Hence by using a selection theorem [21], there exist measurable $\mathcal{F}_t-$adapted processes λ_t and w_t with values, respectively in Λ and W such that:

$$\int_A \widetilde{b}\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(du) = \sum_{i=0}^{d+1} \lambda_i(t)\widetilde{b}(t, x_t, u_i(t))$$

This implies in particular that

$$\int_A b\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(du) = \sum_{i=0}^{d+1} \lambda_i(t) b(t, X_t, E(\Psi(X_t), u_i(t))$$

$$\int_A h\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(du) = \sum_{i=0}^{d+1} \lambda_i(t) h(t, X_t, E(\Psi(X_t), u_i(t))$$

which ends the proof. ■

The next corollary is important in applications. It says that under the so-called Fillipov condition an optimal strict control exists.

Corollary 2.2.1 *Assume that the set*

$$P(t, X_t) = \{(b(t, X_t, E(\Psi(X_t), a), h(t, X_t, E(\Psi(X_t), a))\,;\ a \in A\} \subset \mathbb{R}^{d+1}$$

is convex. Then the relaxed optimal control is realized by a strict control.

Proof. The proof is a direct consequence of Proposition 2.11. Indeed by mimicking the proof of Proposition 2.11, it follows that for each relaxed control μ we have

$$\int_A \widetilde{b}\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(da) \in Conv\,(P(t, X_t))$$

Since $P(t, X_t)$ is convex then $Conv\,(P(t, X_t)) = P(t, X_t)$.

Then applying the same arguments, there exists a measurable $\mathcal{F}_t-$adapted process $u(t)$ such that

$$\int_A b\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(du) = b(t, X_t, u(t))$$

$$\int_A h\,(t, X_t, E(\Psi(X_t), a)\, \mu_t(du) = h(t, X_t, u(t))$$

which implies that X_t is a solution of the MFSDE

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u(t))dt + \sigma(t, X_t, E(\Phi(X_t))dW_t \\ X_0 = x \end{cases}$$

and $J(\mu) = J(u)$. This ends the proof. ■

2.3 Existence of an optimal strong control for linear SDEs

2.3.1 Formulation of the problem

In this section, we assume that the coefficients of our mean field SDE are linear, while the running and final costs remain non linear. Moreover we assume convexity of the instantaneous and terminal cost functions, as well as the action space A. We prove the existence of an optimal strong control, that is a control which is adapted to the initial filtration. Note that for this kind of problems there is no need to use tightness techniques and Skorokhod selection theorem. The techniques used are based on weak convergence techniques in $L^2_{\mathcal{F}}\left(0,T,\mathbb{R}^k\right)$ and Mazur's theorem.

Definition 2.3.1 *Let A be a subset in $\mathbb{R}^d$ called the action space. An admissible control is a measurable, $\mathcal{F}_t-$adapted process with values on the action space A such that:*

$$E\left[\int_0^T |u_t|^2 \, dt\right] < +\infty \tag{2.24}$$

Let us denote $\mathcal{U}_{ad}$ the space of all admissible controls which can be written as:

$$\mathcal{U}_{ad} \triangleq \left\{u \in L^2_{\mathcal{F}}\left(0,T,\mathbb{R}^d\right) \;/\; u(t) \in A, dt- \text{ a.e. } P-a.s\right\}. \tag{2.25}$$

where $L^2_{\mathcal{F}}\left(0,T,\mathbb{R}^d\right)$ is the space of measurable $\mathcal{F}_t-$adapted processes with values in $\mathbb{R}^d$.

Assume that for any ablissible control u, the state of our system is driven by the following linear MFSDE

$$\begin{cases} dX_t = (A.X_t dt + B.E(X_t) + C.u(t))\, dt + (A_1.X_t + B_1.E(X_t) + C_1.u(t))\, dW_t \\ X_0 = x_0 \end{cases} \tag{2.26}$$

where

- A, B, A_1, B_1 are $d \times d$ matrices.
- C, C_1 are $d \times d$ matrices.

The cost functional is given as follows:

$$J(u) = E\left[\int_0^T h(t, X_t, E(X_t), u_t)dt + g(X_T, E(X_T))\right] \tag{2.27}$$

where

$$h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A \to \mathbb{R}$$
$$g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

Let us assume the following.

($\mathbf{H_3}$) The set $A \subset \mathbb{R}^d$ is convex and closed and the functions h and g are convex and for some δ, $k > 0$

$$h(t, x, y, u) \geq \delta \left|u^2\right| - k\ ,\ g(x, y) \geq -k, \text{ for every } (t, x, y, u) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A$$

($\mathbf{H_4}$) The set $A \subset \mathbb{R}^d$ is convex and compact and the functions h and g are convex.

Lemma 2.3.1 *For every admissible control u, equation 2.26 admits a unique strong solution such that for any $p \geq 1$*

$$E\left[\sup_{0\leq t\leq T} |X_t|^p\right] < +\infty. \tag{2.28}$$

Proof. The coefficients of equation 2.26 are linear mappings, then they are globally Lipschitz on the state variable. Then using similar technqiues as in Proposition 2.1, they are also Lipschitz on the marginal distribution of the state process X. Then applying [31], Proposition 1.2, it holds that equation 2.26 has a unique strong solution, such that for any $p \geq 1$, $E\left(\sup_{0\leq t\leq T} |X_t|^p\right) < +\infty$. ■

Existence of an optimal control

The following theorem could be seen as a generalization of [49], Theorem 5.2, to systems driven by mean-field stochatic differential equations.

Theoreme 2.3.1 *Under ($\boldsymbol{H_3}$) or ($\boldsymbol{H_4}$),if the control problem is finite, then it admits an optimal control.*

Proof. Suppose that the control problem is finite ie: $\inf\{J(u), u \in \mathcal{U}_{ad}\} < +\infty$. In particular, this assumption is fulfilled if the running cost $h(t,x,y,u)$ and the final cost $g(x,y)$ have linear growth with respect to (x,y).
Then $\forall \varepsilon > 0, \exists u_\varepsilon \in \mathcal{U}_{ad},\ J(u_\varepsilon) \leq \inf\{J(u), u \in \mathcal{U}_{ad}\} + \varepsilon$
If we take $\varepsilon_j = 1/2^j$, then $\exists u^j \in \mathcal{U}_{ad}, J(u^j) \leq \inf\{J(u), u \in \mathcal{U}_{ad}\} + 1/2^j$
It is clear that $\lim_{j\to+\infty} J(u^j) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$ and then a minimizing sequence exists.
By using assumption $(\mathbf{H_1})$ we obtain:

$$J(u^j) = E\left[\int_0^T h(t, X_t^j, E\left(X_t^j\right), u_t^j)dt + g(X_T^j, E\left(X_T^j\right))\right] \geq E\left[\int_0^T (\delta\left|u^j(t)\right|^2 - K)dt - K)\right] \tag{2.29}$$

Then for each $j \geq 1$

$$E\int_0^T \left|u^j(t)\right|^2 dt \leq KT + K + J(u^j) \tag{2.30}$$

Since the sequence $(J(u^j))$ is convergent, then $\sup_j |J(u^j)| \leq C$. This implies that

$$E\left[\int_0^T \left|u^j(t)\right|^2 dt\right] = \left\|u^j\right\|^2_{L^2_\mathcal{F}} < KT + K + C$$

Therefore, the sequence of admissible controls (u^j) is uniformly bounded in the space $L^2_\mathcal{F}\left(0, T, \mathbb{R}^k\right)$. Alternatively, if we assume $(\mathbf{H}_4)$, then the set A is compact in $\mathbb{R}^d$ and the sequence (u^j) is bounded in $L^2_\mathcal{F}$.
Then assuming either $(\mathbf{H_3})$ or $(\mathbf{H}_4)$, the sequence (u^j) is bounded in $L^2_\mathcal{F}$.
This implies that under assumption $(\mathbf{H_3})$ or $(\mathbf{H}_4)$, (u^j) is relatively compact in $L^2_\mathcal{F}$ equipped with the weak topology. Then there exists a subsequence, still denoted by (u^j) and a process $\overline{u}$, such that (u^j) converges weakly in $L^2_\mathcal{F}$ to $\overline{u}$.
By Mazur's theorem [48], Theorem 2 page 120, there exists a sequence of convex combinations of (u^j) which converges strongly to $\overline{u}$.
This means that there exist real numbers (α_{ij}), with $\alpha_{ij} \geq 0, \sum_{i\geq1}\alpha_{ij} = 1$, α_{ij} are equal to zero except for a finite number, such that if we denote $\widetilde{u}^j = \sum_{i\geq1}\alpha_{ij}.u_{i+j}$, then $\widetilde{u}^j \to \overline{u}$ strongly in $L^2_\mathcal{F}$.

Since $A \subseteq \mathbb{R}^d$ is convex and closed, then $\overline{u}$ is an admissible control belonging to $\mathcal{U}_{ad}$.
Let us denote by $\widetilde{X}^j$ (resp. $\overline{X}$) the solution of the state equation 2.26, associated to the admissible control $\widetilde{u}^j$ (resp. $\overline{u}$). Then, by using classical arguments, from stochastic calculus, one can prove that

$$\widetilde{X}^j \longrightarrow \overline{X} \text{ strongly in } C_{\mathcal{F}}\left([0,T],\mathbb{R}^n\right)$$

where

$$C_{\mathcal{F}}\left([0,T],\mathbb{R}^n\right) = \left\{ X : \Omega \times [0,T] \to \mathbb{R}^n, \mathcal{F}_t - \text{adapted, continuous such that: } E(\sup_{0\le t\le T} |X_t|) < +\infty \right\}$$

Let us give the outlines of the proof.

$$\begin{aligned}(\widetilde{X}_t^j - \overline{X}_t) \quad &= \int_0^t \Big((A(\widetilde{X}_s^j - \overline{X}_s) + B(E(\widetilde{X}_s^j) - E(\overline{X}_s)) + C(\widetilde{u}_s^j - \overline{u}_s) \Big) \, ds \\ &+ \int_0^t \Big(A_1(\widetilde{X}_s^j - \overline{X}_s) + B_1(E(\widetilde{X}_s^j) - E(\overline{X}_s)) + C_1(\widetilde{u}_s^j - \overline{u}_s \Big) \, dW_s \end{aligned} \tag{2.31}$$

Then

$$\begin{aligned}\left(\sup_{s\le t} \left|\widetilde{X}_s^j - \overline{X}_s\right|\right)^2 \quad &\le \int_0^t \|A\|^2 \left(\sup_{0\le v\le s} \left|(\widetilde{X}_v^j - \overline{X}_v\right|\right)^2 + \|B\|^2 \left(\sup_{0\le v\le s} \left|E(\widetilde{X}_v^j) - E(\overline{X}_v)\right|\right)^2 dt \\ &+ \int_0^t \|C\|^2 \left|\widetilde{u}_v^j - \overline{u}_v\right|^2 dt \\ &+ \sup \left(\left| \int_0^t A_1 \left(\widetilde{X}_s^j - \overline{X}_s\right) + B_1 \left(E(\widetilde{X}_s^j) - E(\overline{X}_s)\right) + C_1 \left(\widetilde{u}_s^j - \overline{u}_s\right) dW \right| \right)^2 \end{aligned}$$

By applying the Burkholder-Davis-Gundy inequality to the martingale part, we obtain:

$$E\left[\sup_{s\le T} \left|\widetilde{X}_s^j - \overline{X}_s\right|^2\right] \le C_1 \int_0^t E\left(\sup_{u\le s} \left|\widetilde{X}_u^j - \overline{X}_u\right|^2\right) ds + C_2 E \int_0^t \left|\widetilde{u}_s^j - \overline{u}_s\right|^2 ds. \tag{2.32}$$

If we set $f(t) = E\left[\sup_{s\leq t}\left|\widetilde{X}_s^j - \overline{X}_s\right|^2\right]$, then

$$f(t) \leq C_1\int_0^t f(s)ds + C_2 E\int_0^t \left|\widetilde{u}_s^j - \overline{u}_s\right|^2 ds \tag{2.33}$$

By applying Gronwall's lemma, there exists a positive constant C such that:

$$E\left[\sup_{s\leq T}\left|\widetilde{X}_s^j - \overline{X}_s\right|^2\right] \leq C.E\int_0^t \left|\widetilde{u}_s^j - \overline{u}_s\right|^2 ds \tag{2.34}$$

Since $(\widetilde{u}^j)$ converges to $\overline{u}$ strongly in $L^2_{\mathcal{F}}$, that is $\lim_{j\to\infty} E\int_0^t \left|\widetilde{u}_s^j - \overline{u}_s\right|^2 = 0$, we conclude that

$$\lim_{j\to\infty} E\left[\sup_{s\leq T}\left|\widetilde{X}_s^j - \overline{X}_s\right|^2\right] = 0 \tag{2.35}$$

.Now, let us verify that $\overline{u}$ is an optimal control. The continuity and the convexity of the cost functions h and g imply that

$$\begin{aligned} J(\overline{u}) = \lim_{j\to\infty} J(\widetilde{u}^j) &\leq \lim_{j\to\infty} \sum_{i\geq 1}^{i_j} \alpha_{ij}.J(u^{i+j}) \\ &\leq \lim_{j\to\infty} \sum_{i\geq 1}^{i_j} \alpha_{ij}.\max\left\{J(u^{i+j}), i = 1, ..., i_j\right\} \\ &= \lim_{j\to\infty} \sum_{i=1}^{i_j} \alpha_{ij}.J(u^{j_0+j}) \\ &= \lim_{j\to\infty} J(u^{j_0+j}) = \inf_{u\in\mathcal{U}_{ad}} J(u) \end{aligned} \tag{2.36}$$

because (u^j) is a minimizing sequence. The proof is now complete. ■

Chapter 3

General mean-field stochastic control problems I: Existence

Abstract

This paper is concerned with optimal control problems for systems governed by mean-field stochastic differential equation, in which the control enters both the drift and the diffusion coefficient. We consider the relaxed model in which admissible controls are measure valued and the state process is governed by an orthogonal martingale measure. We prove the existence of an optimal relaxed control. Moreover under some convexity conditions, we show that the optimal control is realized by a strict control.

Key words: Mean-field stochastic differential equation, relaxed control, martingale measure, approximation, tightness, weak convergence.

MSC 2010 subject classifications, 93E20, 60H30.

3.1 Introduction

In this part, we investigate stochastic control problems, where the state is governed by a mean-field stochastic differential equation and the admissible controls are relaxed controls which are measure-valued processes.

We give a precise formulation of this kind of problems in terms of stochastic d=ifferntial equations driven by orthogonal martingale measures. One is tempted to relax the state equation, by replacing simply the drift and diffusion coefficient by their relaxed couterparts ie: the integrals of the drift and diffusion coefficient with respect to the relaxed control, adopting the same method as in deterministic control. As it will be shown in a simple counter example, the suggested "relaxed" state equation is not continuous with respect to the control variable. This implies in particular that the value functions for the original and relaxed problems are not the same. In addition, there is no mean to prove the existence of an optimal control for this model. The fundamental reason is that one has to relax the quadratic variation of the stochastic integral part of the state equation, which is a Lebesgue integral, rather than the stochastic integral itself. Roughly speaking, the idea is to relax the generator of the process, which is intimately linked to the weak solutions of the relaxed stochastic equation, rather than the equation itself. As it will be shown, the stochastic equation associated with the relaxed generator will be governed by a continuous orthogonal martingale measure, rather than a Brownian motion. For this model, we prove that the strict and relaxed control problems have the same value function and that an optimal relaxed control exists. Our result extends in particular [3, 21, 39] to mean field controls. The proof is based on tightness properties of the underlying processes and Skorokhod selection theorem. Moreover, due to the compactness of the action space, we show that the relaxed control could be choosen among the so-called sliding controls, which are convex combinations of Dirac measures. As a consequence and under the so-called Fillipov convexity condition, the optimal relaxed control is shown to be strict.

Mean-field stochastic differential equations are obtained as limits of some interacting particle systems. This kind of approximation result is called "*propagation of chaos*", which says that when the number of particles tends to infinity, the equations defining the evolution of the particles could be replaced by a single equation, called the McKean-Vlasov equation. This mean-field equation, represents in some sense the average behavior of the infinite number of particles, see [43, 31] for details.

Since the earlier papers [34], [28], mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks, management of oil ressources. The main drawback, when dealing with mean-field stochastic control problems, is that the Bellmann principle of optimality does not hold. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see [2, 11, 15, 17, 19, 35]. One can refer also to the recent book [10] and the references therein.

3.2 Existence of optimal relaxed controls

3.2.1 Controlled mean field stochastic differential equations

Let (W_t) is a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Let $\mathbb{A}$ be some compact metric space called the control set.
We study the existence of optimal controls for systems driven non linear mean-field stochastic differential equations (MFSDE in short) of the form:

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t), u_t)dW_t \\ X_0 = x. \end{cases} \tag{3.1}$$

The cost functional over the time interval $[0, T]$ is given by

$$J(u) = E\left(\int_0^T h(t, X_t, E(\varphi(X_t), u_t\right) dt + g(X_T, E\lambda(X_T)), \tag{3.2}$$

where b, σ, l, h, g and ψ are given functions. The control variable u_t called a strict control, is a measurable, $\mathcal{F}_t-$ adapted process with values in the action space $\mathbb{A}$. We denote $\mathcal{U}_{ad}$ the space of strict controls. Let us point out that the probability space and Brownian motion may change with the control u.
The objective is to minimize the cost functional $J(u)$ over the space $\mathcal{U}_{ad}$ ie: find $u^* \in \mathcal{U}_{ad}$ such that $J(u^*) = \inf \{J(u^*), u \in \mathcal{U}_{ad}\}$.
The following assumptions will be in force throughout this paper.
$(\mathbf{H_1})$ Assume that

$$\begin{gathered} b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}^d \\ \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\ \Psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d \end{gathered} \tag{3.3}$$

are bounded continuous functions such that $b(t, ., ., a)$, $\sigma(t, ., ., a)$, $\Psi(.)$ and $\Phi(.)$ are Lipschitz continuous, uniformly in (t, a).
$(\mathbf{H_2})$ Assume that

$$\begin{aligned} &h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{A} \longrightarrow \mathbb{R} \\ &g : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ &\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d \\ &\lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d \end{aligned} \tag{3.4}$$

are bounded continuous functions and $h(t,.,.,a)$ is Lipschiz continuous uniformly in (t,a).

Proposition 3.2.1 *Under assumption* $(\mathbf{H_1})$ *the MFSDE (3.1) has a unique strong solution, such that for each* $p > 0$ *we have* $E(|X_t|^p) < +\infty$.

Proof. Let us define $\overline{b}(t,x,\mu,a)$ and $\overline{\sigma}(t,x,\mu,a)$ on $[0,T]\times\mathbb{R}^d\times\mathbb{M}_1(\mathbb{R}^d)\times\mathbb{A}$ by $\overline{b}(t,x,\mu,a) = b(t,x,\int\Psi(x)d\mu(x),a)$ and $\overline{\sigma}(t,x,\mu,a) = \sigma(t,x,\int\Phi(x)d\mu(x),a)$, where $\mathbb{M}_1(\mathbb{R}^d)$ denotes the space of probability measures in $\mathbb{R}^d$. According to [31] Prop.1.2, it is sufficient to check that $\overline{b}$ and $\overline{\sigma}$ are Lipschitz in (x,μ) where $\mathbb{M}_1(\mathbb{R}^d)$ is equipped with the Wasserstein metric $d(\mu,\nu) = \inf\left\{\left(E^Q|X-Y|^2\right)^{1/2}; Q\in\mathbb{M}_1(\mathbb{R}^d\times\mathbb{R}^d),\ \text{with marginals } \mu,\nu\right\}$. This follows from the Lipschitz continuity of b and σ with respect to (x,y).
Using similar techniques as in [31] Prop.1.2, it holds that for each $p>0$, $E(|X_t|^p) < +\infty$.
■

3.2.2 The relaxed control problem

As it is well known in control theory, in the absence of convexity conditions, an optimal control may fail to exist in the set $\mathcal{U}_{ad}$ of strict controls (see e.g. [23, 36, 44]). This suggests that the set of strict controls is too narrow and should be embedded into a wider class of relaxed controls, with nice compactness properties. For the relaxed model, to be a true extension of the original control problem, must satisfy the following two conditions:
i) The value functions of the original and the relaxed control problems must be equal.
ii) The relaxed control problem must have an optimal solution.
The idea of relaxed control is to replace the $\mathbb{A}$-valued process (u_t) with a $\mathbb{M}_1(\mathbb{A})$-valued process (μ_t), where $\mathbb{M}_1(\mathbb{A})$ is the space of probability measures equipped with the topology of weak convergence. Then (μ_t) may be identified as a random product measure on $[0,T]\times\mathbb{A}$, whose projection on $[0,T]$ coincides with Lebesgue measure.
Let $\mathbb{V}$ be the set of product measures μ on $[0,T]\times\mathbb{A}$ whose projection on $[0,T]$ coincides with the Lebesgue measure dt. It is clear that every μ in $\mathbb{V}$ may be disintegrated as

$\mu = dt.\mu_t(da)$, where $\mu_t(da)$ is a transition probability. The elements of $\mathbb{V}$ are called Young measures in deterministic theory, see [44].
$\mathbb{V}$ as a closed subspace of the space of positive Radon measures $\mathbb{M}_+([0,T] \times \mathbb{A})$, is compact for the topology of weak convergence. In fact it can be proved that it is compact also for the topology of stable convergence, where test functions are measurable, bounded functions $f(t,a)$ continous in a, see [21, 30] for further details.

Definition 3.2.1 *A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.*

Remark 3.2.1 *The set $\mathcal{U}_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying u_t with $dt\delta_{u_t}(da)$.*

Discussion on the relaxation of the state process

A natural question arises: what is the natural SDE associated with a relaxed control. Note that in the deterministic case or in the stochastic case where only the drift is controlled, one has just to replace in equation 3.1 the drift by the same drift integrated against the relaxed control. Now we are in a situation where both the drift and diffusion coefficient are controlled. Let us try a direct relaxation of the original equation. The "relaxed" control problem will be governed by the MFSDE

$$\begin{cases} dX_t = \int_A b(t, X_t, E(\Psi(X_t)), a)\mu_t(da)dt + \int_A \sigma(t, X_t, E(\Psi(X_t)), a)\mu_t(da)dW_t \\ X_0 = x \end{cases}$$

As it will be shown in the sequel, this model does not fullfill the requirements of a true relaxed model. The reason is that the relaxed process is not continuous in the control variable and as a consequence, the value functions of the original and relaxed control problems are not equal. Let us consider the following example. Consider a control problem governed by the following SDE without mean-field terms:

$$\begin{cases} dX_t = u_t.dW_t \\ X_0 = x \end{cases}$$

where the control $u \in \mathcal{U}_{ad}$: the set of measurable functions $u : [0,1] \rightarrow [-1,1]$.
The "relaxed" model will be governed by the equation

$$\begin{cases} dX_t = \int_A a\mu_t(da)dW_t \\ X_0 = x \end{cases}$$

Consider the sequence of Rademacher functions

$$u_n(t) = (-1)^k \text{ if } \tfrac{k}{n} \leq t \leq \tfrac{(k+1)}{n},\ 0 \leq k \leq n-1.$$

Lemma 3.2.1 *Let $dt.\delta_{u_n(t)}(da)$ the relaxed control associated to $u_n(t)$, then the sequence $\left(dt.\delta_{u_n(t)}(da)\right)$ converges weakly to $dt(1/2)(\delta_{-1}+\delta_1)(da)$.*

Proof. See [36] Lemma 1.1, page 20 ■

Remark 3.2.2 *The sequence Rademacher functions is a typical example of a minimising sequence with no limit in the set of stric controls. However its weak limit $dt(1/2)(\delta_{-1}+\delta_1)(da)$ is an optimal relaxed control [36, 44].*

It is clear that $X_t^n = \int_0^t u_n(s).dW_s = \int_0^t \left[\int_A a\delta_{u_n(s)}(da)\right] dW_s$ is a continuous martingale with quadratic variation $\langle X^n, X^n\rangle_t = \int_0^t u_n^2(s).ds = t$. Therefore (X_t^n) is a Brownian motion.
By Lemma 2.4, the sequence of relaxed controls $dt.\delta_{u_n(t)}(da)$ converges weakly in $\mathbb{M}_+([0,T]\times A)$ to $\mu^* = (1/2)dt(\delta_{-1}+\delta_1)(da)$. Let X^* be the relaxed state process corresponding to the limit μ^*, then

$$X^*(t) = \int_0^t \int_A a.(1/2)(\delta_{-1}+\delta_1)(da)dW_t = 0.$$

It is obvious that the sequence of state processes (X_t^n) cannot converge in any way to X_t^*.
Indeed

$$E\left[|X_t^n - X_t^*|^2\right] = E\left[|X_t^n|^2\right] = E\left[\left|\int_0^t u_n(s).dW_s\right|^2\right] = \int_0^t u_n^2(s).ds = t$$

This implies in particular that the state process is not continuous in the control variable and as a byproduct, the value functions of the strict and "relaxed" control problems are not equal. Moreover, even if the set $\mathbb{V}$ is compact, there is no mean to prove existence of an optimal control for this model.

What is the right relaxed state process?

The reason why the proposed model is not a true extension of the original strict control problems, is that the stochastic integral part does not behave as a Lebesgue integral. In fact, one should relax the drift and the quadratic variation of the martingale part, which is a Lebesgue integral.

In the relaxed model, the quadratic variation process must be $\int_0^t \int_A \sigma\sigma^*(t, X_t, E(\Phi(X_t), a)\mu_t(da)dt$, which is more natural than relaxing the stochastic integral itself. Now, one has to look for a square integrable martingale whose quadratic variation is given by $\int_0^t \int_A \sigma\sigma^*(t, X_t, E(\Phi(X_t), a)\mu_t(da)dt$, which is equivalent to the search of an object which is a martingale whose quadratic variation is $dt\mu_t(da)$. This object is an orthogonal martingale measure whose covariance measure is $dt\mu_t(da)$. This is equivalent to the relaxation of the infinitesimal generator associated to the state process.

Following [31] Prop. 1.10, existence of a weak solution of equation 3.1 is equivalent to existence of a solution for the non linear martingale problem:

$$f(X_t) - f(X_0) - \int_0^t L^{P_{X_s}} f(s, X_s, u_s)\, ds \text{ is a } P\text{−martingale},$$

for every $f \in C_b^2$, *for each* $t > 0$, *where* L *is the infinitesimal generator* associated with equation 3.1,

$$L^\nu f(t,x,a) = \frac{1}{2}\sum_{i,j}\left(a_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(t,x,a) + \sum_i \left(b_i \frac{\partial f}{\partial x_i}\right)(t,x,a),$$

$b = b(t, x, \langle \Psi, \nu\rangle, a)$ and $a_{i,j} = \sigma\sigma^*(t, x, \langle \Phi, \nu\rangle, a)$, $\nu \in \mathbb{M}_1(\mathbb{R}^d)$.

The natural relaxed martingale problem associated to the relaxed generator is defined as follows:

$$f(X_t) - f(X_0) - \int_0^t \int_A L^{P_{X_s}} f(s, X_s, a)\, \mu_s(da)ds \text{ is a } P - martingale \tag{3.5}$$

for each $f \in C_b^2$, *for each* $t > 0$.

Now, what is the stochastic differential equation corresponding to the relaxed martingale problem 4.5? The answer is given by the following theorem.

Theoreme 3.2.1 *1)Let P be the solution of the martingale problem 4.5. Then P is the law of a d–dimensional adapted and continuous process X defined on an extension of the space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and solution of the following MFSDE starting at x:*

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, E(\Psi(X_t))\, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt), \\ X_0 = x \end{cases} \tag{3.6}$$

where $M = (M^k)_{k=1}^d$ is a family of d-strongly orthogonal continuous martingale measures, each having intensity $dt\mu_t(da)$.

2) If the coefficients b and σ are Lipschitz in x, y, uniformly in t and a, the SDE (2.6) has a unique pathwise solution.

Proof. 1) The proof is based essentially on the same arguments as in [20] Theorem IV-2 and [31] Prop. 1.10.

2) The coefficients being Lipschitz continuous, following the same steps as in [31] and [20], it is not difficult to prove that Equation 3.6 has a unique solution such that for each $p > 0$ we have $E(|X_t|^p) < +\infty$. ■

Remark 3.2.3 *Note that the othogonal martingale measure corresponding to the relaxed control $dt\mu_t(da)$ is not unique.*

Let us give the precise definition of a martingale measure introduced by Walsh [45], see also [20, 37] for more details.

Definition 3.2.2 *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, and $M(t, B)$ a random process, where $B \in \mathcal{B}(\mathbb{A})$. M is a $(\mathcal{F}_t, P)-$martingale measure if:*

1)For each $B \in \mathcal{B}(\mathbb{A})$, $(M(t,B))_{t\geq 0}$ is a square integrable martingale, $M(0,B)=0$.
2)For every $t>0$, $M(t,.)$ is a $\sigma-$finite L^2-valued measure.
It is called continuous if for each $B \in \mathcal{B}(\mathbb{A})$, $M(t,B)$ is continuous and orthogonal if $M(t,B).M(t,C)$ is a martingale whenever $B \cap C = \phi$.

Remark 3.2.4 *When the martingale measure M is orthogonal, it is proved in [45] the existence of a random positive σ-finite measure $\mu(dt,da)$ on $[0,T]\times\mathbb{A}$ such that $\langle M(.,B), M(.,B)\rangle_t = \mu([0,t]\times B)$ for all $t>0$ and $B \in \mathcal{B}(\mathbb{A})$. $\mu(dt,da)$ is called the covariance measure of M.*

Example Let $\mathbb{A} = \{a_1, a_2, \cdots, a_n\}$ be a finite set. Then every relaxed control $dt\,\mu_t(da)$ will be a convex combination of the Dirac measures $dt\,\mu_t(da) = \sum_{i=1}^n \alpha_t^i\, dt\,\delta_{a_i}(da)$, where for each i, α_t^i is a real–valued process such that $0 \leq \alpha_t^i \leq 1$ and $\sum_{i=1}^n \alpha_t^i = 1$. It is obvious that the solution of the relaxed martingale problem *4.5* is the law of the solution of the SDE

$$dX_t = \sum_{i=1}^n b(t, X_t, E(\Psi(X_t)), a_i)\alpha_t^i dt + \sum_{i=1}^n \sigma(t, X_t, E(\Psi(X_t)), a_i)\left(\alpha_t^i\right)^{1/2} dB_s^i, \qquad X_0 = x, \tag{3.7}$$

where the B^i's are independant d-dimensional Brownian motions, on an extension of the initial probability space. The process M defined by

$$M(A\times[0,t]) = \sum_{i=1}^n \int_0^t \left(\alpha_s^i\right)^{1/2} 1_{\{a_i\in A\}} dB_s^i$$

is in fact an orthogonal continuous martingale measure (cf. [21, 45]) with intensity $\mu_t(da)dt = \sum \alpha_t^i\,\delta_{a_i}(da)dt$. Thus, the SDE 4.7 can be expressed in terms of M and μ as follows:

$$dX_t = \int_{\mathbb{A}} b(t, X_t, E(\Psi(X_t))\, a)\,\mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt)$$

Approximation of the relaxed model

The relaxed control problem is now driven by equation

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, E(\Psi(X_t))\, a)\, \mu_t(da)\, dt + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt), \\ X_0 = x \end{cases} \tag{3.8}$$

and accordingly the relaxed cost functional is given by

$$J(\mu) = E\left(\int_0^T \int_A h(t, X_t, E(\varphi(X_t), a)\mu_t(da)dt + g(X_T, E\lambda(X_T) \right). \tag{3.9}$$

We show in this section that the strict and the relaxed control problems have the same value function. This is based on the chattering lemma and the stability of the state process with respect to the control variable.

Lemma 3.2.2 *(Chattering lemma) i) Let (μ_t) be a relaxed control. Then there exists a sequence of adapted processes (u_t^n) with values in A, such that the sequence of random measures $\left(\delta_{u_t^n}(da)\, dt\right)$ converges in $\mathbb{V}$ to $\mu_t(da)\, dt$, $P-a.s.$*
ii) For any g continuous in $[0,T] \times \mathbb{M}_1(A)$ such that $g(t,.)$ is linear, we have $P-a.s$

$$\lim_{n \to +\infty} \int_0^t g(s, \delta_{u_s^n})ds = \int_0^t g(s, \mu_s)ds \text{ uniformly in } t \in [0,T]. \tag{3.10}$$

Proof. See [21] and [23] Lemma 1 page 152. ■

Proposition 3.2.2 *1) Let $\mu = \mu_t(da)\, dt$ a relaxed control. Then there exist a continuous orthogonal martingale measure $M(dt, da)$ whose covariance measure is given by $\mu_t(da)\, dt$.*
2) If we denote $M^n(t,B) = \int_0^t \int_B \delta_{u_s^n}(da)dW_s$, where (u^n) is defined as in the last Lemma, then for every bounded predictable process $\varphi : \Omega \times [0,T] \times \mathbb{A} \to \mathbb{R}$, such that $\varphi(\omega, t, .)$ is continuous, we have

$$E\left[\left(\int_0^t \int_{\mathbb{A}} \varphi(\omega, t, a) M^n(dt, da) - \int_0^t \int_{\mathbb{A}} \varphi(\omega, t, a) M(dt, da) \right)^2 \right] \to 0 \text{ as } n \longrightarrow +\infty,$$

Proof. See [37] page 18. ■

Proposition 3.2.3 *1) Let X_t, X_t^n be the solutions of state equation (3.6) corresponding to μ and u^n, where μ and u^n are defined as in the chattering lemma. Then*

$$\lim_{n \to \infty} E\left[\sup_{0 \le t \le T} |X_t^n - X_t|^2 \right] = 0. \tag{3.11}$$

2) Let $J(u^n)$ and $J(\mu)$ the expected costs corresponding respectively to u^n and μ. Then there exists a subsequence (u^{n_k}) of (u^n) such that $J(u^{n_k})$ converges to $J(\mu)$.

Proof. 1)Let μ a relaxed control and $\left(dt\delta_{u_t^n}(da)\right)$ the sequence of atomic measures associated to the sequence of strict controls (u^n), as in the last Lemma. Let X_t, X_t^n the corresponding state processes. If we denote $M^n(t,B)=\int_0^t\int_B \delta_{u_s^n}(da)dW_s$, then X^n may be written in a relaxed form $\begin{cases} dX_t^n = \int\limits_A b(t,X_t^n,E(\Psi(X_t^n)),a)\delta_{u_t^n}(da)dt + \int\limits_A \sigma(t,X_t,E(\Phi(X_t)),a)M^n(dt,da) \\ X_0 = x \end{cases}$

We have

$$\begin{aligned}
|X_t - X_t^n| &\leq \left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(du).ds - \int_0^t\int_A b\left(s,X_s^n,E(\Psi(X_s^n),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\left|\int_0^t\int_A \sigma\left(s,X_s,E(\Phi(X_t),a\right)M(ds,da) - \int_0^t\int_A \sigma\left(s,X_s^n,E(\Phi(X_s^n),a\right)M^n(ds,da)\right| \\
&\leq \left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(du).ds - \int_0^t\int_A b\left(s,X_s,E(\Psi(X_s),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\delta_{u_s^n}(da).ds - \int_0^t\int_A b\left(s,X_s^n,E(\Psi(X_s^n),u\right)\delta_{u_s^n}(da)ds\right| \\
&+\sup_{s\leq t}\left|\int_0^s\int_A \sigma\left(v,X_v,E(\Phi(X_v),a\right)M(dv,da) - \int_0^t\int_A \sigma\left(v,X_v,E(\Phi(X_v),a\right)M^n(dv,da)\right| \\
&+\sup_{s\leq t}\left|\int_0^s\int_A \sigma\left(v,X_v,E(\Phi(X_v))\right)M^n(dv,da) - \int_0^t\int_A \sigma\left(v,X_v^n,E(\Phi(X_v^n))\right)M^n(dv,da)\right|
\end{aligned}$$

Then by using Burkholder-Davis-Gundy inequality for the martingale part and the fact that all the functions in equation *(*3.6) are Lipschitz continuous, it holds that

$$E\left(\sup_{0\leq t\leq T}|X_t - X_t^n|^2\right) \leq K\left[\int_0^T E\left(\sup_{0\leq s\leq t}|X_s - X_s^n|^2\right)dt + \varepsilon_n\right] \tag{3.12}$$

where K is a nonnegative constant and

$$\varepsilon_n = E\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_A b\left(s,X_s,E(\Psi(X_t),u\right)\mu_s(da).ds - \int_0^t\int_A b\left(s,X_s,E(\Psi(X_s),a\right)\delta_{u_s^n}(da)ds\right|^2\right) \tag{3.13}$$

$$+E\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_A \sigma\left(s,X_s,E(\Psi(X_t),a\right)\mu_s(da).ds - \int_0^t\int_A \sigma\left(s,X_s,E(\Psi(X_s),a\right)\delta_{u_s^n}(da)ds\right|^2\right) \tag{3.14}$$

By using the chattering lemma and the Lebesgue dominated convergence theorem, it holds that $\lim\limits_{n\to+\infty}\varepsilon_n = 0$. We conclude by using Gronwall's Lemma.

2) Property 1) implies that the sequence (X_t^n) converges to X_t in probability uniformly in t, then there exists a subsequence $(X_t^{n_k})$ which converges to X_t, P -$a.s$ uniformly in t. We have

$$\begin{aligned} \left|J\left(u^{n_k}\right)-J\left(\mu\right)\right| \; & \leq E\left[\int_0^T \int_A \left|h(t,X_t^{n_k},E(\varphi(X_t^n),a)-\; h(t,X_t,E(\varphi(X_t),a)\right|\delta_{u_t^{n_k}}(da)\,dt\right] \\ & +E\left[\left|\int_0^T \int_A h(t,X_t,E(\varphi(X_t),a)\delta_{u_t^{n_k}}(da)\,dt-\int_0^T \int_A h(t,X_t,E(\varphi(X_t),a)\mu_t(da)\,dt\right|\right] \\ & +E\left[\left|g(X_T^{n_k},E(\lambda(X_T^{n_k}))-g(X_T,E(\lambda(X_T))\right|\right] \end{aligned}$$

It follows from the continuity and boundness of the functions h, g, φ and λ with respect to x and y, that the first and third terms in the right hand side converge to 0 . The second term in the right hand side tends to 0 by the weak convergence of the sequence μ^n to μ, the continuity and the boundness of h in the variable a. We use the dominated convergence theorem to conclude. ■

Remark 3.2.5 *As a consequence of the last proposition, it holds that the infimum among relaxed controls is equal to the infimum among strict controls, that is the relaxed model is a true extension of the original control problem.*

3.2.3 Existence of optimal relaxed controls

The following theorem which is the main result of this section, extends [3, 21, 23] to systems driven by mean field SDEs with controlled diffusion coefficient.

Theoreme 3.2.2 *Under assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *there exist an optimal relaxed control.*

The proof is based on some auxiliary Lemmas and will be given later.
Let $(\mu^n)_{n\geq 0}$ be a minimizing sequence, that is $\lim\limits_{n\to\infty} J\left(\mu^n\right) = \inf\limits_{q\in\mathcal{R}} J\left(\mu\right)$ and let X^n be the unique solution of 3.6, associated with μ^n. We will prove that the sequence (μ^n, M^n, X^n) is tight and then show that we can extract a subsequence, which converges in law to a process $(\widehat{\mu}, \widehat{W}, \widehat{X})$, which satisfies the same MFSDE. To finish the proof we show that the sequence of cost functionals $(J(\mu^n))_n$ converges to $J(\widehat{\mu})$ which is equal to $\inf\limits_{\mu\in\mathcal{R}} J\left(\mu\right)$ and then conclude that $(\widehat{\mu}, \widehat{M}, \widehat{X})$ is optimal.

Lemma 3.2.3 *The sequence of distributions of the relaxed controls* $(\mu^n)_n$ *is relatively compact in* $\mathbb{V}$.

Proof. The relaxed controls μ^n are random variables on the space $\mathbb{V}$ which is compact. Then Prohorov's theorem yields that the family of distributions associated to $(\mu^n)_{n\geq 0}$ is tight then it is relatively compact. ■

Lemma 3.2.4 *The family of martingale measures $(M^n)_{n\geq 0}$ is tight in the space $C([0,1],\mathcal{S}')$ of continuous functions from $[0,1]$ into $\mathcal{S}'$, the topological dual of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions.*

Proof. The martingale measures M^n, $n \geq 0$, can be considered as random variables with values in $C([0,1],\mathcal{S}')$ (see [40]). By [40], Lemma 6.3, it is sufficient to show that for every φ in $\mathcal{S}$ the family $(M^n(\varphi), n \geq 0)$ is tight in $C\left([0,T],\mathbb{R}^d\right)$ where $M^n(\omega,t,\varphi) = \int_{\mathbb{A}} \varphi(a) M^n(\omega,t,da)$. Let $p > 1$ and $s < t$, by the Burkholder-Davis-Gundy inequality we have

$$E\left(\left|M_t^n(\varphi) - M_s^n(\varphi)\right|^{2p}\right) \leq C_p\, E\left[\left(\int_s^t \int_{\mathbb{A}} |\varphi(a)|^2 \mu_t^n(da)\, dt\right)^p\right]$$
$$\leq C_p \sup_{a\in\mathbb{A}} |\varphi(a)|^{2p} |t-s|^p = K_p\ |t-s|^p,$$

where K_p is a constant depending only on p. Then, the Kolmogorov tightness criteria in $C\left([0,T],\mathbb{R}^d\right)$ is fulfilled and the sequence $(M^n(\varphi))$ is tight. ■

Lemma 3.2.5 *The sequence $(X^n)_{n\geq 0}$ is tight in the space $C\left([0,T],\mathbb{R}^d\right)$*

Proof. Let $p > 1$ and $s < t$. Using usual arguments from stochastic calculus and the boundness of the coefficients b and σ, it is easy to show that

$$E\left(|X_t^n - X_s^n|^{2p}\right) \leq C_p\ |t-s|^p$$

which yields the tightness of $(X_t^n, n \geq 0)$ *in* $C\left([0,T],\mathbb{R}^d\right)$. ■

Proof. of Theorem 2.14. By using Lemmas 2.15, 2.16 and 2.17, it holds that the sequence of processes (μ^n, M^n, X^n) is tight on the space $\Gamma = \mathbb{V}\times C([0,1],\mathcal{S}')\times C\left([0,T],\mathbb{R}^d\right)$. Then by the Skorokhod representation theorem, there exists a probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$, a sequence $\widehat{\gamma}^n = \left(\widehat{\mu}^n,\widehat{M^n},\widehat{X^n}\right)$ and $\widehat{\gamma} = \left(\widehat{\mu},\widehat{M},\widehat{X}\right)$ defined on this space such that:
(i) for each $n \in \mathbb{N}$, law$(\gamma^n) =$ law$(\widehat{\gamma}^n)$,

(ii) there exists a subsequence $(\widehat{\gamma}^{n_k})$ of $(\widehat{\gamma}^n)$, still denoted by $(\widehat{\gamma}^n)$, which converges to $\widehat{\gamma}$, $\widehat{P}$-$a.s.$ on the space Γ.

This means in particular that the sequence of relaxed controls $(\widehat{\mu}^n)$ converges in the weak topology to $\widehat{\mu}$, $\widehat{P}-a.s.$ and $\left(\widehat{M^n}, \widehat{X^n}\right)$ converges to $\left(\widehat{M}, \widehat{X}\right)$, $\widehat{\mathbb{P}}-a.s.$in $C\left([0,1], \mathcal{S}'\right) \times C\left([0,T], \mathbb{R}^d\right)$.

According to property (i), we get

$$\begin{cases} \widehat{X_t^n} = x + \int_0^t \int_A b\left(s, \widehat{X_s^n}, E(\Psi(\widehat{X_s^n}), a\right) \widehat{\mu}_s^n(da)ds + \int_0^t \int_A \sigma\left(s, \widehat{X_s^n}, E(\Phi(\widehat{X_s^n}), a\right) \widehat{M^n}(ds, da), \\ \widehat{X_0^n} = x. \end{cases} \tag{3.15}$$

The coefficients b, σ, Ψ and Φ being Lipschitz continuous in (x, y),then according to property (ii) and using similar arguments as in [42] page 32, it holds that

$$\int_0^t \int_A b\left(s, \widehat{X_s^n}, E(\Psi(\widehat{X_s^n}), a\right) \widehat{\mu}_s^n(da)ds \text{ converges in probability to } \int_0^t \int_A b\left(s, \widehat{X_s}, E(\Psi(\widehat{X_s}), a\right) \widehat{\mu}_s(da)ds$$

and

$$\int_0^t \int_A \sigma\left(s, \widehat{X_s^n}, E(\Phi(\widehat{X_s^n}), a\right) \widehat{M^n}(ds, da) \text{ converges in probability to } \int_0^t \int_A \sigma\left(s, \widehat{X_s}, E(\Phi(\widehat{X_s}), a\right) \widehat{M}(ds$$

b and σ being Lipschitz continuous, then $\widehat{X}$ is the unique solution of the MFSDE

$$\begin{cases} \overline{X}_t = x + \int_0^t \int_A b\left(s, \widehat{X_s}, E(\Psi(\widehat{X_s}), a\right) \widehat{\mu}_s(da)ds + \int_0^t \int_A \sigma\left(s, \widehat{X_s}, E(\Phi(\widehat{X_s}), a\right) \widehat{M}(ds, da), \\ \overline{X}_0^n = x. \end{cases} \tag{3.16}$$

To finish the proof of Theorem 2.14, it remains to check that $\widehat{\mu}$ is an optimal control.According to above properties (i)-(ii) and assumption $(\mathbf{H}_2)$, we have

$$\begin{aligned}
\inf_{\mu\in\mathcal{R}} J(\mu) &= \lim_{n\to\infty} J(\mu^n), \\
&= \lim_{n\to\infty} E\left[\int_0^T \int_A h(t, X_t^n, E(\varphi(X_t^n), a)\mu_t^n(da)dt + g(X_T^n, E\lambda(X_T^n))\right] \\
&= \lim_{n\to\infty} \widehat{E}\left[\int_0^T \int_A h\left(t, \widehat{X_t^n}, E(\varphi(\widehat{X_t^n}), a\right)\widehat{\mu}_t^n(da)dt + g(\widehat{X_T^n}, E\lambda(\widehat{X_T^n}))\right] \\
&= \widehat{E}\left[\int_0^T \int_A h\left(t, \widehat{X_t}, E(\varphi(\widehat{X_t}), a\right)\widehat{\mu}_t(da)dt + g(\widehat{X_T}, E\lambda(\widehat{X_T}))\right].
\end{aligned}$$

Hence $\widehat{\mu}$ is an optimal control. ■

Remark 3.2.6 *The proof of the last Theorem is based on tightness and weak convergence techniques. Then it is possible to prove it by using the non linear martingale problem and following the same steps as in [21] without using the pathwise representation of the solution.*

We prove that in the next proposition that we can restrict the investigation for an optimal relaxed control to the class of so-called sliding controls also known as chattering controls, having the form

$$\mu_t = \sum_{i=1}^{p} \alpha_i(t)\delta_{u_i(t)}(da), u_i(t) \in \mathbb{A}, \alpha_i(t) \geq 0 \text{ and } \sum_{i=1}^{p} \alpha_i(t) = 1. \tag{3.17}$$

where $\alpha_i(t)$ and $u_i(t)$ are adapted stochastic processes.

Proposition 3.2.4 *Let μ be a relaxed control and X the corresponding state process. Then one can choose a sliding control*

$$\nu_t = \sum_{i=1}^{p} \alpha_i(t)\delta_{u_i(t)}(da),\ u_i(t) \in A,\ \alpha_i(t) \geq 0 \textit{ and } \sum_{i=1}^{p} \alpha_i(t) = 1 \tag{3.18}$$

such that

1) X is a solution of the controlled MFSDE

$$\begin{cases} dX_t &= \sum_{i=1}^{p}\alpha_i(t)b(t,X_t,E(\Psi(X_t)),u_i(t))dt + \sum_{i=1}^{p}\alpha_i(t)^{1/2}\sigma(t,X_t,E(\Phi(X_t),u_i(t))dW_t^i \\ X_0 &= x \end{cases} \tag{3.19}$$

where (W^i) are independant Brownian motions defined on an extension of the probability space..

2) $J(\mu) = J(\nu)$.

Proof. Let Λ denote the $d+d^2+1$-dimensional simplex

$\Lambda = \left\{ \lambda = (\lambda_0, \lambda_1, ..., \lambda_{d+d^2+1})\,;\ \lambda_i \geq 0;\ \sum_{i=0}^{d+d^2+1} \lambda_i = 1 \right\}$

and W the $(d+d^2+2)$-cartesian product of the set$\mathbb{A}$ $W = \{w = (u_0, u_1, ..., u_{d+d^2+1})\,;\ u_i \in \mathbb{A}\}$

Define the function

$g(t,\lambda,w) = \sum_{i=0}^{d+d^2+1} \lambda_i \widetilde{b}(t,X_t,E(\Psi(X_t),u_i) - \int_A \widetilde{b}\,(t,X_t,E(\Psi(X_t),a)\,\mu_t(da)$

where $t \in [0,T]\,, \lambda \in \Lambda, w \in W$ and $\widetilde{b}(t,X_t,E(\Psi(X_t),u_i) = \begin{pmatrix} b(t,X_t,E(\Psi(X_t),u_i) \\ \sigma\sigma^*(t,X_t,E(\Phi(X_t),u_i) \\ h(t,x_t,E(\varphi(X_t),u_i) \end{pmatrix}$

Let $\widetilde{b}(t,X_t,E(\Psi(X_t),u_i)$, $i = 0,1,...,d+d^2+1$, be the subset of $(d+1)$ arbitrary points

in $P(t,X_t)$ where

$P(t,X_t) = \{(b(t,X_t,E(\Psi(X_t),a),\ \sigma\sigma^*(t,X_t,E(\Phi(X_t),u_i),\ h(t,X_t,E(\Psi(X_t),a))\,;\ a \in \mathbb{A}\} \subset \mathbb{R}^{d+d^2+1}$

Then the convex hull of this set is the collection of all points of the form

$\sum_{i=0}^{d+d^2+1} \lambda_i \widetilde{b}(t,X_t,E(\Psi(X_t),u_i)$

If μ is a relaxed control, then $\int_A \widetilde{b}\,(t,X_t,E(\Psi(X_t),a)\,\mu_t(da) \in Conv\,(P(t,X_t))$, the convex hull of $P(t,X_t)$. Therefore it follows from Carathéodory's Lemma (which says that the convex hull of a d-dimensional set M coincides with the union of the convex hulls of $d+1$ points of M), that for each $(\omega,t) \in \Omega \times [0,T]$ the equation $g(t,\lambda,\omega) = 0$ admits at least one solution. Moreover the set

$\left\{ (\omega,\lambda,w) \in \Omega \times \Lambda \times W :\ \sum_{i=0}^{d+d^2+1} \lambda_i \widetilde{b}(t,X_t,E(\Psi(X_t),u_i) = \int_A \widetilde{b}\,(t,x_t,E(\Psi(x_t),a)\,\mu_t(da) \right\}$

is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(\mathbb{A}^{d+1})$ with non empty $\omega-$sections for each ω.

Hence by using a measurable selection theorem [21], there exist measurable $\mathcal{F}_t-$adapted processes λ_t and w_t with values, respectively in Λ and W such that:

$$\int_A \widetilde{b}\,(t, X_t, E(\Psi(X_t), a)\,\mu_t(du) = \sum_{i=0}^{d+d^2+1} \lambda_i(t)\widetilde{b}(t, x_t, u_i(t)).$$

Then it is easy to verify that the process defined by its drift $\sum_{i=0}^{d+d^2+1} \lambda_i(t) b(t, X_t, E(\Psi(X_t), u_i(t))$ and its quadratic variation $\sum_{i=0}^{d+d^2+1} \lambda_i(t)\sigma\sigma^*(t, X_t, E(\Psi(X_t), u_i(t))$ is the solution of the MFSDE 3.19, defined (possibly on an extension of the initial probability space because of the possible degeneracy of the matrix $\sigma\sigma^*$). ■

Corollary 3.2.1 *Assume that the set*

$$P(t, X_t) = \{(b(t, X_t, E(\Psi(X_t), a), h(t, X_t, E(\Psi(X_t), a))\,;\; a \in A\} \subset \mathbb{R}^{d+d^2+1}$$

is convex. Then the relaxed optimal control is realized by a strict control.

Proof. Using Proposition 2.18, it follows that for each relaxed control μ we have

$$\int_A \widetilde{b}\,(t, X_t, E(\Psi(X_t), a)\,\mu_t(da) \in Conv\,(P(t, X_t))$$

Since $P(t, X_t)$ is convex then $Conv\,(P(t, X_t)) = P(t, X_t)$. Then applying the same arguments as in Prop. 2.18, there exists a measurable $\mathcal{F}_t-$adapted process u_t such that

$$\int_A \widetilde{b}\,(t, X_t, E(\Psi(X_t), a)\,\mu_t(du) = \widetilde{b}(t, X_t, u_t)$$

which implies that X_t is a solution of the MFSDE

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t), u_t)dW_t \\ X_0 = x \end{cases}$$

and $J(\mu) = J(u)$. This ends the proof. ■

3.3 Conclusion

We have proved existence of an optimal relaxed control, for mean-field stochastic control problems, where both the drift and diffusion coefficient are controlled. The natural stochastic equation corresponding to a relaxed control is a stochastic equation driven by an orthogonal martingale measure. As we have proved in a counter-example, replacing the *drift and diffusion coefficient by their relaxed counterparts in the relaxed equation, does not lead to a true extension of the original problem. In fact, the case where the diffusion coefficient is controlled is not a direct extension of the deterministic case and reflects the stochastic nature of the problem.*

Chapter 4

General mean-field stochastic control problems II: the relaxed maximum principle

Abstract
We consider optimal control problems for systems governed by mean-field stochastic differential equation, where the control enters both the drift and the diffusion coefficient. We consider the relaxed model, in which admissible controls are measure-valued processes and the natural relaxed state process is governed by stochastic differential equation driven by an orthogonal martingale measure, whose covariance measure is the relaxed control.We establish optimality necessary conditions, in terms of two adjoint processes extending Peng's maximum principle.

Key words: Mean-field stochastic differential equation, relaxed control, martingale measure,
adjoint process, stochastic maximum principle, variational principle.
MSC 2010 subject classifications, 93E20, 60H30.

4.1 Introduction

We consider mean-field control problems, where the state process is governed by a mean-field stochastic differential equation (MFSDE in short). In these equations the drift and diffusion coefficient are controlled and depend not only on the state but also on the distribution of the state. The present paper is the natural continuation of [7], in which we have proved the existence of an optimal solution for the relaxed model. We refer also to [6] for the case where only the drift is controlled. The relaxed control problem is a natural extension of the original problem, in the sense that they have the same value function. Moreover admissible controls are measure valued processes and the state process is governed by a MFSDE, driven by an orthogonal martingale measure. In the present article, we establish necessary conditions for optimality in the form of a relaxed stochastic maximum principle, obtained via the first and second order adjoint processes, extending Peng's maximum principle [41] to mean field control problems and [11] to relaxed controls. The advantage of our result is that the maximum pinciple applies to a natural class of controls, which is the closure of the class of strict controls, for which we know that an optimal control exists. The proof of the main result is based on the approximation of the relaxed control problem by a sequence of strict control problems. Then Ekeland's variational principle is applied to get necessary conditions of near-optimality for the sequence of nearly optimal strict controls. The result is obtained by a passage to the limit in the state equation as well as in the adjoint processes. The resulting first and second order adjoint processes are solutions of linear backward SDEs driven by a Brownian motion and an orthogonal square integrable martingale. The other advantage of our result is that it is given via an approximation procedure, so that it could be convenient for numerical computation.

Note that in [16] (A. Chala, *The relaxed optimal control problem for Mean-Field SDEs systems and application*, Automatica **50** (2014) 924–930), the author has addressed the same problem and obtained a maximum principle by using only the first order adjoint process. In the case where the optimal control is a strict control and there are no mean-field terms, this result seems to improve fundamentally Peng's maximum principle [41], obtained earlier with two adjoint processes. But there are many counter-examples (see [49], page 117), for which the first order adjoint process only, is not sufficient to get necessary conditions. In fact, the "relaxed" model considered in [16], even if it seems natural, does not fulfill the minimal requirements of a true extension of the original problem. As it was proved in [7] by a simple counter example, the "relaxed" state process considered is not continuous in the control variable and as a byproduct, the original and relaxed control problems are not equivalent. In fact the problem lies in the type of relaxation itself. The author has established a stochastic maximum principle for a problem where admissible controls are measure valued and the coefficients are linear in the control variable, in the same convex framework, as in Bensoussan [9]. Therefore it is not surprising that the resulting maximum principle is given via only the first order adjoint process. However, this problem cannot, in any case, be considered as the relaxed model for the original control problem.

Mean-field stochastic differential equations describe limits of interacting particle systems, as the number of particles tends to infinity. This property which is called "*propagation of chaos*", says that all the limiting particles are independent and satisfy the same MFSDE, called the Mc Kean-Vlasov equation. It represents the average behavior of the infinite number of particles, see [43, 31] for details. Since the pioneering papers [34], [28], mean-field control theory has interested many researchers, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil ressources [1, 10, 15]. It should be pointed out that the Bellmann optimality principle does not hold for mean-field stochastic control problems. For this kind of problems, the stochastic maximum principle, provides a powerful tool, see [2, 11, 17, 19, 35, 38, 47]. For classical control problems, the stochastic maximum principle has been investigated first by Kushner [32], then by Haussmann [24] for feed-back controls by using Girsanov theorem. The case of controlled diffusion coefficient has been treated by Bensoussan [9], by making convex perturbations and obtained a weak maximum principle, by using only the first order adjoint process. The general non convex case has been solved by Peng [41], in the framework of the modern theory of backward stochastic differential equations (BSDEs). It should be pointed out that Peng's maximum principle was obtained via the first and second order adjoint processes. Relaxed classical control problems have been investigated in [3, 39].We refer to the book by Yong and Zhou [49] for a systematic study of the suject and a complete list of references.

4.2 Assumptions and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions and (W_t) a $(\mathcal{F}_t, P)-$Brownian motion. Let $\mathbb{A}$ be some compact metric space called the action space. A strict control (u_t) is a measurable, $\mathcal{F}_t-$ adapted process with

values in the action space $\mathbb{A}$. We denote $\mathcal{U}_{ad}$ the space of strict controls.
The state process corresponding to a strict control is the unique solution, of the mean-field stochastic differential equations (MFSDE)

$$\begin{cases} dX_t = b(t, X_t, E(X_t), u_t)dt + \sigma(t, X_t, E(X_t), u_t)dW_t \\ X_0 = x. \end{cases} \tag{4.1}$$

The cost functional associated to a strict control u is given by

$$J(u) = E\left(\int_0^T h(t, X_t, E(X_t), u_t\right) dt + g(X_T, E(X_T)). \tag{4.2}$$

The coefficients in the state equation as well as in the cost functional are of mean-field type, in the sense that they depend not only on the state process, but also on its marginal law through its expectation.
The objective is to minimize $J(u)$ over the space $\mathcal{U}_{ad}$, that is find $u^* \in \mathcal{U}_{ad}$ such that $J(u^*) = \inf \{J(u^*), u \in \mathcal{U}_{ad}\}$.
The following assumptions will be in force throughout this paper.
$(\mathbf{H_1})$ Assume that

$$\begin{aligned} b &: [0,T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R} \\ \sigma &: [0,T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R} \end{aligned} \tag{4.3}$$

are bounded continuous functions such that $b(t,.,.,a)$ and $\sigma(t,.,.,a)$ are twice continuously differentiable with respect to (x, y). Assume also that the derivatives of b, σ, up to the second order with respect to (x, y) are bounded and continuous in (x, y, a).
$(\mathbf{H_2})$ Assume that

$$\begin{aligned} h &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R} \\ g &: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \end{aligned} \tag{4.4}$$

are bounded continuous functions such that $h(t,.,.,a)$ and $g(.,.)$ are twice continuously differentiable with respect to (x, y). Assume also that the derivatives of h, g, up to the second order with respect to (x, y) are bounded and continuous in (x, y, a).
The coefficients are assumed to be one dimensional as in [11] to avoid heavy notations in the definition of adjoint processes. In fact, the results remain valid for multidimensional

processes.

Under assumption ($\mathbf{H_1}$), according to [31] Prop.1.2, for each $u \in \mathcal{U}_{ad}$ the MFSDE (4.1) has a unique strong solution, such that for each $p > 0$ we have $E(|X_t|^p) < +\infty$. Moreover the cost functional is well defined.

4.3 The relaxed control problem

4.3.1 The space of relaxed controls

In the absence of Fillipov type convexity conditions, an optimal control may fail to exist in the set $\mathcal{U}_{ad}$ of strict controls (see e.g. [23]). Let us consider a deterministic example. Minimize $J(u) = \int_0^T (X^u(t))^2 \, dt$ over the set $\mathcal{U}_{ad}$ of measurable functions $u : [0,T] \to \{-1,1\}$, where $X^u(t)$ is the solution of $dX^u(t) = u(t)dt$, $X(0) = 0$. We have $\inf_{u \in \mathcal{U}_{ad}} J(u) = 0$.

Indeed, consider the sequence of Rademacher functions:

$$u_n(t) = (-1)^k \text{ if } \tfrac{kT}{n} \leq t \leq \tfrac{(k+1)T}{n},\ 0 \leq k \leq n-1.$$

Then clearly $|X^{u_n}(t)| \leq 1/n$ and $|J(u_n)| \leq T/n^2$ which implies that $\inf_{u \in \mathcal{U}_{ad}} J(u) = 0$. There is however no control $\widehat{u}$ such that $J(\widehat{u}) = 0$. If this would have been the case, then for every t, $X^{\widehat{u}}(t) = 0$. This in turn would imply that $\widehat{u}_t = 0$, which is impossible.

The problem is that the sequence (u_n) has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality. If we identify $u_n(t)$ with the Dirac measure $\delta_{u_n(t)}(du)$, then $(dt\delta_{u_n(t)}(du))_n$ converges weakly to $(dt/2)\cdot[\delta_{-1} + \delta_1](du)$.

The idea is to embedd the set $\mathcal{U}_{ad}$ into the set $\mathcal{R}$ of relaxed controls.

Let $\mathbb{V}$ be the set of product measures μ on $[0,T] \times \mathbb{A}$ whose projection on $[0,T]$ coincides with the Lebesgue measure dt. It is clear that every μ in $\mathbb{V}$ may be disintegrated as $\mu = dt.\mu_t(da)$, where $\mu_t(da)$ is a transition probability.

Then $\mathbb{V}$ equipped with the topology of stable convergence is a compact metric space, where test funtions are measurable, bounded functions $f(t,a)$ continous in a, see [21, 30] for further details.

Definition 4.3.1 *A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable* *with respect to* $(\mathcal{F}_t)$ *and such that for each* t, $1_{(0,t]}.\mu$ *is* $\mathcal{F}_t-$*measurable. Let us denote* $\mathcal{R}$ *the space of relaxed controls.*

Remark 4.3.1 *The set* $\mathcal{U}_{ad}$ *of strict controls is embedded into the set* $\mathcal{R}$ *of relaxed controls by identifying* u_t *with the random measure* $dt\delta_{u_t}(da)$.

4.3.2 The relaxed state equation

Following [31] Prop. 1.10, existence of a weak solution of equation 4.1 associated to a strict control u is equivalent to the existence of a solution for the non linear martingale problem:

$$f(X_t) - f(X_0) - \int_0^t L^{P_{X_s}} f(s, X_s, u_s)\, ds \text{ is a } P-\text{martingale},$$

for every $f \in C_b^2$, *for each* $t > 0$, *where* L *is the infinitesimal generator* associated with equation 4.1,

$$L^{\nu} f(t,x,a) = \frac{1}{2}\sum_{i,j}\left(a_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(t,x,a) + \sum_i \left(b_i \frac{\partial f}{\partial x_i}\right)(t,x,a),$$

$b = b(t, x, \langle y, \nu\rangle, a)$ and $a_{i,j} = \sigma\sigma^*(t, x, \langle y, \nu\rangle, a)$, $\nu \in \mathbb{M}_1(\mathbb{R}^d)$.
Therefore, the natural generator associated to a relaxed control is given $\int_A L^{\nu} f(s,x,a)\, \mu_s(da)ds$ and accordingly the relaxed martingale problem is defined as follows:

$$f(X_t) - f(X_0) - \int_0^t \int_A L^{P_{X_s}} f(s, X_s, a)\, \mu_s(da)ds \text{ is a } P - \text{martingale} \tag{4.5}$$

for each $f \in C_b^2$, *for each* $t > 0$.
The following theorem gives a pathwise reprsentation of the solution of the relaxed martingale problem, in terms of a mean-field stochastic differential equation driven by an orthogonal martingale measure.

Theoreme 4.3.1 *1)Let* P *a the solution of the martingale problem 4.5. Then* P *is the law of a* d*–dimensional adapted and continuous process* X *defined on an extension of the space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *and solution of the following MFSDE starting at* x:

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, E(X_t)\, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, E(X_t), a)\, M(da, dt), \\ X_0 = x \end{cases} \tag{4.6}$$

where $M = (M^k)_{k=1}^d$ *is a family of* d*-strongly orthogonal continuous martingale measures, each having intensity* $dt\mu_t(da)$*.*
2) If the coefficients b *and* σ *are Lipschitz in* x*,* y*, uniformly in* t *and* a*, the SDE (2.6) has a unique pathwise solution.*

Proof. 1) The proof is based essentially on the same arguments as in [20] Theorem IV-2 and [31] Prop. 1.10.
2) The coefficients being Lipschitz continuous, following the same steps as in [31] and [20], it is not difficult to prove that Equation 4.6 has a unique solution such that for each $p > 0$ we have $E(|X_t|^p) < +\infty$. ■

Remark 4.3.2 *i) Note that the othogonal martingale measure corresponding to the relaxed control* $dt\mu_t(da)$ *is not unique.*
ii) From now on, the probability space is an extension of the initial probability space. The Brownian motion (W_t) *remains a Brownian motion on this new probability space, but the filtration is no longer the natural filtration of* (W_t)*.*

Martingale measures have been introduced by Walsh [45], see also [20, 37] for more details.

Definition 4.3.2 *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *be a filtered probability space, and* $M(t, B)$ *a random process, where* $B \in \mathcal{B}(\mathbb{A})$*.* M *is a* $(\mathcal{F}_t, P)-$*martingale measure if:*
1)For each $B \in \mathcal{B}(\mathbb{A}), (M(t, B))_{t\geq 0}$ *is a square integrable martingale,* $M(0, B) = 0$*.*
2)For every $t > 0$*,* $M(t, .)$ *is a* $\sigma-$*finite* L^2*-valued measure.*
It is called continuous if for each $B \in \mathcal{B}(\mathbb{A})$*,* $M(t, B)$ *is continuous and orthogonal if* $M(t, B).M(t, C)$ *is a martingale whenever* $B \cap C = \phi$*.*

Remark 4.3.3 *When the martingale measure* M *is orthogonal, it is proved in [45] the existence of a random positive* σ*-finite measure* $\mu(dt, da)$ *on* $[0, T]\times\mathbb{A}$ *such that* $\langle M(., B), M(., B)\rangle_t = \mu([0, t] \times B)$ *for all* $t > 0$ *and* $B \in \mathcal{B}(\mathbb{A})$*.* $\mu(dt, da)$ *is called the covariance measure of* M*.*

Example Let $\mathbb{A} = \{a_1, a_2, \cdots, a_n\}$ be a finite set. Then every relaxed control $dt\, \mu_t(da)$ will be a convex combination of the Dirac measures $dt\, \mu_t(da) = \sum_{i=1}^n \alpha_t^i\, dt\, \delta_{a_i}(da)$, where for each i, α_t^i is a real–valued process such that $0 \leq \alpha_t^i \leq 1$ and $\sum_{i=1}^n \alpha_t^i = 1$. It is obvious that the solution of the relaxed martingale problem *4.5* is the law of the solution of the SDE

$$dX_t = \sum_{i=1}^{n} b(t, X_t, E(\Psi(X_t)), a_i)\alpha_t^i dt + \sum_{i=1}^{n} \sigma(t, X_t, E(\Psi(X_t)), a_i) \left(\alpha_t^i\right)^{1/2} dW_s^i, \qquad X_0 = x, \tag{4.7}$$

where the W^i's are independant Brownian motions, on an extension of the initial probability space. The process M defined by

$$M([0,t] \times A) = \sum_{i=1}^{n} \int_0^t \left(\alpha_s^i\right)^{1/2} 1_{\{a_i \in A\}} dB_s^i$$

is in fact an orthogonal continuous martingale measure (cf. [21, 45]) with intensity $\mu_t(da)dt = \sum \alpha_t^i \delta_{a_i}(da)dt$. Thus, the SDE 4.7 can be expressed in terms of M and μ as follows:

$$dX_t = \int_{\mathbb{A}} b(t, X_t, E(\Psi(X_t))\, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt)$$

4.4 Maximum principle for relaxed control problems

From now on, the relaxed control problem is defined via the relaxed state process, which satisfies the following relaxed MFSDE

$$\begin{cases} dX_t = \int_A b(t, X_t, E(X_t), a)\mu_t(da)dt + \int_A \sigma(t, X_t, E(X_t), a)M(dt, da) \\ X_0 = x, \end{cases} \tag{4.8}$$

and the relaxed cost functional is given by

$$J(u) = E\left[\int_0^T \int_A h(t, X_t, E(X_t), a)\mu_t(da)dt + g(X_T, E(X_T))\right]. \tag{4.9}$$

$M(dt, da)$ is a vector of continuous orthogonal martingale measures with common covariance measure $\mu_t(da)dt$.

It is proved in [7] that, the relaxed control problem is a natural extension of the strict

control problem, in the sense that the value functions of the strict and relaxed control problems are equal. Morever, the relaxed control problem admits an optimal control.
In this section, we'll derive necessary conditions for optimality, satisfied by an optimal relaxed control. We begin by deriving necessary conditions for near optimality satisfied by the minimizing sequence of stict controls, which converge to the relaxed control. Then we pass to the limit in the state equation as well as in the adjoint processes to obtain the relaxed maximum principle. To achieve this program, the following approximation lemmas will play a key role in the sequel.

Lemma 4.4.1 *(Chattering lemma) i) Let (μ_t) be a relaxed control. Then there exists a sequence of adapted processes (u_t^n) with values in $\mathbb{A}$, such that the sequence of random measures $\left(\delta_{u_t^n}(da)\, dt\right)$ converges weakly in $\mathbb{V}$ to $\mu_t(da)\, dt$, $P-a.s.$*
ii) For any g continuous in $[0,T]\times \mathbb{M}_1(\mathbb{A})$ such that $g(t,.)$ is linear, we have $P-a.s$

$$\lim_{n\to+\infty}\int_0^t g(s,\delta_{u_s^n})ds = \int_0^t g(s,\mu_s)ds \text{ uniformly in } t\in[0,T]. \tag{4.10}$$

Proof. See [21] and [23] Lemma 1 page 152. ■

Proposition 4.4.1 *1) Let $\mu=\mu_t(da)\, dt$ a relaxed control. Then there exist a continuous orthogonal martingale measure $M(dt,da)$ whose covariance measure is given by $\mu_t(da)\, dt$.*
2) If we denote $M^n(t,B)=\int_0^t\int_B \delta_{u_s^n}(da)dW_s$, where (u^n) is defined as in the last Lemma, then for every bounded predictable process $\varphi:\Omega\times[0,T]\times\mathbb{A}\to\mathbb{R}$, such that $\varphi(\omega,t,.)$ is continuous, we have

$$E\left[\left(\int_0^t\int_{\mathbb{A}}\varphi(\omega,t,a)M^n(dt,da)-\int_0^t\int_{\mathbb{A}}\varphi(\omega,t,a)M(dt,da)\right)^2\right]\to 0 \text{ as } n\longrightarrow+\infty,$$

Proof. See [37] page 18. ■

Proposition 4.4.2 *1) Let X_t, X_t^n be the solutions of state equation (4.6) corresponding to μ and u^n, where μ and u^n are defined as in the chattering lemma. Then*

$$\lim_{n\to\infty}E\left[\sup_{0\le t\le T}\left|X_t^n-X_t\right|^2\right]=0. \tag{4.11}$$

2) Let $J(u^n)$ and $J(\mu)$ the expected costs corresponding respectively to u^n and μ. Then there exists a subsequence (u^{n_k}) of (u^n) such that $J(u^{n_k})$ converges to $J(\mu)$.

Proof. See [7] ■

4.4.1 Necessary conditions for near optimality

Let $\mu = dt\mu_t(da)$ be an optimal relaxed control and X be the corresponding state process solution of 4.6. According to the optimality of μ and the chattering lemma, there exists a sequence $(u_n) \subset \mathcal{U}_{ad}$ such that:

$$J(u^n) = J(\mu^n) \leq \inf \{J(\mu);\ \mu \in \mathcal{R}\} + \varepsilon_n,$$

where $\mu^n = dt\delta_{u_t^n}(da)$ and $\lim\limits_{n \to +\infty} \varepsilon_n = 0$.
In this section, we give necessary conditions for near-optimality satisfied by the minimizing sequence (u^n). Let us recall Ekeland's variational principle [18]..

Lemma 4.4.2 *(Ekeland) Let (V, d) be a complete metric space and $F : V \longrightarrow \mathbf{R} \cup \{+\infty\}$ be lower-semicontinuous and bounded from below. Given $\epsilon > 0$, suppose $u^\epsilon \in V$ satisfies $F(u^\epsilon) \leq \inf \{F(v)\ ; v \in V\} + \epsilon$. Then for any $\lambda > 0$, there exists $v \in V$ such that:*
i) $F(v) \leq F(u^\epsilon)$
ii) $d(u^\epsilon, v) \leq \lambda$
iii) $\forall w \neq v\ ;\ F(v) < F(w) + \varepsilon/\lambda.d(w, v)$.

For u, v in $\mathcal{U}_{ad}$, define $d(u, v) = P \otimes dt \{(\omega, t) \in \Omega \times [0, T]\,;\ u(\omega, t) \neq v(\omega, t)\}$,where $P \otimes dt$ is the product measure of P and the Lebesgue measure. It is clear that d defines a metric in $\mathcal{U}_{ad}$.

Lemma 4.4.3 *i) $(\mathcal{U}_{ad}, d)$ is a complete metric space.*
ii) For any $p \geq 1$, there exists $M > 0$ such that for any $u, v \in \mathcal{U}_{ad}$

$$E\left[\sup_{0 \leq t \leq T} |X_t^u - X_t^v|^{2p}\right] \leq M.\left(d(u, v)\right)^{1/2},$$

where X_t^u, X_t^v are the solutions of 4.1 corresponding to u and v.
iii) The cost functional $J : (\mathcal{U}_{ad}, d) \longrightarrow \mathbb{R}$ is continuous. More precisely if u, v are two elements in $\mathcal{U}_{ad}$ then

$$|J(u) - J(v)| \leq C.\left(d(u, v)\right)^{1/2}$$

Proof. The proof goes as in [50] Lemma 3.1 and uses classical arguments from stochastic calculus, such as Burkholder-Davis-Gundy and Hölder's inequalities and Gronwall lemma. The fact that the coefficients are of mean-field type and depend on the expectation of the solution, does not add new difficulties. ■

Let us define the Hamiltonian of the system associated to a random variable X

$$H(t,X,u,p,q) = b(t,X,E(X),u).p + \sigma(t,X,E(X),u).q - h(t,X,E(X),u) \tag{4.12}$$

For any strict control $u \in \mathcal{U}$, we denote (p,q) and (P,Q) the first and second order adjoint processes satisfying the following backward SDEs

$$\begin{cases} dp(t) = & -\left[b_x(t)p(t) + E\left(b_y(t)p(t)\right) + \sigma_x(t)q(t) + E\left(\sigma_y(t)q(t)\right)\right. \\ & \left. -h_x(t) - E\left(h_y(t)\right)\right] dt + q(t)dW_t + dM_t \\ p(T) = & -g_x(T) - E\left(g_y(T)\right) \end{cases} \tag{4.13}$$

$$\begin{cases} -dP(t) = & -\left[2b_x(t)P(t) + \sigma_x^2(t)P(t) + 2\sigma_x(t)Q(t) + H_{xx}(t)\right]dt \\ & +Q(t)dW_t + dN_t \\ P(T) = & -g_{xx}(x(T)) \end{cases} \tag{4.14}$$

where $X(t)$ is the state process associated with u, $f_x(t) = f_x(t,X_t,E(X_t),u_t)$ for $f = b$, σ, h and

$$H_{xx}(t,X,u,p,q) = b_{xx}(t,X,E(X),u).p + \sigma_{xx}(t,X,E(X),u).q - h_{xx}(t,X,E(X),u).$$

M and N are square integrable martingales which are orthogonal to the Brownian motion and are parts of the solutions. The appearance of such martingales is due to the fact that $(\mathcal{F}_t)$ is not necessarily the Brownian filtration.

Equation 4.13 is a mean field backward stochastic differential equation (MFBSDE), whose driver is Lipshitz continuous, then by [14] Theorem 3.1, it has a unique $\mathcal{F}_t-$adapted solution (p,q,M) such that:

$$E\left[\sup_{0\leq t\leq T} |p(t)|^2 + \int_0^T |q(t)|^2\, dt + [M,M]_T\right] < +\infty \tag{4.15}$$

Note that in [14] Theorem 3.1, $(\mathcal{F}_t)$ is the Brownian filtration. Considering general filtrations on which a Brownian motion is defined does not bring additional difficulties in the proof of existence and uniqueness (see e.g [22] Theorem 5.1, page 54).
Equation 4.14 is a classical backward stochastic differential equation, whose driver is Lipshitz continuous, then by [22] Theorem 5.1, it has a unique $\mathcal{F}_t-$adapted solution (P, Q, N) such that:

$$E\left[\sup_{0\leq t\leq T}|P(t)|^2+\int_0^T|Q(t)|^2\,dt+[N,N]_T\right]<+\infty$$

The following lemma is a stability result of the adjoints processes with respect to the control variable.

Lemma 4.4.4 *For any $0<\alpha<1$ and $1<p<2$ satisfying $(1+\alpha)<2$, there exists a constant $C_1=C_1(\alpha,p)>0$ such that for any strict controls u, u' along with the corresponding trajectories X, X' and the solutions (p,q,P,Q,M,N), (p',q',P',Q',M',N') of the backward SDEs 4.13 and 4.14, the following estimates hold*

$$E\left[\int_0^T\left(|p(t)-p'(t)|^p+|q(t)-q'(t)|^p\right)dt+[M-M',M-M']_T^{p/2}\right]\leq C_1 d(u,u')^{\frac{\alpha p}{2}}$$
$$E\left[\int_0^T\left(|P(t)-P'(t)|^p+|Q(t)-Q'(t)|^p\right)dt+[N-N',N-N']_T^{p/2}\right]\leq C_2 d(u,u')^{\frac{\alpha p}{2}}$$

Proof. The proof goes as in [50] Lemma 3.2. The only difference is that the driver is of mean-field type. But this does not add new difficulties, as the driver is linear and then Lipschitz in the state variable as well as in its expectation. ■

Necessary conditions for near optimality

The $\mathcal{H}-$function or generalized Hamiltonian (see [49] page 118), associated with a strict control u and its state process X is defined as follows:

$$\mathcal{H}^{(X(.),u(.))}(t,Y,E(Y),a)=H(t,Y,E(Y),a,p(t),q(t)-P(t).\sigma(t,X_t,E(X_t),u(t)))$$
$$-\tfrac{1}{2}\sigma^2(t,Y,E(Y),a)P(t)$$

where $(p(t),q(t))$, $(P(t),Q(t))$ are solutions of the adjoint equations 4.13 and 4.14.
The next proposition gives necessary conditions for near-optimality satisfied by the minimizing sequence (u^n) (ie . $(\mu^n)=\left(dt\delta_{u_t^n}(da)\right)$ converges to the optimal relaxed control $dt\mu_t(da)$).

Proposition 4.4.3 *Let u^n be an admissible strict control such that*

$$J(u^n) = J(\mu^n) \leq \inf\{J(\mu);\ \mu \in \mathcal{R}\} + \varepsilon_n,$$

then there exist adapted (p^n, q^n, M^n) and (P^n, Q^n, N^n), solutions of the adjoint equations 4.13 and 4.14, corresponding to the admissible pair (u^n, X^n) such that:

$$E\left(\int_0^T \mathcal{H}^{(X^n(t),u^n(t))}(t, X^n(t), u^n(t))dt\right) \geq \sup_{a\in A} E\left(\int_0^T \mathcal{H}^{(X^n(t),u^n(t))}(t, X^n(t), a)dt\right) - \varepsilon^{1/3} \tag{4.16}$$

Proof. According to Lemma 4.5, the cost functional $J(u)$ is continuous with respect to the topology induced by the metric d. Then by applying Ekeland's variational principle for u^n with $\lambda = \varepsilon^{2/3}$, there exists an admissible control v^n such that
$d(u^n, v^n) \leq \varepsilon^{2/3}$,
$\widehat{J}(v^n) \leq \widehat{J}(u)$ for all $u \in \mathcal{U}$,
$\widehat{J}(u) = J(u) + \varepsilon^{1/3} d(u, v^n)$.
The control v_n which is ε_n-optimal is in fact optimal for the new cost functional $\widehat{J}(u)$. We proceed as in the classical mean-field maximum principle [11] to derive a maximum principle for v^n. Let $t_0 \in (0,1)$, $a \in \mathbb{A}$ and define the spike variation of $v_n(t)$

$$v_\delta^n = \begin{cases} a \text{ on } (t_0, t_0+\delta) \\ v_n(t) \text{ otherwise} \end{cases}$$

The fact that $\widehat{J}(v^n) \leq \widehat{J}(u)$ and $d(v^n, v_\delta^n) \leq \delta$ imply that
$J(v_\delta^n) - J(v^n) \geq -\varepsilon_n^{1/3}\,\delta$.
Proceeding as in [11], we can expand $Y_\delta^n(.)$ (the solution of 4.1 corresponding to v_δ^n) to the second order, to get the following inequality

$$E\left[\int_{t_0}^{t_0+\delta} \tfrac{1}{2}\left(\sigma(t, Y^n(t), a) - \sigma(t, Y^n(t), v^n\right)^2 P_t^n + p_t^n \left(b(t, Y^n(t), a) - b(t, Y^n(t), v^n)\right)\right.$$
$$+ q_t^n \left(\sigma(t, Y^n(t)(t), a) - \sigma(t, Y^n(t), v^n)\right)$$
$$\left. + \left(h(t, Y^n(t), a) - h(t, Y^n(t), v^n)\right)dt\right] + o(\delta) \geq -\varepsilon_n\,\delta,$$

where $Y^n(t)$ is the state process (solution of 4.1) corresponding to the control v^n and (p^n, q^n) and (P^n, Q^n) are the first and second order adjoint processes, solutions of 4.13

and 4.14 corresponding to (v^n, Y^n).
The variational inequality is obtained for v^n by dividing by δ and tending δ to 0.
The same claim can be proved for u^n by using the stability of the state equations and the adjoint processes with respect to the control variable (Lemma 4.5 and Lemma 4.6) ■

Remark 4.4.1 *The variational inequality (4.7) can be proved with the supremum over $a \in A$ replaced by the supremum over $u \in \mathcal{U}_{ad}$ by simply putting $u(t)$ in place of a in the definition of the strong perturbation.*

4.4.2 The relaxed maximum principle

We know that the relaxed control problem has an optimal solution μ. Let $X(.)$ be the corresponding optimal state process. Let (p, q, M) and (P, Q, N) the solutions of the first and second order adjoint equations, associated with the optimal relaxed pair (μ, X).

$$\begin{cases} dp(t) = & -\left[\overline{b}_x(t)p(t) + E\left(\overline{b}_y(t)p(t)\right) + \overline{\sigma}_x(t)q(t) + E\left(\overline{\sigma}_y(t)q(t)\right)\right. \\ & \left. -\overline{h}_x(t) - E\left(\overline{h}_y(t)\right)\right]dt + q(t)dW_t + dM_t \\ p(T) = & -\overline{g}_x(T) - E\left(\overline{g}_y(T)\right) \end{cases} \tag{4.17}$$

$$\begin{cases} -dP(t) = & -\left[2\overline{b}_x(t)P(t) + \overline{\sigma}_x^2(t)P(t) + 2\overline{\sigma}_x(t)Q(t) + \overline{H}_{xx}(t)\right]dt \\ & +Q(t)dW_t + dN_t \\ P(T) = & -\overline{g}_{xx}(x(T)) \end{cases} \tag{4.18}$$

where we denote $\overline{f}(t) = f(t, x(t), \mu(t)) = \int_A f(t, x(t), a)\mu(t, da)$ and f stands for b_x, σ_x, h_x, b_y, σ_y, h_y, H_{xx}.
(M_t) and (N_t) are square integrable martingales which are orthogonal to the Brownian motion (W_t).
The drivers of the BSDEs 4.17 and 4.18 being Lipschitz continuous, then by [14] Theorem 3.1 and [22] Theorem 5.1, they admit unique solutions (p, q, M) and (P, Q, N) satisfying:

$$E\left[\sup_{0\le t\le T} |p(t)|^2 + \int_0^T |q(t)|^2 dt + [M, M]_T\right] < +\infty,$$
$$E\left[\sup_{0\le t\le T} |P(t)|^2 + \int_0^T |Q(t)|^2 dt + [N, N]_T\right] < +\infty.$$

Define the generalized Hamiltonian function associated with the optimal pair $(\mu, X(.))$ and the corresponding adjoint processes,

$$\mathcal{H}^{(X(.),\mu(.))}(t,Y,E(Y),a)=H(t,Y,E\left(Y\right),a,p(t),q(t)-P(t).\sigma(t,X_t,E(X_t),\mu(t)))$$
$$-\tfrac{1}{2}\sigma^2(t,Y,E\left(Y\right),a)P(t)$$

The main result of this paper is the following.

Theoreme 4.4.1 ***(Relaxed maximum principle)***

Let (μ, x) be an optimal relaxed pair, then there exist adapted pairs (p, q) and (P, Q), solutions of the adjoint equations 4.17 and 4.18 respectively, such that

$$E\left(\int_0^1 \mathcal{H}^{(X(t),\mu(t))}(t,X(t),\mu(t))dt\right)=\sup_{a\in A}E\left(\int_0^1 \mathcal{H}^{(X(t),\mu(t))}(t,X(t),a)dt\right) \tag{4.19}$$

The proof of Theorem 4.9 is based will be given later.

Corollary 4.4.1 *Under the same conditions as in Theorem 4.9 it holds that*

$$E\left(\int_0^1 \mathcal{H}^{(X(t),\mu(t))}(t,X(t),\mu(t))dt\right)=\sup_{\upsilon\in\mathbb{P}(A)}\int_0^T E\left[\mathcal{H}^{(X(t),\mu(t))}(t,X(t),\upsilon)\right]dt \tag{4.20}$$

where $\mathcal{H}^{(x(t),\mu(t))}(t,x(t),\upsilon)=\int_{\mathbb{A}}\mathcal{H}^{(x(t),\mu(t))}(t,x(t),a)\upsilon(da)$.

Proof. The inequality

$\sup\limits_{\mu\in\mathbb{P}(\mathbb{A})}\int_0^T E\left[\mathcal{H}^{(x(t),\mu(t))}(t,x(t),\upsilon)\right]dt\geq\sup\limits_{a\in\mathbb{A}}E\left(\int_0^1 \mathcal{H}^{(x(t),\mu(t))}(t,x(t),a)dt\right)$

is obvious. Indeed it suffices to take $\mu=\delta_a$ where a is any element of $\mathbb{A}$. Now if $\upsilon\in\mathbb{P}\left(A\right)$ is a probability measure on $\mathbb{A}$, then

$\int_0^T E\left[\mathcal{H}^{(x(t),\mu(t))}(t,x(t),\upsilon)\right]dt\in conv\left\{E\left(\int_0^1 \mathcal{H}^{(x(t),\mu(t))}(t,x(t),a)dt\right),a\in\mathbb{A}\right\}$

Hence, by using a result on convex analysis, it holds that

$\int_0^T E\left[\mathcal{H}^{(x(t),\mu(t))}(t,x(t),\upsilon)\right]dt\leq\sup\limits_{u\in\mathbb{A}}E\left(\int_0^1 \mathcal{H}^{(x(t),\mu(t))}(t,x(t),a)dt\right)$. ■

Remark . Since $\mathbb{P}(\mathbb{A})$ is a subspace of $\mathbb{V}$ whose elements are constant (in (ω,t)) relaxed controls, then 4.20 may be replaced by

$$E\left(\int_0^1 \mathcal{H}^{(x(t),\mu(t))}(t,x(t),\mu(t))dt\right)=\sup_{\upsilon\in\mathbb{V}}\int_0^T E\left[\mathcal{H}^{(x(t),\mu(t))}(t,x(t),\upsilon(t))\right]dt \tag{4.21}$$

Corollary 4.4.2 *(The Pontriagin relaxed maximum principle).*

If (μ, X) denotes an optimal relaxed pair, then there exists a Lebesgue negligible subset N such that for any t not in N

$$\mathcal{H}^{(X(t),\mu(t))}(t, x(t), \mu(t)) = \sup_{\upsilon \in \mathbb{P}(A)} \mathcal{H}^{(x(t),\mu(t))}(t, x(t), \upsilon), P - a.s. \tag{4.22}$$

Proof. Let $\theta \in \left]0, T\right[$ and $B \in \mathcal{F}_\theta$, for small $h > 0$ define the relaxed control

$$\mu_t^h = \begin{cases} \upsilon 1_B & \text{for } \theta < t < \theta + h \\ \widehat{\mu}_t & \text{otherwise.} \end{cases}$$

where υ is a probability measure on A. It follows from 4.20 that

$$1/h \int_\theta^{\theta+h} E\left[1_B \mathcal{H}^{(X(t),\mu(t))}(t, x(t), \mu(t))\right] dt \geq \ 1/h \int_\theta^{\theta+h} E\left[1_B \mathcal{H}^{(X(t),\mu(t))}(t, x(t), \upsilon)\right] dt$$

Therefore passing at the limit as h tends to zero, we obtain

$$E\left[1_B \mathcal{H}^{(X(\theta),\mu(\theta))}(\theta, x(\theta), \mu(\theta))\right] \geq E\left[1_B \mathcal{H}^{(X(\theta),\mu(\theta))}(\theta, x(\theta), \upsilon)\right]$$

for any θ not in some Lebesgue null set N.

The last inequality is valid for all $B \in \mathcal{F}_\theta$, then for any bounded $\mathcal{F}_\theta$-measurable random variable F it holds that

$$E\left[F \mathcal{H}^{(X(t),\mu(t))}(t, x(t), \mu(t))\right] \geq E\left[F \mathcal{H}^{(X(t),\mu(t))}(t, x(t), \upsilon)\right]$$

which leads to

$$E\left[\mathcal{H}^{(X(\theta),\mu(\theta))}(\theta, x(\theta), \mu(\theta))/\mathcal{F}_\theta\right] \geq E\left[\mathcal{H}^{(X(\theta),\mu(\theta))}(\theta, x(\theta), \upsilon)/\mathcal{F}_\theta\right]$$

The result follows from the measurability with respect to $\mathcal{F}_\theta$ of the quantities inside the conditional expectation. ∎

The proof of theorem (4.6) is based on the following lemma.

Lemma 4.4.5 *Let (p^n, q^n), (P^n, Q^n) (resp. (p, q), (P, Q)) be the solutions of first and second order adjoint equations 4.13 and 4.13 associated with the pair (u^n, X^n), (resp. solutions of first and second order adjoint equations 4.17 and 4.18 associated to (μ, X)), then it holds that*

$$i) \lim_{n \to +\infty} E\left[\int_0^T \left(|p^n(t) - p(t)|^2 + |q^n(t) - q(t)|^2\right) dt + [M^n - M, M^n - M]_T\right] = 0$$

$$ii) \lim_{n \to +\infty} E\left[\int_0^T \left(|P^n(t) - P(t)|^2 + |Q^n(t) - Q(t)|^2\right) dt + [N^n - N, N^n - N]_T\right] = 0$$

$$iii) \lim_{n \to +\infty} E\left(\int_0^1 \mathcal{H}^{(X^n(t),u^n(t))}(t, X^n(t), u^n(t)) dt\right) = E\left(\int_0^1 \mathcal{H}^{(X(t),\mu(t))}(t, X(t), \mu(t)) dt\right)$$

Proof. i)Let us write down the drivers of the first order adjoint equations *4.13 and* 4.17 corresponding to (u^n, X^n) and (μ, X).

$H^n(t,p_t^n,q_t^n) = -b_x^n(t)p^n(t)+E\left(b_y^n(t)p^n(t)\right)+\sigma_x^n(t)q^n(t)+E\left(\sigma_y^n(t)q^n(t)\right)-h_x^n(t)-E\left(h_y^n(t)\right)$

$H(t,p_t,q_t) = -\overline{b}_x(t)p(t)+E\left(\overline{b}_y(t)p(t)\right)+\overline{\sigma}_x(t)q(t)+E\left(\overline{\sigma}_y(t)q(t)\right)-\overline{h}_x(t)-E\left(\overline{h}_y(t)\right)$

where

$f^n(t) = f(t,X_t^n,E(X_t^n),u_t^n) = \displaystyle\int_{\mathbb{A}} f(t,X_t^n,E(X_t^n),a)\delta_{u_t^n}(da)$ for $f = b_x,\ \sigma_x,\ h_x, b_y, \sigma_y, h_y$.

$\overline{f}(t) = f(t,X(t),E(X_t),\mu(t)) = \int_A f(t,X(t),E(X_t),a)\mu(t,da)$ where f stands for b_x, σ_x, h_x,

b_y, σ_y, h_y.

By using a subtil stability result of Hu and Peng [27], Theorem 2.1, it is sufficient to show that:

$\lim_{n\to\infty} E\left[\left|\int_t^T (H^n(t,p_t,q_t)-H(t,p_t,q_t))\,dt\right|^2\right] = 0$

We have

$$\begin{aligned}\left|\int_t^T (H^n(t,p_t,q_t)-H(t,p_t,q_t))\,dt\right| &\leq \left|\int_t^T \left(b_x^n(t)-\overline{b}_x(t)\right)p(t)dt\right| + \left|\int_t^T E\left[\left(b_y^n(t)-\overline{b}_y(t)\right)p(t)\right]dt\right| \\ &+\left|\int_t^T \left(\sigma_x^n(t)-\overline{\sigma}_x(t)\right)q(t)dt\right| + \left|\int_t^T E\left[\left(\sigma_y^n(t)-\overline{\sigma}_y(t)\right)q(t)\right]dt\right| \\ &+\left|\int_t^T \left(h_x^n(t)-\overline{h}_x(t)\right)dt\right| + \left|\int_t^T E\left(h_y^n(t)-\overline{h}_y(t)\right)dt\right|\end{aligned} \tag{4.23}$$

Let us treat the first term in the right hand side .

$$\begin{aligned}\int_t^T \left(b_x^n(t)-\overline{b}_x(t)\right)p(t)dt &= \int_t^T \left(\int_{\mathbb{A}} b_x(t,X_t^n,E(X_t^n),a)\delta_{u_t^n}(da) - \int_A b_x(t,X_t,E(X_t),a)\mu_t(da)\right)p(t)dt \\ &= \int_t^T \left(\int_{\mathbb{A}} b_x(t,X_t^n,E(X_t^n),a)\delta_{u_t^n}(da) - \int_A b_x(t,X_t,E(X_t),a)\delta_{u_t^n}(da)\right)p(t)dt \\ &+\int_t^T \left(\int_{\mathbb{A}} b_x(t,X_t,E(X_t),a)\delta_{u_t^n}(da) - \int_A b_x(t,X_t,E(X_t),a)\mu_t(da)\right)p(t)dt\end{aligned} \tag{4.24}$$

The facts that b_x is Lipshitz continuous in (x,y) and (X_t^n) converges to X_t uniformly in t in probability imply that the first term in the right hand side of **??** converges in probability to 0.

Furthermore $E\left(\sup\limits_{0\leq t\leq T}|p(t)|^2\right) < +\infty$, then $\sup\limits_{0\leq t\leq T}|p(t)| < +\infty$, $P-a.s$, which implies the existence of a $P-$negligible set N, such that for each $\omega \notin N$, there exist $M(\omega) < +\infty$ s.t $\sup\limits_{0\leq t\leq T}|p(t)| \leq M(\omega)$. In particular for each $\omega \notin N$, the function $b_x(t,X_t,E(X_t),a)p(t).1_{[0,t]}$

is a measurable bounded function in (t,a) and continuous in a, therefore it is a test function for the stable convergence. Hence by using the fact that $\left(\delta_{u_t^n}(da)\,dt\right)$ *converges in* $\mathbb{V}$ *to* $\mu_t(da)\,dt$, $P-a.s.$, it follows that the second term in the right hand side tends to 0, $P-a.s.$ We conclude by using the Lebesgue dominated convergence theorem.

The other terms containing $p(t)$ can be handled by using the same techniques.

The terms in 4.23 containing $q(t)$ can be treated similarily. However one should pay a little more attention as $q(t)$ is just square integrable (in (t,ω)). More precisely

$$\begin{aligned}\left|\int_t^T \left(\sigma_x^n(t)-\overline{\sigma}_x(t)\right)q(t)dt\right| &\leq \left|\int_t^T \left(\sigma_x^n(t)-\overline{\sigma}_x(t)\right)q(t)1_{\{|q(t)|\leq N\}}dt\right| \\ &+\left|\int_t^T \left(\sigma_x^n(t)-\overline{\sigma}_x(t)\right)q(t)1_{\{|q(t)|\geq N\}}dt\right|\end{aligned}$$

The first integral in the right hand side may be handled by using similar argument as precedently as the function $\left(\sigma_x^n(t)-\overline{\sigma}_x(t)\right)q(t)1_{\{|q(t)|\leq N\}}$ is measurable bounded and continuous in a. The second term tends to 0 by Tchebychev's inequality, using the square integrability of $q(t)$.

ii) and iii) are proved by using similar arguments. ■

Proof. of Theorem 4.9 ■

The result is proved by passing to the limit in inequality 4.16 and using Lemma 4.12.

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