République Algérienne Démocratique et Populaire

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

UNIVERSITÉ MOHAMED KHIDER, BISKRA

FACULTÉ des SCIENCES EXACTES et des SCIENCES de la NATURE et de la VIE

DÉPARTEMENT DE MATHÉMATIQUES

Thèse présentée en vue de l'obtention du Diplôme de**:**

Doctorat en Mathématiques

Option: **Probabilités et Statistiques**

Par

Mr. **ALIA ISHAK**

Titre :

Contrôle inconsistant et jeux différentiels stochastiques

Membres du Comité d'Examen :

Brahim Mezerdi	Professeur	UMK Biskra	**Président**
Farid Chighoub	Maitre de conférences A	UMK Biskra	**Rapporteur**
Salah-Eddine Rebiai	Professeur	ULK Batna	**Examinateur**
Nabil Khelfallah	Maitre de conférences A	UMK Biskra	**Examinateur**

06 Juin 2016

DEdicace

Je dédie cette thèse à mes parents car mon éducation est le fruit de leurs ardents efforts.

REMERCIEMENTS

Je tiens tout d'abord à remercier chaleureusement mon directeur de thèse, Dr. Farid Chighoub, pour la qualité remarquable de son encadrement. Il m'a en effet guidé pendant toutes ces années, se rendant toujours disponible et relançant ma motivation lorsque celle-ci diminuait. Il a fait preuve à la fois d'énormément de pédagogie et de psychologie pour me faire avancer tout en me poussant peu à peu vers une plus grande autonomie.

Je suis très heureux que le Pr. Brahim Mezerdi m'ait fait l'honneur d'être le président du jury de ma thèse.

Je tiens à exprimer ma reconnaissance au Dr. Nabil Khelfallah pour avoir accepté d'examiner cette thèse et de faire partie du jury.

Je souhaite aussi adresser ma gratitude au Pr. Salah Eddine Rebiai de l'université de Batna, pour avoir accepté d'examiner mon travail et de faire partie du jury.

Je ne peux oublier toutes les autres personnes qui ont contribué au bon déroulement de cette thèse. Je pense notamment à Ayesha Souhail pour les échanges intéressants que nous avons eus.

Enfin, un grand merci également pour les membres du laboratoire de mathématiques appliquées pour leurs accueil chaleureux et cela depuis le début de ma thèse.

Abstract

This thesis presents two independent research topics, the first one being divided into three distinct problems. Those topics all use stochastic control methods in order to solve, in different contexts, stochastic optimal control problems which are time inconsistent.

In the first part, we formulate a general time-inconsistent stochastic linear-quadratic (LQ) control problem. The time-inconsistency arises from the presence, of a quadratic term of the expected state as well as a state-dependent term in the objective functional. Due to time inconsistency, we consider the problem within a game theoretic framework and we seek equilibrium, instead of optimal, solution. We derive a necessary and sufficient condition, for equilibrium controls, in the form of a maximum principle, which is also applied to solve the mean-variance portfolio problem.

In the second part, we study optimal investment and reinsurance problem for mean–variance insurers within a game theoretic framework and aims to seek the corresponding equilibrium strategies. Specially, the insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market, where the surplus of the insurers is assumed to follow a jump–diffusion model and the financial market consists of one risk-free asset and multiple risky assets whose price processes are modelled by a geometric Lévy processes. By solving a flow of FBSDEs, we obtain the equilibrium strategy among all the open-loop controls for this time inconsistent control problem.

In the third part, we investigate the Merton portfolio management problem in the context of non-exponential discounting. This gives rise to time-inconsistency of the decision maker. Open loop Nash equilibrium are considered. The model is solved for different utility functions and the results are compared.

Finally, the fourth part is concerned with necessary as well as sufficient conditions for near-optimality to stochastic impulse control problems of mean-field type. Necessary conditions for a control to be near-optimal are derived, using Ekeland's variational principle and some stability results on the state and adjoint processes, with respect to the control variable. In a second step, we show that the necessary conditions for near-optimality are in fact sufficient for near-optimality provided some concavity conditions are fulfilled.

Keys words. time inconsistency, mean-field control problem, hyperbolic discounting, stochastic systems with jumps, stochastic maximum principle, equilibrium control, near-optimal controls, stochastic LQ control, mean-variance criterion, Portfolio optimization, Merton problem.

Résumé

Cette thèse présente deux sujets de recherche indépendants, le premier étant décliné sous la forme de trois problèmes distincts. Ces différents sujets ont en commun d'appliquer des méthodes de contrôle stochastique à des problèmes qui sont inconsistants dans le sens où le principe d'optimalité de Bellman n'est pas satisfait.

Dans une première partie, nous formulons un problème de contrôle stochastique inconsistant de type linéaire-quadratique (LQ). L'inconsistance découle de la présence, d'un terme quadratique de l'espérance conditionnelle de l'état ainsi que d'un terme dépendant de l'état initiale dans la fonction objective. En raison de l'inconsistance, nous considérons le problème dans un cadre théorique de jeu et nous cherchons les solutions d'équilibres de Nash. Nous dérivons une condition nécessaire et suffisante, pour les contrôles équilibres, sous la forme d'un principe du maximum, qui est également appliqué pour résoudre le problème de choix du portefeuille sous le critère moyenne-variance.

Dans la seconde partie, nous étudions le problème de choix de stratégies investissement-réassurance optimales pour les assureurs sous le critère moyenne –variance. Tout comme dans la première partie, nous amenons le problème dans un cadre théorique de jeu et on s'intéresse aux stratégies d'équilibres correspondants. En particulier, les assureurs sont autorisés à acheter de la réassurance proportionnelle, d'acquérir de nouvelles entreprises et d'investir dans un marché financier, où le surplus des assureurs est supposé suivre un modèle avec sauts et le marché financier se compose d'un actif sans risque et d'une multitude d'actifs risqués dont les processus du prix sont modélisés par des processus de Lévy. En résolvant un système stochastique nous obtenons la stratégie d'équilibre entre toutes les stratégies en boucle ouverte.

Dans la troisième partie, nous étudions le problème de gestion de portefeuille de Merton dans le cadre de l'escompte non-exponentielle. Cela donne lieu à l'inconsistance des choix optimaux du décideur. Nous caractérisons les stratégies d'équilibres de Nash et nous obtenons des solutions explicites dans le cas de l'utilité logarithmique, l'utilité puissance, et l'utilité exponentielle.

Enfin, dans la quatrième partie, on s'intéresse aux modèles de contrôle stochastiques de type à champ moyen où la variable de contrôle comporte deux composantes, la première étant absolument continue et la seconde est un processus d'impulsion par morceaux. En utilisant le principe variationnel de Ekeland ainsi que certains résultats de stabilité sur le processus d'état et les processus adjoints, par rapport à la variable de contrôle, on dérive des conditions nécessaires pour les contrôle près-optimaux (Near-optimal). Dans un second temps, nous montrons que les conditions nécessaires sont en fait suffisantes pour les contrôles près-optimaux si quelques conditions de concavité sont satisfaites.

Mots Clés. Inconsistance, problèmes à champ-moyen, escompte hyperbolique, system stochastique avec sauts, principe du maximum stochastique, contrôles équilibres, contrôles près-optimaux, problèmes linéaire-quadratique, critère moyenne-variance, problème de Merton.

Contents

Introduction

Stochastic optimal control theory can be described as the study of strategies to optimally influence a stochastic system $X(.)$ with dynamics evolving over time according to a stochastic differential equation (SDE), defined on some complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$. The influence on the system is modeled as a $\{\mathcal{F}_t\}_{t\geq 0}$ $-$adapted process, $u(.)$, called the control which is allowed to take values in some separable metric space (U, d) which is known as the action space. For a control to be optimal, it should minimize (or maximize) some expected objective functional, which depends on the whole controlled state of the system $X(.)$ and the control $u(.)$ over some time interval $[0, T]$. The infimum of the cost functional is known as the value function (as a function of the initial time and state). This optimization problem is infinite dimensional, since we are minimizing (or maximizing) a functional over the space of processes.

Optimal control theory essentially consists of different methods of reducing the original problem which is infinite dimensional to a more manageable problem. The two main methods are dynamic programming, introduced by Bellman in 1950, and the Pontryagin's maximum principle. As for dynamic programming, it is essentially a mathematical technique for making a sequence of interrelated decisions. By considering a family of control problems with different initial times and states and establishing relationships between them, via dynamic programming principle (DPP), one obtains a nonlinear second-order partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation. Solving this equation gives the value function, after which a finite dimensional maximization problem can be solved. On the other hand, the maximum principle gives necessary conditions for optimality by perturbing an optimal control on a small time interval of length ε. Performing a Taylor expansion with respect to ε and then sending ε to zero one obtains a variational inequality. By duality the maximum principle is obtained. It states that any optimal control must solve the Hamiltonian system associated with the control problem. The Hamiltonian system involves a linear differential equation, with terminal conditions, called the adjoint equation, and a (finite dimensional) maximization problem.

1. Time inconsistency

In a typical dynamic programming problem setup, when a controller wants to optimize an objective function by choosing the best plan, he is only required to decide his current action. This is because dynamic programming principle, or Belman's optimality principle, assumes that the future incarnations of the controller are going to solve the remaining part of today's problem and act optimally when future comes. However, in many problems, the DPP does not hold, meaning that an optimal control selected at some initial pair (of time and state) might not remain optimal as time goes. In such problems, the future incarnations of the controller may have changed preferences or tastes, or would want to make decisions based on different objective functions, effectively acting as opponents of the current self of the controller. The dilemma described above is called dynamic inconsistency, which has been noted and studied by economists for many years, mainly in the context of non-exponential (or hyperbolic) type discount functions. In [88], Strotz demonstrated that when a discount function was applied to consumption plans, one could favour a certain plan at the beginning, but later switch preference to another plan. This would hold true for most types of discount functions, the only exception being the exponential. Nevertheless, exponential discounting is the default setting in most literatures, as none of the other types could produce explicit solutions. Results from experimental studies contradict this assumption indicating that the discount rates for the near future are much lower than the discount rates for the time further away in future, and therefore a hyperbolic type discount function would be more realistic. see, for example, Loewenstein and Prelec [66].

In addition to the non-exponential discounted utility maximization, the mean-variance (MV) optimization problems, introduced by Markowitz [67], is another important example of time inconsistent problems. The idea of mean-variance criterion is that it quantifies the risk using the variance, which enables decision makers to seek the highest return after evaluating their acceptable risk level. However, due to the presence of a non-linear function of the expectation in the objective functional, the mean-variance criterion lacks the iterated expectation property. Hence, continuous-time and multi-period mean-variance problems are time-inconsistent.

Other types of time-inconsistency do exist as well. In the literature (see e.g. [17]), there have been listed three possible scenarios where time inconsistency would occur in stochastic continuous time control problems. More specifically, given an objective function of the following form

$$J\left(t,x,u\left(.\right)\right)=\mathbb{E}^{t}\left[\int_{t}^{T}\lambda\left(s-t\right)f\left(x,s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)ds+\lambda\left(T-t\right)h\left(x,X\left(T\right),\mathbb{E}^{t}\left[X\left(T\right)\right]\right)\right],$$

where $T>0$, $(t,x)\in[0,T]\times\mathbb{R}^{n}$, $\lambda(.)$, $f(.)$ and $h(.)$ are given functions, $u(s)\in U$ is the control action applied at time s, and $X(.)=X(.;u(.))$ is some controlled state process which solves the following SDE, driven by a standard Brownian motion defined on some filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P}\right)$

$$\begin{cases} dX\left(s\right)=b\left(s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)dt+\sigma\left(s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)dW\left(s\right),\ s\in\left[t,T\right], \\ X\left(t\right)=x, \end{cases}$$

the optimization for $J(t,x,.)$ is a time-inconsistent problem if:

- The discount function $\lambda(.)$ is not of exponential type, e.g. a hyperbolic discount function;

- the coefficients b, σ, f and/or h are non linear functions of the marginal conditional probability law of the controlled state process, e.g. mean field control problems;

- initial state x appears in the objective function, e.g. a utility function that depends on the initial state x.

The mean-field models were initially proposed to study the aggregate behaviour of a large number of mutually interacting particles in diverse areas of physical science, such as statistical mechanics, quantum mechanics and quantum chemistry. Roughly speaking, the mean-field models describe the complex interactions of individual "agents" (or particles) through a medium, namely the mean-field term, which describes the action and reaction between the "agents". In a recent paper Lasry and Lions [60] extended the application of the mean-field models to economics and finance, where they considered N-player stochastic differential games, proved the existence of the Nash equilibrium points and derived rigorously the mean-field limit equations as N goes to infinity.

In all the three cases, listed above, the standard HJB equations cannot be derived since the usual formulation requires an argument about the value function (process) being a supermartingale for arbitrary controls and being a martingale at optimum, which does not hold here.

2. Approaches to handle time inconsistency

In light of the non-applicability of standard DPP on these problems, there are two basic ways of handling (various forms of) time inconsistency in optimal control problems.

2.1. The strategy of pre-commitment

One possibility is to study the pre-committed problem: we fix one initial point, like for example $(0, x_0)$, and then try to find the control process $\bar{u}(.)$ which optimizes $J(0, x_0, .)$. We then simply disregard the fact that at a later points in time such as $(s, X(s; 0, x_0, \bar{u}(.)))$ the control $\bar{u}(.)$ will not be optimal for the functional $J(s, X(s; \bar{u}(.)), .)$. Kydland and Prescott [59] indeed argue that a pre-committed strategy may be economically meaningful in certain circumstances. In the context of MV optimization problem, pre-committed optimal solution have been extensively investigated in different situations. [83] is probably the earliest paper that studies a pre-committed MV model in a continuous-time setting (although he considers only one single stock with a constant risk-free rate), followed by [10]. In a discrete-time setting, [64] developed an embedding technique to change the originally time-inconsistent MV problem into a stochastic LQ control problem. This technique was extended in [112], along with an indefinite stochastic linear– quadratic control approach, to the continuous-time case. Further extensions and improvements are carried out in, among many others, [62], [61], [15], and [98]. Markowitz's problem with transaction cost is recently solved in [34]. For general mean field control problems, Andersson and Djehiche [7], Li [63] , and Buckdahn et al. [23] derived a mean field type stochastic maximum principle to characterize "pre-committed" optimal control where both the state dynamics and the cost functional are of a mean-field type. The linear-quadratic optimal control problem for mean-field SDEs has been studied by Yong [104]. The maximum principle for a jump-diffusion mean-field model has been investigated in Shen and Siu [86] and Chighoub and Mezerdi [31].

2.2. Game theoretic approach

Another approach to handle time inconsistency in dynamic decision making problems is by considering time-inconsistent problems as non-cooperative games in which decisions at every instant of time are selected as if various players at each instant of time are intended to maximize or minimize their own objective functions; Nash equilibriums are therefore considered instead of optimal solutions, see e.g. [17], [33], [39], [40], [51], [58], [78], [80], [88], [45], [102] and [103]. In the context of non-exponential type discount functions, Strotz [88], was the first who used this game perspective to handle the dynamic time-inconsistent decision problem on the deterministic Ramsay problem. Then by capturing the idea of non-commitment, by letting the commitment period being infinitesimally small, he introduced a primitive notion of Nash equilibrium strategy. Further work which extend [88] are [58], [78], [80] and [45]. In order to study the optimal investment-consumption problem under hyperbolic discount functions, in both, deterministic and stochastic framework, Ekeland and Lazrak [39] and Ekeland and Pirvu [40] provided a formal definition of feedback Nash equilibrium controls in continuous time setting. Further extensions of Ekeland and Pirvu's work can be found in Björk and Murguci [17], and Ekeland et al. [41]. Recently, Yong [103], provided an alternative approach for studying a general discounting time inconsistent optimal control problem in continuous time setting by considering a discrete time counterpart. Following Yong's approach Q. Zhaoa et al. [107] studied the consumption-investment problem with a general discount function and a logarithmic utility function. In the context of MV optimization problem, Basak and Chabakauri [11] first investigated the equilibrium solutions for continuous-time Markowitz's mean-variance portfolio selection problem. Böjrk et al. [18] studied the mean-variance portfolio selection with state dependent risk aversion. In a non-Markovian framework, a time-consistent strategy is obtained for the mean-variance portfolio selection by Hu et al. [51], followed by Czichowsky [33].

Concerning equilibrium strategies for mean field optimal control problems. Following the approach developed in [51], Bensoussan et al. [14] investigate time-inconsistent stochastic LQ problem of mean field type. Yong [102] investigate closed-loop Nash equilibrium strategies for general time-inconsistent stochastic LQ problem for mean-field type stochastic differential equation by adopting a discretization conter part. Dehiche and Huang [37] derived a Pontryagin's type stochastic maximum principle to characterize equilibrium control where both the state dynamics and the cost functional are of a mean-field type.

3. Organization of Thesis

This PhD dissertation presents two independent research topics about stochastic control problems which, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. For the first topic, we attack these problems by viewing them within a game theoretic framework, and we look for open-loop Nash equilibrium solutions. The second one is concerned with the characterization of pre-committed near-optimal controls for general stochastic control problems where both the state dynamics and the cost functional are of a mean-field type.

The structure of the dissertation is as follows:

In **Chapter 1**, we discuss a class of stochastic linear quadratic dynamic decision problems of a general time-inconsistent type, in the sense that, it does not satisfy the Bellman optimality principle. More precisely, the dependence of the running and the terminal costs in the objective functional on some general discounting coefficients, as well as on some quadratic terms of the conditional expectation of the state process, makes the problem time-inconsistent. Open-loop Nash equilibrium controls are then constructed instead of optimal controls, this has been accomplished through the stochastic maximum principle approach that includes a flow of forward-backward stochastic differential equations under a maximum condition. Then by decoupling the flow of the adjoin process, we derive an explicit representation of the equilibrium strategies in feedback form. As an application, we study some concrete examples. We emphasize that, this method can provide the necessary and sufficient conditions to characterize the equilibrium strategies. While most existing results which are based on the extended HJB techniques can create only the sufficient condition to characterize the equilibrium strategies. The results obtained in this chapter, extend some ones obtained in Hu et al. [51] and Yong [102].

In **Chapter 2**, we study the equilibrium reinsurance/new business and investment strategy for mean-variance insurers with constant risk aversion. The insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market, where the surplus of the insurers is assumed to follow a jump-diffusion model and the financial market consists of one riskless asset and a multiple risky assets whose price processes are driven by Poisson random measures and an independent Brownian

motions. By using a version of the stochastic maximum principle approach, we characterize the open loop equilibrium strategies via a stochastic system which consists of a flow of forward-backward stochastic differential equations (FBSDEs in short) and an equilibrium condition. Then by decoupling the flow of FSBDEs, an explicit representation of an equilibrium solution is derived as well as it corresponding objective function value. The results obtained, in this chapter, cover the ones obtained in [110] and [111].
In **Chapter 3**, we revisit time inconsistent consumption-investment problem with a general discount function and a general utility function in a non-Markovian framework. The coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. We adopt a variational method to characterize equilibrium strategies in terms of the unique solutions of a flow of BSDEs. When the coefficients in the problem are all deterministic, we find an explicit equilibrium solutions in feedback from via some parabolic PDE. Our results generalize the ones obtained in [87] and [40].
In **Chapter 4,** we discuss stochastic control models which are described by a stochastic differential equation of mean-field type, in the sense that the coefficients are permitted to depend on the state process as well as of its expected value. The control variable has two components, the first being absolutely continuous and the second is a piecewise impulse process which is not necessarily increasing. Necessary and sufficient conditions for a control to be near optimal are studied in the form of stochastic maximum principle by using Ekeland's variational principle, which allows to produce two approximate variational inequalities in integral form. The first inequality is constructed by the spike variation technique in terms of the $\mathcal{H}$-function employed for absolutely continuous part of all near optimal control. The second one is defined in term of the first order adjoint process by using a convex perturbation technique for all near optimal impulse controls.

Chapter 5 concludes the thesis with some final remarks and proposes some future research directions.

4. Relevant Papers

The content of this thesis was the subject of the following papers:

1. Farid Chighoub, Ayesha Sohail, Ishak Alia, *Near-optimality conditions in mean-field control models involving continuous and impulse controls.* Nonlinear Studies, 22(4), (2015).
2. Ishak Alia, Farid Chighoub, Ayesha Sohail, *The Maximum Principle in Time-Inconsistent LQ Equilibrium.* arXiv:1505.04674v1, (submitted).
3. Ishak Alia, Farid Chighoub, Ayesha Sohail, *A Characterization of Equilibrium Strategies in Continuous-Time Mean-Variance Problems for Insurers.* (submitted).

4. Ishak Alia, Farid Chighoub, Ayesha Sohail, *Open Loop Equilibrium Strategies in General Discounting Merton Portfolio Problem.* (Preprint).

Notation

- $\mathbb{R}^{n\times m}$: the set of $n \times m$ real matrices.
- S^n : the set of $n \times n$ symmetric real matrices.
- $C^\top$: the transpose of the vector (or matrix) C.
- $\langle .,. \rangle$: the inner product in some Euclidean space.
- For a function f, we denote by f_x (resp. f_{xx}) the gradient or Jacobian (resp. the Hessian) of f with respect to the variable x.

For any Euclidean space $H = \mathbb{R}^n$, $\mathbb{R}^{n\times m}$ or S^n with Frobenius norm $|.|$ we let for any $t \in [0,T]$

- $\mathbb{L}^p(\Omega,\mathcal{F}_t,\mathbb{P};H) := \{\xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t - \text{measurable, with } \mathbb{E}[|\xi|^p] < \infty\}$, for any $p \geq 1$.
- $\mathbb{L}^2(Z,\mathcal{B}(Z),\theta;H) := \left\{ r(.) : Z \to H \mid r(.) \text{ is } \mathcal{B}(Z) - \text{measurable, with } \int_Z |r(z)|^2 \theta(dz) < \infty \right\}.$
- $\mathcal{S}^2_{\mathcal{F}}(t,T;H) := \Big\{ X(.) : [t,T]\times\Omega \to H \mid X(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]} - \text{adapted,}$
$$s \mapsto X(s) \text{ is càdlàg, with } \mathbb{E}\left[\sup_{s\in[t,T]} |X(s)|^2\right] < \infty \Big\}.$$
- $\mathcal{C}^2_{\mathcal{F}}(t,T;H) := \Big\{ X(.) : [t,T]\times\Omega \to H \mid X(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]} - \text{adapted,}$
$$s \mapsto X(s) \text{ is continuous, with } \mathbb{E}\left[\sup_{s\in[t,T]} |X(s)|^2\right] < \infty \Big\}.$$
- $\mathcal{L}^\infty_{\mathcal{F}}(t,T;H) := \left\{ c(.) : [t,T]\times\Omega \to H \mid c(.)\, is\, (\mathcal{F}_s)_{s\in[t,T]} - \text{adapted,with } \mathbb{E}\left[\sup_{s\in[t,T]} |c(s)|\right] < \infty. \right\}.$
- $\mathcal{L}^2_{\mathcal{F}}(t,T;H) := \left\{ q(.) : [t,T]\times\Omega \to H \mid q(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]} - \text{adapted, with } \mathbb{E}\left[\int_t^T |q(s)|^2 ds\right] < \infty \right\}.$
- $\mathcal{L}^2_{\mathcal{F},p}(t,T;H) := \left\{ u(.) : [t,T]\times\Omega \to H \mid u(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]} - \text{predictible, with } \mathbb{E}\left[\int_t^T |u(s)|^2 ds\right] < \infty \right\}.$
- $\mathcal{L}^{\theta,2}_{\mathcal{F},p}([t,T]\times Z;H) := \Big\{ R(.,.) : [t,T]\times\Omega\times Z \to H \mid R(.) \text{ is measurable and } (\mathcal{F}_s)_{s\in[t,T]} - \text{predictable,}$
$$\text{with } \mathbb{E}\left[\int_t^T \int_Z |R(s,z)|^2 \theta(dz)\, ds\right] < \infty \Big\}.$$
- $\mathcal{C}([0,T];H) := \{ f : [0,T] \to H \mid f(.) \ is \text{ continuous} \}.$
- $\mathcal{D}[0,T] := \{(t,s) \in [0,T]\times[0,T], \text{ such that } s \geq t\}.$
- $\mathcal{C}(\mathcal{D}[0,T];H) := \{ f(.,.) : \mathcal{D}[0,T] \to H \mid f(.,.) \ is \text{ continuous} \}.$
- $\mathcal{C}^{0,1}(\mathcal{D}[0,T];H) := \left\{ f(.,.) : \mathcal{D}[0,T] \to H \mid f(.,.) \text{ and } \frac{\partial f}{\partial s}(.,.) \text{ are continuous} \right\}.$
- $\mathcal{C}^{1,2}([0,T]\times\mathbb{R};\mathbb{R}) := \left\{ f(.,.) : [0,T]\times\mathbb{R} \to \mathbb{R} \mid f(.,x) \text{ is } C^1 \text{ in } t \text{ and } f(t,.) \text{ is } C^2 \text{ in } x \right\}.$

Chapter 1

The Maximum Principle in Time-Inconsistent LQ Equilibrium Control Problem for Jump Diffusions

Stochastic optimal control problems with linear dynamics and quadratic stage costs are one of the most important classes of optimal control ones. They have wide applications in engineering and financial mathematics, etc. A major approach for studying such stochastic control problems is the dynamic programming principle which expresses the optimal policy in terms of an optimization problem involving the value function (or a sequence of value functions in the time-varying case). The proof of the dynamic programming principle is technical and has been studied by different methods. The value function can be created, by an iteration connecting to the Bellman operator, which maps functions on the state space into functions on the state space and involves an expectation and a minimization step.

A number of studies have been devoted to this topic, Wu and Wang [96] discussed a kind of stochastic LQ problem for system driven by a Brownian motion and an independent Poisson jump process and a linear feedback regulator for the optimal control problem is given by the solution of a generalized Riccati equation system. In view of completing of squares technique, Hu and Øksendal [53] studied the stochastic LQ problem for a general stochastic differential equation with random coefficients, under partial information. Meng [68] investigate the stochastic maximum principle in LQ control problem for multidimensional stochastic differential equation driven by a Brownian motion and a Poisson random martingale measure and obtain the existence and uniqueness result for a class of backward stochastic Riccati equations. For more information on LQ control models for stochastic dynamic systems, we refer to [91], [104], and [112].

To the best of our knowledge, there is little work in the literature concerning equilibrium strategy for time-inconsistent LQ control problems. In [101] Yong studied a general discounting time-inconsistent deterministic LQ model, and he derived a closed-loop equilibrium strategies, via a forward ordinary differential equation coupled with a backward Riccati-Volterra integral equation. Hu et al. [51] investigate open loop equilibrium strategies for time inconsistent LQ control problem with random coefficients by adopting a Pontryagin type stochastic maximum principle approach, we refer to [94] for partially observed time inconsistent recursive LQ optimization problem. Yong [102] investigate a time-inconsistent stochastic LQ problem for mean-field type stochastic differential equation and closed-loop solutions are presented by means of multi-person differential games, the limit of which leads to the equilibrium Riccati equation. In [14] Bensoussan et al. investigate time-inconsistent stochastic LQ problem of mean field type. As far as we know, there is no literature on the time-inconsistent stochastic linear-quadratic optimal control problems incorporating stochastic jumps.

Motivated by these points, this thesis studies optimality conditions for time-inconsistent linear quadratic stochastic control problem, in the sense that, it does not satisfy the Bellman optimality principle, since a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair. Different from [51], [102], and [14] in which the noise is driven only by a Brownian motion, in our LQ model the state evolves according to a SDE, when the noise is driven by a multidimensional Brownian motion and an independent Poisson point process. The objective functional includes the cases of hyperbolic discounting, as well as, the continuous-time Markowitz's mean-variance portfolio selection problem, with state-dependent risk aversion.

Our objective is to investigate a characterization of Nash equilibrium controls instead of optimal controls. The novelty of this work lies in the fact that, our calculations are not limited to the exponential discounting framework, the time-inconsistency of the LQ optimal control in this situation, is due to the presence of some general discounting coefficients, involving the so-called hyperbolic discounting situations. In addition, the presence of some quadratic terms of the expected controlled state process, in both the running cost and the terminal cost, make also the problem time-inconsistent. This can be motivated by the reward term in the mean-variance portfolio choice model.

We accentuate that, our model covers some class of time-inconsistent stochastic LQ optimal control problem studied by [51], and some relevant cases appeared in [102]. Moreover, we have defined the equilibrium controls in open-loop sense, in a manner similar to [51], which is different from the feedback form, see e.g. [40], [11] , [17], and [103].

The rest of the chapter is organized as follows. In Section 1, we describe the model and formulate the objective. In Section 2 we present the first main result of this work (Theorem 1.2.1), which characterizes the equilibrium control via a stochastic system, which involves a flow of forward-backward stochastic differential equation with jumps (FBSDEJ in short), along with some equilibrium conditions. In Section 3, by decoupling the flow of the FBSDEJ, we investigate a feedback representation of the equilibrium control, via some class of ordinary differential equations. Section 4 is devoted to some applications, we solve a continuous time mean–variance portfolio selection model and some one-dimensional general discounting LQ problems.

1.1 Problem setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. We assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by a d-dimensional standard Brownian motion $(W(t))_{t\in[0,T]}$ and an independent Poisson measure N on $[0,T] \times Z$ where $Z \subseteq \mathbb{R} - \{0\}$. We assume that the compensator of N has the form $\mu(dt, dz) = \theta(dz)\, dt$ for some positive and $\sigma-$finite Levy measure on Z, endowed with its Borel $\sigma-$field $\mathcal{B}(Z)$. We suppose that $\int_Z 1 \wedge |z|^2 \theta(dz) < \infty$ and write $\tilde{N}(dt, dz) = N(dt, dz) - \theta(dz)\, dt$ for the compensated jump martingale random measure of N. Obviously, we have

$$\mathcal{F}_t = \sigma\left[\int\int_{A\times(0,t]} N(ds, dz); s \leq t, A \in \mathcal{B}(Z)\right] \vee \sigma[W(s); s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\theta-$null sets, and $\sigma_1 \vee \sigma_2$ denotes the $\sigma-$field generated by $\sigma_1 \cup \sigma_2$.

We consider a n-dimensional non homogeneous linear controlled jump diffusion system

$$\begin{cases} dX(s) = \{A(s)X(s) + B(s)u(s) + b(s)\}\, ds + \sum_{j=1}^{d} \{C_j(s)X(s) + D_j(s)u(s) + \sigma_j(s)\}\, dW^j(s) \\ \qquad\qquad + \int_Z \{E(s,z)X(s-) + F(s,z)u(s) + c(s,z)\}\, \tilde{N}(ds, dz), \;\; s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{1.1.1}$$

where $(t, \xi, u(.)) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F},p}(t, T; \mathbb{R}^m)$. Under some conditions, for any initial situation (t, ξ) and any admissible control $u(.)$ the state equation is uniquely solvable, we denote by $X(s) = X^{t,\xi}(s; u(.))$ its solution, for $s \in [t,T]$. Different controls $u(.)$ will lead to different solutions $X(.)$. Note that $\mathcal{L}^2_{\mathcal{F},p}(t, T; \mathbb{R}^m)$ is the space of all admissible strategies over $[t,T]$.

Our aim is to minimize the following expected discounted cost functional

$$\begin{aligned} &J\left(t,\xi,u\left(.\right)\right) \\ &= \mathbb{E}^{t}\left[\int_{t}^{T}\frac{1}{2}\left(\left\langle Q\left(t,s\right)X\left(s\right),X\left(s\right)\right\rangle + \left\langle \bar{Q}\left(t,s\right)\mathbb{E}^{t}\left[X\left(s\right)\right],\mathbb{E}^{t}\left[X\left(s\right)\right]\right\rangle + \left\langle R\left(t,s\right)u\left(s\right),u\left(s\right)\right\rangle\right)ds \right. \\ &\qquad + \left\langle \mu_{1}\left(t\right)\xi+\mu_{2}\left(t\right),X\left(T\right)\right\rangle \\ &\qquad \left. + \frac{1}{2}\left(\left\langle G\left(t\right)X\left(T\right),X\left(T\right)\right\rangle + \left\langle \bar{G}\left(t\right)\mathbb{E}^{t}\left[X\left(T\right)\right],\mathbb{E}^{t}\left[X\left(T\right)\right]\right\rangle\right)\right], \end{aligned} \tag{1.1.2}$$

over $u\left(.\right)\in\mathcal{L}_{\mathcal{F},p}^{2}\left(t,T;\mathbb{R}^{m}\right)$,where $X\left(.\right)=X^{t,\xi}\left(.;u\left(.\right)\right)$ and $\mathbb{E}^{t}\left[.\right]=\mathbb{E}\left[.\,|\mathcal{F}_{t}\right].$

We need to impose the following assumptions about the coefficients

(H1) The functions $A\left(.\right),C_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n\times n}$, $B\left(.\right),D_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n\times m}$, $b\left(.\right),\sigma_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n}$, $E\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n\times n}$, $F\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n\times m}$, and $c\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n}$ are continuous and bounded. The coefficients on the cost functional satisfy

$$\begin{cases} Q\left(.,.\right),\bar{Q}\left(.,.\right)\in C\left(\mathcal{D}\left[0,T\right];S^{n}\right), \\ R\left(.,.\right)\in C\left(\mathcal{D}\left[0,T\right];S^{m}\right), \\ G\left(.\right),\bar{G}\left(.\right)\in C\left(\left[0,T\right];S^{n}\right), \\ \mu_{1}\left(.\right)\in C\left(\left[0,T\right];\mathbb{R}^{n\times n}\right), \\ \mu_{2}\left(.\right)\in C\left(\left[0,T\right];\mathbb{R}^{n}\right). \end{cases}$$

(H2) The functions $R\left(.,.\right),Q\left(.,.\right)$ and $G\left(.\right)$ satisfy

$$R\left(t,t\right)\geq0,\ G\left(t\right)>0,\text{for }t\in\left[0,T\right],\ \text{and }Q\left(t,s\right)\geq0,\ \text{for }\left(t,s\right)\in\mathcal{D}\left[0,T\right].$$

Under **(H1)** for any $\left(t,\xi,u\left(.\right)\right)\in\left[0,T\right]\times\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{n}\right)\times\mathcal{L}_{\mathcal{F},p}^{2}\left(t,T;\mathbb{R}^{m}\right)$, the state equation (1.2.1) has a unique solution $X\left(.\right)\in\mathcal{S}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{n}\right)$, see for example [68]. Moreover, we have the following estimate

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|X\left(s\right)\right|^{2}\right]\leq K\left(1+\mathbb{E}\left[\left|\xi\right|^{2}\right]\right),$$

for some positif constant K. The optimal control problem can be formulated as follows.

Problem (LQJ). *For any given initial pair* $\left(t,\xi\right)\in\left[0,T\right]\times\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{n}\right)$*, find a control* $\bar{u}\left(.\right)\in\mathcal{L}_{\mathcal{F},p}^{2}\left(t,T;\mathbb{R}^{m}\right)$ *such that*

$$J\left(t,\xi,\bar{u}\left(.\right)\right)=\inf_{u\left(.\right)\in\mathcal{L}_{\mathcal{F},p}^{2}\left(t,T;\mathbb{R}^{m}\right)}J\left(t,\xi,u\left(.\right)\right).$$

Remark 1.1.1 *1) The dependence of the weighting matrices of the current time t, the dependence of the terminal cost on the current state ξ and the presence of quadratic terms of the expected controlled state process in the cost functional make the Problem (LQJ) time-inconsistent.*
2) One way to get around the time-inconsistency issue is to consider only precommitted controls (i.e., the controls are optimal only when viewed at the initial time).

1.1.1 An example of time-inconsistent optimal control problem

We present a simple illustration of stochastic optimal control problem which is time-inconsistent. Our aim is to show that the classical SMP approach is not efficient in the study of this problem if it's viewed as time-consistent. For $n = d = m = 1$, consider the following controlled SDE starting from $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{cases} dX^{t,x}(s) = bu(s)\,ds + \sigma dW(s), \ s \in [t, T], \\ X^{t,x}(t) = x, \end{cases} \tag{1.1.3}$$

where b and σ are real constants. The cost functional is given by

$$J(t, x, u(.)) = \frac{1}{2}\mathbb{E}\left[\int_t^T |u(s)|^2\,ds + h(t)\left(X^{t,x}(T) - x\right)^2\right], \tag{1.1.4}$$

where $h(.) : [0, T] \to (0, \infty)$, is a general deterministic non-exponential discount function satisfying $h(0) = 1$, $h(s) \geq 0$ and $\int_0^T h(t)\,dt < \infty$. We want to address the following stochastic control problem.
Problem (E). *For any given initial pair $(t, x) \in [0, T] \times \mathbb{R}$, find a control $\bar{u}(.) \in \mathcal{L}^2_{\mathcal{F}}(t, T; \mathbb{R})$ such that*

$$J(t, x, \bar{u}(.)) = \inf_{u(.) \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R})} J(t, x, u(.)).$$

At a first stage, we consider the Problem (E) as a standard time consistent stochastic linear quadratic problem. Since $J(t, x, .)$ is convex and coercive, there exists then a unique optimal control for this problem for each fixed initial pair $(t, x) \in [0, T] \times \mathbb{R}$. Notice that the usual Hamiltonian associated to this problem is $\mathbb{H} : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ such that for every $(s, y, v, p, q) \in [0, T] \times \mathbb{R}^4$ we have

$$\mathbb{H}(s, y, v, p, q) = pbv + \sigma q - \frac{1}{2}v^2,$$

Let $u^{t,x}(.)$ be an admissible control for $(t, x) \in [0, T] \times \mathbb{R}$. Then the corresponding first order and second order adjoint equations are given respectively by

$$\begin{cases} dp^{t,x}(s) = q^{t,x}(s)\, dW(s), \; s \in [t,T], \\ p^{t,x}(T) = -h(t)\left(X^{t,x}(T) - x\right), \end{cases}$$

and

$$\begin{cases} dP^{t,x}(s) = Q^{t,x}(s)\, dW(s), \; s \in [t,T], \\ P^{t,x}(T) = -h(t), \end{cases}$$

the last equation has only the solution $\left(P^{t,x}(s), Q^{t,x}(s)\right) = \left(-h(t), 0\right), \forall s \in [t,T]$.

Note that, the corresponding $\mathcal{H}$-function is given by

$$\mathcal{H}(s,y,v) = \mathbb{H}\left(s,y,v,p^{t,x}(s), q^{t,x}(s)\right) = p^{t,x}(s)\, bv + \sigma q^{t,x}(s) - \frac{1}{2}v^2,$$

which is a concave function of v. Then according to the sufficient condition of optimality, see e.g. Theorem 5.2 pp 138 in [105], for any fixed initial pair $(t,x) \in [0,T] \times \mathbb{R}$, Problem (E) is uniquely solvable with an optimal control $\bar{u}^{t,x}(.)$ having the representation

$$\bar{u}^{t,x}(s) = b\bar{p}^{t,x}(s), \; \forall s \in [t,T],$$

such that the process $\left(\bar{p}^{t,x}(.), \bar{q}^{t,x}(.)\right)$ is the unique adapted solution to the BSDE

$$\begin{cases} d\bar{p}^{t,x}(s) = \bar{q}^{t,x}(s)\, dW(s), \; s \in [t,T], \\ \bar{p}^{t,x}(T) = -h(t)\left(\bar{X}^{t,x}(s) - x\right). \end{cases}$$

By stadard arguments we can show that the processes $\left(\bar{p}^{t,x}(.), \bar{q}^{t,x}(.)\right)$ are explicitly given by

$$\begin{cases} \bar{p}^{t,x}(s) = -M^t(s)\left(\bar{X}^{t,x}(s) - x\right), \; s \in [t,T], \\ \bar{q}^{t,x}(s) = -\sigma M^t(s), \; s \in [t,T], \end{cases}$$

where $\bar{X}^{t,x}(.)$ is the solution of the state equation corresponding to $\bar{u}^{t,x}(.)$, given by

$$\begin{cases} d\bar{X}^{t,x}(s) = b^2\bar{p}^{t,x}(s)\, ds + \sigma dW(s), \; s \in [t,T], \\ \bar{X}^{t,x}(t) = x. \end{cases}$$

and

$$M^t(s) = \frac{h(t)}{b^2 h(t)(T-s) + 1}, \; \forall s \in [t,T].$$

A simple computation show that

$$\overline{u}^{t,x}(s) = -\frac{bh(t)}{b^2 h(t)(T-s)+1}\left(\bar{X}^{t,x}(s) - x\right),\ \forall s \in [t,T],$$

clearly we have

$$\overline{u}^{t,x}(s) \neq 0,\ \forall s \in (t,T]. \tag{1.1.5}$$

In the next stage, we will prove that the Problem (E) is time-inconsistent, for this we first fix the initial data $(t,x) \in [0,T] \times \mathbb{R}$. Note that, if we assume that the Problem (E) is time-consistent, in the sense that for any $r \in [t,T]$ the restriction of $\bar{u}^{t,x}(.)$ on $[r,T]$ is optimal for Problem (E) with initial pair $\left(r, \bar{X}^{t,x}(r)\right)$, however as Problem (E) is uniquely solvable for any initial pair, we should have then $\forall r \in (t,T]$

$$\bar{u}^{t,x}(s) = \bar{u}^{r,\bar{X}^{t,x}(r)}(s) = -\frac{bh(r)}{b^2 h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s) - \bar{X}^{t,x}(r)\right), \forall s \in [r,T],$$

where $\bar{X}^{r,\hat{X}^{t,x}(r)}(.)$ solves the SDE

$$\begin{cases} d\bar{X}^{r,\bar{X}^{t,x}(r)}(s) = b^2 \dfrac{h(r)}{b^2 h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s) - \bar{X}^{t,x}(r)\right) ds + \sigma dW(s), \forall s \in [r,T], \\ \bar{X}^{r,\bar{X}^{t,x}(r)}(r) = \bar{X}^{t,x}(r). \end{cases}$$

In particular by the uniqueness of solution to the state SDE we should have

$$\bar{u}^{t,x}(r) = -\frac{bh(r)}{b^2 h(r)(T-r)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(r) - \bar{X}^{t,x}(r)\right) = 0,$$

is the only optimal solution of the Problem (E), this contradict (1.1.5). Therefore, the Problem (E) is not time-consistent, and more precisely, the solution obtained by the classical SMP is wrong and the problem is rather trivial since the only optimal solution equal to zero.

1.2 Characterization of equilibrium strategies

The purpose of this thesis is to characterize open-loop Nash equilibriums instead of optimal controls. We use the game theoretic approach to handle the time inconsistency in the same perspective as Ekeland and Lazrak [39], and Bjork and Murgoci [17]. Let us briefly describe the game perspective that we need to consider, as follows.

- We consider a game with one player at each point t in $[0,T]$. This player represents the incarnation of the controller at time t and is referred to as "player t".

- The $t-th$ player can control the system only at time t by taking his/her strategy $u(t,.):\Omega\rightarrow\mathbb{R}^m$.
- A control process $u(.)$ is then viewed as a complete description of the chosen strategies of all players in the game.
- The reward to the player t is given by the functional $J\left(t,X(t),u_{|[t,T]}(.)\right)$, which depends only on the restriction of the control $u(.)$ to the time interval $[t,T]$.

For the above description, we define the concept of a " Nash equilibrium point (control)" of the game. This is an admissible control process $\hat{u}(.)$ satisfying the following condition; For an arbitrary point t in time, suppose that every player s, such that $s>t$, will use the strategy $\hat{u}(s)$. Then the optimal choice for player t is that, he/she also uses the strategy $\hat{u}(t)$.

Nevertheless, the problem with this "definition", is that the individual player t does not really influence the outcome of the game at all. He/she only chooses the control at the single point t, and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. Therefore, to characterize open-loop Nash equilibriums, which have not to be necessary feedback, we follow [51] who suggests the following formal definition inspired by [40] and [17].

Noting that, for brevity, in the rest of the paper, we suppress the subscript (s) for the coefficients $A(s), B(s), b(s), C_j(s), D_j(s), \sigma_j(s)$, and we use the notation $\varrho(z)$ instead of $\varrho(s,z)$ for $\varrho=E,F$ and c. In addition, sometimes we simply call $\hat{u}(.)$ an equilibrium control instead of open-loop Nash equilibrium control when there is no ambiguity.

Following [51], we first consider an equilibrium by local spike variation, given an admissible control $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$. For any $t\in[0,T]$, $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$ and for any $\varepsilon\in[0,T-t)$, define

$$u^{\varepsilon}(s)=\begin{cases}\hat{u}(s)+v, & \text{for } s\in[t,t+\varepsilon),\\ \hat{u}(s), & \text{for } s\in[t+\varepsilon,T],\end{cases}\tag{1.2.1}$$

we have the following definition.

Definition 1.2.1 (Open-loop Nash equilibrium) *An admissible strategy $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ is an open-loop Nash equilibrium control for Problem (LQJ) if*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right)\right\}\geq 0,\tag{1.2.2}$$

for any $t\in[0,T]$, and $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$. The corresponding equilibrium dynamics solves the following

SDE with jumps

$$\begin{cases} d\hat{X}(s) = \left\{ A\hat{X}(s) + B\hat{u}(s) + b \right\} ds + \sum_{j=1}^{d} \left\{ C_j \hat{X}(s) + D_j \hat{u}(s) + \sigma_j \right\} dW^j(s) \\ \qquad\qquad + \int_Z \left\{ E(z) \hat{X}(s-) + F(z) \hat{u}(s) + c(z) \right\} \tilde{N}(ds, dz), \;\; s \in [0,T], \\ \hat{X}_0 = x_0. \end{cases}$$

1.2.1 The flow of adjoint equations

We introduce the adjoint equations involved in the stochastic maximum principle which characterize the open-loop Nash equilibrium controls of Problem (LQJ). First, define the Hamiltonian $\mathbb{H} : \mathcal{D}[0,T] \times \mathbb{L}^1(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{L}^2(Z, \mathcal{B}(Z), \theta; \mathbb{R}^n) \to \mathbb{R}$ by

$$\begin{aligned} &\mathbb{H}(t, s, X, u, p, q, r(.)) \\ &= \langle p, AX + Bu + b \rangle + \sum_{j=1}^{d} \langle q_j, D_j X + C_j u + \sigma_j \rangle - \frac{1}{2} \langle R(t,s) u, u \rangle \\ &\quad + \int_Z \langle r(z), E(z) X + F(z) u + c(z) \rangle \theta(dz) - \frac{1}{2} \left(\langle Q(t,s) X, X \rangle + \left\langle \bar{Q}(t,s) \mathbb{E}^t[X], \mathbb{E}^t[X] \right\rangle \right). \end{aligned} \tag{1.2.3}$$

Let $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ and denote by $\hat{X}(.) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ the corresponding controlled state process. For each $t \in [0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$, and satisfied by the triplet of processes $(p(.;t), q(.;t), r(.,.;t))$ as follows

$$\begin{cases} dp(s;t) = -\left\{ A^\top p(s;t) + \sum_{j=1}^{d} C_j^\top q_j(s;t) + \int_Z E(z)^\top r(s,z;t)\theta(dz) - Q(t,s)\hat{X}(s) \right. \\ \qquad\qquad \left. - \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right] \right\} ds + \sum_{j=1}^{d} q_j(s;t)\, dW^j(s) + \int_Z r(s,z;t)\tilde{N}(ds,dz), \;\; s \in [t,T], \\ p(T;t) = -G(t)\hat{X}(T) - \bar{G}(t)\mathbb{E}^t\left[\hat{X}(T)\right] - \mu_1(t)\hat{X}(t) - \mu_2(t), \end{cases} \tag{1.2.4}$$

where $q(.;t) = (q_1(.;t), ..., q_d(.;t))$.

Similarly, we introduce the second order adjoint equation defined on the time interval $[t,T]$, and satisfied by the triplet of processes $(P(.;t), \Lambda(.;t), \Gamma(.,.;t))$ as follows

$$\begin{cases} dP(s;t) = -\Big\{A^\top P(s;t) + P(s;t)A + \sum_{j=1}^d \big(C_j^\top P(s;t) C_j + \Lambda_j(s;t) C_j \\ \qquad + C_j^\top \Lambda_j(s;t)\big) + \int_Z E(z)^\top (\Gamma(s,z;t) + P(s;t)) E(z) \theta(dz) \\ \qquad + \int_Z \Gamma(s,z;t) E(z) \theta(dz) + \int_Z E(z)^\top \Gamma(s,z;t) \theta(dz) - Q(t,s)\Big\} ds \\ \qquad + \sum_{j=1}^d \Lambda_j(s;t) dW_s^j + \int_Z \Gamma(s,z;t) \tilde{N}(ds,dz), \; s \in [t,T], \\ P(T;t) = -G(t), \end{cases} \tag{1.2.5}$$

where $\Lambda(.;t) = (\Lambda_1(.;t), ..., \Lambda_d(.;t))$. Under **(H1)** the BSDE (1.2.4) is uniquely solvable in $\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{n\times d}\right) \times \mathcal{L}^{\theta,2}_{\mathcal{F},p}([t,T]\times Z;\mathbb{R}^n)$, see e.g. [68]. Moreover there exists a constant $K>0$ such that

$$\mathbb{E}\left[\sup_{t\le s\le T} |p(s;t)|^2_{\mathbb{R}^n}\right] + \mathbb{E}\left[\int_t^T |q(s;t)|^2_{\mathbb{R}^{n\times d}} ds\right] + \mathbb{E}\left[\int_t^T \int_Z |r(s,z;t)|^2_{\mathbb{R}^n} \theta(dz) ds\right] \le K\left(1 + |x_0|^2\right). \tag{1.2.6}$$

In an other hand, noting that the final data of the equation (1.2.5) is deterministic, it is straightforward to look at a deterministic solution. In addition we have the following representation

$$\begin{cases} dP(s;t) = -\Big\{A^\top P(s;t) + P(s;t)A + \sum_{j=1}^d C_j^\top P(s;t) C_j \\ \qquad + \int_Z E(z)^\top P(s;t) E(z) \theta(dz) - Q(t,s)\Big\} ds, \; s \in [t,T], \\ P(T;t) = -G(t), \end{cases} \tag{1.2.7}$$

which is a uniquely solvable matrix-valued ordinary differential equation.

Next, for each $t \in [0,T]$, associated with the 6-tuple $\left(\hat{u}(.), \hat{X}(.), p(.;t), q(.,t), r(.,.;t), P(.;t)\right)$ we define the $\mathcal{H}_t$-function as follows

$$\begin{aligned} \mathcal{H}_t(s,X,u) &= \mathbb{H}(t,s,X,\hat{u}(s)+u, p(s;t), q(s;t), r(s,.;t)) \\ &\quad + \frac{1}{2} u^\top \left\{\sum_{j=1}^d D_j^\top P(s;t) D_j + \int_Z F(z)^\top P(s;t) F(z) \theta(dz)\right\} u, \end{aligned} \tag{1.2.8}$$

where $(s,X,u) \in [t,T] \times \mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n) \times \mathbb{R}^m$. In the rest of the paper, we will keep the following notation, for $(s,t) \in \mathcal{D}[0,T]$

$$\delta\mathbb{H}(t;s) = \mathbb{H}\left(t,s,\hat{X}(s), \hat{u}(s)+u, p(s;t), q(s;t), r(s,.;t)\right) - \mathbb{H}\left(t,s,\hat{X}(s), \hat{u}(s), p(s;t), q(s;t), r(s,.;t)\right).$$

1.2.2 A stochastic maximum principle for equilibrium controls

In this subsection, we present a version of Pontryagin's stochastic maximum principle which characterize the equilibrium controls of the Problem (LQJ). We derive the result by using the second order Taylor expansion in the special form spike variation (1.2.1). Here, we don't assume the non-negativity condition about the matrices Q, G and R as in [51] and [102].

The following theorem is the first main result of this work, it's providing a necessary and sufficient condition to characterize the open-loop Nash equilibrium controls for time-inconsistent Problem (LQJ).

Theorem 1.2.1 (Stochastic Maximum Principle For Equilibriums) *Let **(H1)** holds. Then an admissible control $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ is an open-loop Nash equilibrium, if and only if, for almost all $t \in [0,T]$, there exist a unique triplet of adapted processes $(p(.;t), q(.;t), r(.,.;t))$ which satisfy the BSDE* (1.2.4) *and a deterministic matrix-valued function $P(.;t)$ which satisfies the ODE* (1.2.7)*, such that the following condition holds, for all $u \in \mathbb{R}^m$,*

$$\delta \mathbb{H}(t;t) + \frac{1}{2} u^\top \left\{ \sum_{j=1}^{d} D_j^\top P(t;t) D_j + \int_Z F(z)^\top P(t;t) F(z) \theta(dz) \right\} u \leq 0, \ \mathbb{P}-a.s. \tag{1.2.9}$$

Or equivalently, we have the following two conditions. The first order equilibrium condition

$$R(t,t)\hat{u}(t) - B^\top p(t;t) - \sum_{j=1}^{d} D_j^\top q_j(t;t) - \int_Z F(z)^\top r(t,z;t)\theta(dz) = 0, \ a.e. t \in [0,T], \ \mathbb{P}-a.s., \tag{1.2.10}$$

and the second order equilibrium condition

$$R(t,t) - \sum_{j=1}^{d} D_j^\top P(t;t) D_j - \int_Z F(z)^\top P(t;t) F(z)\theta(dz) \geq 0, \ a.e. t \in [0,T], \ \mathbb{P}-a.s. \tag{1.2.11}$$

Remark 1.2.1 *Note that for each $t \in [0,T]$,* (1.2.4) *and* (1.2.5) *are backward stochastic differential equations. So, as we consider all t in $[0,T]$, all their corresponding adjoint equations form essentially a "flow" of BSDEs. Moreover, there is an additional constraint* (1.2.9) *which is equivalent to the conditions* (1.2.10) *and* (1.2.11) *that acts on the flow only when $s = t$, while the Pontryagin's stochastic maximum principle for optimal control involves only one system of forward-backward stochastic differential equation.*

Proof of the Theorem 1.2.1

Our goal now, is to give a proof of the Theorem 1.2.1. The main idea is still based on the variational techniques in the same spirit of proving the stochastic Pontryagin's maximum principle [90].

Let $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ be an admissible control and $\hat{X}(.)$ the corresponding controlled process solution to the state equation. Consider the perturbed control $u^\varepsilon(.)$ defined by the spike variation (1.2.1) for some fixed arbitrary $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$ and $\varepsilon \in [0, T-t)$. Denote by $\hat{X}^\varepsilon(.)$ the solution of the state equation corresponding to $u^\varepsilon(.)$. Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, see e.g. [90], we have

$$\hat{X}^\varepsilon(s) - \hat{X}(s) = y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s), \; s \in [t,T], \tag{1.2.12}$$

where $y^{\varepsilon,v}(.)$ and $z^{\varepsilon,v}(.)$ solve the following linear stochastic differential equations, respectively

$$\begin{cases} dy^{\varepsilon,v}(s) = Ay^{\varepsilon,v}(s)\,ds + \sum_{j=1}^{d}\left\{C_j y^{\varepsilon,v}(s) + D_j v 1_{[t,t+\varepsilon)}(s)\right\} dW^j(s) \\ \qquad + \int_Z \left\{E(z)\, y^{\varepsilon,v}(s-) + F(z)\, v 1_{[t,t+\varepsilon)}(s)\right\} \tilde{N}(ds,dz), \; s \in [t,T], \\ y^{\varepsilon,v}(t) = 0, \end{cases} \tag{1.2.13}$$

and

$$\begin{cases} dz^{\varepsilon,v}(s) = \left\{Az^{\varepsilon,v}(s) + Bv1_{[t,t+\varepsilon)}(s)\right\} ds + \sum_{j=1}^{d} C_j z^{\varepsilon,v}(s)\, dW^j(s) \\ \qquad + \int_Z E(z)\, z^{\varepsilon,v}(s-)\, \tilde{N}(ds,dz), \; s \in [t,T], \\ z^{\varepsilon,v}(t) = 0. \end{cases} \tag{1.2.14}$$

First, we present the following technical lemma needed later in this study.

Lemma 1.2.1 *Under assumption **(H1)**, the following estimates hold*

$$\mathbb{E}^t\left[y^\varepsilon(s)\right] = 0, \; a.e. \; s \in [t,T] \; \text{ and } \sup_{s\in[t,T]}\left|\mathbb{E}^t\left[z^\varepsilon(s)\right]\right|^2 = O\left(\varepsilon^2\right), \tag{1.2.15}$$

$$\mathbb{E}^t \sup_{s\in[t,T]}\left|y^\varepsilon(s)\right|^2 = O(\varepsilon) \; \text{ and } \mathbb{E}^t \sup_{s\in[t,T]}\left|z^\varepsilon(s)\right|^2 = O\left(\varepsilon^2\right). \tag{1.2.16}$$

Moreover, we have the equality

$$\begin{aligned} & J\left(t, \hat{X}(t), u^\varepsilon(.)\right) - J\left(t, \hat{X}(t), \hat{u}(.)\right) \\ & = -\mathbb{E}^t\left[\int_t^T \left\{\delta\mathbb{H}(t;s) + \frac{1}{2}v^\top\left(\sum_{j=1}^{d} D_j^\top P(s;t)\, D_j + \int_Z F(z)^\top P(s;t)\, F(z)\,\theta(dz)\right) v\right\} 1_{[t,t+\varepsilon)}(s)\, ds\right] + o(\varepsilon). \end{aligned} \tag{1.2.17}$$

Proof. Let $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$ and $\varepsilon \in [0, T-t)$. Since $\mathbb{E}^t\left[y^{\varepsilon,v}(.)\right]$ and $\mathbb{E}^t\left[z^{\varepsilon,v}(.)\right]$ solve the following ODEs, respectively,

$$\begin{cases} d\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right] = A\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right]ds,\ s\in[t,T], \\ \mathbb{E}^t\left[y^{\varepsilon,v}(t)\right]=0, \end{cases}$$

and

$$\begin{cases} d\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right] = \left\{A\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right]+B\mathbb{E}^t\left[v\right]1_{[t,t+\varepsilon)}(s)\right\}ds,\ s\in[t,T], \\ \mathbb{E}^t\left[z^{\varepsilon,v}(t)\right]=0. \end{cases}$$

Thus, it is clear that $\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right]=0$, *a.e.* $s\in[t,T]$. According to Gronwall's inequality we have $\sup_{s\in[t,T]}\left|\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right]\right|^2=O\left(\varepsilon^2\right)$. Moreover, by Lemma 2.1. in [90], we obtain (1.2.16).
Now, we can calculate the difference

$$\begin{aligned}
&J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right)\\
&=\mathbb{E}^t\Bigg[\int_t^T\Big\{\Big\langle Q(t,s)\hat{X}(s)+\bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right],y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\Big\rangle\\
&\qquad\qquad+\frac{1}{2}\left\langle Q(t,s)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right),y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle\\
&\qquad\qquad+\frac{1}{2}\left\langle\bar{Q}(t,s)\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right],\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right]\right\rangle\\
&\qquad\qquad+\left\langle R(t,s)\hat{u}(s),v\right\rangle 1_{[t,t+\varepsilon)}(s)+\frac{1}{2}\left\langle R(t,s)v,v\right\rangle 1_{[t,t+\varepsilon)}(s)\Big\}ds\\
&\qquad+\frac{1}{2}\left\langle G(t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right),y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\\
&\qquad+\Big\langle G(t)\hat{X}(T)+\bar{G}(t)\mathbb{E}^t\left[\hat{X}(T)\right]+\mu_1(t)\hat{X}(t)+\mu_2(t),y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\Big\rangle\\
&\qquad+\frac{1}{2}\left\langle\bar{G}(t)\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right],\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]\right\rangle\Bigg].
\end{aligned}$$

In an other hand, from **(H1)** and (1.2.15) – (1.2.16) the following estimate holds

$$\begin{aligned}
&\mathbb{E}^t\Bigg[\int_t^T\frac{1}{2}\left\langle\bar{Q}(t,s)\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right],\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right]\right\rangle ds\\
&\qquad+\frac{1}{2}\left\langle\bar{G}(t)\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right],\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]\right\rangle\Bigg]=o(\varepsilon). \qquad (1.2.18)
\end{aligned}$$

Then, from the terminal conditions in the adjoint equations, it follows that

$$\begin{aligned}
& J\left(t, \hat{X}(t), u^{\varepsilon}(.)\right) - J\left(t, \hat{X}(t), \hat{u}(.)\right) \\
& = \mathbb{E}^t\left[\int_t^T \left\{\left\langle Q(t,s)\hat{X}(s) + \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right], y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right\rangle\right.\right. \\
& \qquad\qquad + \frac{1}{2}\left\langle Q(t,s)\left(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right), y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right\rangle \\
& \qquad\qquad \left. + \left\langle R(t,s)\hat{u}(s), v\right\rangle 1_{[t,t+\varepsilon)}(s) + \frac{1}{2}\left\langle R(t,s)v, v\right\rangle 1_{[t,t+\varepsilon)}(s)\right\} ds \\
& \qquad \left. - \left\langle p(T;t), y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T)\right\rangle - \frac{1}{2}\left\langle P(T;t)\left(y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T)\right), y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T)\right\rangle\right] \\
& \qquad + o(\varepsilon).
\end{aligned} \tag{1.2.19}$$

Now, by applying Ito's formula to $s \mapsto \langle p(s;t), y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\rangle$ on $[t,T]$, we get

$$\begin{aligned}
& \left\langle p(T;t), y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T)\right\rangle \\
& = \int_t^T \left\{(Bv)^\top p(s;t) 1_{[t,t+\varepsilon)}(s) + \left(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right)^\top \left(Q(t,s)\hat{X}(s) + \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right]\right)\right. \\
& \qquad \left. + \sum_{j=1}^d (D_j v)^\top q_j(s;t) 1_{[t,t+\varepsilon)}(s) + \int_Z (F(z)v)^\top r(s,z;t) 1_{[t,t+\varepsilon)}(s)\theta(dz)\right\} ds \\
& + \sum_{j=1}^d \int_t^T \left\{\left(C_j\left(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right) + D_j v 1_{[t,t+\varepsilon)}(s)\right)^\top p(s;t) + \left(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right)^\top q_j(s;t)\right\} dW^j(s) \\
& + \int_t^T \int_Z \left\{\left(E(z)\left(y^{\varepsilon,v}(s-) + z^{\varepsilon,v}(s-)\right) + F(z)v1_{[t,t+\varepsilon)}(s)\right)^\top p(s;t)\right. \\
& \qquad\qquad + \left(y^{\varepsilon,v}(s-) + z^{\varepsilon,v}(s-)\right)^\top r(s,z;t) + \\
& \qquad\qquad \left. \left(E(z)\left(y^{\varepsilon,v}(s-) + z^{\varepsilon,v}(s-)\right) + F(z)v1_{[t,t+\varepsilon)}(s)\right)^\top r(s,z;t)\right\} \tilde{N}(ds,dz).
\end{aligned} \tag{1.2.20}$$

Again, by applying Ito's formula to $s \mapsto \langle P(s;t)(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)), y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\rangle$ on $[t,T]$, we get

$$\begin{aligned}
&\langle P(T;t)(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\rangle \\
&= \int_t^T \Big\{ 2(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top P(s;t) Bv1_{[t,t+\varepsilon)}(s) + (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top Q(t,s)(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)) \\
&\qquad + \sum_{j=1}^d \Big(2(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top C_j^\top P(s;t) D_j v1_{[t,t+\varepsilon)}(s) + v^\top D_j^\top P(s;t) D_j v1_{[t,t+\varepsilon)}(s)\Big) \\
&\qquad + \int_Z \Big\{ 2(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top E(z)^\top P(s;t) F(z) v1_{[t,t+\varepsilon)}(s) \\
&\qquad\qquad + v^\top F(z)^\top P(s;t) F(z) v1_{[t,t+\varepsilon)}(s)\theta(dz)\Big\} ds \\
&\quad + 2\sum_{j=1}^d \int_t^T \Big\{ (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top P(s;t)\big(C_j(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)) + D_j v1_{[t,t+\varepsilon)}(s)\big)\Big\} dW^j(s) \\
&\quad + \int_t^T \int_Z \Big\{ 2(y^{\varepsilon,v}(s-)+z^{\varepsilon,v}(s-))^\top P(s;t)\big(E(z)(y^{\varepsilon,v}(s-)+z^{\varepsilon,v}(s-)) + F(z)v1_{[t,t+\varepsilon)}(s)\big) \\
&\qquad\qquad \big(E(z)(y^{\varepsilon,v}(s-)+z^{\varepsilon,v}(s-)) + F(z)v1_{[t,t+\varepsilon)}(s)\big)^\top P(s;t)(E(z)(y^{\varepsilon,v}(s-)+z^{\varepsilon,v}(s-)) \\
&\qquad\quad + F(z)v1_{[t,t+\varepsilon)}(s)\Big\} \tilde{N}(ds,dz),
\end{aligned} \tag{1.2.21}$$

Moreover, we conclude from **(H1)** together with (1.2.15) – (1.2.16) that

$$\begin{aligned}
&\mathbb{E}^t\left[\int_t^T (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top P(s;t) Bv1_{[t,t+\varepsilon)}(s)\,ds\right] = o(\varepsilon), \\
&\mathbb{E}^t\left[\int_t^T (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top C_j^\top P(s;t) D_j v1_{[t,t+\varepsilon)}(s)\,ds\right] = o(\varepsilon), \\
&\mathbb{E}^t\left[\int_t^T \int_Z (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top E(z)^\top P(s;t) F(z) v1_{[t,t+\varepsilon)}(s)\theta(dz)\,ds\right] = o(\varepsilon).
\end{aligned} \tag{1.2.22}$$

By taking the conditional expectation in (1.2.20) and (1.2.21), then by invoking (1.2.22) it holds that

$$\begin{aligned}
&\mathbb{E}^t\left[\langle p(T;t), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\rangle\right] \\
&= \mathbb{E}^t\left[\int_t^T \Big\{ v^T B^\top p(s;t) 1_{[t,t+\varepsilon)}(s) + (y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^\top \Big(Q(t,s)\hat{X}(s) + \bar{Q}(t,s)\mathbb{E}^t\big[\hat{X}(s)\big]\Big) \right. \\
&\qquad \left. + \sum_{j=1}^d v^T D_j^\top q_j(s;t) 1_{[t,t+\varepsilon)}(s) + \int_Z v^T F(z)^\top r(s,z;t) 1_{[t,t+\varepsilon)}(s)\theta(dz)\Big\} ds\right],
\end{aligned} \tag{1.2.23}$$

and

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}^{t}\left[\left\langle P\left(T;t\right)\left(y^{\varepsilon,v}\left(T\right)+z^{\varepsilon,v}\left(T\right)\right),y^{\varepsilon,v}\left(T\right)+z^{\varepsilon,v}\left(T\right)\right\rangle\right]\\
&=\frac{1}{2}\mathbb{E}^{t}\left[\int_{t}^{T}\left\{\left(y^{\varepsilon,v}\left(s\right)+z^{\varepsilon,v}\left(s\right)\right)^{\top}Q\left(t,s\right)\left(y^{\varepsilon,v}\left(s\right)+z^{\varepsilon,v}\left(s\right)\right)\right.\right.\\
&\qquad+\sum_{j=1}^{d}v^{\top}D_{j}^{\top}P\left(s;t\right)D_{j}v1_{[t,t+\varepsilon)}\left(s\right)\\
&\qquad\left.\left.+\int_{Z}v^{\top}F\left(z\right)^{\top}P\left(s;t\right)F\left(z\right)v1_{[t,t+\varepsilon)}\left(s\right)\theta\left(dz\right)\right\}ds\right]+o\left(\varepsilon\right).
\end{aligned}\tag{1.2.24}$$

By taking (1.2.23) and (1.2.24) in $(1.2.19)$, it follows that

$$\begin{aligned}
&J\left(t,\hat{X}\left(t\right),u^{\varepsilon}\left(.\right)\right)-J\left(t,\hat{X}\left(t\right),\hat{u}\left(.\right)\right)\\
&=-\mathbb{E}^{t}\left[\int_{t}^{T}\left\{v^{\top}B^{\top}p\left(s;t\right)+\sum_{j=1}^{d}v^{\top}D_{j}^{\top}q_{j}\left(s,t\right)+\frac{1}{2}\sum_{j=1}^{d}v^{\top}D_{j}^{\top}P\left(s;t\right)D_{j}v\right.\right.\\
&\qquad-v^{\top}R\left(t,s\right)\hat{u}\left(s\right)-\frac{1}{2}v^{\top}R\left(t,s\right)v\\
&\qquad\left.\left.+\int_{Z}\left(r\left(s,z;t\right)^{\top}F\left(z\right)v+\frac{1}{2}v^{\top}F\left(z\right)^{\top}P\left(s;t\right)F\left(z\right)v\right)\theta\left(dz\right)\right\}1_{[t,t+\varepsilon)}\left(s\right)ds\right]+o\left(\varepsilon\right),
\end{aligned}$$

which is equivalent to $(1.2.17)$. ■

Now, we are ready to give a proof of Theorm 1.2.1

Proof of Theorem 1.2.1. Given an open-loop Nash equilibrium $\hat{u}(.)$, then for any $t\in[0,T]$ and $v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{m}\right)$, we have clearly

$$\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}\left(t\right),\hat{u}\left(.\right)\right)-J\left(t,\hat{X}\left(t\right),u^{\varepsilon}\left(.\right)\right)\right\}\leq0,$$

which leads from (1.2.17) to

$$\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\mathbb{E}^{t}\left[\int_{t}^{T}\left\{\delta\mathbb{H}\left(t;s\right)+\frac{1}{2}v^{\top}\left(\sum_{j=1}^{d}D_{j}^{\top}P\left(s;t\right)D_{j}+\int_{Z}F\left(z\right)^{\top}P\left(s;t\right)F\left(z\right)\theta\left(dz\right)\right)v\right\}1_{[t,t+\varepsilon)}\left(s\right)ds\right]\leq0,$$

from which we deduce for almost all $t\in[0,T]$ and $v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{m}\right)$,

$$\delta\mathbb{H}\left(t;t\right)+\frac{1}{2}v^{\top}\left(\sum_{j=1}^{d}D_{j}^{\top}P\left(t;t\right)D_{j}+\int_{Z}F\left(z\right)^{\top}P\left(t;t\right)F\left(z\right)\theta\left(dz\right)\right)v\leq0,\ \mathbb{P}-a.s,$$

Therefore, the inequality (1.2.9) is ensured by stetting $v\equiv u$ for an arbitrarily $u\in\mathbb{R}^{m}$.

Conversely, given an admissible control $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$. Suppose that, for almost all $t \in [0,T]$, the variational inequality (1.2.9) holds. Then for any $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$ it yields

$$\delta\mathbb{H}(t;t) + \frac{1}{2}v^\top \left(\sum_{j=1}^{d} D_j^\top P(t;t) D_j + \int_Z F(z)^\top P(t;t) F(z)\theta(dz)\right) v \leq 0,\ \mathbb{P}-a.s,$$

consequently, for all $t \in [0,T]$ and $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{\delta\mathbb{H}(t;s) + \frac{1}{2}v^\top\left(\sum_{j=1}^{d} D_j^\top P(s;t) D_j + \frac{1}{2}\int_Z F(z)^\top P(s;t) F(z)\theta(dz)\right)v\right\}ds\right] \leq 0.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}(t),\hat{u}(.)\right) - J\left(t,\hat{X}(t),u^\varepsilon(.)\right)\right\} \leq 0.$$

$\forall\ t \in [0,T]$ and $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$. Thus $\hat{u}(.)$ is an equilibrium control.

Easy manipulations show that the variational inequality (1.2.9) is equivalent to

$$\mathcal{H}_t\left(t,\hat{X}(t),0\right) = \max_{u\in\mathbb{R}^m}\mathcal{H}_t\left(t,\hat{X}(t),u\right),$$

then (1.2.10) and (1.2.11) follow respectively from the following first order and second order conditions at the maximum point $u = 0$ for the quadratic function $\mathcal{H}_t\left(t,\hat{X}(t),u\right)$

$$\mathcal{D}_u\mathcal{H}_t\left(t,\hat{X}(t),0\right) = 0 \text{ and } \mathcal{D}_u^2\mathcal{H}_t\left(t,\hat{X}(t),u\right) \leq 0,$$

where we denote by $\mathcal{D}_u\mathcal{H}_t$ (resp. $\mathcal{D}_u^2\mathcal{H}_t$) the gradient (resp. the Hessian) of $\mathcal{H}_t$ with respect to the variable u. Then, the required result is directly follows. ■

In Theorem 1.2.1, in view of condition (1.2.11), as long as the term

$$-\sum_{j=1}^{d} D_j^\top P(t;t) D_j - \int_Z F(z)^\top P(t;t) F(z)\theta(dz),$$

for each $t \in [0,T]$ is sufficiently positive definite, the necessary and sufficient condition for equilibriums might still be satisfied even if $R(t,t)$ is negative. This is different from [51] and [102] where the authors have assumed the non-negativity of the matrices Q, G and R in order to state their stochastic maximum principle for open-loop Nash equilibriums. Moreover, in the case where $Q(t,s) \geq 0$ for every $s \in [t,T]$, and $G(t) \geq 0$, it follows that the solution of the second order adjoint equation satisfies $P(t;t) \leq 0$, then if further we have $R(t,t) \geq 0$, the condition that

$$R(t,t) - \sum_{j=1}^{d} D_j^\top P(t;t) D_j - \int_Z F(z)^\top P(t;t) F(z) \theta(dz) \geq 0,$$

is obviously satisfied. Therefore, we summarize the main theorem into the following Corollary.

Corollary 1.2.1 *Let **(H1)-(H2)** hold. Then an admissible control $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ is an equilibrium control, if and only if, for almost all $t \in [0,T]$, there exists a triple of adapted processes $(p(.;t), q(.;t), r(.,.;t))$ which satisfies the BSDE* (1.2.4), *with only the first order condition* (1.2.10) *holds.*

1.3 Linear feedback stochastic equilibrium control

In this section, we consider only the case where the Brownian motion is one-dimensional ($d = 1$) for simplicity of presentation. There is no essential difficulty with the multidimensional Brownian motions. All the indices j will then be dropped. Our goal is to obtain a state feedback representation of an equilibrium control for Problem (LQJ) via some class of ordinary differential equations.

We first consider the following system of coupled generalized Riccati equations[1], for $(t,s) \in \mathcal{D}[0,T]$

$$\begin{cases} 0 = \dfrac{\partial M}{\partial s}(t,s) + M(t,s)A + A^\top M(t,s) + C^\top M(t,s)C + \displaystyle\int_Z E(z)^\top M(t,s)E(z)\theta(dz) \\ \quad - \left(M(t,s)B + C^\top M(t,s)D + \displaystyle\int_Z E(z)^\top M(t,s)F(z)\theta(dz) \right) \Psi(s) + Q(t,s), \\ 0 = \dfrac{\partial \bar{M}}{\partial s}(t,s) + \bar{M}(t,s)A + A^\top \bar{M}(t,s) - \bar{M}(t,s)B\Psi(s) + \bar{Q}(t,s), \\ 0 = \dfrac{\partial \Upsilon}{\partial s}(t,s) + A^\top \Upsilon(t,s), \\ 0 = \dfrac{\partial \varphi}{\partial s}(t,s) + \left(M(t,s) + \bar{M}(t,s) \right)(b - B\psi(s)) + A^\top \varphi(t,s) + C^\top M(t,s)(\sigma - D\psi(s)) \\ \quad + \displaystyle\int_Z E(z)^\top M(t,s)(c(z) - F(z)\psi(s))\theta(dz), \\ M(t,T) = G(t),\ \bar{M}(t,T) = \bar{G}(t),\ \Upsilon(t,T) = \mu_1(t),\ \varphi(t,T) = \mu_2(t), t \in [0,T], \end{cases} \tag{1.3.1}$$

where

$$\det\left(R(t,t) + D^\top M(t,t)D + \int_Z F(z)^\top M(t,t)F(z)\theta(dz) \right) \neq 0,\ t \in [0,T],$$

[1]Strictly speaking, these are not Riccati equations in the usual sense as they are not symmetric. However, we still use the term so as to see the connection and difference between time-inconsistent and time-consistent LQ control problems.

and $\Psi(t)$ and $\psi(t)$, for $t\in[0,T]$ are given by

$$\begin{cases}\Theta(t)=\Big(R(t,t)+D^{\top}M(t,t)D+\int_Z F(z)^{\top}M(t,t)F(z)\theta(dz)\Big)^{-1},\\ \Psi(t)=\Theta(t)\Big\{B^{\top}\left(M(t,t)+\bar{M}(t,t)+\Upsilon(t,t)\right)+D^{\top}M(t,t)C+\displaystyle\int_Z F(z)^{\top}M(t,t)E(z)\theta(dz)\Big\},\\ \psi(t)=\Theta(t)\Big\{B^{\top}\varphi(t,t)+D^{\top}M(t,t)\sigma+\displaystyle\int_Z F(z)^{\top}M(t,t)c(z)\theta(dz)\Big\},\end{cases}\tag{1.3.2}$$

Theorem 1.3.1 *Let **(H1)-(H2)** hold. If there exists a solution to the system* (1.3.1). *Then Problem(LQJ) has an equilibrium control that can be represented by the state feedback form:*

$$\hat{u}(t)=-\Psi(t)\hat{X}(t)-\psi(t),\ \ a.e.t\in[0,T].\tag{1.3.3}$$

Proof. Suppose that $\hat{u}(.)$ is an equilibrium control and denote by $\hat{X}(.)$ the corresponding controlled process. Then in view of Corollary 1.2.1, there exist an adapted processes $\Big(\hat{X}(.),(p(.;t),q(.;t),r(.,.;t))_{t\in[0,T)}\Big)$ solution to the following flow of forward-backward SDE with jumps, parametrized by $t\in[0,T]$

$$\begin{cases}d\hat{X}(s)=\Big\{A\hat{X}(s)+B\hat{u}(s)+b\Big\}ds+\Big\{C\hat{X}(s)+D\hat{u}(s)+\sigma\Big\}dW(s)\\ \qquad\qquad+\displaystyle\int_Z\Big\{E(z)\hat{X}(s-)+F(z)\hat{u}(s)+c(z)\Big\}\tilde{N}(ds,dz),\ \ s\in[0,T],\\ dp(s;t)=-\Big\{A^{\top}p(s;t)+C^{\top}q(s;t)+\displaystyle\int_Z E(z)^{\top}r(s,z;t)\theta(dz)-Q(t,s)\hat{X}(s)\\ \qquad\qquad-\bar{Q}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big]\Big\}ds+q(s;t)dW(s)+\displaystyle\int_Z r(s,z;t)\tilde{N}(ds,dz),\ \ 0\le t\le s\le T,\\ \hat{X}_0=x_0,\ p(T;t)=-G(t)\hat{X}(T)-\bar{G}(t)\mathbb{E}^t\Big[\hat{X}(T)\Big]-\mu_1(t)\hat{X}(t)-\mu_2(t),\ \ t\in[0,T],\end{cases}\tag{1.3.4}$$

such that the following condition holds

$$R(t,t)\hat{u}(t)-B^{\top}p(t;t)-D^{\top}q(t;t)-\int_Z F(z)^{\top}r(t,z;t)\theta(dz)=0,\ \ \mathbb{P}-a.s,\ \ a.e.t\in[0,T].\tag{1.3.5}$$

Now, to solve the above stochastic system, we conjecture that $\hat{X}(.)$ and $p(.;t)$ for $t\in[0,T)$ are related by the following relation

$$p(s;t)=-M(t,s)\hat{X}(s)-\bar{M}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big]-\Upsilon(t,s)\hat{X}(t)-\varphi(t,s),\ \ (t,s)\in\mathcal{D}[0,T],\tag{1.3.6}$$

for some deterministic functions $M(.,.),\bar{M}(.,.),\Upsilon(.,.)\in C^{0,1}(\mathcal{D}[0,T],\mathbb{R}^{n\times n})$ and $\varphi(.,.)\in C^{0,1}(\mathcal{D}[0,T],\mathbb{R}^{n})$ such that

$$M(t,T)=G(t),\ \ \bar{M}(t,T)=\bar{G}(t),\ \ \Upsilon(t,T)=\mu_1(t),\ \varphi(t,T)=\mu_2(t),\ \ t\in[0,T].\tag{1.3.7}$$

Applying Itô's formula to (1.3.6) and using (1.3.4), it yields

$$\begin{aligned}
dp(s;t) = &\Big\{-\frac{\partial M}{\partial s}(t,s)\hat{X}(s) - \frac{\partial \bar{M}}{\partial s}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big] - \frac{\partial \Upsilon}{\partial s}(t,s)\hat{X}(t) - \frac{\partial \varphi}{\partial s}(t,s) \\
&- M(t,s)\Big(A\hat{X}(s) + Bu(s) + b\Big) - \bar{M}(t,s)\Big(A\mathbb{E}^t\Big[\hat{X}(s)\Big] + B\mathbb{E}^t[u(s)] + b\Big)\Big\}ds \\
&- M(t,s)\Big(C\hat{X}(s) + D\hat{u}(s) + \sigma\Big)dW(s) \\
&- \int_Z M(t,s)\Big(E(z)\hat{X}(s-) + F(z)\hat{u}(s) + c(z)\Big)\tilde{N}(ds,dz), \\
= &-\Big\{A^\top p(s;t) + C^\top q(s;t) + \int_Z E(z)^\top r(s,z;t)\theta(dz) - Q(t,s)\hat{X}(s) \\
&-\bar{Q}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big]\Big\}ds + q(s;t)dW(s) + \int_Z r(s,z;t)\tilde{N}(ds,dz),\ s\in[t,T], \qquad (1.3.8)
\end{aligned}$$

from which we deduce

$$q(s;t) = -M(t,s)\Big(C\hat{X}(s) + D\hat{u}(s) + \sigma\Big),\ \text{a.e.} s\in[t,T], \qquad (1.3.9)$$

$$r(s,z;t) = -M(t,s)\Big(E(z)\hat{X}(s) + F(z)\hat{u}(s) + c(z)\Big),\ \text{a.e.} s\in[t,T]. \qquad (1.3.10)$$

We put the above expressions of $q(s;t)$ and $r(s,z;t)$ into (1.3.5), then

$$\begin{aligned}
0 = &R(t,t)\hat{u}(t) + B^\top\Big(\big(M(t,t) + \bar{M}(t,t) + \Upsilon(t,t)\big)\hat{X}(t) + \varphi(t,t)\Big) \\
&+ D^\top M(t,t)\Big(C\hat{X}(t) + D\hat{u}(t) + \sigma\Big) \\
&+ \int_Z F(z)^\top M(t,t)\Big(E(z)\hat{X}(t) + F(z)\hat{u}(t) + c(z)\Big)\theta(dz),\ a.e.t\in[0,T].
\end{aligned}$$

Subsequently, we obtain with the above notations

$$\Theta(t)^{-1}\Big(\hat{u}(t) + \Psi(t)\hat{X}(t) + \psi(t)\Big) = 0,\ a.e.t\in[0,T].$$

Hence (1.3.3) holds, and for any $(t,s)\in\mathcal{D}[0,T]$, we have

$$\mathbb{E}^t[\hat{u}(s)] = -\Psi(s)\mathbb{E}^t\Big[\hat{X}(s)\Big] - \psi(s). \qquad (1.3.11)$$

Next, comparing the ds term in (1.3.8) , then by using the expressions (1.3.3) and (1.3.11), we obtain

$$\begin{aligned}
0 = &\left\{ \frac{\partial M}{\partial s}(t,s) + M(t,s)A + A^{\top} M(t,s) + C^{\top} M(t,s) C + \int_Z E(z)^{\top} M(t,s) E(z) \theta(dz) \right. \\
&\left. - \left(M(t,s) B + C^{\top} M(t,s) D + \int_Z E(z)^{\top} M(t,s) F(z) \theta(dz) \right) \Psi(s) + Q(t,s) \right\} \hat{X}(s) \\
&+ \left\{ \frac{\partial \bar{M}}{\partial s}(t,s) + \bar{M}(t,s) A + A^{\top} \bar{M}(t,s) - \bar{M}(t,s) B \Psi(s) + \bar{Q}(t,s) \right\} \mathbb{E}^t \left[\hat{X}(s) \right] \\
&+ \left\{ \frac{\partial \Upsilon}{\partial s}(t,s) + A^{\top} \Upsilon(t,s) \right\} \hat{X}(t) \\
&+ \frac{\partial \varphi}{\partial s}(t,s) + \left(M(t,s) + \bar{M}(t,s) \right) (b - B\psi(s)) + A^{\top} \varphi(t,s) \\
&+ C^{\top} M(t,s) (\sigma - D\psi(s)) + \int_Z E(z)^{\top} M(t,s) (c(z) - F(z)\psi(s)) \theta(dz).
\end{aligned}$$

This suggests that the functions $M(.,.), \bar{M}(.,.), \Upsilon(.,.)$ and $\varphi(.,.)$ solve the system (1.3.1).

Note that, we can check that $\Psi(.)$ and $\psi(.)$ in (1.3.2) are both uniformly bounded. Then the following linear SDEJ,

$$\begin{cases}
d\hat{X}(s) = \left\{ (A - B\Psi(s)) \hat{X}(s) + b - B\psi(s) \right\} ds \\
\qquad + \left\{ (C - D\Psi(s)) \hat{X}(s) + \sigma - D\psi(s) \right\} dW(s) \\
\qquad + \int_Z \left\{ (E(z) - F(z)\Psi(s)) \hat{X}(s-) + c(z) - F(z)\psi(s) \right\} \widetilde{N}(ds,dz), \text{ for } s \in [0,T], \\
\hat{X}(0) = x_0,
\end{cases}$$

is uniquely solvable, and the following estimate holds

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left| \hat{X}(s) \right|^2 \right] \leq K \left(1 + |x_0|^2 \right).$$

So the control $\hat{u}(.)$ defined by (1.3.3) is admissible. ∎

Remark 1.3.1 *Note that, the verification theorem (Theorem 1.3.1) assumes the existence of a solution to the system* (1.3.1). *However, since the ODEs which should be solved by* $M(.,.)$ *and* $\bar{M}(.,.)$ *do not have a symmetry structure. The general solvability for this type of ODEs when* $(n > 1)$ *remains an outstanding open problem. We will see in the next section two examples in the case when* $n = 1$*, this case is important, especially in financial applications as will be confirmed by the mean–variance portfolio selection model. Also, we remark that a special feature of the case when* $n = 1$ *is that the state* $X(.)$ *is one-dimensional, so are the unknowns* $M(.,.), \bar{M}(.,.), \Upsilon(.,.)$ *and* $\varphi(.,.)$ *of the system* (1.3.1). *This makes it easier to solve* (1.3.1).

1.4 Some applications

1.4.1 Mean-variance portfolio selection problem

In this subsection, we discuss the continuous-time Markowitz's mean-variance portfolio selection problem. We apply Theorem 1.3.1 to obtain a state feedback representation of an equilibrium control for the problem. In the absence of Poisson random jumps this problem is discussed in [51].
The problem is formulated as follows: We consider a financial market, in which two securities are traded continuously. One of them is a bond, with price $S^0(s)$ at time $s \in [0,T]$ governed by

$$dS^0(s) = S^0(s)\, r(s)\, ds,\ S^0(0) = s_0 > 0. \tag{1.4.1}$$

There is also a stock with unit price $S^1(s)$ at time $s \in [0,T]$ governed by

$$dS^1(s) = S^1(s-)\left(\alpha(s)\, ds + \beta(s)\, dW(s) + \int_Z \gamma(s,z)\, \widetilde{N}(ds,dz)\right),\ S^1(0) = s^1 > 0. \tag{1.4.2}$$

where $r : [0,T] \to (0,\infty)$, $\alpha, \beta : [0,T] \to \mathbb{R}$ and $\gamma : [0,T] \times Z \to \mathbb{R}$ are assumed to be deterministic, continuous, and bounded such that $\alpha(s) > r(s)$ and $\gamma(s,z) \geq -1$. We also assume a uniform ellipticity condition as follow $\sigma(t)^2 + \int_Z \gamma(t,z)^2\, \theta(dz) \geq \delta$, *a.e*, for some $\delta > 0$. For an investor, a portfolio $\pi(.)$ is a process represents the amount of money invested in the stock. The wealth process $X^{x_0,\pi(.)}(.)$ corresponding to initial capital $x_0 > 0$, and portfolio $\pi(.)$, satisfies then the equation

$$\begin{cases} dX(s) = (r(s)\, X(s) + \pi(s)(\alpha(s) - r(s)))\, ds + \pi(s)\, \beta(s)\, dW(s) \\ \qquad\qquad + \pi(s) \displaystyle\int_Z \gamma(s,z)\, \widetilde{N}(ds,dz),\ \text{for } t \in [0,T], \\ X(0) = x_0. \end{cases} \tag{1.4.3}$$

As time evolves, we need to consider the controlled stochastic differential equation parametrized by $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ and satisfiied by $X(.)$,

$$\begin{cases} dX(s) = (r(s)\, X(s) + \pi(s)(\alpha(s) - r(s)))\, ds + \pi(s)\, \beta(s)\, dW(s) \\ \qquad\qquad + \pi(s) \displaystyle\int_Z \gamma(s,z)\, \widetilde{N}(ds,dz),\ \text{for } s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{1.4.4}$$

The objective is to maximize the conditional expectation of terminal wealth $\mathbb{E}^t[X(T)]$, and at the same time to minimize the conditional variance of the terminal wealth $\mathrm{Var}^t[X(T)]$, over controls $\pi(.)$ valued in

$\mathbb{R}$. Then, the mean-variance portfolio optimization problem is denoted as: minimizing the cost $J(t,\xi,.)$, given by

$$J(t,\xi,\pi(.)) = \frac{1}{2}\mathrm{Var}^t[X(T)] - (\mu_1(t)\xi + \mu_2(t))\mathbb{E}^t[X(T)], \tag{1.4.5}$$

subject to (1.4.4). Here $\mu_1, \mu_2 : [0,T] \to [0,\infty)$, are some deterministic, continuous and bounded functions. The above model covers the one in [51], since, in our case, the weight between the conditional variance and the conditional expectation depends on the current wealth level, as well as, the current time, while in [51], the weight depends on the current wealth level only. Hence, in the above model, there are three different sources of time-inconsistency. Moreover, the above model is mathematically a special case of the general LQ problem formulated earlier in this paper, with $n = d = m = 1$. Then we can apply Theorem 1.3.1 to obtain a Nash equilibrium control.

The optimal control problem associated with (1.4.4) and (1.4.5) is equivalent to minimize

$$J(t,\xi,u(.)) = \frac{1}{2}\left(\mathbb{E}^t\left[X(T)^2\right] - \mathbb{E}^t[X(T)]^2\right) - (\mu_1(t)\xi + \mu_2(t))\mathbb{E}^t[X(T)]$$

subject to (1.4.4). Denote

$$\rho(t) = \frac{(\alpha(t) - r(t))^2}{\beta(t)^2 + \int_Z \gamma(t,z)^2\,\theta(dz)}.$$

Thus, the system (1.3.1) reduces to

$$\begin{cases} \dfrac{\partial M}{\partial s}(t,s) + \left\{2r(s) - \dfrac{\rho(s)}{M(s,s)}\left(M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\right)\right\} M(t,s) = 0, \ \ (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \bar{M}}{\partial s}(t,s) + \left\{2r(s) - \dfrac{\rho(s)}{M(s,s)}\left(M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\right)\right\} \bar{M}(t,s) = 0, \ \ (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \Upsilon}{\partial s}(t,s) + r(s)\Upsilon(t,s) = 0, \ \ (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \varphi}{\partial s}(t,s) + r(s)\varphi(t,s) = 0, \ \ (t,s) \in \mathcal{D}[0,T], \\ M(t,T) = 1, \ \bar{M}(t,T) = -1, \ \Upsilon(t,T) = -\mu_1(t), \ \varphi(t,T) = -\mu_2(t), \ t \in [0,T]. \end{cases} \tag{1.4.6}$$

Clearly, if $M(.,.)$ and $\bar{M}(.,.)$ are solutions to the first and the second equations, respectively, in (1.4.6), then $\tilde{M}(.,.) = \left(\bar{M} + M\right)(.,.)$ solves the following ODE

$$\begin{cases} \dfrac{\partial \tilde{M}}{\partial s}(t,s) + \left\{2r(s) - \dfrac{\rho(s)}{M(s,s)}\left(\tilde{M}(s,s) + \Upsilon(s,s)\right)\right\} \tilde{M}(t,s) = 0, \ \forall (t,s) \in \mathcal{D}[0,T], \\ \tilde{M}(t,T) = 0, \ t \in [0,T], \end{cases} \tag{1.4.7}$$

which is equivalent to

$$\tilde{M}(t,s) = \tilde{M}(t,T)\, e^{\int_s^T \left(2r(\tau) - \frac{\rho(\tau)}{M(\tau,\tau)}\left(\tilde{M}(\tau,\tau)+\Upsilon(\tau,\tau)\right)\right)d\tau},$$

from the boundary condition in (1.4.7), it yields

$$\bar{M}(t,s) + M(t,s) = \tilde{M}(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T].$$

Moreover, we remark that all data of the ODEs which should be solved by $M(.,.)$ and $\bar{M}(.,.)$ are not influenced by t, thus (1.4.6) reduces to

$$\begin{cases} \dfrac{dM}{ds}(s) + 2r(s)\, M(s) - \rho(s)\,\Upsilon(s,s) = 0,\ \forall s \in [0,T], \\ \bar{M}(s) = -M(s),\ \forall s \in [0,T], \\ \dfrac{\partial \Upsilon}{\partial s}(t,s) + r(s)\,\Upsilon(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \varphi}{\partial s}(t,s) + r(s)\,\varphi(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T], \\ M(T) = 1,\ \Upsilon(t,T) = -\mu_1(t),\ \varphi(t,T) = -\mu_2(t),\ \forall t \in [0,T]. \end{cases} \tag{1.4.8}$$

which is explicitly solved by

$$\begin{cases} M(s) = e^{2\int_s^T r(\tau)d\tau}\left\{1 + \displaystyle\int_s^T e^{-\int_\tau^T r(l)dl}\mu_1(\tau)\,\rho(\tau)\,d\tau\right\},\ s \in [0,T], \\ \bar{M}(s) = -e^{2\int_s^T r(\tau)d\tau}\left\{1 + \displaystyle\int_s^T e^{-\int_\tau^T r(l)dl}\mu_1(\tau)\,\rho(\tau)\,d\tau\right\},\ s \in [0,T], \\ \Upsilon(t,s) = -\mu_1(t)\, e^{\int_s^T r(\tau)d\tau},\ (t,s) \in \mathcal{D}[0,T], \\ \varphi(t,s) = -\mu_2(t)\, e^{\int_s^T r(\tau)d\tau},\ (t,s) \in \mathcal{D}[0,T]. \end{cases} \tag{1.4.9}$$

In view of Theorem 1.3.1, the representation of the Nash equilibrium control (1.3.3) then gives

$$\hat{\pi}(s) = -\Psi(s)\,\hat{X}(s) - \psi(s),\ s \in [0,T], \tag{1.4.10}$$

where, $\forall s \in [0,T]$

$$\Psi(s) = \frac{\rho(s)}{(\alpha(s) - r(s))}\frac{\Upsilon(s,s)}{M(s)} \text{ and } \psi(s) = \frac{\rho(s)}{(\alpha(s) - r(s))}\frac{\varphi(s,s)}{M(s)}.$$

The corresponding equilibrium dynamics solves the SDEJ,

$$\begin{cases} d\hat{X}(s) = \Big\{(r(s) - \Psi(s)(\alpha(s) - r(s)))\hat{X}(s) - \psi(s)(\alpha(s) - r(s))\Big\} ds \\ \qquad\qquad - \Big(\Psi(s)\hat{X}(s) + \psi(s)\Big)\Big\{\beta(s)\, dW(s) + \int_Z \gamma(s,z)\widetilde{N}(ds,dz)\Big\}, \text{ for } s \in [0,T], \\ \hat{X}(0) = x_0. \end{cases}$$

Special cases and relationship to other works

Equilibrium investment strategies for mean–variance models have been studied in [11], [17], [18] and [51], among others in different frameworks. In this paragraph, we will compare our results with some existing ones in literature. First, suppose that the price process of the risky asst do not have jumps, i.e $\gamma(s,z) = 0$ a.e.

Special case 1. When $\mu_1(t) \equiv 0$ and $\mu_2(t) \equiv \mu_2 > 0$. In this case the objective is equivalent to Basak and Chabakauri [11] and Bjork and Murguci [17] in which the equilibrium is defined within the class of feedback controls. Moreover the equilibrium strategy $\hat{\pi}(.)$ given in our study by (1.4.10) changes to

$$\hat{\pi}(s) = \mu_2 \frac{(\alpha(s) - r(s))}{\beta(t)^2} e^{-\int_s^T r(\tau)d\tau}, \; s \in [0,T].$$

It is worth pointing out that the above equilibrium solution is the same form as that obtained in Bjork and Murguci [17] by solving the extended HJB equations.

Special case 2. Suppose that $\mu_1(t) \equiv \mu_2 > 0$ and $\mu_2(t) \equiv \mu_2 \equiv 0$. In this case, the equilibrium strategy $\hat{\pi}(.)$ given by expressions (1.4.10) changes to

$$\hat{\pi}(s) = \frac{\mu_1(\alpha(s) - r(s))}{\left(1 + \mu_1 \int_s^T e^{-\int_\tau^T r(l)dl}\rho(\tau)\, d\tau\right)\beta(s)^2} e^{-\int_s^T r(\tau)d\tau}\hat{X}(s),$$

which is the same as the solution obtained in Hu et al [51], with one risky asset.

1.4.2 General discounting LQ regulator

In this subsection, we consider an example of a general discounting time-inconsistent LQ model. The objective is to minimize the expected cost functional, that is earned during a finite time horizon

$$J(t,\xi,u(.)) = \frac{1}{2}\mathbb{E}^t\left[\int_t^T |u(s)|^2\, ds + h(t)|X(T) - \xi|^2\right] \tag{1.4.11}$$

where $h(.) : [0,T] \rightarrow (0,\infty)$, is a general deterministic non-exponential discount function satisfying $h(0) = 1$, $h(s) \geq 0$ and $\int_0^T h(t)\,dt < \infty$. Subject to a controlled one dimontional SDE, parametrized by $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$,

$$\begin{cases} dX(s) = \{aX(s) + bu(s)\}\,ds + \sigma dW(s) + \displaystyle\int_Z c(z)\,\tilde{N}(ds,dz), \ s \in [0,T], \\ X(t) = \xi, \end{cases} \tag{1.4.12}$$

where a, b are real constants and $c : Z \rightarrow \mathbb{R}$ is assumed to be deterministic and bounded. As mentioned in [17], this is a time-inconsistent version of the classical linear quadratic regulator, we want control the system so that the final sate $X(T)$ is close to ξ while at the same time we keep the control energy (formalized by the running cost) small. Note that, here the time-inconsistency is due to the fact that the terminal cost depends explicitly on the current state ξ as well as the current time t. Hence there are two different sources of time-inconsistency. For this example, the system (1.3.1) reduces to

$$\begin{cases} \dfrac{\partial M}{\partial s}(t,s) + 2aM(t,s) - b^2 M(t,s)\left\{M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\right\} = 0, \ \forall (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \bar{M}}{\partial s}(t,s) + 2a\bar{M}(t,s) - b^2 \bar{M}(t,s)\left\{M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\right\} = 0, \ \forall (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \Upsilon}{\partial s}(t,s) + a\Upsilon(t,s) = 0, \ \forall (t,s) \in \mathcal{D}[0,T], \\ \dfrac{\partial \varphi}{\partial s}(t,s) + a\varphi(t,s) - b^2\left\{M(t,s) + \bar{M}(t,s)\right\}\varphi(s,s) = 0, \ \forall (t,s) \in \mathcal{D}[0,T], \\ M(t,T) = h(t), \ \bar{M}(t,T) = 0, \ \Upsilon(t,T) = h(t), \ \varphi(t,T) = 0, \ \forall t \in [0,T], \end{cases} \tag{1.4.13}$$

obviously, $\Upsilon(.,.)$ is explicitely given by

$$\Upsilon(t,s) = h(t)\exp\{a(T-s)\}, \ \forall (t,s) \in \mathcal{D}[0,T]. \tag{1.4.14}$$

Moreover, we can check that $M(.,.), \bar{M}(.,.)$ and $\varphi(.,.)$ solve $(1.4.13)$, if and only if, they solve the following system of integral equations

$$\begin{cases} M(t,s) = M(t,T)\,e^{\int_s^T \left\{2a - b^2\left(M(\tau,\tau) + \bar{M}(\tau,\tau) + \Upsilon(\tau,\tau)\right)\right\}d\tau}, \ \forall (t,s) \in \mathcal{D}[0,T]. \\ \bar{M}(t,s) = \bar{M}(t,T)\,e^{\int_s^T \left\{2a - b^2\left(M(\tau,\tau) + \bar{M}(\tau,\tau) + \Upsilon(\tau,\tau)\right)\right\}d\tau}, \ \forall (t,s) \in \mathcal{D}[0,T]. \\ \varphi(t,s) = \varphi(t,T)\,e^{a(T-s)} - b^2 \displaystyle\int_s^T e^{a(\tau - s)}\left(M(t,\tau) + \bar{M}(t,\tau)\right)\varphi(\tau,\tau)\,d\tau, \ \forall (t,s) \in \mathcal{D}[0,T], \end{cases} \tag{1.4.15}$$

on the other hand, we have $\bar{M}(t,T)=\varphi(t,T)=0$, then (1.4.15) reduces to

$$\begin{cases} M(t,s)=M(t,T)\,e^{\int_s^T\left\{2a-b^2(M(r,r)+\Upsilon(r,r))\right\}dr},\ \forall(t,s)\in\mathcal{D}[0,T]. \\ \bar{M}(t,s)=0,\ \forall(t,s)\in\mathcal{D}[0,T], \\ \varphi(t,s)=-b^2\displaystyle\int_s^T e^{a(\tau-s)}M(t,\tau)\,\varphi(\tau,\tau)\,d\tau,\ \forall(t,s)\in\mathcal{D}[0,T]. \end{cases} \tag{1.4.16}$$

It is clear that if $M(.,.)$ is the solution of the first equation in $(1.4.16)$, then

$$\varphi(s,s)=-b^2\int_s^T e^{a(\tau-s)}M(s,\tau)\,\varphi(\tau,\tau)\,d\tau,\ \forall s\in[0,T],$$

thus, there exists some constant $L>0$ such that $|\varphi(s,s)|\leq L\displaystyle\int_s^T|\varphi(\tau,\tau)|\,d\tau$, then by Gronwall Lemma, we conclude that $\varphi(s,s)=0,\forall s\in[0,T]$. Therefore $\varphi(t,s)=0,\ \forall(t,s)\in\mathcal{D}[0,T]$, is the unique solution to the last equation in the system $(1.4.16)$.

Now, it's remains to solve the first equation in the system $(1.4.16)$. It is easy to check that the first equation in the system (1.4.16) is equivalent to

$$\begin{cases} \dfrac{\partial M}{\partial s}(t,s)+2aM(t,s)-b^2M(t,s)\{M(s,s)+\Upsilon(s,s)\}=0,(t,s)\in\mathcal{D}[0,T], \\ M(t,T)=h(t). \end{cases} \tag{1.4.17}$$

We try a solution of the form $M(t,s)=h(t)\,N(s)$, we finde that $N(.)$ should solve the following ODE

$$\begin{cases} \dfrac{dN}{ds}(s)+\left(2a+b^2\Upsilon(s,s)\right)N(s)-b^2h(s)\,N(s)^2=0,s\in[0,T] \\ N(T)=1, \end{cases} \tag{1.4.18}$$

We put $N(s)=\dfrac{1}{y(s)}$, the equation (1.4.18) leads to

$$\begin{cases} \dfrac{dy}{ds}(s)-\left(2a+b^2\Upsilon(s,s)\right)y(s)+b^2h(s)=0,\ \ s\in[0,T] \\ y(T)=1, \end{cases}$$

which is explicitly solvable by

$$y(s)=e^{-\int_s^T\left\{2a+b^2\Upsilon(\tau,\tau)\right\}d\tau}\left(1+b^2\int_s^T e^{\int_\tau^T\left\{2a+b^2\Upsilon(l,l)\right\}dl}h(\tau)\,d\tau\right),\ \ s\in[0,T],$$

thus

$$M(t,s)=h(t)\frac{e^{\int_s^T\left\{2a+b^2\Upsilon(\tau,\tau)\right\}d\tau}}{1+b^2\int_s^T e^{\int_\tau^T\{2a+b^2\Upsilon(l,l)\}dl}h(\tau)\,d\tau},\ \ (t,s)\in\mathcal{D}[0,T]$$

In view of Theorem 1.3.1, the representation (1.3.3) of the Nash equilibrium control, then gives

$$\hat{u}(s)=-b\left\{\Upsilon(s,s)+M(s,s)\right\}\hat{X}(s)\,,\ \forall s\in[0,T]\,, \tag{1.4.19}$$

and the corresponding equilibrium dynamics solves the SDEJ

$$\begin{cases} d\hat{X}(s)=\left\{a-b^2\left(\Upsilon(s,s)+M(s,s)\right)\right\}\hat{X}(s)\,ds+\sigma dW(s)+\int_Z c(z)\,\tilde{N}(ds,dz)\,,\ s\in[0,T]\,,\\ X(0)=x_0. \end{cases} \tag{1.4.20}$$

To conclude this section let us present the following remark.

Remark 1.4.1 *The Problem (E) given by the subsection 1.1.1, is in fact shown to be a particular case of the general discounting LQ regulator model, formulated earlier in this paragraph, in the case when* $a=0$, $c(z)\equiv 0$, *and the initial data* $\xi=x$, *this leads to the following representation of the Nash equilibrium control of this problem ,*

$$\hat{u}(s)=-b\left(h(s)+M(s,s)\right)\hat{X}(s)\,,\forall s\in[0,T]\,,$$

where $M(t,s)$ *is given by,*

$$M(t,s)=h(t)\frac{e^{\int_s^T\left\{b^2\Upsilon(\tau,\tau)\right\}d\tau}}{1+b^2\int_s^T e^{\int_\tau^T\{b^2\Upsilon(l,l)\}dl}h(\tau)\,d\tau},\ \ for\ \ (t,s)\in\mathcal{D}[0,T]\,,$$

and the corresponding equilibrium dynamics solves the SDE

$$\begin{cases} d\hat{X}(s)=-b^2\left\{h(s)+M(s,s)\right\}\hat{X}(s)\,ds+\sigma dW(s)\,,\ s\in[0,T]\,,\\ X(0)=x_0. \end{cases}$$

This in fact, the equilibrium solution of the Problem (E).

Chapter 2

A Characterization of Equilibrium Strategies in Continuous-Time Mean-Variance Problems for Insurers

In the recent decades, the risk models for insurers that can control and manage their risk by means of some business activities to optimize some objectives have received remarkable attention. Browne [22] first obtained the optimal investment strategy which maximizes the exponential utility of terminal wealth, where the surplus process of the insurer is modelled by a geometric Brownian motion. Yang and Zhang [100] followed by Wang [93] considered the same optimal investment problem, where the surplus process of the insurer is modelled, respectively, by a jump-diffusion process and an increasing pure jump process. Moreover, Xu et al. [99], Cao and Wan [28] and Gu et al [47] have investigated the optimal investment and reinsurance strategies for the insurers to optimize the expected utility of the terminal wealth in different situations.

In addition to the expected utility maximization, the mean-variance criterion, introduced by Markowitz [67], is another important objective function to the optimal investment and reinsurance problems for insurers. The idea of mean-variance criterion is that it quantifies the risk using the variance, which enables insurers to seek the highest return after evaluating their acceptable risk level. Bäuerle [12] considered the optimal proportional reinsurance problem under the mean-variance criterion where the surplus process of an insurer is modelled by the classical Cramér–Lundberg (CL) model, and solves this problem by adopting the stochastic control approach. Bai and Zhang [9] studied the optimal reinsurance/new business and investment strategy for the mean-variance problem where the surplus process is modelled respectively, by

the classical risk model and a diffusion model. For related works we refer to Delong and Gerrard [36], Zeng et al. [109] , Zeng and Li [108].
However, there is little work in the literature concerning equilibrium strategies for optimal investment and reinsurance problems under the mean-variance criterion. Zeng and Li [110] are the first to investigate Nash equilibrium strategies for dynamic mean-variance insurers with constant risk aversion, where the surplus of insurers is only modelled by the diffusion model and the price processes of the risky assets are only driven by geometric Brownian motions. The work [65] studied the case with state dependent risk aversion and they derived equilibrium strategies via some class of well posed integral equations. Zeng et al [111] considered the equilibrium investment and reinsurance strategies for mean-variance insurers with constant risk aversion where both the surplus process and the risky asset's price process follow a geometric Lévy processes.

Our objective in this thesis is to characterize equilibrium investment and reinsurance strategies for the mean-variance insurers with constant risk aversion, where the surplus process is modelled by a geometric Levi process and the financial market consists of one risk-free asset and multiple risky assets whose price processes follow jump-diffusion processes. Different from most of the existing literature, on this topic, [110], [111], [65] where a feedback equilibrium strategies are derived via a very complicated (extended) Hamilton–Jacobi–Bellman equation, the novelty of this work is that: by means of the variational method, we derive a necessary and sufficient conditions to characterize the equilibrium investment and reinsurance strategies via a stochastic system, which involves a flow of forward-backward stochastic differential equation with jumps (FBSDEJ in short), along with some equilibrium condition. Then by decoupling the flow of the FBSDEJ, we derive an explicit representation of the equilibrium strategies. We accentuate that, this method can provide the necessary and sufficient conditions to characterize the equilibrium strategies. While the extended HJB techniques studied in [110] and [111] can create only the sufficient condition to characterizes the equilibrium strategies.

The rest of the chapter is organized as follows. In Section 1, we formulate the problem and give necessary notations and preliminaries. In Section 2 we present the first main result of this work and we derive some explicit representation of the equilibrium investment and reinsurance strategies. Section 3 is devoted to some comparisons with some existing ones in literature.

2.1 The model and problem formulation

Throughout this chapter, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. We use $\mathrm{diag}(C)$ for the diagonal matrix with the elements of a vector C on the diagonal. For some Euclidean space $\mathbb{R}^m$ with Frobenius norm $|.|$, we denote $\mathbf{0}_{\mathbb{R}^m}$ the null vector. $\mathbb{R}^\star$ denotes $\mathbb{R} - \{0\}$. In addition, we define the following space of processes, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, as

- $\mathcal{L}^{\mu,2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^*;\mathbb{R}^m)$: the space of $\mathbb{R}^m-$valued, $(\mathcal{F}_s)_{s\in[t,T]}-$predictable processes $R(.,.)$, with

$$\|R(.,.)\|^2_{\mathcal{L}^{\mu,2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^*;\mathbb{R}^m)} = \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^\star} R(s,z)^\top \mathrm{diag}(\mu(dz))\, R(s,z)\, ds\right] < \infty,$$

for any positive and $\sigma-$finite Lévy measure $\mu(dz) = (\mu_1(dz), \mu_2(dz), ..., \mu_m(dz))^\top$.

2.1.1 Financial Market

Suppose that there is a financial market in which $n+1$ assets (or securities) are traded continuously. One of them is a bond, with price $S_0(s)$ at time $s \in [0,T]$ governed by

$$dS_0(s) = \rho_0(s)\, S_0(s)\, ds,\ \ S_0(0) = s_0 > 0. \tag{2.1.1}$$

where $\rho_0 : [0,T] \to (0,+\infty)$ is deterministic function which represents the risk-free rate. The other n assets are called risky stocks, whose price processes $S_1(.), ..., S_n(.)$ satisfy the following jump-diffusion stochastic differential equations

$$\begin{cases} dS_i(s) = S_i(s-)\left(\alpha_i(s)\, ds + \sum\limits_{j=1}^n \beta_{ij}(s)\, dW_j(s) + \sum\limits_{j=1}^n \int_{\mathbb{R}^\star} \gamma_{ij}(s,z)\,(N_j(ds,dz) - \mu_j(dz)\, ds)\right), \\ S_i(0) = s_i > 0. \end{cases} \tag{2.1.2}$$

where $\alpha_i : [0,T] \to \mathbb{R}$, $\beta_{ij} : [0,T] \to \mathbb{R}$ and $\gamma_{ij} : [0,T]\times\mathbb{R}^\star \to \mathbb{R}$ are deterministic functions, such that $\forall s \in [0,T]$, $\alpha_i(s) \geq \rho_0(s)$, $W(.) = (W_1(.), ..., W_n(.))^\top$ is a n-dimensional standard Brownian motion, $N(.,.) = (N_1(.,.), N_2(.,.), ..., N_n(.,.))^\top$ is an n-dimensional Poisson random measure on the measurable space $([0,T]\times\mathbb{R}^*, \mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^*))$. For $j = 1,2,...,n$, the compensator of $N_j(ds,dz)$ has the form $\mu_j(ds,dz) = \mu_j(dz)\, ds$ for some positive and $\sigma-$finite Lévy measure $\mu_j(dz)$ on $\mathbb{R}^*$ endowed with it's Borel $\sigma-$field $\mathcal{B}(\mathbb{R}^*)$ such that $\int_{\mathbb{R}^*} 1 \wedge z^2 \mu_j(dz) < \infty$. Denote by $\mu(dz) = (\mu_1(dz), ..., \mu_n(dz))^\top$ the

n-dimensional Lévy measure. We suppose that $W(.)$ and $N(.,.)$ are independent, and write $\widetilde{N}_j(.,.) = N_j(.,.) - \mu_j(.,.)$ for the compensated jump martingale random measure of $N_j(.,.)$.

2.1.2 Surplus process

Next we introduce the insurance risk model. Consider an insurer whose surplus process (without reinsurance and investment) is described, by the following jump–diffusion model

$$dR(s) = cds + \beta_0 dW_0(s) - d\left\{\sum_{i=1}^{L(s)} Y_i\right\}, \tag{2.1.3}$$

where $c > 0$ is the premium rate, β_0 is a positive constant, $W_0(.)$ is a one-dimensional standard Brownian motion, $L(s)$ is a Poisson process with intensity $\lambda > 0$, representing the number of claims occurring up time s, Y_i is the size of the i-th claim and $\{Y_i\}_{i\in\mathbb{N}-\{0\}}$ are assumed to be independent and identically distributed positive random variables with common distribution $\mathbb{P}_Y$ having finite firs and second moments $m_Y = \int_0^{+\infty} y\mathbb{P}_Y(dy)$ and $\sigma_Y = \int_0^{+\infty} y^2\mathbb{P}_Y(dy)$, respectively. The term $\beta_0 dW_0(s)$ can be regarded as the uncertainty from the premium income of the insurer.

We assume that $W(.)$, $W_0(.)$, $N(.,.)$,and $\sum_{i=1}^{L(.)} Y_i$ are independent. The premium rate c is assumed to be calculated via the expected value principle, i.e. $c = (1+\eta)\lambda m_Y$ with safety loading $\eta > 0$. We refer the readers to [9], [36], [111] and references therein for more information about the above model.

Suppose that the insurer can purchase proportional reinsurance or acquire new business (for example, acting as a reinsurer of other insurers, see Bäuerle [12]) at each moment in order to control the insurance business risk. Let $a(s)$ the retention level of reinsurance or new business acquired at time $s \in [0,T]$. When $a(s) \in [0,1]$, it corresponds to a proportional reinsurance cover and shows that the cedent should divert part of the premium to the reinsurer at the rate of $(1-a(s))(\theta_0+1)\lambda m_Y$, where θ_0 is the relative safety loading of the reinsurer satisfying $\theta_0 \geq \eta$. Meanwhile, for each claim occurring at time s, the reinsurer pays $100(1-a(s))\%$ of the claim, while the insurer pays the rest. The case where $a(s) \in (1,+\infty)$ corresponds to acquiring new business. The process $a(.)$ is called a reinsurance strategy. Incorporating purchasing proportional reinsurance and acquiring new business into the surplus process, then the expression (2.2.3) becomes

$$dR^{a(s)}(s) = \{\eta - \theta_0 + (1+\theta_0)a(s)\}\lambda m_Y ds + \beta_0 a(s)dW_0(s) - a(s)d\left\{\sum_{i=1}^{L(s)} Y_i\right\}.$$

We refer the readers for example to [111] and references therein for more information about the above model.

Note that, the compound Poisson process $\sum_{i=1}^{L(.)} Y_i$ can also be defined through a random measure $N_0(.,.)$ as

$$\sum_{i=1}^{L(s)} Y_i = \int_0^s \int_{\mathbb{R}^\star} z N_0(dr, dz),$$

where $N_0(.,.)$ is a finite Poisson random measure on the space $[0,T] \times \mathbb{R}^*$ endowed with it's Borel $\sigma-$field $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^*)$, with a compensator having the form $\mu_0(dz)\, ds = \lambda \mathbb{P}_Y(dz)\, ds$, see e.g. [73]. We denote $\widetilde{N}_0(dr, dz) = N_0(dr, dz) - \mu_0(dz)\, dr$ for the compensated jump martingale random measure of $N_0(dr, dz)$. Obviously, we have

$$\int_{\mathbb{R}^\star} z\mu_0(dz)\, ds = \lambda \int_{\mathbb{R}^\star} z\mathbb{P}_Y(dz)\, ds = \lambda m_Y ds.$$

Hence, the dynamics for the surplus process becomes

$$dR^{a(s)}(s) = \left\{ (\eta - \theta_0 + \theta_0 a(s)) \lambda m_Y + a(s) \int_{\mathbb{R}^\star} z\mu_0(dz) \right\} ds + \beta_0 a(s)\, dW_0(s) - a(s) \int_{\mathbb{R}^\star} z N_0(dr, dz),$$

equivalently, we obtain

$$dR^{a(s)}(s) = (\eta - \theta_0 + \theta_0 a(s)) \lambda m_Y ds + \beta_0 a(s)\, dW_0(s) - a(s) \int_{\mathbb{R}^\star} z\tilde{N}_0(dr, dz). \tag{2.1.4}$$

2.1.3 Wealth process

Starting from an initial capital $x_0 > 0$ at time 0, the insurer is allowed to dynamically purchase proportional reinsurance, acquire new business and invest in the financial market. A trading strategy is an $(n+1)$-dimensional stochastic process $\pi(.) = (a(.), b_1(.), ..., b_n(.))^\top$, where $a(s)$ represents the retention level of reinsurance or new business acquired at time $s \in [0,T]$, and $b_i(s)$ represents the amount invested in the *i-th* risky stock at time $s \in [0,T]$. The dollar amount invested in the bond at time s is $X^{x_0,\pi(.)}(s) - \sum_{j=1}^{n} b_i(s)$, where $X^{x_0,\pi(.)}(.)$ is the wealth process associated with the strategy $\pi(.)$ and the initial capital x_0. Then the evolution of $X^{x_0,\pi(.)}(.)$ can be described as

$$\begin{cases} dX^{x_0,\pi(.)}(s) = dR^{a(.)}(s) + \left\{ X^{x_0,\pi(.)}(s) - \sum_{i=1}^{n} b_i(s) \right\} \dfrac{dS_0(s)}{S_0(s)} + \sum_{i=1}^{n} b_i(s) \dfrac{dS_i(s)}{S_i(s-)}, \text{ for } s \in [0,T], \\ X(0) = x_0. \end{cases}$$

Accordingly, the wealth process solves the following SDE with jumps

$$\begin{cases} dX^{x_0,\pi(.)}(s) = \left\{\rho_0(s) X^{x_0,\pi(.)}(s) + (\delta + \theta_0 a(s)) \lambda m_Y + b(s)^\top \rho(s)\right\} ds + \beta_0 a(s) dW_0(s) \\ \qquad + b(s)^\top \beta(s) dW(s) - a(s) \int_{\mathbb{R}^*} z \widetilde{N}_0(ds, dz) \\ \qquad + \int_{\mathbb{R}^*} b(s)^\top \gamma(s,z) \widetilde{N}(ds, dz), \text{ for } s \in [0,T], \\ X(0) = x_0. \end{cases} \tag{2.1.5}$$

where $b(s) = (b_1(s), ..., b_n(s))^\top$, $\rho(s) = (\alpha_1(s) - \rho_0(s), ..., \alpha_n(s) - \rho_0(s))^\top$, $\beta(s) = (\beta_{ij}(s))_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$, $\gamma(s,z) = (\gamma_{ij}(s,z))_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$, and $\delta = \eta - \theta_0$. It is worth pointing out that the above model covers the one in [111], since, in our case, we consider the case of multiple assets whose price processes are general jump–diffusion processes, while in [111] the authors consider the case of one risky stok whose price process is modelled by a geometric Lévy process.

As time evolves, we need to consider the controlled stochastic differential equation parametrized by $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ and satisfied by $X(.) = X^{t,\xi}(.; \pi(.))$

$$\begin{cases} dX(s) = \left\{\rho_0(s) X(s) + (\delta + \theta_0 a(s)) \lambda m_Y + b(s)^\top \rho(s)\right\} ds + \beta_0 a(s) dW_0(s) \\ \qquad + b(s)^\top \beta(s) dW(s) - a(s) \int_{\mathbb{R}^*} z \widetilde{N}_0(dt, dz) \\ \qquad + \int_{\mathbb{R}^*} b(s)^\top \gamma(s,z) \widetilde{N}(ds, dz), \text{ for } s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{2.1.6}$$

In this paper, a trading strategy $\pi(.) = (a(.), b_1(.), ..., b_n(.))^\top$ is said to be admissible if it is an $(\mathcal{F}_s)_{s\in[0,T]}$ −predictable and square-integrable process with values in $\mathbb{R}^{n+1}$. Therefore $\mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ is the space of all admissible strategies.

2.1.4 Assumptions on the coefficients

We impose the following assumptions about the coefficients of the state equation

(H1) *The functions $\rho_0(.), \alpha(.), \beta(.)$ and $\gamma(.,.)$ are continuous such that*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^*} tr\left[\gamma(t,z)^\top \operatorname{diag}(\mu(dz)) \gamma(t,z)\right] < +\infty.$$

(H2) *We also assume a uniform ellipticity condition as follows*

$$\beta(s)\beta(s)^\top + \gamma(s,z) \operatorname{diag}(\mu(dz)) \gamma(s,z)^\top \geq \epsilon I_n,\ ds - a.e.$$

for some $\epsilon > 0$.

By Lemma 2.1 in [68], under **(H1)**, for any $(t, \xi, \pi(.)) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, the state equation (2.1.6) has a unique solution $X(.) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R})$, we also have the following estimate

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X(s)|^2\right] \leq K\left(1+\mathbb{E}\left[|\xi|^2\right]\right), \tag{2.1.7}$$

for some positive constant K. In particular for $t=0$ and $\pi(.) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, the state equation (2.1.5) has a unique solution $X(.) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ with the following estimate holds

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|X(s)|^2\right] \leq K\left(1+|x_0|^2\right). \tag{2.1.8}$$

2.1.5 Mean–variance criterion

The objective of the insurer at time $t \in [0,T]$ is to achieve a balance between conditional variance and conditional expectation of terminal wealth; namely, to choose a strategy $\bar{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ so as to minimize

$$J(t,\xi,\pi(.)) = \frac{1}{2}\mathrm{Var}^t\left[X(T)\right] - \frac{1}{\gamma}\mathbb{E}^t\left[X(T)\right], \tag{2.1.9}$$

over $\mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, subject to (2.1.6), where $\gamma > 0$ denotes the constant risk aversion, $\mathbb{E}^t[.] = \mathbb{E}[.\,|\mathcal{F}_t]$ is the conditional expectation with respect to $\mathcal{F}_t$, and $\mathrm{Var}^t[.] = \mathrm{Var}[.\,|\mathcal{F}_t]$ is the conditional variance with respect to $\mathcal{F}_t$. This problem can be viewed as a dynamic optimal control problem, since the objective of the insurer updates as state (t,ξ) changes.

If we define the processes $W^\star(.)$ and $\widetilde{N}^\star(.,.)$ by $W^\star(.) = (W_0(.), W_1(.), .., W_n(.))^\top$ and $\widetilde{N}^\star(.,.) = \left(\widetilde{N}_0(.,.), \widetilde{N}_1(.,.), .., \widetilde{N}_n(.,.)\right)^\top$, respectively, then the optimal control problem associated with (2.1.6) and (2.1.9) is equivalent to minimize

$$J(t,\xi,\pi(.)) = \frac{1}{2}\left(\mathbb{E}^t\left[X(T)^2\right] - \mathbb{E}^t\left[X(T)\right]^2\right) - \frac{1}{\gamma}\mathbb{E}^t\left[X(T)\right], \tag{2.1.10}$$

subject to

$$\begin{cases} dX(s) = \left(\rho_0(s)X(s) + \pi(s)^\top \mathbf{B}(s) + \kappa\right)ds + \pi(s)^\top \mathbf{D}(s)\,dW^\star(s) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^\star} \pi(s)^\top \mathbf{F}(s,z)\,\widetilde{N}^\star(ds,dz), \text{ for } s \in [t,T], \\ X(t) = \xi, \end{cases} \tag{2.1.11}$$

where $\mathbf{B}(s) = (\lambda m_Y \theta_0, \alpha_1(s) - \rho_0(s), .., \alpha_n(s) - \rho_0(s))^\top$, $\kappa = \delta \lambda m_Y$, and we shall write

$$\mathbf{D}(s) = \begin{pmatrix} \beta_0 & 0 & . & . & 0 \\ 0 & \beta_{11}(s) & . & . & \beta_{1n}(s) \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \beta_{n1}(s) & . & . & \beta_{nn}(s) \end{pmatrix},$$

$$\mathbf{F}(s,z) = \begin{pmatrix} -z & 0 & . & . & 0 \\ 0 & \gamma_{11}(s,z) & . & . & \gamma_{1n}(s,z) \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \gamma_{n1}(s,z) & . & . & \gamma_{nn}(s,z) \end{pmatrix},$$

and

$$\mu^*(dz) = (\mu_0(dz), \mu_1(dz), ..., \mu_n(dz))^\top.$$

2.2 Characterization of equilibrium strategies

It is well known that the problem described above turn out to be time inconsistent in the sense that, it does not satisfy the Bellman optimality principle, which more precisely says that if for some fixed initial point $(0, x_0)$ we determine the control $\bar{\pi}(.)$ which minimize $J(0, x_0, .)$, then at some later point $(t, \bar{X}(t))$ the control $\bar{\pi}(.)$ will no longer be optimal for the functional $J(t, \bar{X}(t), .)$. We refer the readers to Björk and Murgoci [17] for a more detailed discussion. Since lack of time consistency, the notion "optimality" needs to be defined in an appropriate way. Following [51], we adopt the concept of open loop Nash equilibrium solution, which is, for any $t \in [0,T]$, optimal "infinitesimally" via spike variation.

Given an admissible strategy $\hat{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^{n+1})$. For any $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1})$ and for any $\varepsilon \in [0, T-t)$, define

$$\pi^\varepsilon(s) = \begin{cases} \hat{\pi}(s) + v, & \text{for } s \in [t, t+\varepsilon), \\ \hat{\pi}(s), & \text{for } s \in [t+\varepsilon, T], \end{cases} \tag{2.2.1}$$

we have the following definition.

Definition 2.2.1 (Open-loop Nash equilibrium) *An admissible strategy* $\hat{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^{n+1})$ *is* *an open-loop Nash equilibrium strategy if*

$$\lim_{\varepsilon\downarrow 0}\inf\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}(t),\pi^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{\pi}(.)\right)\right\}\geq 0, \tag{2.2.2}$$

for any $t\in[0,T]$, *and* $v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{n+1}\right)$. *The corresponding equilibrium wealth process solves the following SDE with jumps*

$$\begin{cases} d\hat{X}(s)=\left(\rho_{0}(s)\hat{X}(s)+\hat{\pi}(s)^{\top}\mathbf{B}(s)+\kappa\right)ds+\hat{\pi}(s)^{\top}\mathbf{D}(s)\,dW^{\star}(s)\\ \qquad\qquad +\int_{\mathbb{R}^{\star}}\hat{\pi}(s)^{\top}\mathbf{F}(s,z)\,\widetilde{N}^{\star}(ds,dz)\,,\ \text{for } s\in[0,T]\,,\\ \hat{X}(0)=x_{0}, \end{cases} \tag{2.2.3}$$

Remark 2.2.1 *The above definition of Nash equilibrium strategy is different from the one in [65], [111] and [110]. Since an equilibrium strategy here is defined in the class of open-loop strategies, while in the most existing literature only feedback strategies are considered. In addition, in the above definition, the perturbation of the strategy* $\hat{\pi}(.)$ *in* $[t,t+\varepsilon)$ *will not change* $\hat{\pi}(.)$ *in* $[t+\varepsilon,T]$, *which is not the case with feedback strategies.*

2.2.1 The flow of adjoint equations

In the sequel, we present a general necessary and sufficient condition for equilibriums. We derive this condition by a second-order expansion in spike variation. First, we introduce the adjoint equations involved in the characterization of equilibrium strategies. Let $\hat{\pi}(.)\in\mathcal{L}^{2}_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ and denote by $\hat{X}(.)\in\mathcal{S}^{2}_{\mathcal{F}}(0,T;\mathbb{R})$ the corresponding wealth process. For each $t\in[0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$ and satisfied by the processes $(p(.;t),q(.;t),r(.,.;t))$ as follows

$$\begin{cases} dp(s;t)=-\rho_{0}(s)\,p(s;t)\,ds+q(s;t)^{\top}dW^{\star}(s)+\int_{\mathbb{R}^{*}}r(s,z;t)^{\top}\widetilde{N}^{\star}(ds,dz)\,,\ s\in[t,T]\,,\\ p(T;t)=-\hat{X}(T)+\mathbb{E}^{t}\left[\hat{X}(T)\right]+\dfrac{1}{\gamma}, \end{cases} \tag{2.2.4}$$

where $q(.;t)=(q_{0}(.;t),q_{1}(.;t),...,q_{n}(.;t))^{\top}$, and $r(.,.;t)=(r_{0}(.,.;t),r_{1}(.,.;t),...,r_{n}(.,.;t))^{\top}$. By Lemma 2.2 in [68] , under (**H1**), equation (2.3.4) is uniquely solvable in the space $\mathcal{S}^{2}_{\mathcal{F}}(t,T;\mathbb{R})\times\mathcal{L}^{2}_{\mathcal{F}}\left(t,T;\mathbb{R}^{n+1}\right)\times\mathcal{L}^{\mu^{*},2}_{\mathcal{F},p}\left([t,T]\times\mathbb{R}^{\star};\mathbb{R}^{n+1}\right)$. Moreover there exists a constant $K>0$ such that we have the following estimate

$$\|p(.;t)\|^{2}_{\mathcal{S}^{2}_{\mathcal{F}}(t,T;\mathbb{R})}+\|q(.;t)\|^{2}_{\mathcal{L}^{2}_{\mathcal{F}}(t,T;\mathbb{R}^{n+1})}+\|r(.,.;t)\|^{2}_{\mathcal{L}^{\mu^{*},2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^{*};\mathbb{R}^{n+1})}\leq K\left(1+x_{0}^{2}\right). \tag{2.2.5}$$

The second order adjoint equation is defined on the time interval $[0,T]$ and satisfied by the processes $(P(.), \Lambda(.), \Gamma(.,.))$ as follows

$$\begin{cases} dP(s) = -2\rho_0(s) P(s) ds + \Lambda(s)^\top dW^\star(s) + \int_{\mathbb{R}^\star} \Gamma(s,z)^\top \widetilde{N}^\star(ds,dz), \ s \in [0,T], \\ P(T) = -1. \end{cases} \tag{2.2.6}$$

where $\Lambda(.) = (\Lambda_0(.), \Lambda_1(.), ..., \Lambda_n(.))^\top$, and $\Gamma(.,.) = (\Gamma_0(.,.), \Gamma_1(.,.), ..., \Gamma_n(.,.))^\top$. In the other hand, noting that the final data of the equation (2.2.6) is deterministic, it is straightforward to look at a deterministic solution. In addition we have the following representation

$$\begin{cases} dP(s) = -2\rho_0(s) P(s) ds, \ s \in [0,T], \\ P(T) = -1. \end{cases} \tag{2.2.7}$$

Clearly $P(s) \equiv -e^{\int_s^T 2\rho_0(\tau)d\tau}$, hence the solution of (2.2.6) is explicitly given by the triplet

$$(P(s), \Lambda(s), \Gamma(s,z)) = \left(-e^{\int_s^T 2\rho_0(\tau)d\tau}, \mathbf{0}_{\mathbb{R}^{n+1}}, \mathbf{0}_{\mathbb{R}^{n+1}}\right), \ \forall (s,z) \in [0,T] \times \mathbb{R}^\star. \tag{2.2.8}$$

Remark 2.2.2 *For each $t \in [0,T)$ be fixed, the adjoint equation (2.2.4) is a backward stochastic differential equation with jumps (BSDEJ for short) which has a unique solution $(p(.;t), q(.;t), r(.,.;t)) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^{n+1}) \times \mathcal{L}^{\mu^*,2}_{\mathcal{F},p}([t,T] \times \mathbb{R}^*; \mathbb{R}^{n+1})$.*

Next, for any $t \in [0,T]$ associated to the 5-tuple $\left(\hat{u}(.), \hat{X}(.), p(.;t), q(.;t), r(.,.;t)\right)$ we define for $s \in [t,T]$

$$\mathbf{U}(s;t) = \mathbf{B}(s) p(s;t) + \mathbf{D}(s) q(s;t) + \int_{\mathbb{R}^\star} \mathbf{F}(s,z) \operatorname{diag}(\mu^\star(dz)) r(s,z;t). \tag{2.2.9}$$

2.2.2 A necessary and sufficient condition for equilibrium strategies

The following theorem is the first main result of this work, it provides a necessary and sufficient condition to characterize the open-loop Nash equilibrium controls for the time inconsistent minimization problem (2.1.9) subject to the dynamics (2.1.11).

Theorem 2.2.1 *Let **(H1)-(H2)** hold. Given an admissible strategy $\hat{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^{n+1})$, let for any $t \in [0,T]$, $(p(.;t), q(.;t), r(.,.;t)) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^{n+1}) \times \mathcal{L}^{\mu^*,2}_{\mathcal{F},p}([t,T] \times \mathbb{R}^\star; \mathbb{R}^{n+1})$ be the unique solution to the BSDE (2.2.4). Then $\hat{\pi}(.)$ is an open-loop Nash equilibrium, if and only if, the following condition holds*

$$\mathbf{U}(t;t) = 0, \ d\mathbb{P}-a.s., \ dt - a.e., \tag{2.2.10}$$

where $\mathbf{U}(.;.)$ *is given by* (2.3.9).

Our goal now, is to give a proof of the Theorem 2.2.1. The main idea is based on the variational techniques in the same spirit of proving the stochastic Pontryagin's maximum principle for equilibriums in [51] and [52]. Note that in [51] and [52] the authors studied the Brownian case only.
Let $\hat{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ be an admissible strategy and $\hat{X}(.)$ the corresponding controlled process. Consider the perturbed strategy $\pi^{\varepsilon}(.)$ defined by the spike variation (2.2.1) for some fixed arbitrary $t \in [0,T]$, $v \in \mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{n+1}\right)$ and $\varepsilon \in [0,T-t)$. Denote by $\hat{X}^{\varepsilon}(.)$ the solution of the state equation corresponding to $\pi^{\varepsilon}(.)$. Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, see e.g. [90], we have

$$\hat{X}^{\varepsilon}(s) - \hat{X}(s) = y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s),\ s \in [t,T], \tag{2.2.11}$$

where $y^{\varepsilon,v}(.)$ and $z^{\varepsilon,v}(.)$ solve the following linear stochastic differential equations, respectively

$$\begin{cases} dy^{\varepsilon,v}(s) = \{\rho_0(s)\, y^{\varepsilon,v}(s)\}\, ds + 1_{[t,t+\varepsilon)}(s)\, v^{\top}\mathbf{D}(s)\, dW^{\star}(s) \\ \qquad\qquad + 1_{[t,t+\varepsilon)}(s) \displaystyle\int_{\mathbb{R}^{\star}} v^{\top}\mathbf{F}(s,z)\, \widetilde{N}^{\star}(ds,dz),\ s \in [t,T], \\ y^{\varepsilon,v}(t) = 0, \end{cases} \tag{2.2.12}$$

and

$$\begin{cases} dz^{\varepsilon,v}(s) = \left\{\rho_0(s)\, z^{\varepsilon,v}(s) + v^{\top}\mathbf{B}(s)\, 1_{[t,t+\varepsilon)}(s)\right\} ds,\ s \in [t,T], \\ z^{\varepsilon,v}(t) = 0. \end{cases} \tag{2.2.13}$$

We need to the following two lemmas

Lemma 2.2.1 *Under assumption **(H1)**, the following estimates hold*

$$\mathbb{E}^t\left[y^{\varepsilon}(s)\right] = 0,\ ds - a.e. \text{ and } \sup_{s\in[t,T]}\left|\mathbb{E}^t\left[z^{\varepsilon}(s)\right]\right|^2 = O\left(\varepsilon^2\right), \tag{2.2.14}$$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]}\left|y^{\varepsilon}(s)\right|^2\right] = O(\varepsilon) \text{ and } \mathbb{E}^t\left[\sup_{s\in[t,T]}\left|z^{\varepsilon}(s)\right|^2\right] = O\left(\varepsilon^2\right). \tag{2.2.15}$$

Moreover, we have the equality

$$\begin{aligned} &J\left(t,\hat{X}(t),\pi^{\varepsilon}(.)\right) - J\left(t,\hat{X}(t),\hat{\pi}(.)\right) \\ &= -\int_t^{t+\varepsilon}\left\{\left\langle\mathbb{E}^t\left[\mathbf{U}(s;t)\right],v\right\rangle ds + \frac{1}{2}\left\langle\mathbf{H}(s)\,v,v\right\rangle\right\} ds + o(\varepsilon). \end{aligned} \tag{2.2.16}$$

where

$$\mathbf{H}(s) = -e^{\int_s^T 2\rho_0(\tau)d\tau}\left(\mathbf{D}(s)\mathbf{D}(s)^\top + \int_{\mathbb{R}^\star}\mathbf{F}(s,z)\,diag(\mu^*(dz))\,\mathbf{F}(s,z)^\top\right). \tag{2.2.17}$$

Proof. Let $t \in [0,T]$, $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$ and $\varepsilon \in [0, T-t)$. Since $\mathbb{E}^t\left[y^{\varepsilon,v}(.)\right]$ and $\mathbb{E}^t\left[z^{\varepsilon,v}(.)\right]$ solve the following ODEs, respectively

$$\begin{cases} d\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right] = \rho_0(s)\,\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right]ds,\; s\in[t,T], \\ \mathbb{E}^t\left[y^{\varepsilon,v}(t)\right] = 0, \end{cases} \tag{2.2.18}$$

and

$$\begin{cases} d\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right] = \left\{\rho_0(s)\,\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right] + \mathbb{E}^t\left[v^\top\right]\mathbf{B}(s)\,1_{[t,t+\varepsilon)}(s)\right\}ds,\; s\in[t,T], \\ \mathbb{E}^t\left[z^{\varepsilon,v}(t)\right] = 0. \end{cases} \tag{2.2.19}$$

Thus, it is clear that $\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right] = 0$, *a.e.* $s \in [t,T]$. According to Gronwall's inequality we have $\sup_{s\in[t,T]}\left|\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right]\right|^2 = O\left(\varepsilon^2\right)$. The estimation (2.2.15) is a direct consequence of Lemma 2.1. in [?]. To prove (2.2.16), we consider the difference

$$\begin{aligned} &J\left(t,\hat{X}(t),u^\varepsilon(.)\right) - J\left(t,\hat{X}(t),\hat{u}(.)\right) \\ &= \mathbb{E}^t\left[\frac{1}{2}\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2 + \left(\hat{X}(T) - \mathbb{E}^t\left[\hat{X}(T)\right] - \frac{1}{\gamma}\right)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)\right. \\ &\qquad \left. - \frac{1}{2}\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]^2\right], \end{aligned} \tag{2.2.20}$$

according to the estimations (2.2.14) and (2.2.15) the following valuation holds

$$\frac{1}{2}\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]^2 = o(\varepsilon).$$

Then, from the terminal conditions in the adjoint equations, it follows that

$$\begin{aligned} &J\left(t,\hat{X}(t),\pi^\varepsilon(.)\right) - J\left(t,\hat{X}(t),\hat{\pi}(.)\right) \\ &= -\mathbb{E}^t\left[p(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right) + \frac{1}{2}P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2\right] + o(\varepsilon). \end{aligned} \tag{2.2.21}$$

Now, by applying Ito's formula to $s \mapsto p(s;t)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)$ on $[t,T]$, we get

$$\begin{aligned} &\mathbb{E}^t\left[p(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)\right] \\ &= \mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{v^\top\mathbf{B}(s)\,p(s;t) + v^\top\mathbf{D}(s)\,q(s;t) + \int_{\mathbb{R}^\star} v^\top\mathbf{F}(s,z)\,\mathrm{diag}(\mu^\star(dz))\,r(s,z;t)\right\}ds\right]. \end{aligned} \tag{2.2.22}$$

Again, by applying Ito's formula to $s \mapsto P(s;t)(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))^2$ on $[t,T]$, we get

$$\begin{aligned}&\mathbb{E}^t\left[P(T;t)(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T))^2\right]\\&=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{2v^\top\mathbf{B}(s)(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))P(s,t)\right.\right.\\&\qquad\left.\left.+v^\top\left(\mathbf{D}(s)\mathbf{D}(s)^\top+\int_{\mathbb{R}^\star}\mathbf{F}(s,z)\operatorname{diag}(\mu^\star(dz))\mathbf{F}(s,z)^\top\right)vP(s,t)\right\}ds\right].\end{aligned}\tag{2.2.23}$$

Moreover, we conclude from **(H1)** together with (2.2.14) – (2.2.15) that

$$\mathbb{E}^t\left[\int_t^{t+\varepsilon}(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s))P(s;t)v^\top\mathbf{B}(s)\,ds\right]=o(\varepsilon).$$

By taking (2.2.22) and (2.2.23) in (2.2.20), it follows that

$$\begin{aligned}&J\left(t,\hat{X}(t),\pi^\varepsilon(.)\right)-J\left(t,\hat{X}(t),\hat{\pi}(.)\right)\\&=-\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{v^\top\mathbf{B}(s)p(s;t)+v^\top\mathbf{D}(s)q(s,t)+\int_{\mathbb{R}^\star}v^\top\mathbf{F}(s,z)\operatorname{diag}(\mu^\star(dz))r(s,z;t)\right.\right.\\&\qquad\left.\left.+\frac{1}{2}v^\top\left(\mathbf{D}(s)\mathbf{D}(s)^\top+\int_{\mathbb{R}^\star}\mathbf{F}(s,z)\operatorname{diag}(\mu^\star(dz))\mathbf{F}(s,z)^\top\right)vP(s,t)\right\}ds\right]+o(\varepsilon),\end{aligned}\tag{2.2.24}$$

which is equivalent to (2.2.16).■

Lemma 2.2.2 *The following two statements are equivalent*

1) $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}^t[\mathbf{U}(s;t)]\,ds=0,\ d\mathbb{P}-a.s,\ \forall t\in[0,T].$

2) $\mathbf{U}(t;t)=0,\ d\mathbb{P}-a.s,\ dt-a.e.$

Proof. We put $\alpha(s)\equiv e^{\int_s^T-\rho_0(\tau)d\tau}$. Define for $t\in[0,T]$ and $s\in[t,T]$

$$(\bar{p}(s;t),\bar{q}(s;t),\bar{r}(s,z;t))\equiv-\left(\left(\alpha(s)p(s;t)-\mathbb{E}^t\left[\hat{X}(T)\right]-\frac{1}{\gamma}\right),\alpha(s)q(s;t),\alpha(s)r(s,z;t)\right).$$

Then for any $t\in[0,T]$, in the interval $[t,T]$, the tiple $(\bar{p}(.;t),\bar{q}(.;t),\bar{r}(.,.;t))$ satisfies

$$\begin{cases}d\bar{p}(s;t)=\bar{q}(s;t)^\top dW^\star(s)+\int_{\mathbb{R}^\star}\bar{r}(s,z;t)^\top\widetilde{N}^\star(ds,dz),\ s\in[t,T],\\ \bar{p}(T;t)=\hat{X}(T).\end{cases}\tag{2.2.25}$$

Moreover, it is clear that from the uniqueness of solutions to (2.2.25), we have

$$\left(\bar{p}\left(s;t_1\right),\bar{q}\left(s;t_1\right),\bar{r}\left(s,z;t_1\right)\right)=\left(\bar{p}\left(s;t_2\right),\bar{q}\left(s;t_2\right),\bar{r}\left(s,z;t_2\right)\right),$$

for any $t_1, t_2, s \in [0,T]$ such that $0 < t_1 < t_2 < s < T$. Hence, the solution $\left(\bar{p}\left(.;t\right),\bar{q}\left(.;t\right),\bar{r}\left(.,.;t\right)\right)$ does not depend on t. Thus we denote the solution of (2.2.25) by $\left(\bar{p}\left(.\right),\bar{q}\left(.\right),\bar{r}\left(.,.\right)\right)$.

We have then, for any $t \in [0,T]$, and $s \in [t,T]$

$$\left(p\left(s;t\right),q\left(s;t\right),r\left(s,z;t\right)\right)=-\alpha\left(s\right)^{-1}\left(\left(\bar{p}\left(s\right)-\mathbb{E}^t\left[\hat{X}\left(T\right)\right]-\frac{1}{\gamma}\right),\bar{q}\left(s\right),\bar{r}\left(s,z\right)\right). \tag{2.2.26}$$

Now using (2.2.26) we have, for any $t \in [0,T]$, and $s \in [t,T]$

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left|\mathbf{U}\left(s;t\right)-\mathbf{U}\left(s;s\right)\right|ds\right]&=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left|\mathbf{B}\left(s\right)\alpha\left(s\right)^{-1}\left\{\mathbb{E}^t\left[\hat{X}\left(T\right)\right]-\mathbb{E}^s\left[\hat{X}\left(T\right)\right]\right\}\right|ds\right]\\&\leq C\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left|\mathbb{E}^t\left[\hat{X}\left(T\right)\right]-\mathbb{E}^s\left[\hat{X}\left(T\right)\right]\right|ds\right]\\&=0,\end{aligned}$$

where we have used the fact that, the last quantity is a right continuous function of s, $dt-a.e.$ and vanishes at $s=t$. Thus

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{U}\left(s;t\right)ds\right]=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{U}\left(s;s\right)ds\right]. \tag{2.2.27}$$

From the above equality, it is clear that if 2) holds, then

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{U}\left(s;t\right)ds\right]=0.\ d\mathbb{P}-a.s,$$

Conversely, according to Lemma 3.5 in [52], if 1) holds then

$$\mathbf{U}\left(s;s\right)=0,\ d\mathbb{P}-a.s,\ ds-a.e.$$

This completes the proof.■

Now, we are ready to give a proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. Given an admissible strategy $\hat{\pi}(.) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ for which (2.2.10) holds, according to Lemma 2.2.2 we have for any $t \in [0,T]$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{U}(s;t)\right] ds = 0.$$

Then for any $t \in [0,T]$ and for any $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J\left(t, \hat{X}(t), u^{\varepsilon}(.)\right) - J\left(t, \hat{X}(t), \hat{u}(.)\right) \right\} \\
&= -\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left\{ \left\langle \mathbb{E}^t \left[\mathbf{U}(s;t)\right], v \right\rangle ds + \frac{1}{2} \left\langle \mathbf{H}(s) v, v \right\rangle \right\} ds \\
&= -\frac{1}{2} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left\langle \mathbf{H}(s) v, v \right\rangle ds \\
&= -\frac{1}{2} \left\langle \mathbf{H}(t) v, v \right\rangle \\
&\geq 0,
\end{aligned}$$

where we have used the fact that, under assumption **(H2)**, $\langle \mathbf{H}(t) v, v \rangle \leq 0$. Hence $\hat{\pi}(.)$ is an equilibrium strategy.

Conversely, assume that $\hat{\pi}(.)$ is an equilibrium strategy. Then, by (2.2.2) together with (2.2.16) ,for any $(t,\pi) \in [0,T] \times \mathbb{R}^{n+1}$ the following inequality holds

$$\lim_{\varepsilon \downarrow 0} \left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{U}(s;t)\right] ds, \pi \right\rangle + \frac{1}{2} \left\langle \mathbf{H}(t) \pi, \pi \right\rangle \leq 0. \tag{2.2.28}$$

Now, we define

$$\Psi(t,\pi) = \left\langle \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{U}(s;t)\right] ds, \pi \right\rangle + \frac{1}{2} \left\langle \mathbf{H}(t) \pi, \pi \right\rangle, \ \forall (t,\pi) \in [0,T] \times \mathbb{R}^{n+1}.$$

Easy manipulations show that the inequality (2.2.18) is equivalent to

$$\Psi(t,0) = \max_{\pi \in \mathbb{R}^{n+1}} \Psi(t,\pi), \ d\mathbb{P} - a.s, \forall t \in [0,T]. \tag{2.2.29}$$

It is easy to prove that the maximum condition (2.2.19) leads to the following condition, $\forall t \in [0,T]$

$$D_{\pi} \Psi(t,0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{U}(s;t)\right] ds = 0, \ d\mathbb{P} - a.s. \tag{2.2.30}$$

According to Lemma 2.2.2, the expression (2.2.10) follows immediately. This completes the proof.■

2.2.3 An explicit representation of the equilibrium control

In the next, we will find the (equilibrium) efficient frontier of the mean-variance problem. The key point in the explicit resolution of the problem is that the adjoint process may be separated into functions of time and state variables. Then, on needs only to solve some linear ODEs in order to completely determine the equilibrium control, and the corresponding equilibrium value function. Then by standard arguments, we will prove the following result

Theorem 2.2.2 *Let **(H1)-(H2)** hold. The stochastic mean-variance control problem* (2.1.9) *subject to the SDE* (2.1.11) , *has an open-loop Nash equilibrium solution*

$$\hat{a}(s) = \frac{(\lambda m_Y \theta_0)\, e^{-\int_s^T \rho_0(\tau)d\tau}}{\gamma \left(\beta_0^2 + \int_0^{+\infty} z^2 \mu_0(dz)\right)}, \tag{2.2.31}$$

$$\hat{b}(s) = \frac{1}{\gamma}\left(\beta(s)\beta(s)^\top + \int_{\mathbb{R}^\star} \gamma(s,z)\, diag(\mu(dz))\, \gamma(s,z)^\top\right)^{-1} \rho(s)\, e^{-\int_s^T \rho_0(\tau)d\tau}, \tag{2.2.32}$$

Moreover, the associated expected terminal wealth is

$$\mathbb{E}\left[\hat{X}(T)\right] = x_0 e^{\int_0^T \rho_0(\tau)d\tau} + \kappa \int_0^T e^{\int_\tau^T \rho_0(l)dl} d\tau + \frac{1}{\gamma}\int_0^T \Phi(\tau)\, d\tau, \tag{2.2.33}$$

and the corresponding variance of the terminal wealth is

$$\mathrm{Var}\left[\hat{X}(T)\right] = \frac{1}{\gamma^2}\int_0^T \Phi(\tau)\, d\tau, \tag{2.2.34}$$

where $\forall t \in [0,T]$

$$\Phi(t) = \mathbf{B}(t)^\top \left(\mathbf{D}(t)\mathbf{D}(t)^\top + \int_{\mathbb{R}^\star} \mathbf{F}(t,z)\, diag(\mu^\star(dz))\, \mathbf{F}(t,z)^\top\right)^{-1} \mathbf{B}(t). \tag{2.2.35}$$

Proof. The result of the previous subsections leads to the following flow of forward and backward stochastic differential system with jumps (parametrized by t),

$$\begin{cases} d\hat{X}(s) = \left(\rho_0(s)\hat{X}(s) + \hat{\pi}(s)^\top \mathbf{B}(s) + \kappa\right) ds + \hat{\pi}(s)^\top \mathbf{D}(s)\, dW^\star(s) \\ \qquad\qquad + \int_{\mathbb{R}^\star} \hat{\pi}(s)^\top \mathbf{F}(s,z)\, \widetilde{N}^\star(ds,dz),\ s \in [0,T], \\ dp(s;t) = -\rho_0(s)\, p(s;t)\, ds + q(s;t)^\top dW^\star(s) + \int_{\mathbb{R}^\star} r(s,z;t)^\top \widetilde{N}^\star(ds,dz),\ 0 \le t \le s \le T, \\ \hat{X}(0) = x_0,\ p(T;t) = -\left(\hat{X}(T) - \mathbb{E}^t\left[\hat{X}(T)\right]\right) + \frac{1}{\gamma},\ \text{for } t \in [0,T], \end{cases} \tag{2.2.36}$$

with the equilibrium condition

$$\mathbf{B}(t)\,p(t;t)+\mathbf{D}(t)\,q(t;t)+\int_{\mathbb{R}^{\star}}\mathbf{F}(t,z)\operatorname{diag}(\mu^{\star}(dz))\,r(t,z;t)=0,\ d\mathbb{P}-a.s.,\ dt-a.e. \tag{2.2.37}$$

Inspired by [51], we consider the following Ansatz:

$$p(s;t)=-M(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\varphi(s),\ \forall 0\leq t\leq s\leq T \tag{2.2.38}$$

for some deterministic functions $M(.),\varphi(.)\in C^{1}([0,T],\mathbb{R})$ such that, $M(T)=1$ and $\varphi(T)=\dfrac{1}{\gamma}$. We would like to determine the equations that $M(.)$ and $\varphi(.)$ should satisfy. To this end we differentiate (2.2.38) we get

$$dp(s;t)=-\frac{dM}{ds}(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\frac{d\varphi}{ds}(s)-M(s)\,d\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right), \tag{2.2.39}$$

we remark that

$$d\mathbb{E}^{t}\left[\hat{X}(s)\right]=\left(\rho_{0}(s)\,\mathbb{E}^{t}\left[\hat{X}(s)\right]+\mathbb{E}^{t}\left[\hat{\pi}(s)^{\top}\right]\mathbf{B}(s)+\kappa\right)ds,$$

then

$$\begin{aligned}d\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)&=\left(\rho_{0}(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\left(\hat{\pi}(s)-\mathbb{E}^{t}\left[\hat{\pi}(s)\right]\right)^{\top}\mathbf{B}(s)\right)ds\\&\quad+\hat{\pi}(s)^{\top}\mathbf{D}(s)\,dW(s)+\int_{\mathbb{R}^{\star}}\hat{\pi}(s)^{\top}\mathbf{F}(s,z)\,\widetilde{N}(ds,dz),\end{aligned} \tag{2.2.40}$$

Now, invoking (2.2.39) and (2.2.40), then by comparing with (2.2.36), we easily check that

$$\begin{aligned}&-\rho_{0}(s)\left(-M(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\varphi(s)\right)\\&=-\frac{dM}{ds}(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\frac{d\varphi}{ds}(s)\\&\quad-M(s)\left\{\rho_{0}(s)\left(\hat{X}(s)-\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)+\left(\hat{\pi}(s)-\mathbb{E}^{t}\left[\hat{\pi}(s)\right]\right)^{\top}\mathbf{B}(s)\right\},\end{aligned} \tag{2.2.41}$$

also we get

$$\left(q(s;t),r(s,z;t)\right)=\left(-M(s)\,\mathbf{D}(s)^{\top}\hat{\pi}(s),-M(s)\,\mathbf{F}(s,z)^{\top}\hat{\pi}(s)\right). \tag{2.2.42}$$

Moreover by taking (2.2.38) and (2.2.42) in (2.2.37), we obtain

$$\mathbf{B}(t)\,\varphi(t)-M(t)\left(\mathbf{D}(t)\,\mathbf{D}(t)^{\top}+\int_{\mathbb{R}^{\star}}\mathbf{F}(t,z)\operatorname{diag}(\mu^{\star}(dz))\,\mathbf{F}(t,z)^{\top}\right)\hat{\pi}(t)=0.$$

Subsequently, we obtain that $\hat{\pi}(.)$ admits the following representation

$$\hat{\pi}(t) = \frac{1}{M(t)} \Theta(t) \mathbf{B}(t) \varphi(t). \tag{2.2.43}$$

where

$$\Theta(t) = \left(\mathbf{D}(t) \mathbf{D}(t)^\top + \int_{\mathbb{R}^\star} \mathbf{F}(t,z) \operatorname{diag}(\mu^\star(dz)) \mathbf{F}(t,z)^\top \right)^{-1}, \ \forall t \in [0,T].$$

Note that $\forall t \in [0,T]$, $\Theta(t)$ can be represented as follows

$$\Theta(t) = \begin{pmatrix} \dfrac{1}{\left(\beta_0^2 + \int_0^{+\infty} z^2 \mu_0(dz)\right)} & \mathbf{0}_{\mathbb{R}^n}^\top \\ \mathbf{0}_{\mathbb{R}^n} & \left(\beta(t)\beta(t)^\top + \int_{\mathbb{R}^\star} \gamma(t,z) \operatorname{diag}(\mu(dz)) \gamma(t,z)^\top\right)^{-1} \end{pmatrix}.$$

Next, from (2.2.41) and (2.2.43) we obtain

$$\left(-\frac{dM}{ds}(s) - 2\rho_0(s) M(s)\right)\left(\hat{X}(s) - \mathbb{E}^t\left[\hat{X}(s)\right]\right) + \frac{d\varphi}{ds}(s) + \rho_0(s)\varphi(s) = 0. \tag{2.2.44}$$

This suggests that the functions $M(.)$, and $\varphi(.)$ solve the following system of equations,

$$\begin{cases} \dfrac{dM}{ds}(s) + 2\rho_0(s) M(s) = 0, \ s \in [0,T], \\ \dfrac{d\varphi}{ds}(s) + \rho_0(s)\varphi(s) = 0, \ \forall s \in [0,T], \\ M(T) = 1, \ \varphi(T) = \dfrac{1}{\gamma}, \end{cases} \tag{2.2.45}$$

which is explicitely solved by

$$\begin{cases} M(s) = e^{2\int_s^T \rho_0(\tau) d\tau}, \ \forall s \in [0,T], \\ \varphi(s) = \dfrac{1}{\gamma} e^{\int_s^T \rho_0(\tau) d\tau}, \ \forall s \in [0,T]. \end{cases} \tag{2.2.46}$$

From (2.2.43) we obtain

$$\begin{pmatrix} \hat{a}(s) \\ \hat{b}(s) \end{pmatrix} = \frac{1}{\gamma} \Theta(t) \begin{pmatrix} \lambda m_Y \theta_0 \\ \rho(s) \end{pmatrix} e^{-\int_s^T \rho_0(\tau) d\tau}, \tag{2.2.47}$$

It follows immediately that the open loop Nash equilibrium solution is given by (2.2.31) and (2.2.32).

Next we derive the efficient frontier of the mean-variance problem. Substituting the equilibrium solution

(2.2.43) into the wealth process results

$$\begin{cases} d\hat{X}(s) = \left(\rho_0(s)\hat{X}(s) + \frac{1}{\gamma}e^{-\int_s^T \rho_0(\tau)d\tau}\Phi(s) + \kappa\right)ds + \frac{1}{\gamma}e^{-\int_s^T \rho_0(\tau)d\tau}\mathbf{B}(s)^\top \Theta(s)\mathbf{D}(s)dW^\star(s) \\ \qquad + \frac{1}{\gamma}e^{-\int_s^T \rho_0(\tau)d\tau}\int_{\mathbb{R}^\star}\mathbf{B}(s)^\top \Theta(s)\mathbf{F}(s,z)\widetilde{N}^\star(ds,dz), \text{ for } s\in[0,T], \\ \hat{X}(0) = x_0, \end{cases} \tag{2.2.48}$$

By taking expectations on both sides of (2.2.48), we represent $\mathbb{E}\left[\hat{X}(s)\right]$ as follows

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s)\right] = \left\{\rho_0(s)\mathbb{E}\left[\hat{X}(s)\right] + \kappa + \frac{1}{\gamma}e^{-\int_s^T \rho_0(\tau)d\tau}\Phi(s)\right\}ds, \ s\in[0,T], \\ \mathbb{E}\left[\hat{X}(0)\right] = x_0, \end{cases} \tag{2.2.49}$$

A variation of constant formula yields

$$\mathbb{E}\left[\hat{X}(s)\right] = x_0 e^{\int_0^s \rho_0(\tau)d\tau} + \kappa\int_0^s e^{\int_\tau^s \rho_0(l)dl}d\tau + \frac{1}{\gamma}\int_0^s \Phi(\tau)d\tau, \ s\in[0,T], \tag{2.2.50}$$

which implies (2.2.43). Now, applying Ito's formula to $\hat{X}(s)^2$. Taking the expectation, we conclude that $\mathbb{E}\left[\hat{X}(s)^2\right]$ satisfies the following linear ordinary differential equation

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s)^2\right] = \left\{2\rho_0(s)\mathbb{E}\left[\hat{X}(s)^2\right] + 2\left(\frac{1}{\gamma}e^{-\int_s^T \rho_0(\tau)d\tau}\Phi(s) + \kappa\right)\mathbb{E}\left[\hat{X}(s)\right] \right. \\ \qquad \left. + \frac{1}{\gamma^2}e^{-\int_s^T 2\rho_0(\tau)d\tau}\Phi(s)\right\}ds, \text{ for } s\in[0,T], \\ \hat{X}(0)^2 = x_0^2, \end{cases} \tag{2.2.51}$$

a simple computations show that $\mathrm{Var}\left[\hat{X}(s)\right] = \mathbb{E}\left[\hat{X}(s)^2\right] - \mathbb{E}\left[\hat{X}(s)\right]^2$ satisfies the following linear ordinary differential equation

$$\begin{cases} d\mathrm{Var}\left[\hat{X}(s)\right] = \left\{2\rho_0(s)\mathrm{Var}\left[\hat{X}(s)\right] + \frac{1}{\gamma^2}e^{-2\int_s^T \rho_0(\tau)d\tau}\Phi(s)\right\}ds, \ s\in[0,T] \\ \mathrm{Var}\left[\hat{X}(0)\right] = 0. \end{cases} \tag{2.2.52}$$

which leads to the representation

$$\mathrm{Var}\left[\hat{X}(s)\right] = \frac{1}{\gamma^2}\int_0^s \Phi(\tau)d\tau, \ s\in[0,T], \tag{2.2.53}$$

which implies (2.2.34). This completes the proof.■

Remark 2.2.3 *The objective function value of the equilibrium strategy* $\hat{\pi}(.)$ *is*

$$J(0,x_0,\hat{\pi}(.)) = -\frac{1}{\gamma}\left(x_0 e^{\int_0^T \rho_0(\tau)d\tau} + \kappa\int_0^T e^{\int_\tau^T \rho_0(\tau)d\tau}d\tau + \frac{1}{2\gamma}\int_0^T \Phi(\tau)\,d\tau\right).$$

2.3 Special cases and relationship to other works

Equilibrium reinsurance and investment strategies for mean–variance models with constant risk aversion have been studied in [110] and [111] among others in different frameworks. In this section, we will compare our results with some existing ones in literature.

2.3.1 Special case 1

In the case where the surplus of the insurers is modelled by (2.1.4) , the financial market consists of one risk-free asset whose price process is given by (2.1.1), and only one risky asset whose price process is modelled by the geometric Lévy process

$$dS_1(s) = S_1(s-)\left(\tilde{\alpha}(s)\,ds + \beta(s)\,dW_1(s) + d\sum_{i=1}^{\tilde{L}(s)} Z_i\right),\ S_1(0) = s_1 > 0. \tag{2.3.1}$$

where $\tilde{\alpha},\beta : [0,T] \to \mathbb{R}$ are assumed to be deterministic and continuous functions. $W_1(.)$ is a one-dimensional standard Brownian motion, $\tilde{L}(s)$ representing the number of the jumps of the risky asset's price occurring up time s is a Poisson process with intensity $\lambda_Z > 0$, Z_i is the size of the ith jump amplitude of the risky asset's price and $\{Z_i\}_{i\in\mathbb{N}-\{0\}}$, are assumed to be i.i.d. random variables taking values in $[-1,+\infty)$ with common distribution $\mathbb{P}_Z$ having finite first and second moments $m_Z = \int_{-1}^{\infty} z\mathbb{P}_Z(dz)$ and $\sigma_Z = \int_{-1}^{\infty} z^2\mathbb{P}_Z(dz)$, respectively. The process $\sum_{i=1}^{\tilde{L}(s)} Z_i$ can also be defined through a random measure $N_1(ds,dz)$ as

$$\sum_{i=1}^{\tilde{L}(s)} Z_i = \int_0^s \int_{-1}^{\infty} zN_1(dr,dz),$$

where $N_1(.,.)$ is a Poisson random measure with a random compensator having the form $\mu_1(dz)\,ds = \lambda_Z\mathbb{P}_Z(dz)\,ds$. We recall that $\tilde{N}_1(ds,dz) = N_1(ds,dz) - \mu_1(dz)\,ds$ defines the compensated jump martingale random measure of $N_1(.,.)$. A trading strategy $\pi(.)$ is described by a two-dimensional stochastic processes $(a(.),b(.))$, where $a(s)$ represents the retention level of reinsurance or new business acquired at time $s\in[0,T]$. $b(s)$ represents the amount invested in the risky stock at time s. The dynamics of the wealth process $X(.) = X^{t,\xi}(.;\pi(.))$ which corresponds to an admissible strategy $\pi(.) = (a(.),b(.))$ and initial pair $(t,\xi)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$ can be described by

$$\begin{cases} dX(s) = \{\rho_0(s) X(s) + (\delta + \theta_0 a(s)) \lambda m_Y + \rho(s) b(s)\} ds + \beta_0 a(s) dW_0(s) \\ \qquad + \beta(s) b(s) dW_1(s) - a(s) \int_0^{+\infty} z \tilde{N}_0(ds, dz) \\ \qquad + b(s) \int_{-1}^{+\infty} z \tilde{N}_1(ds, dz), \text{ for } s \in [t, T], \\ X(t) = \xi. \end{cases} \tag{2.3.2}$$

where $\rho(s) = \tilde{\alpha}(s) - \rho_0(s) + \lambda_Z m_Z$ and $\delta = \eta - \theta_0$. In this case the objective is exactly the same as Zeng et al [111] in which the equilibrium is defined within the class of feedback controls. Moreover, the equilibrium strategy $\left(\hat{a}(.), \hat{b}(.)\right)$ given in our paper by the expressions (2.2.31) and (2.2.32) change to

$$\hat{a}(s) = \frac{\lambda m_Y \theta_0 e^{-\int_s^T \rho_0(\tau) d\tau}}{\gamma(\beta_0^2 + \lambda \sigma_Y)}, \ s \in [0, T], \tag{2.3.3}$$

$$\hat{b}(s) = \frac{(\tilde{\alpha}(s) - \rho_0(s) + \lambda_Z m_Z) e^{-\int_s^T \rho_0(\tau) d\tau}}{\gamma\left(\beta(s)^2 + \lambda_Z \sigma_Z\right)}, \ s \in [0, T]. \tag{2.3.4}$$

Which coincide with the ones obtained in Zeng et al [111] by solving an extended Hamilton–Jacobi–Bellman (HJB) equations.

2.3.2 Special case 2

Now, suppose that the surplus of the insurer is modelled the classical Cramér–Lundberg (CL) model (.i.e. the model (2.1.4) where $\beta_0 = 0$), and we assume that the financial market consists of one risk-free asset whose price process is given by (2.1.1), and only one risky asset whose price process do not have jumps and is modelled by a diffusion process

$$dS_1(s) = S_1(s-) (\tilde{\alpha}(s) ds + \beta(s) dW_1(s)), \ S_1(0) = s_1 > 0. \tag{2.3.5}$$

where $\tilde{\alpha}, \beta : [0, T] \to \mathbb{R}$ are assumed to be deterministic and continuous functions. $W_1(.)$ is a one-dimensional standard Brownian motion. In this case, the dynamics of the wealth process $X(.) = X^{t,\xi}(.; \pi(.))$ which corresponds to an admissible strategy $\pi(.) = (a(.), b(.))$ and initial pair $(t, \xi) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ can be described by

$$\begin{cases} dX(s) = \{\rho_0(s) X(s) + (\delta + \theta_0 a(s)) \lambda m_Y + \rho(s) b(s)\} ds + \beta(s) b(s) dW_1(s) \\ \qquad - a(s) \int_0^{+\infty} z \tilde{N}_0(ds, dz), \text{ for } s \in [t, T], \\ X(t) = \xi. \end{cases} \tag{2.3.6}$$

where $\rho(s) = \tilde{\alpha}(s) - \rho_0(s)$, and $\delta = \eta - \theta_0$. The equilibrium strategy $\left(\hat{a}(.), \hat{b}(.)\right)$ given by the expressions (2.2.31) and (2.2.32) change to

$$\hat{a}(s) = \frac{m_Y \theta_0 e^{-\int_s^T \rho_0(\tau) d\tau}}{\gamma \lambda \sigma_Y}, \ s \in [0, T], \tag{2.3.7}$$

$$\hat{b}(s) = \frac{(\tilde{\alpha}(s) - \rho_0(s)) e^{-\int_s^T \rho_0(\tau) d\tau}}{\gamma \beta(s)^2}, \ s \in [0, T]. \tag{2.3.8}$$

It is worth pointing out that the above equilibrium solutions are the same as those obtained in Zeng and Li [110] by solving some extended HJB equations.

Remark 2.3.1 *On comparing the results of this special case with the ones in the special case 1, we find the following facts:*

1) In the special case 2, the insurer will purchase less reinsurance or acquire more new business.

2) Comparing between $\frac{(\tilde{\alpha}(s) - \rho_0(s) + \lambda_Z m_Z)}{\gamma\left(\beta(s)^2 + \lambda_Z \sigma_Z^2\right)}$ *and* $\frac{(\tilde{\alpha}(s) - \rho_0(s))}{\gamma \beta(s)^2}$, *by simple computations, we have if* $\frac{m_Z}{\sigma_Z^2} > \frac{\tilde{\alpha}(s) - \rho_0(s)}{\beta(s)^2}$ *then the insurer in the second case will invest less money in the risky asset. Moreover, if* $\frac{m_Z}{\sigma_Z^2} < \frac{\tilde{\alpha}(s) - \rho_0(s)}{\beta(s)^2}$, *he/she will invest more money in the risky asset.*

2.3.3 Special case 3

We conclude this section with the case where he insurer is not allowed to purchase reinsurance or acquire new business, i.e. $a(s) \equiv 1$, and the financial market consists of one risk-free asset whose price process is given by (2.1.1), and only one risky asset whose price process is modelled by the diffusion process (2.3.5). In this case a trading strategy $\pi(.)$ reduces to a one-dimensional stochastic processes $b(.)$, where $b(s)$ represents the amount invested in the risky stock at time s. The dynamics of the wealth process $X(.) = X^{t,\xi}(.; \pi(.))$ which corresponds to an admissible investment strategy $b(.)$ and initial pair $(t, \xi) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ can be described by

$$\begin{cases} dX(s) = \{\rho_0(s) X(s) + \eta \lambda m_Y + \rho(s) b(s)\} ds + \beta(s) b(s) dW_1(s) \\ \qquad\qquad + \beta_0 b(s) dW_0(s) - \int_0^{+\infty} z \tilde{N}_0(ds, dz), \text{ for } s \in [t, T], \\ X(t) = \xi, \end{cases} \tag{2.3.9}$$

where $\rho(s) = \tilde{\alpha}(s) - \rho_0(s)$. Similar to the previous section, for the investment-only case we can derive the equilibrium strategy which is described as

$$\hat{b}(s) = \frac{(\tilde{\alpha}(s) - \rho_0(s))\, e^{-\int_s^T \rho_0(\tau)d\tau}}{\gamma\beta(s)^2}, \ \ s \in [0, T]. \tag{2.3.10}$$

This essentially covers the solution obtained by Bjök and Murgoci [17] by solving some extended HJB equations.

Chapter 3

Open Loop Equilibrium Strategies in General Discounting Merton Portfolio Problem

Discounted utility maximization is one of the most frequent problems in financial mathematics and has been considered by numerous authors. Landmark papers are the ones by Ramsey [81] in 1928 and Samuelson [84] in 1937. There is by now a very rich literature. In [69], and [70] Merton was the first to formulate the continuous-time portfolio-consumption model and apply the dynamic programming approach. Merton's results and technique have been further extended and developed by many authors since then. Among them we mention Richard [82], Breeden [21], and Fleming et al [44]. Another approach to solve the problem is the martingale method studied by Pliska [79], and Karatzas-Lehoczky-Shreve [55].

The common assumption in the literature cited above is that the discount rate is constant over time which leads to the discount function be exponential. However, Results from experimental studies contradict this assumption indicating that the discount rates for the near future are much lower than the discount rates for the time further away in future. Ainslie [3] performed empirical studies on human and animal behaviour and found that discount functions are almost hyperbolic, that is they decrease like a negative power of time rather than an exponential. Loewenstein and Prelec [66] present four drawbacks of exponential discounting and propose a model which accounts for them. They analyze implications for savings behaviour and estimation of discount rates.

As soon as discount function is non-exponential, the discounted utility models become time-inconsistent in the sens that they do not admit a Bellman's optimality principle. Consequently, research on equilibrium strategies for time inconsistent discounted utility maximization is a focus and attracts many scholars attentions. Recently, an extended (non local) HJB equation is derived in Marín-Solano and Navas [87] which solves the optimal consumption and investment problem with non-constant discount rate for both naive and sophisticated agents. The similar problem is also considered by another approach in Ekeland and Pirvu [40], they characterize the equilibrium policies through the solutions of a flow of BSDEs, and they show, with special form of the discount factor, this BSDE reduces to a system of two ODEs which has a solution. Another approach to the time-inconsistent discounted utility models is developed by Yong [103]. In Yong's paper, a sequence of multi-person hierarchical differential games is studied first and then the time-consistent equilibrium strategy and equilibrium value function are obtained by taking limit.

The purpose of this thesis is to develop on the existing theory concerning the study of equilibrium solutions to time inconsistents consumption-investment problem with a general discount function and a general utility function. The novelty of this work is that, here, we consider this problem in a non-Markovian framework. More specifically, the coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. To our best knowledge, the literature on the time-inconsistent Merton problem in a non-Markovian model is rather limited. We consider the definition of equilibrium strategies in the sense of open-loop one. Motivated by the works Hu et al [52] and Ekland and Pirvu [40], by means of the variational method, we characterize the open-loop equilibrium consumption-investment strategies via a stochastic system, which involves a flow of forward-backward stochastic differential equation along with some equilibrium condition. We accentuate that our work covers some results obtained in Marín-Solano and Navas [87] and [40], since here we provide the necessary and sufficient conditions to characterize the equilibrium strategies and we consider the discount function in fairly general form.

The rest of this chapter is organized as follows. The next section is devoted to the formulation of our problem. In Section 2, we apply the spike variation technique to derive a flow of FBSDEs and a necessary and sufficient condition for equilibrium controls. Based on this general result, we solve in Section 3 the case when all the coefficients are deterministic and we give some comparisons with some existing results in literature.

3.1 Problem formulation

Suppose that all stochastic processes and random variables are defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

3.1.1 Investment-consumption strategies and wealth processes

Consider an individual facing the intertemporal consumption and portfolio problem where the market environment consists of one riskless and d risky securities. The risky securities are stocks and their prices are modelled as Ito processes. Namely, for $i = 1, 2, .., d$, the price $S_i(s)$, $s \in [0, T]$, of the i-th risky asset satisfies

$$dS_i(s) = S_i(s)\left(\mu_i(s)\,ds + \sum_{j=1}^{d} \sigma_{ij}(s)\,dW_j(s)\right), \tag{3.1.1}$$

with $S_i(0) > 0$, for $i = 1, 2, ..., d$. The process $W(.) = (W_1(.), ..., W_d(.))^\top$ is a d-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration $(\mathcal{F}_t)_{t\in[0,T]}$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$.

The coefficients $r_i(.)$ and $\sigma_i(.) = (\sigma_{i1}(.), .., \sigma_{id}(.))$, for $i = 1, .., d$, are $(\mathcal{F}_t)_{t\in[0,T]}$ -progressively measurable processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. For brevity, we use $\mu(s) = (\mu_1(s), \mu_2(s), .., \mu_d(s))$ to denote the the drift rate vector, and denote by $\sigma(s) = (\sigma_{ij}(s))_{1\leq i,j\leq d}$ the random volatility matrix.

The riskless asset, the savings account, has the price process $B(s)$ at time $s \in [0, T]$ governed by

$$dB(s) = r_0(s)\,B(s)\,ds, \;\; B(0) = 1, \tag{3.1.2}$$

where $r_0(.)$ is $(\mathcal{F}_t)_{t\in[0,T]}$ -progressively measurable process with values in $[0, \infty)$ which represents the interest rate. We assume that $\mu_i(t) > r_0(t) > 0$, for $i = 1, 2, .., d$, $ds-a.e.$, $d\mathbb{P}-a.s.$ This is a very natural assumption, since otherwise nobody is willing to invest in the risky stocks.

Starting from an initial capital $x_0 > 0$ at time 0, during the time horizon $[0, T]$, the decision maker is allowed to dynamically investing in the stocks as well as in the bond, and consuming. A consumption-investment strategy is described by an $(d + 1)$-dimensional stochastic process $u(.) = (c(.), u_1(.), .., u_d(.))^\top$, where $c(s)$ represents the consumption rate at time for $s \in [0, T]$ and $u_i(s)$, for $i = 1, 2, .., d$, represents the amount invested in the *i-th* risky stock at time $s \in [0, T]$. The process $u_I(.) = (u_1(.), ..., u_d(.))^\top$ is called an investment strategy. The dollar amount invested in the bond at time s is $X^{x_0,u(.)}(s) - \sum_{i=1}^{d} u_i(s)$, where $X^{x_0,u(.)}(.)$ is the wealth process associated with the strategy $u(.)$ and the initial capital x_0. The evolution of $X^{x_0,u(.)}(.)$ can be described as

$$\begin{cases} dX^{x_0,u(.)}(s) = \left(X^{x_0,u(.)}(s) - \sum_{i=1}^{d} u_i(s)\right)\dfrac{dB(s)}{B(s)} + \sum_{i=1}^{d} u_i(s)\dfrac{dS_i(s)}{S_i(s)} - c(s)\,ds, \text{ for } s \in [0, T], \\ X^{x_0,u(.)}(0) = x_0. \end{cases}$$

Accordingly, the wealth process solves the following SDE

$$\begin{cases} dX^{x_0,u(.)}(s) = \left\{ r_0(s) X^{x_0,u(.)}(s) + u_I(s)^\top r(s) - c(s) \right\} ds + u_I(s)^\top \sigma(s) dW(s), \text{ for } s \in [0,T], \\ X^{x_0,u(.)}(0) = x_0. \end{cases} \tag{3.1.3}$$

where $r(s) = (\mu_1(s) - r_0(s), ..., \mu_d(s) - r_0(s))^\top$, see e.g [105] for more information. As time evolves, we need to consider the controlled stochastic differential equation parameterized by $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ and satisfied by $X(.) = X^{t,\xi}(.; u(.))$,

$$\begin{cases} dX(s) = \left\{ r_0(s) X^{x_0,u(.)}(s) + u_I(s)^\top r(s) - c(s) \right\} ds + u_I(s)^\top \sigma(s) dW(s), \text{ for } s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{3.1.4}$$

Definition 3.1.1 (Admissible Strategy) *A consumption-investment strategy* $u(.) = \left(c(.), u_I(.)^\top \right)^\top$ *is said be admissible over* $[t,T]$ *if* $u(.) \in \mathcal{L}^\infty_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^d\right)$, *and equation* (3.1.4) *has a unique solution* $X(.) = X^{t,\xi}(.; \pi(.))$, *for any* $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$.

We impose the assumption about the coefficients:

(H1) The processes $r_0(.), r(.)$, and $\sigma(.)$, are uniformly bounded. We also assume a uniform ellipticity condition as follows

$$\sigma(s)\sigma(s)^\top \geq \epsilon I_n, \ a.e, \ \mathbb{P}-a.s.$$

for some $\epsilon > 0$, *where* I_n denotes the identity matrice of $\mathbb{R}^{n \times n}$.

Under **(H1)**, for any $(t,\xi,u(.)) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \mathcal{L}^\infty_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^d\right)$, the state equation (3.1.4) has a unique solution $X(.) \in \mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R})$. Moreover, we have the following estimate

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X(s)|^2 \right] \leq K\left(1 + \mathbb{E}\left[|\xi|^2\right]\right), \tag{3.1.5}$$

for some positive constant K. In particular for $t = 0$ and $u(.) = \left(c(.), u_I(.)^\top \right)^\top \in \mathcal{L}^\infty_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^d\right)$, equation (3.1.3) has a unique solution $X(.) \in \mathcal{C}^2_{\mathcal{F}}(0,T;\mathbb{R})$ with the following estimate holds

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |X(s)|^2 \right] \leq K\left(1 + |x_0|^2\right). \tag{3.1.6}$$

3.1.2 General discounted Utility Function

In order to evaluate the performance of an consumption-investment strategy, the decision maker uses an expected utility criterion. Then, for any $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ the investment-consumption optimization problem is denoted as: maximizing, the utility function $J(t,\xi,.)$ given by,

$$J(t,\xi,u(.)) = \mathbb{E}^t \left[\int_t^T \lambda(s-t)\varphi(c(s))\,ds + \lambda(T-t)\,h(X(T)) \right], \tag{3.1.7}$$

over $u(.) \in \mathcal{L}^{\infty}_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^d\right)$, subject to $(3.1.4)$, where $\mathbb{E}^t[.] = \mathbb{E}[.\,|\mathcal{F}_t]$ and $\varphi, h : \mathbb{R} \to \mathbb{R}$ are strictly increasing, strictly concave twice continuously differentiable functions. $\varphi(c(s))$ represents the instantaneous utility from consumption $c(s)$ and $\lambda(T-t)\,h(X(T))$ is the (general discounted) utility that is derived from bequests. $\lambda : [0,T]\times\Omega \to \mathbb{R}$ is a stochastic adapted process which represents the hyperbolic stochastic discount function satisfying $\lambda(0) = 1$, $\lambda(s) > 0$ $ds-a.e$, $d\mathbb{P}-a.s$, and $\int_0^T \mathbb{E}[\lambda(s)]\,ds < \infty$. We also impose a technical assumption on $\lambda(.)$

(H2) There exists a constant $C > 0$ such that $|\lambda(s) - \lambda(t)| \leq C|s-t|$, $d\mathbb{P}-a.s$, for any $t,s \in [0,T]$.

Remark 3.1.1 *Assumption (**H2**) is satisfied by many discount functions, such as exponential discount functions, mixture of exponential functions and hyperbolic discount functions.*

If we denote $\mathbf{B}(s) = \left(-1, r(s)^\top\right)^\top$, $\mathbf{D}(s) = \begin{pmatrix} 0 & \mathbf{0}^\top_{\mathbb{R}^d} \\ \mathbf{0}_{\mathbb{R}^d} & \sigma(s) \end{pmatrix}$ and we write $W^*(s) = \left(0, W(s)^\top\right)^\top$. Then the optimal control problem associated with (3.1.4) and (3.1.7) is equivalent to maximize

$$J(t,\xi,u(.)) = \mathbb{E}^t \left[\int_t^T \lambda(s-t)\varphi\left(\mathbf{e}^\top u(.)\right) ds + \lambda(T-t)\,h(X(T)) \right], \tag{3.1.8}$$

where $\mathbf{e} = \left(1, \mathbf{0}^\top_{\mathbb{R}^d}\right)^\top$, subject to

$$\begin{cases} dX(s) = \left\{ r_0(s)X(s) + u(s)^\top \mathbf{B}(s) \right\} ds + u(s)^\top \mathbf{D}(s)\,dW^*(s), \text{ for } s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{3.1.9}$$

3.2 Equilibrium strategies

It is well known that, the problem described above turn out to be time inconsistent in the sense that, it does not satisfy the Bellman optimality principle, since a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair. For a more detailed discussion see Ekeland and Pirvu [40] and Yong [103]. Since lack of time consistency, we consider open-loop

Nash equilibrium controls instead of optimal controls. As in [51], we first consider an equilibrium by local spike variation, given an admissible consumption-investment strategy $\hat{u}(.) \in \mathcal{L}_{\mathcal{F}}^{\infty}(t,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{d}\right)$. For any $t \in [0,T]$, $v \in \mathbb{L}^{1}(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \mathbb{L}^{2}\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{d}\right)$ and for any $\varepsilon > 0$, define

$$u^{\varepsilon}(s) = \begin{cases} \hat{u}(s) + v, & \text{for } s \in [t, t+\varepsilon), \\ \hat{u}(s), & \text{for } s \in [t+\varepsilon, T], \end{cases} \tag{3.2.1}$$

we have the following definition.

Definition 3.2.1 (Open-loop Nash equilibrium) *An admissible strategy $\hat{u}(.) \in \mathcal{L}_{\mathcal{F}}^{\infty}(t,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{d}\right)$ is an open-loop Nash equilibrium strategy if*

$$\lim_{\varepsilon \downarrow 0} \inf \frac{1}{\varepsilon}\left\{ J\left(t, \hat{X}(t), u^{\varepsilon}(.)\right) - J\left(t, \hat{X}(t), \hat{u}(.)\right)\right\} \leq 0, \tag{3.2.2}$$

for any $t \in [0,T]$ and $v \in \mathbb{L}^{1}(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \mathbb{L}^{2}\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{d}\right)$. The corresponding equilibrium wealth process solves the following SDE

$$\begin{cases} d\hat{X}(s) = \left\{ r_0(s)\hat{X}(s) + \hat{u}(s)^{\top}\mathbf{B}(s)\right\} ds + \hat{u}(s)^{\top}\mathbf{D}(s)\, dW^{*}(s), \ \text{for } s \in [t,T], \\ \hat{X}(t) = \xi. \end{cases} \tag{3.2.3}$$

3.2.1 A necessary and sufficient condtion for equilibrium controls

Our objective is to present a necessary and sufficient condition for equilibriums. In the same spirit of the precedent chapters, we derive this condition by a second-order expansion in spike variation. First, we introduce the adjoint equations involved in the characterization of open-loop Nash equilibrium controls. Let $\hat{u}(.) = \left(\hat{c}(.), \hat{u}_I(.)^{\top}\right)^{\top} \in \mathcal{L}_{\mathcal{F}}^{\infty}(0,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{d}\right)$ and denote by $\hat{X}(.) \in \mathcal{C}_{\mathcal{F}}^{2}(0,T;\mathbb{R})$ the corresponding wealth process. For each $t \in [0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$, and satisfied by the pair of processes $(p(.;t), q(.;t))$ as follows

$$\begin{cases} dp(s;t) = -r_0(s)\, p(s;t)\, ds + q(s;t)^{\top} dW(s), \ s \in [t,T], \\ p(T;t) = \lambda(T-t)\dfrac{dh}{dx}\left(\hat{X}(T)\right), \end{cases} \tag{3.2.4}$$

where $q(.;t) = (q_1(.;t), ..., q_d(.;t))^{\top}$. Under **(H1)**, equation (3.2.4) is uniquely solvable in $\mathcal{C}_{\mathcal{F}}^{2}(t,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{d}\right)$. Moreover there exists a constant $K > 0$ such that, for any $t \in [0,T]$, we have the following estimate

$$\|p(.;t)\|^{2}_{\mathcal{C}_{\mathcal{F}}^{2}(t,T;\mathbb{R})} + \|q(.;t)\|^{2}_{\mathcal{L}^{2}(t,T;\mathbb{R}^{d})} \leq K\left(1 + x_0^{2}\right). \tag{3.2.5}$$

The second order adjoint equation is defined on the time interval $[t,T]$ and satisfied by the pair of processes $(P(.;t), Q(.;t)) \in \mathcal{C}_{\mathcal{F}}^2(t,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(t,T;\mathbb{R}^d)$ as follows

$$\begin{cases} dP(s;t) = -2r_0(s) P(s;t) ds + Q(s;t)^\top dW(s), \ s \in [t,T], \\ P(T;t) = \lambda(T-t) \dfrac{d^2h}{dx^2}\left(\hat{X}(T)\right) \end{cases} \tag{3.2.6}$$

where $Q(.;t) = (Q_1(.;t), ..., Q_d(.;t))^\top$. Under **(H1)** the above BSDE has unique solution $(P(.;t), Q(.;t)) \in \mathcal{C}_{\mathcal{F}}^2(t,T;\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(t,T;\mathbb{R}^d)$. Moreover we have the following representation for $P(.;t)$

$$P(s;t) = \mathbb{E}^s\left[\lambda(T-t)\frac{d^2h}{dx^2}\left(\hat{X}(T)\right)e^{\int_s^T 2r_0(\tau)d\tau}\right], \ s \in [t,T]. \tag{3.2.7}$$

Indeed, if we define the function $\Phi(s,.)$ for each $s \in [0,T]$, as the fundamental solution of the following linear ODE

$$\begin{cases} d\Phi(s,\tau) = r_0(\tau)\Theta(s,\tau)d\tau, \ \tau \in [s,T], \\ \Phi(s,s) = 1. \end{cases} \tag{3.2.8}$$

Then, we apply Ito's formula to $\tau \to P(\tau;t)\Theta(s,\tau)^2$ on $[s,T]$ and by taking the conditional expectations we obtain (3.2.7). Note that since $\frac{d^2h}{dx^2}\left(\hat{X}(T)\right) \le 0$, then $P(s;t) \le 0 \ ds-a.e.$

Proposition 3.2.1 *Let **(H1)** holds, for any* $t \in [0,T]$, $v \in \mathbb{L}^1(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d\right)$, *and* $\varepsilon \in [0,T-t)$, *we have the following equality*

$$J\left(t,\hat{X}(t),u^\varepsilon(.)\right) - J\left(t,\hat{X}(t),u(.)\right) = \int_t^{t+\varepsilon}\mathbb{E}^t\left[\langle\mathbf{H}(s;t),v\rangle + \frac{1}{2}\langle\mathbf{L}(s;t)v,v\rangle\right]ds + o(\varepsilon) \tag{3.2.9}$$

where for $\tilde{q}(s;t) = \left(0, q(s;t)^\top\right)^\top$

$$\mathbf{H}(s;t) \triangleq p(s;t)\mathbf{B}(s) + \mathbf{D}(s)\tilde{q}(s;t) + \lambda(s-t)\frac{d\varphi}{dc}\left(\mathbf{e}^\top\hat{u}(s)\right)\mathbf{e}, \tag{3.2.10}$$

and

$$\mathbf{L}(s;t) \triangleq \begin{pmatrix} \lambda(s-t)\dfrac{d^2}{dc^2}\varphi\left(\mathbf{e}^\top\left(\hat{u}(s)+\theta v 1_{[t,t+\varepsilon)}\right)\right) & \mathbf{0}_{\mathbb{R}^n}^\top \\ \mathbf{0}_{\mathbb{R}^n} & \sigma(s)\sigma(s)^\top P(s;t) \end{pmatrix} \tag{3.2.11}$$

with $\theta \in (0,1)$.

Proof. Denote by $\hat{X}^\varepsilon(.)$ the solution of the state equation corresponding to $u^\varepsilon(.)$. Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, see e.g. [105], we have

$$\hat{X}^{\varepsilon}(s) - \hat{X}(s) = y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s), \; s \in [t,T], \tag{3.2.12}$$

where $y^{\varepsilon,v}(.)$ and $z^{\varepsilon,v}(.)$ solve the following linear stochastic differential equations, respectively

$$\begin{cases} dy^{\varepsilon,v}(s) = \{r_0(s)\, y^{\varepsilon,v}(s)\}\, ds + 1_{[t,t+\varepsilon)}(s)\, v^{\top}\mathbf{D}(s)\, dW^*(s), \; s \in [t,T], \\ y^{\varepsilon,v}(t) = 0, \end{cases} \tag{3.2.13}$$

and

$$\begin{cases} dz^{\varepsilon,v}(s) = \left\{r_0(s)\, z^{\varepsilon,v}(s) + v^{\top}\mathbf{B}(s)\, 1_{[t,t+\varepsilon)}(s)\right\} ds, \; s \in [t,T], \\ z^{\varepsilon,v}(t) = 0. \end{cases} \tag{3.2.14}$$

Moreover, by Theorem 4.4 in [105], the following estimates hold

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} |y^{\varepsilon}(s)|^2\right] = O(\varepsilon) \text{ and } \mathbb{E}^t\left[\sup_{s\in[t,T]} |z^{\varepsilon}(s)|^2\right] = O\left(\varepsilon^2\right). \tag{3.2.15}$$

In addition, we have the following equality

$$\begin{aligned} &J\left(t, \hat{X}(t), u^{\varepsilon}(.)\right) - J\left(t, \hat{X}(t), \hat{u}(.)\right) \\ &= \mathbb{E}^t\left[\lambda(T-t)\left(\frac{dh}{dx}\left(\hat{X}(T)\right)(y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T)) + \frac{1}{2}\frac{d^2h}{dx^2}\left(\hat{X}(T)\right)(y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T))^2\right)\right] \\ &+ \mathbb{E}^t\left[\int_t^T \lambda(s-t)\left(\varphi\left(\mathbf{e}^{\top}u^{\varepsilon}(s)\right) - \varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)\right) ds\right] + o(\varepsilon) \\ &= \mathbb{E}^t\left[p(T;t)(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)) + \frac{1}{2}P(s;t)(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s))^2\right] \\ &+ \mathbb{E}^t\left[\int_t^T \lambda(s-t)\left(\varphi\left(\mathbf{e}^{\top}u^{\varepsilon}(s)\right) - \varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)\right)\right] + o(\varepsilon). \end{aligned} \tag{3.2.16}$$

Now, by applying Ito's formula to $s \mapsto p(s;t)(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s))$ on $[t,T]$, we get

$$\mathbb{E}^t\left[p(T;t)(y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T))\right] = \mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{v^{\top}\mathbf{B}(s)\, p(s;t) + v^{\top}\mathbf{D}(s)\, \widetilde{q}(s;t)\right\} ds\right]. \tag{3.2.17}$$

Again, by applying Ito's formula to $s \mapsto P(s;t)(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s))^2$ on $[t,T]$, we get

$$\begin{aligned} &\mathbb{E}^t\left[P(T;t)(y^{\varepsilon,v}(T) + z^{\varepsilon,v}(T))^2\right] \\ &= \mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{2v^{\top}(y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s))\left(\mathbf{B}(s)\, P(s,t) + \mathbf{D}(s)\, \widetilde{Q}(s,t)\right)\right.\right. \\ &\qquad \left.\left. + v^{\top}\left(\mathbf{D}(s)\, \mathbf{D}(s)^{\top}\right) vP(s,t)\right\} ds\right], \end{aligned} \tag{3.2.18}$$

where $\widetilde{Q}(s;t)=\left(0,Q(s;t)^{\top}\right)^{\top}$. In the other hand, we conclude from **(H1)** together with (3.2.15) that

$$\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)\left(\mathbf{B}(s)P(s,t)+\mathbf{D}(s)\widetilde{Q}(s,t)\right)ds\right]=o(\varepsilon). \tag{3.2.19}$$

By taking (3.2.17) and (3.2.18) in (3.2.16), it follows that

$$\begin{aligned}&J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right)\\&=\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\left\{v^{\top}\mathbf{B}(s)p(s;t)+v^{\top}\mathbf{D}(s)\widetilde{q}(s;t)+\frac{1}{2}v^{\top}\mathbf{D}(s)\mathbf{D}(s)^{\top}vP(s,t)\right\}ds\right]\\&+\mathbb{E}^{t}\left[\int_{t}^{T}\lambda(s-t)\left(\varphi\left(\mathbf{e}^{\top}u^{\varepsilon}(s)\right)-\varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)\right)ds\right]+o(\varepsilon).\end{aligned} \tag{3.2.20}$$

Now, applying second order Taylor-Lagrange expansion to $\varphi\left(\mathbf{e}^{\top}u^{\varepsilon}(s)\right)-\varphi\left(\mathbf{e}^{\top}u(s)\right)$ we find

$$\begin{aligned}\varphi\left(\mathbf{e}^{\top}u^{\varepsilon}(s)\right)-\varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)&=\varphi\left(\mathbf{e}^{\top}\hat{u}(s)+\mathbf{e}^{\top}v1_{[t,t+\varepsilon)}\right)-\varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)\\&=v^{\top}\mathbf{e}\frac{d\varphi\left(\mathbf{e}^{\top}\hat{u}(s)\right)}{dc}1_{[t,t+\varepsilon)}+\frac{1}{2}v^{\top}\mathbf{e}\frac{d^{2}\varphi}{dc^{2}}\left(\mathbf{e}^{\top}\left(\hat{u}(s)+\theta v1_{[t,t+\varepsilon)}\right)\right)\mathbf{e}^{\top}v1_{[t,t+\varepsilon)}\end{aligned} \tag{3.2.21}$$

By taking (3.2.21) in (3.2.20), it follows that

$$\begin{aligned}&J\left(t,\hat{X}(t),u^{\varepsilon}(s)\right)-J\left(t,\hat{X}(t),\hat{u}(s)\right)\\&=-\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\left\{v^{\top}\mathbf{B}(s)p(s;t)+v^{\top}\mathbf{D}(s)\widetilde{q}(s;t)+\frac{1}{2}v^{\top}\mathbf{D}(s)\mathbf{D}(s)^{\top}vP(s,t)\right\}ds\right]\\&+\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\lambda(s-t)\left(v^{\top}\mathbf{e}\frac{d\varphi}{dc}\left(\mathbf{e}^{\top}\hat{u}(s)\right)+\frac{1}{2}v^{\top}\mathbf{e}\frac{d^{2}\varphi}{dc^{2}}\left(\mathbf{e}^{\top}\left(\hat{u}(s)+\theta v1_{[t,t+\varepsilon)}\right)\right)\mathbf{e}^{\top}v\right)ds\right]+o(\varepsilon),\end{aligned} \tag{3.2.22}$$

which is equivalent to (3.2.9).■

Now, we presents the following technical lemma needed later in this study.

Lemma 3.2.1 *Under assumptions **(H1)-(H2)**. The following two statements are equivalent*

i) $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\mathbb{E}^{t}\left[\mathbf{H}(s;t)\right]ds=0,\ d\mathbb{P}-a.s,\ \forall t\in[0,T].$

ii) $\mathbf{H}(t;t)=0,\ d\mathbb{P}-a.s,\ dt-a.e.$

Proof. We set up $\alpha(s)=e^{\int_{s}^{T}-r_{0}(\tau)d\tau}$. Now, we define for $t\in[0,T]$ and $s\in[t,T]$

$$\left(\bar{p}(s;t),\bar{q}(s;t)\right)\equiv\frac{1}{\lambda(T-t)}\alpha(s)\left(p(s;t),q(s;t)\right).$$

then for any $t \in [0,T]$, in the interval $[t,T]$, the pair $(\bar{p}(.;t), \bar{q}(.;t))$ satisfies

$$\begin{cases} d\bar{p}(s;t) = \bar{q}(s;t)^\top dW(s), \ s \in [t,T], \\ \bar{p}(T;t) = \dfrac{dh}{dx}\left(\hat{X}(T)\right), \end{cases} \tag{3.2.23}$$

Moreover, it is clear that from the uniqueness of solutions to (3.2.23), we have $(\bar{p}(s;t_1), \bar{q}(s;t_1)) = (\bar{p}(s;t_2), \bar{q}(s;t_2))$, for any $t_1, t_2, s \in [0,T]$ such that $0 < t_1 < t_2 < s < T$. Hence, the solution $(\bar{p}(.;t), \bar{q}(.;t))$ does not depend on t. Thus we denote the solution of (3.2.23) by $(\bar{p}(.), \bar{q}(.))$.
We have then, for any $t \in [0,T]$, and $s \in [t,T]$

$$(p(s;t), q(s;t)) = \lambda(T-t)\,\alpha(s)^{-1}(\bar{p}(s), \bar{q}(s)). \tag{3.2.24}$$

Now using (3.2.24) we have, under **(H2)**, for any $t \in [0,T]$ and $s \in [t,T]$

$$\begin{aligned} |p(s;t) - p(s;s)| &= \left|(\lambda(T-t) - \lambda(T-s))\,\alpha(s)^{-1}\bar{p}(s)\right| \\ &\leq |s-t|\left|\alpha(s)^{-1}\bar{p}(s)\right|, \end{aligned} \tag{3.2.25}$$

and

$$|q(s;t) - q(s;s)| \leq |s-t|\left|\alpha(s)^{-1}\bar{q}(s)\right|.$$

From which, we have for any $t \in [0,T]$,

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}|\mathbf{H}(s;t) - \mathbf{H}(s;s)|\,ds\right] \\ &\leq C\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}|s-t|\left|\bar{p}(s) + \bar{q}(s) + \mathbf{e}^\top\frac{d\varphi}{dc}\left(\mathbf{e}^\top\hat{u}(s)\right)\right|ds\right] \\ &\leq C\lim_{\varepsilon\downarrow 0}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left|\bar{p}(s) + \bar{q}(s) + \mathbf{e}^\top\frac{d\varphi}{dc}\left(\mathbf{e}^\top\hat{u}(s)\right)\right|ds\right] \\ &= 0. \end{aligned}$$

Thus

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{H}(s;t)\,ds\right] = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{H}(s;s)\,ds\right]. \tag{3.2.26}$$

From the above equality, it is clear that if ii) holds, then

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathbf{H}(s;t)\,ds\right] = 0.\ d\mathbb{P}-a.s,$$

Conversely, according to Lemma 3.5 in [52], if i) holds then

$$\mathbf{H}\left(s;s\right)=0,\ d\mathbb{P}-a.s,\ ds-a.e.$$

This completes the proof.■

The following theorem is main result of this work, it's provides a necessary and sufficient condition for equilibriums.

Theorem 3.2.1 *Let **(H1)-(H2)** hold. Given an admissible strategy* $\hat{u}\left(.\right)\in\mathcal{L}_{\mathcal{F}}^{\infty}\left(0,T;\mathbb{R}\right)\times\mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{d}\right)$, *let for any* $t\in\left[0,T\right],\ \left(p\left(.;t\right),q\left(.;t\right)\right)\in\mathcal{C}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}\right)\times\mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{d}\right)$ *be the unique solution to the BSDE* (3.3.4). *Then* $\hat{u}\left(.\right)$ *is an equilibrium consumption-investment strategy, if and only if, the following condition holds*

$$\mathbf{H}\left(t;t\right)=0,\ d\mathbb{P}\text{−}a.s.,\ dt-a.e., \tag{3.2.27}$$

Proof. Given an admissible strategy $\hat{u}\left(.\right)\in\mathcal{L}_{\mathcal{F}}^{\infty}\left(0,T;\mathbb{R}\right)\times\mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{d}\right)$ for which (3.2.27) holds, according to Lemma 3.2.1 we have, for any $t\in\left[0,T\right]$,

$$\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\mathbb{E}^{t}\left[\mathbf{H}\left(s;t\right)\right]ds=0.$$

Then for any $t\in\left[0,T\right]$ and for any $v\in\mathbb{L}^{1}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}\right)\times\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{d}\right)$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}\left(t\right),u^{\varepsilon}\left(.\right)\right)-J\left(t,\hat{X}\left(t\right),\hat{u}\left(.\right)\right)\right\}\\
&=\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\left\{\left\langle\mathbb{E}^{t}\left[\mathbf{H}\left(s;t\right)\right],v\right\rangle ds+\frac{1}{2}\left\langle\mathbb{E}^{t}\left[\mathbf{L}\left(s;t\right)\right]v,v\right\rangle\right\}ds\\
&=\frac{1}{2}\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\left\langle\mathbb{E}^{t}\left[\mathbf{L}\left(s;t\right)\right]v,v\right\rangle ds\\
&\leq0,
\end{aligned}$$

where we have used the fact that, under the concavity condition of $\varphi\left(.\right)$ and $h\left(.\right)$, it follows $\left\langle\mathbf{L}\left(s;t\right)v,v\right\rangle\leq0$. Hence $\hat{u}\left(.\right)$ is an equilibrium strategy.

Conversely, assume that $\hat{u}\left(.\right)$ is an equilibrium strategy. Then, by (3.3.2) together with (3.3.11), for any $\left(t,u\right)\in\left[0,T\right]\times\mathbb{R}\times\mathbb{R}^{d}$ the following inequality holds

$$\lim_{\varepsilon\downarrow0}\left\langle\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\mathbb{E}^{t}\left[\mathbf{H}\left(s;t\right)\right]ds,u\right\rangle+\lim_{\varepsilon\downarrow0}\left\langle\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon}\mathbb{E}^{t}\left[\mathbf{L}\left(s;t\right)\right]dsu,u\right\rangle\leq0. \tag{3.2.28}$$

Now, we define $\forall (t,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\Phi(t,u) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\langle \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{H}(s;t) \right] ds, u \right\rangle + \frac{1}{2} \lim_{\varepsilon \downarrow 0} \left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{L}(s;t) \right] dsu, u \right\rangle .$$

Clearly $\Phi(.,.)$ is well defined. Moreover, easy manipulations show that the inequality (3.2.28) is equivalent to

$$\Phi(t,0) = \max_{u \in \mathbb{R} \times \mathbb{R}^d} \Psi(t,u), \ d\mathbb{P} - a.s, \forall t \in [0,T]. \tag{3.2.29}$$

It is easy to prove that the maximum condition (3.2.29) leads to the following condition, $\forall t \in [0,T]$

$$D_u \Phi(t,0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[\mathbf{H}(s;t) \right] ds = 0, \ d\mathbb{P} - a.s. \tag{3.2.30}$$

According to Lemma 3.2.1, the expression (3.2.27) follows immediately. This completes the proof.■

3.3 Equilibrium When Coefficients are Deterministic

Theorem 3.2.1 shows that one can obtain equilibrium consumption-investment strategies by solving the system of FBSDEs which are not standard since a "flow" of unknowns $(p(.;t), q(.;t))_{t \in [0,T]}$ is involved. Moreover, there is an additional constraints that act on the "diagonal" (i.e. when $s = t$) of the flow. As far as we know, the unique solvability of this type of equations remains an open problem. However, we are able to solve quite thoroughly this problem when the parameters $r_0(.), r(.), \sigma(.)$ and $\lambda(.)$ are all deterministic functions.

In this section, let us look at the Merton's portfolio problem with general discounting and deterministic parameters. The results in this section are comparable with the results obtained in [40], [87] and [103].
Suppose that $\hat{u}(.) = \left(\hat{c}(.), \hat{u}_I(.)^\top \right)^\top$ is an equilibrium control and denote by $\hat{X}(.)$ the corresponding wealth process. Then in view of Theorem 3.2.1 there exist an adapted processes $\left(\hat{X}(.), (p(.;t), q(.;t))_{t \in [0,T]} \right)$ solution to the following flow of forward-backward SDEs, parameterized by $t \in [0,T]$

$$\begin{cases} dX(s) = \left\{ r_0(s) \hat{X}(s) + \hat{u}_I(s)^\top r(s) - \hat{c}(s) \right\} ds + \hat{u}_I(s)^\top \sigma(s) dW(s), \ s \in [0,T], \\ dp(s;t) = -r_0(s) p(s;t) ds + q(s,t)^\top dW(s), \ 0 \le t \le s \le T, \\ \hat{X}(0) = x_0, \ p(T;t) = \lambda(T-t) \dfrac{dh}{dx} \left(\hat{X}(T) \right), \ t \in [0,T], \end{cases} \tag{3.3.1}$$

whith the following conditions hold

$$-p(s;s)+\dot{\varphi}(\hat{c}(s))=0,\ \text{a.e. } s\in[0,T], \tag{3.3.2}$$

$$p(s;s)r(s)+\sigma(s)q(s;s)=0,\ \text{a.e. } s\in[0,T], \tag{3.3.3}$$

where $\dot{\varphi}(c)$ denotes $\frac{d\varphi}{dc}(c)$. From the terminal condition in the first order adjoint process we consider the following Ansatz

$$p(s;t)=\lambda(T-t)\,\theta\left(s,\hat{X}(s)\right),\ \forall 0\leq t\leq s\leq T \tag{3.3.4}$$

for some deterministic function $\theta(.,.)\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ such that $\theta(T,x)=\frac{dh}{dx}(x)$.
Applying Itô's formula to (3.3.4) and using the second equation in (3.3.1), it yields

$$\begin{aligned}dp(s;t)&=\lambda(T-t)\Big\{\theta_s\left(s,\hat{X}(s)\right)ds+\theta_x\left(s,\hat{X}(s)\right)\left(\hat{X}(s)r_0(s)+\hat{u}_I(s)^\top r(s)-\hat{c}(s)\right)ds\\&\quad+\frac{1}{2}\theta_{xx}\left(s,\hat{X}(s)\right)\hat{u}_I(s)^\top\sigma(s)\sigma(s)^\top\hat{u}_I(s)\,ds+\theta_x\left(s,\hat{X}(s)\right)\hat{u}_I(s)^\top\sigma(s)\,dW(s)\Big\},\\&=-r_0(s)\lambda(T-t)\,\theta\left(s,\hat{X}(s)\right)ds+q(s,t)^\top dW(s).\end{aligned} \tag{3.3.5}$$

from which we deduce

$$q(s,t)=\lambda(T-t)\,\theta_x\left(s,\hat{X}(s)\right)\sigma(s)^\top\hat{u}_I(s) \tag{3.3.6}$$

We put the above expressions of $p(s;t)$ and $q(s;t)$ into (3.3.2) and (3.3.3), then

$$\lambda(T-s)\,\theta\left(s,\hat{X}(s)\right)-\dot{\varphi}(\hat{c}(s))=0,$$

and

$$\theta_x\left(s,\hat{X}(s)\right)\sigma(s)\sigma(s)^\top\hat{u}_I(s)=-r(s)\,\theta\left(s,\hat{X}(s)\right),$$

which leads to

$$\hat{c}(s)=\dot{\varphi}^{-1}\left(\lambda(T-s)\,\theta\left(s,\hat{X}(s)\right)\right),ds-a.e. \tag{3.3.7}$$

$$\hat{u}_I(s)=-\Sigma(s)r(s)\frac{\theta\left(s,\hat{X}(s)\right)}{\theta_x\left(s,\hat{X}(s)\right)},\ ds-a.e. \tag{3.3.8}$$

where $\dot{\varphi}^{-1}(.)$ denotes the inverse function of $\dot{\varphi}(.)=\frac{d\varphi}{dc}(.)$, and $\Sigma(s)\equiv\left(\sigma(s)\sigma(s)^\top\right)^{-1}$.

Next, comparing the ds term in (3.3.5) , then by using the expressions (3.3.7) and (3.3.8), we obtain

$$\theta_s\left(s,\hat{X}(s)\right)+\theta_x\left(s,\hat{X}(s)\right)\left(r_0(s)\hat{X}(s)-r(s)^\top\Sigma(s)r(s)\frac{\theta\left(s,\hat{X}(s)\right)}{\theta_x\left(s,\hat{X}(s)\right)}-\dot{\varphi}^{-1}\left(\lambda(T-s)\theta\left(s,\hat{X}(s)\right)\right)\right)$$
$$+\frac{1}{2}\theta_{xx}\left(s,\hat{X}(s)\right)r(s)^\top\Sigma(s)r(s)\left(\frac{\theta\left(s,\hat{X}(s)\right)}{\theta_x\left(s,\hat{X}(s)\right)}\right)^2 ds+r_0(s)\theta\left(s,\hat{X}(s)\right)=0,$$

which suggests that $\theta(.,.)$ solves the following parabolic backward partial differential equation

$$\begin{cases}\theta_s(s,x)+\theta_x(s,x)\left(r_0(s)x-r(s)^\top\Sigma(s)r(s)\dfrac{\theta(s,x)}{\theta_x(s,x)}-\dot{\varphi}^{-1}\left(\lambda(T-s)\theta(s,x)\right)\right)\\ +\frac{1}{2}\theta_{xx}(s,x)r(s)^\top\Sigma(s)r(s)\left(\dfrac{\theta(s,x)}{\theta_x(s,x)}\right)^2+\theta(s,x)r_0(s)=0,\quad(t,x)\in[0,T]\times\mathbb{R},\\ \theta(T,x)=h_x(x)\end{cases}\tag{3.3.9}$$

We summarize the above into the following theorem

Theorem 3.3.1 *Let **(H1)-(H2)** hold. If there exists a classical solution* $\theta(.,.)\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ *to the PDE* (3.4.9) *such that*

(i) The stochastic differential equation

$$\begin{cases}d\hat{X}(s)=\left\{r_0(s)\hat{X}(s)-r(s)^\top\Sigma(s)r(s)\dfrac{\theta\left(s,\hat{X}(s)\right)}{\theta_x\left(s,\hat{X}(s)\right)}-\dot{\varphi}^{-1}\left(\lambda(T-s)\theta\left(s,\hat{X}(s)\right)\right)\right\}ds\\ \qquad\qquad-\dfrac{\theta\left(s,\hat{X}(s)\right)}{\theta_x\left(s,\hat{X}(s)\right)}r(s)^\top\Sigma(s)\sigma(s)dW(s),\\ \hat{X}(0)=x_0.\end{cases}$$

define a unique solution $\hat{X}(.)$.

(ii) The strategy $\hat{c}(.)$ *and* $\hat{u}_I(.)$ *given by* (3.3.7) , *and* (3.3.8) , *respectively, are well defined and admissible. Then* $\hat{u}(.)=\left(\hat{c}(.),\hat{u}_I(.)^\top\right)^\top$ *is an equilibrium consumption-investment strategy.*

Remark 3.3.1 *Equation* (3.3.9) *is comparable with the ones in Solano and Navas [87] and Ekland and Pirvu [40], in which the equilibrium is defined within the class of feedback controls.*

Special cases

Equilibrium investment-consumption strategies for Merton's portfolio problem with general discounting and deterministic parameters have been studied in [87], [40] and [103] among others in different frameworks. In this subsection, we discuss some special cases in which the function $\theta(t,x)$ may be separated into functions of time and state variables. Then, on needs only to solve some linear ODEs in order to completely determine the equilibrium strategies. We will compare our results with some existing ones in literature.

Logarithmic Utility Function. First of all, let us analyse the case where $\varphi(c) = \ln(c)$ and $h(x) = a\ln(x)$, with $a > 0$.

In this case the PDE (3.3.9) reduces to

$$\begin{cases} \theta_s(s,x) + \theta_x(s,x)\left(r_0(s)x - r(s)^\top \Sigma(s) r(s) \dfrac{\theta(s,x)}{\theta_x(s,x)} - \dot{\varphi}^{-1}(\lambda(T-s)\theta(s,x))\right) \\ +\frac{1}{2}\theta_{xx}(s,x) r(s)^\top \Sigma(s) r(s) \left(\dfrac{\theta(s,x)}{\theta_x(s,x)}\right)^2 ds + \theta(s,x) r_0(s) = 0, \\ \theta(T,x) = a\dfrac{1}{x}. \end{cases} \tag{3.3.10}$$

Because of the terminal condition, we put

$$\theta(s,x) = aM(s)\frac{1}{x}, \tag{3.3.11}$$

where, $M(.) \in \mathcal{C}([0,T],\mathbb{R})$. Clearly we have

$$\theta_x(s,x) = -aM(s)\frac{1}{x^2} \text{ and } \theta_{xx}(s,x) = aM(s)\frac{2}{x^3}. \tag{3.3.12}$$

Substituting (3.3.11) and (3.3.12) in (3.3.10), we get

$$\begin{cases} \dfrac{dM(s)}{ds} + \dfrac{1}{a\lambda(T-s)} = 0, \\ M(T) = 1, \end{cases} \tag{3.3.13}$$

which is explicitly solved by

$$M(s) = 1 + \int_s^T \frac{1}{a\lambda(T-l)} dl.$$

In view of Theorem 3.3.1, the representation of the Nash equilibrium strategies (3.3.7)-(3.3.8) then give

$$\hat{c}(s) = \frac{1}{a\lambda(T-s) + \int_s^T \frac{\lambda(T-s)}{\lambda(T-l)} dl} \hat{X}(s) \tag{3.3.14}$$

$$\hat{u}_I(s) = \Sigma(s) r(s) \hat{X}(s), \tag{3.3.16}$$

Remark 3.3.2 *If we consider the special form of the discount function, suggested by Karp [56],*

$$\lambda(s-t) \equiv e^{-\int_0^{s-t} \delta(\tau) d\tau},$$

where $\delta : [0,T] \to \mathbb{R}^+$ is the instantaneous discount rate. In this case the objective is exactly the same as Solano and Navas [87]. Moreover, the equilibrium strategy $(\hat{c}(.), \hat{u}_I(s))$ given by the expressions (3.3.14) *and* (3.3.15) *change to*

$$\hat{u}_I(s) = \Sigma(s) r(s) \hat{X}(s), \tag{3.3.14}$$

$$\hat{c}(s) = \frac{1}{a\lambda(T-s) + \int_s^T e^{-\int_{T-l}^{T-s} \delta(\tau) d\tau} dl} \hat{X}(s), \tag{3.3.15}$$

which is comparable with the ones obtained in Solano and Navas [87] by solving an extended Hamilton–Jacobi–Bellman equations.

Potential Utility Function (CRRA model). Now, we consider the case where $\varphi(c) = \frac{c^\gamma}{\gamma}$, and $h(x) = a\frac{x^\gamma}{\gamma}$, with $a > 0$ and $\gamma \in (0,1)$. In this case the PDE (3.3.9) reduces to

$$\begin{cases} \theta_s(s,x) + \theta_x(s,x)\left(r_0(s)x - r(s)^\top \Sigma(s) r(s) \dfrac{\theta(s,x)}{\theta_x(s,x)} - \dot{\varphi}^{-1}(\lambda(T-s)\theta(s,x))\right) \\ +\dfrac{1}{2}\theta_{xx}(s,x) r(s)^\top \Sigma(s) r(s) \left(\dfrac{\theta(s,x)}{\theta_x(s,x)}\right)^2 ds + \theta(s,x) r_0(s) = 0, \\ \theta(T,x) = ax^{\gamma-1}. \end{cases} \tag{3.3.16}$$

We put

$$\theta(s,x) = aN(s) x^{\gamma-1}, \tag{3.3.17}$$

$$\theta_x(s,x) = a(\gamma-1) N(s) x^{\gamma-2}, \tag{3.3.18}$$

$$\theta_{xx}(s,x) = a(\gamma-1)(\gamma-2) N(s) x^{\gamma-3}. \tag{3.3.19}$$

For some deterministic function $N(.) \in \mathcal{C}([0,T], \mathbb{R})$. Substituting $(3.3.17), (3.3.18)$ and $(3.3.19)$ in $(3.3.16)$, we obtain

$$\begin{cases} \dfrac{dN(s)}{ds} + \left\{\gamma r_0(s) + \dfrac{1}{2}\dfrac{\gamma}{(1-\gamma)} r(s)^\top \Sigma(s) r(s) \right. \\ \qquad\qquad \left. + (1-\gamma) N(s)^{\frac{1}{\gamma-1}} (a\lambda(T-s))^{\frac{1}{\gamma-1}} \right\} N(s) = 0, \\ N(T) = 1. \end{cases} \tag{3.3.20}$$

Define

$$K(s) \equiv \gamma r_0(s) + \frac{1}{2}\frac{\gamma}{(1-\gamma)} r(s)^\top \Sigma(s) r(s).$$

and

$$A(s) \equiv (1-\gamma)(a\lambda(T-s))^{\frac{1}{\gamma-1}}.$$

It is an easy exercise to check that $N(.)$ solves $(4.3.20)$, if and only if, it solves the following integral equation

$$N(s) = e^{\int_s^T \left(K(\tau)+A(\tau)N(\tau)^{\frac{1}{\gamma-1}}\right)d\tau}. \tag{3.3.21}$$

We have the following Theorem.

Theorem 3.3.2 *Equation* (3.3.21) *has a unique solution in* $\mathcal{C}\left([0,T],\left[1,\tilde{M}\right]\right)$ *with* $\tilde{M}$ *having the following estimate*

$$\tilde{M} = e^{T\left(\|K(.)\|_\infty + \|A(.)\|_\infty\right)}.$$

Proof. For a constant $\beta > 0$, to be fixed later, we introduce the following norm, for $f(.,.) \in C([0,T],\mathbb{R})$

$$\|f\|_{\infty,\beta} = \sup_{s\in[0,T]} \left|e^{-\beta(T-s)} f(s)\right|,$$

it is easy to check that $e^{-\beta T}\|f\|_\infty \leq \|f\|_{\infty,\beta} \leq \|f\|_\infty$, for every $f \in \mathcal{C}([0,T],\mathbb{R})$, hence the norm $\|.\|_{\infty,\beta}$ is equivalent to $\|.\|_\infty$ on the Banach space $\mathcal{C}([0,T],\mathbb{R})$.We introduce the following nonlinear operator, $\tilde{L}[.]:\mathcal{C}([0,T],\mathbb{R}^+) \to \mathcal{C}([0,T],\mathbb{R}^+)$, such that for all $f(.) \in C([0,T],\mathbb{R}^+)$, we have

$$\tilde{L}[f](s) = e^{\int_s^T \left(K(\tau)+A(\tau)f(\tau)^{\frac{1}{\gamma-1}}\right)d\tau}.$$

Since we have $\tilde{L}[f](s) \geq 1$, $ds-a.e.$ Then, it is an easy to check that, our problem is equivalent to a fixed point problem for the operator $\tilde{L}[.]$ in the closed subset $\mathcal{C}\left([0,T],\left[1,\tilde{M}\right]\right)$ of the Banach space $\left(\mathcal{C}([0,T],\mathbb{R}),\|.\|_{\infty,\beta}\right)$.

1) **Existence of solution.** It is clear that $\tilde{L}[.]$ is well defined. Now, consider $f_1, f_2 \in C\left([0,T],\left[1,\tilde{M}\right]\right)$, we have

$$\tilde{L}[f_1](s) - \tilde{L}[f_2](s) = e^{\int_s^T K(\tau)d\tau}\left(e^{\int_s^T A(\tau)f_1(\tau)^{\frac{1}{\gamma-1}}d\tau} - e^{\int_s^T A(\tau)f_2(\tau)^{\frac{1}{\gamma-1}}d\tau}\right). \tag{3.3.22}$$

Moreover, since $K(.)$ and $A(.)$ are uniformly bounded and form the Lipschitz condition $|e^x - e^y| \leq$

$c_1 |x-y|, \forall x,y \in \left[0, \tilde{M}\right]$,

$$\left| e^{\int_s^T A(\tau) f_1(\tau)^{\frac{1}{\gamma-1}} d\tau} - e^{\int_s^T A(\tau) f_2(\tau)^{\frac{1}{\gamma-1}} d\tau} \right| \leq K_1 \int_s^T \left| f_1(\tau)^{\frac{1}{\gamma-1}} - f_2(\tau)^{\frac{1}{\gamma-1}} \right| d\tau.$$

for some constant $K_1 > 0$. In the other hand, from $\left| x^{\frac{1}{\gamma-1}} - y^{\frac{1}{\gamma-1}} \right| \leq c_2 |x-y|, \forall x,y \in \left[1, \tilde{M}\right]$, we deduce

$$\left| \tilde{L}[f_1](s) - \tilde{L}[f_2](s) \right| \leq K_2 \int_s^T |f_1(\tau) - f_2(\tau)| d\tau, \tag{3.3.23}$$

for some constant $K_2 > 0$. Thus

$$\begin{aligned} e^{-\beta(T-s)} \left| \tilde{L}[f_1](t,s) - \tilde{L}[f_2](t,s) \right| &\leq e^{-\beta(T-s)} K_2 \int_s^T |f_1(\tau) - f_2(\tau)| d\tau \\ &= e^{-\beta(T-s)} K_2 \int_s^T e^{\beta(T-\tau)} e^{-\beta(T-\tau)} |f_1(\tau) - f_2(\tau)| d\tau \\ &\leq \frac{K_2 \left(1 - e^{-\beta(T-s)}\right)}{\beta} \|f_1 - f_2\|_{\infty,\beta}, \end{aligned}$$

hence

$$\left\| \tilde{L}[f_1] - \tilde{L}[f_2] \right\|_{\infty,\beta} \leq \frac{K\left(1 - e^{-\beta T}\right)}{\beta} \|f_1 - f_2\|_{\infty,\beta}.$$

Therefore $\tilde{L}[.]$ is a contraction mapping for β large enough.

2) **Uniqueness of solution.** Let $f_1, f_2 \in C(D[0,T], \mathbb{R}^+)$ be two solutions, then

$$f_1(s) = \tilde{L}[f_1](s) \text{ and } f_2(s) = \tilde{L}[f_2](s), \ \forall s \in [0,T].$$

From (3.3.23) we have

$$|f_1(s) - f_2(s)| \leq K_2 \int_s^T |f_1(\tau) - f_2(\tau)| d\tau, \forall s \in [0,T],$$

therefore, by Gronwall Lemma, we conclude that $|f_1(s) - f_2(s)| = 0, \forall s \in [0,T]$.

This completes the proof.

Iin view of Theorem 3.3.1, the representation of the Nash equilibrium strategies (3.3.7)-(3.3.8) give

$$\hat{c}(s) = (a\lambda (T-s) N(s))^{\frac{1}{\gamma-1}} \hat{X}(s), \ a.e.t \in [0,T], \tag{3.3.24}$$

$$\hat{u}_I(s) = \Sigma(s) r(s) \frac{\hat{X}(s)}{(1-\gamma)}, \ a.e.t \in [0,T], \tag{3.3.25}$$

which is comparable with the ones obtained by Solano and Navas [87] , Ekland and Pirvu [40] and Yong [103].

Exponential Utility Function.

Finally, we consider the case where $\varphi(c) = -\frac{e^{-\gamma c}}{\gamma}$ and $h(x) = -a\frac{e^{-\gamma x}}{\gamma}$, with $a, \gamma > 0$.
The PDE (3.3.9) becomes

$$\begin{cases} \theta_s(s,x) + \theta_x(s,x)\left(r_0(s)\,x - r(s)^\top \Sigma(s)\, r(s)\frac{\theta(s,x)}{\theta_x(s,x)} - \dot{\varphi}^{-1}\left(\lambda(T-s)\,\theta(s,x)\right)\right) \\ +\frac{1}{2}\theta_{xx}(s,x)\, r(s)^\top \Sigma(s)\, r(s)\left(\frac{\theta(s,x)}{\theta_x(s,x)}\right)^2 ds + \theta(s,x)\, r_0(s) = 0, \\ \theta(T,x) = ae^{-\gamma x}. \end{cases} \tag{3.3.26}$$

We try a solution of the form

$$\theta(s,x) = ae^{-\gamma(L(s)x+G(s))}. \tag{3.3.27}$$

where $L(.)$,$G(.) \in C([0,T],\mathbb{R})$. Clearly we have

$$\theta_x(s,x) = -\gamma L(s)\, ae^{-\gamma(L(s)x+G(s))}, \tag{3.3.28}$$

$$\theta_{xx}(s,x) = (\gamma L(s))^2\, ae^{-\gamma(L(s)x+G(s))}. \tag{3.3.29}$$

Substituting (3.3.27) , (3.3.28) and (3.3.29) in (3.3.26) , we get

$$\begin{aligned} &-\gamma\frac{dL(s)}{ds}x - \gamma\dot{G}(s) - \gamma L(s)\, r_0(s)\, x - \frac{1}{2\gamma}r(s)^\top \Sigma(s)\, r(s) \\ &-L(s)\ln\left(\frac{\lambda(T-s)\,a}{\lambda(0)}\right) + \gamma L(s)^2 x + \gamma L(s)\, G(s) + r_0(s) = 0. \end{aligned}$$

This suggests that the functions $L(.)$ and $G(.)$ should solve the following system of equations

$$\begin{cases} \frac{dL(s)}{ds} = -r_0(s)\, L(s) + L(s)^2,\ s \in [0,T], \\ \frac{dG(s)}{ds} = L(s)\, G(s) - \frac{1}{\gamma}L(s)\ln\left(\frac{\lambda(T-s)\,a}{\lambda(0)}\right) - \frac{1}{2\gamma}r(s)^\top \Sigma(s)\, r(s) + \frac{r_0(s)}{\gamma},\ s \in [0,T], \\ L(T) = 1,\ G(T) = 0, \end{cases} \tag{3.3.30}$$

which is explicitly solvable by

$$\begin{cases} L(s) = \dfrac{e^{\int_s^T r_0(\tau)d\tau}}{1+\int_s^T e^{\int_l^T r_0(\tau)d\tau} dl}, \\ G(s) = e^{-\int_s^T L(\tau)d\tau} \int_s^T e^{\int_l^T L(\tau)d\tau} \left(\dfrac{1}{\gamma} L(l) \ln(\lambda(T-l)a) + \dfrac{1}{2\gamma} r(s)^\top \Sigma(s) r(s) - \dfrac{r_0(l)}{\gamma} \right) ds. \end{cases} \tag{3.3.31}$$

The representation of the Nash equilibrium strategies (3.3.7)-(3.3.8) give

$$\hat{c}(s) = \frac{1}{\gamma} \ln\left[\frac{1}{\lambda(T-s)}\right] + L(s)\hat{X}(s) + G(s), \tag{3.3.30}$$

$$\hat{u}_I(s) = \Sigma(s) r(s) \frac{1}{\gamma L(s) a}, \ a.e.t \in [0,T]. \tag{3.3.31}$$

Remark 3.3.3 *The equilibrium solutions* $(3.3.30)-(3.3.31)$ *are comparable with the ones obtained in Solano and Navas [87] by solving an extended HJB equations.*

Chapter 4

Near-Optimality Conditions in Mean-Field Control Models Involving Continuous and Impulse Controls

The chief goal of this chapter is to study the stochastic maximum principle approach for near-optimality, where the state of the system under consideration is governed by a controlled SDE, of mean-field type, in which the coefficients depend on the state of the solution process as well as of its expected value. Moreover, the cost functional is also of mean-field type. More specifically, the dynamics of the controlled system is driven by

$$\begin{cases} dx_t = & b\left(t, x_t, \mathbb{E}\left[x_t\right], u_t\right) dt + \sigma\left(t, x_t, \mathbb{E}\left[x_t\right], u_t\right) dW_t + G_t d\xi_t, \\ x_s = & a. \end{cases} \tag{4.1.1}$$

Where $\xi_t = \sum_{i\geq 1} \xi_i 1_{[\tau_i, T[}\left(t\right), t \leq T$, is a piecewise process. Here $\{\tau_i\}_{i\geq 1}$ is a fixed sequence of increasing $\mathcal{F}_t-$stopping times, each ξ_i is an $\mathcal{F}_{\tau_i}-$measurable random variable, and $\left(\mathcal{F}_t\right)_{t\leq T}$ is the natural filtration generated by the Brownian motion. This mean-field SDE is obtained as the mean-square limit, when

$n \to \infty$, of a system of interacting particles of the form

$$dx_t^{i,n} = b\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n} x_t^{j,n}, u_t\right)dt + \sigma\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n} x_t^{j,n}, u_t\right)dW_t^i + \sum_{j=1}^{m} G_t^{ij} d\xi_t^j,$$

where $\left(W_t^i\right)_{i\geq 1}$ is a collection of independent Brownian motions, see e.g. [89]. The objective of the controller is to minimize the expected cost functional, which depends on the control inputs to the system

$$J\left(u\right) = \mathbb{E}\left[\int_s^T f\left(t, x_t, \mathbb{E}x_t, u_t\right)dt + g\left(x_T, \mathbb{E}x_T\right) + \sum_{i\geq 1} l\left(\tau_i, \xi_i\right)\right], \tag{4.1.2}$$

over the set of the admissible controls. In contrast with the standard stochastic control problems for stochastic control of Itô diffusions, the coefficients in the state equation (4.1.1) and the cost functional (4.1.2) involve the expected value of the solution. Problems of this type occur in many applications, for example in the continuous-time Markowitz's mean-variance portfolio selection model, where one should minimize an objective function involving a quadratic function of the expected value, due to the variance term, see for example [7]. Note that related to the subject of SDEs of a mean-field type, in this sense, is a large literature on the approximation of McKean-Vlasov SDEs, PDEs, and certain generalizations of them, through interacting particle systems, see for example [1], [2] and [20]. The main difficulty when facing a general mean-field controlled diffusion is that, the setting is non-Markovian, and hence, the dynamic programming principle and the characterization of the Hamilton-Jacobi-Belman equations based on the law of iterated expectations on J does not hold in general. The notion of the stochastic maximum principle provides a powerful tool for handling this problem, see [7] and [23].

In the recent years, stochastic impulse control problems have also received considerable research attention due to wide application in a number of different areas. For example, they can be used for portfolio optimization problems with transaction costs, see e.g. [35] and [74], and for optimal control of exchange rates between different currencies see [27]. For a comprehensive survey of the theory of impulse control and its applications, one is referred to [57], [72] and [106].

The maximum principle for stochastic optimal control problem by which a necessary or a sufficient condition of optimality can be obtained by duality theory, involves the so-called adjoint process, which solves a linear backward stochastic differential equation (BSDE in short). Some results about the first order stochastic maximum principle for controlled diffusion processes are discussed, see e.g. [16], [43], [26], [8] and [13]. The second order stochastic maximum principle for optimal controls of nonlinear dynamics was developed via spike variation method by Peng [77]. These conditions are described in terms of two adjoint processes, which are linear backward SDE's. Existence and uniqueness for solutions to such equations with nonlinear coefficients has been treated by Pardoux and Peng [76].

In Andersson & Djehiche [7] and Li [63], the stochastic maximum principle (SMP in short) is proved for mean-field stochastic control problem where both the state dynamics and the cost functional are of a mean-field type. This SMP is obtained as an extension of the Bensoussan approach [13]. Some works that cover the controlled jump diffusion processes are discussed in [31] and [71]. The notion of mean-field BSDE appears in [24] and [25]. Equations of this type are essentially a generalization of a BSDE which allows the generator term to be an expectation of certain nonlinear function. In Buckdahn, Djehiche & Li [24], a general notion of mean-field BSDE associated with a mean-field SDE is obtained in a natural way as limit of some highly dimensional system of FBSDE governed by a d-dimensional Brownian motion, and influenced by positions of a large numbre of other particles. The study of the mean-field BSDE in a Markovian framework, associated with a mean-field SDE is given in [25]. By combining classical BSDE theory with particular arguments for mean-field BSDE it was shown that this mean-field BSDE discribes the viscosity solution of a nonlocal PDE.

Other type of control problems that are also important both from a theoretical and applied viewpoint are called the near-optimality problems. In fact, as well documented in Zhou [114], the near-optimal controls have several attractive features. First, optimal controls may not even exist in many situations. So it becomes very important to study near-optimal controls which are always exist and much easier to be obtained than optimal ones, both analytically and numerically. Moreover, since there are many more candidates for near-optimal controls it is likely to choose suitable ones, that are suitable for analysis and implementation. For example, Sethi and Zhou [85] showed that a near-optimal control can be found in certain class of the so-called threshold-type policies in optimal production controls for a stochastic two-machine flowshop, while, this models may involve very complicated switching that is very difficult to obtain their optimal controls.

Second, many practical systems are complicated that it is simply impossible to obtain their optimal controls. A commontly used approach is to approximate the original optimal control problems by simpler ones and then construct controls of the approximating problems. This idea has been applied to the hierarchical controls of manufacturing systems [113]. Note also that, in general, optimal feedback controls of linear systems are not continuous on the state, making it very difficult to handle analytically. However, one can modify this kind of controls into Lipschitz continuous controls with a small loss in the value function.

For deterministic control problems, the first result on necessary conditions for near-optimality has been proved in Ekeland [38], see also Zhou [115], by using Ekeland's variational principle. Their necessary conditions were derived only for some near-optimal controls. Maximum principle for near-optimal controls problems for systems deriven by Itô SDE with an uncontrolled diffusion coefficient, have been studied in [43], which are used to explore the stochastic maximum principle for optimal control in the situations where the coefficients of the state dynamics and the cost functional are nonsmooth, see [8] for more detail. We also would like to mention the work of Zhou [114] for the same problem in the setting where the control domain is non-convex, and the diffusion coefficient in the state SDE contain the control variable. For related works under different situations we refer to [30] for jump-diffusion systems, [49] and [50] for systems deriven by a forward-backward stochastic differential equation (FBSDE in short). Zhou [114] showed that any near-optimal control (in terms of a small parameter ε) nearly maximizes the $\mathcal{H}$-function in the integral form. Under certain concavity conditions, the near-maximum condition of the $\mathcal{H}$-function in the integral form is sufficient for near-optimality.

The purpose of the present thesis is to make a first attempt to study the near-optimal controls for system driven by a stochastic differential equation of mean-field type. The main contribution is the developments of necessary and sufficient conditions for all near-optimal controls. More specifically, according to Ekeland's variational principle we shall create two approximate variational inequalities in integral forme, with an error of order of "almost" $\epsilon^{\frac{1}{3}}$, the first is in terms of the $\mathcal{H}-$function obtained by the spike variation technique, see e.g. [23], used for all near-optimal absolutely continuous part of the control. In addition, we use a convex perturbation, see e.g. [97], for all near-optimal impulse controls to obtain the second variational inequality in terms of the first order adjoint process. Compared with the references [48] and [114], this paper mainly has two advantages stated as follows. First, it should be noted that, the impulse control used in this paper is different to singular one, which has been studied in [48], since the singular control is assumed to be a increasing process, while the impulse one is a piecewise process which is note necessarily increasing. Second, we generalize results in [114] by allowing both continuous and impulse controls, at least in the mean-field setting.

This chapter is organized as follows. The assumptions, notations and formulation of the problem are given in section 2. By using some stochastic results for diffusion processes of mean-field type and some properties of impulse controls, the study of the continuity property of the state equation and the adjoint processes with respect to an appropriate metric in the set of admissible controls is given in section 3. Then the necessary conditions for all near-optimal controls, which are the first main result, are stated and proved in this section. Section 4 is devoted to the sufficient conditions for near optimality.

4.1 Assumptions and problem formulation

This section sets out the notation and the assumptions that are supposed to hold in the sequel.

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P}\right)$ be a probability space such that $\mathcal{F}_0$ contains the $\mathbb{P}-$null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horison T, and $(\mathcal{F}_t)_{t\leq T}$ satisfies the usual assumptions. We assume that, $(\mathcal{F}_t)_{t\leq T}$ is generated by a one dimensional standard Brownian motion W for notational simplicity. Let $\{\tau_i\}_{i\geq 1}$ be a sequence of increasing $\mathcal{F}_t$-stopping times such that $\tau_i \uparrow +\infty$, and let $\{\xi_i\}_{i\geq 1}$ be a given sequence of random variables, such that each ξ_i is a V-valued $\mathcal{F}_{\tau_i}-$mesurable, where V is a nonempty convex subset of $\mathbb{R}^n$. It's worth noting that, the assumption $\tau_i \uparrow +\infty$ implies that at most finitely many impulses may occur on $[0,T]$.

Definition 4.1.1 *Let U is a non empty subset of $\mathbb{R}^n$. An admissible control is a pair of measurable, adapted processes $u : [0,T] \times \Omega \to U$, and $\xi : [0,T]\times\Omega \to V$, such that*

1. *u is absolutely continuous, and $\xi = \sum_{i\geq 1} \xi_i 1_{[\tau_i,T]}$ is a piecewise process, and $\xi_{0-} = 0$,*
2. *$\mathbb{E} \sup\limits_{t\in[0,T]} |u_t|^2 + \mathbb{E}\sum_{i\geq 1} |\xi_i|^2 < \infty$.*

We denote by $\mathcal{I} = \mathcal{U}\times\mathcal{V}$ the set of all admissible controls. Here $\mathcal{U}$ (resp. $\mathcal{V}$) represents the set of the admissible controls u (resp. ξ).

Notation. In this chapter, any element $x \in \mathbb{R}^n$ will be identified to a column vector with i-th component, and the norm $|x| = |x_1| + ... + |x_n|$. The scalar product of any two vectors x and y on $\mathbb{R}^n$ is denoted by $x.y$

In what follows, C represents a generic constant, which can be different from line to line.

Let us consider the following stochastic control problem:

For $(u,\xi) \in \mathcal{I}$, suppose the state of a controlled diffusion in $\mathbb{R}^n$ is described by the following stochastic differential equation

$$\begin{cases} dx_t = & b\left(t, x_t, \mathbb{E}x_t, u_t\right) dt + \sigma\left(t, x_t, \mathbb{E}x_t, u_t\right) dW_t + G_t d\xi_t, \\ x_s = & a, \end{cases} \tag{4.2.1}$$

where $(s,a) \in [0,T)\times\mathbb{R}^n$ be given, representing the initial time and initial state respectively, of the system.

Suppose the cost functional has the form

$$J\left(u,\xi\right) = \mathbb{E}\left[\int_s^T f\left(t, x_t, \mathbb{E}x_t, u_t\right) dt + g\left(x_T, \mathbb{E}x_T\right) + \sum_{i\geq 1} l\left(\tau_i, \xi_i\right)\right], \tag{4.2.2}$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. Here $b : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U \to$

$\mathbb{R}^n$, $G : [0, T] \to \mathbb{R}^{n\times m}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $l : [0, T] \times V \to \mathbb{R}$ are measurable functions in (t, x, y, u).

The following assumptions will be in force throughout this chapter.

(H1) For each $(t, x, y, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U$. The maps b, σ, and f are twice continuously differentiable in (x, y) , and all derivatives are bounded. There exists a constants $M > 0$ such that, for $h = b, \sigma$, and f

$$\begin{aligned} &|h(t,x,y,u) - h(t,x',y',u)| + |h_x(t,x,y,u) - h_x(t,x',y',u)| + |h_y(t,x,y,u) - h_y(t,x',y',u)| \\ &\leq M(|x-x'| + |y-y'|), \qquad (4.2.3) \\ &|h(t,x,y,u)| \leq M(1+|x|+|y|). \qquad (4.2.4) \end{aligned}$$

(H2) g is twice continuously differentiable in (x, y) and all derivatives are bounded. There exists a constants $M > 0$ such that

$$\begin{aligned} &|g(x,y) - g(x',y')| + |g_x(x,y) - g_x(x',y')| + |g_y(x,y) - g_y(x',y')| \\ &\leq M(|x-x'| + |y-y'|), \qquad (4.2.5) \\ &|g(x,y)| \leq M(1+|x|+|y|). \qquad (4.2.6) \end{aligned}$$

(H3) For any $\tau \in [0, T]$, l is continuous and is continuously differentiable in ξ. There exists a constants $M > 0$ such that

$$|l_\xi(\tau,\xi) - l_\xi(\tau,\eta)| \leq M|\xi - \eta|.$$

Under the above hypothesis, the SDE (4.2.1) has a unique strong solution, see [25], and by a standard argument it is easy to show that for any $p > 0$,

$$\mathbb{E}\left[\sup_{s\leq t\leq T} |x_t|^p\right] < \infty, \qquad (4.2.7)$$

and the functional J is well defined from $\mathcal{I}$ into $\mathbb{R}$.

Remark 4.1.1 *The above assumptions can be made weaker, see e.g. [75], where the usual Lipschitz assumption on the drift term is weakened by certain dissipativity conditions, allowing polynomial growth. But we do not focus on this subject here.*

4.1.1 Adjoint Processes and Maximum Principle

Define the usual Hamiltonian for $(t,x,y,u,p,q) \in [s,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n$, by

$$\mathbb{H}(t,x,y,u,p,q) = -f(t,x,y,u) - p.b(t,x,y,u) - q.\sigma(t,x,y,u). \tag{4.2.8}$$

For any $(u,\xi) \in \mathcal{I}$ and the corresponding state trajectory x, we define the first-order adjoint process ψ as the solution to the following linear mean-field BSDE.

$$\begin{cases} d\psi_t = & -\Big\{ b_x(t,x_t,\mathbb{E}x_t,u_t)^\top \psi_t + \mathbb{E}\Big(b_y(t,x_t,\mathbb{E}x_t,u_t)^\top \psi_t \Big) + \sigma_x(t,x_t,\mathbb{E}x_t,u_t)^\top \phi_t \\ & + \mathbb{E}\Big(\sigma_y(t,x_t,\mathbb{E}x_t,u_t)^\top \phi_t \Big) + f_x(t,x_t,\mathbb{E}x_t,u_t) + \mathbb{E}f_y(t,x_t,\mathbb{E}x_t,u_t) \Big\} dt + \phi_t dW_t, \\ \psi_T = & g_x(x_T,\mathbb{E}x_T) + \mathbb{E}g_y(x_T,\mathbb{E}x_T). \end{cases} \tag{4.2.9}$$

Note that under assumptions $(4.2,3)-(4.2,6)$ the equation (4.2.9) admits a unique $\mathcal{F}_t$-adapted solution $(\psi,\phi) \in \mathbb{R}^n \times \mathbb{R}^n$, see theorem 3.1 in [25]. This BSDE reduces to the standard one, when the coefficients b,σ,f, and g do not explicitly depend on expected value of the diffusion process. The second-order adjoint process Ψ is the one satisfying the following BSDE, that appears in Peng's SMP [77].

$$\begin{cases} d\Psi_t = & -\Big\{ b_x(t,x_t,\mathbb{E}x_t,u_t)^\top \Psi_t + \Psi_t.b_x(t,x_t,\mathbb{E}x_t,u_t) + \sigma_x(t,x_t,\mathbb{E}x_t,u_t)^\top \Phi_t + \Phi_t.\sigma_x(t,x_t,\mathbb{E}x_t,u_t) \\ & + \sigma_x(t,x_t,\mathbb{E}x_t,u_t)^\top \Psi_t \sigma_x(t,x_t,\mathbb{E}x_t,u_t) + \mathbb{H}_{xx}(t,x_t,\mathbb{E}x_t,u_t,\psi_t,\phi_t) \Big\} dt + \Phi_t dW_t, \\ \Psi_T = & g_{xx}(x_T,\mathbb{E}x_T) + \mathbb{E}g_{yy}(x_T,\mathbb{E}x_T). \end{cases} \tag{4.2.10}$$

Under assumptions $(4.2.3)-(4.2.6)$ the classical linear BSDE (4.2.10) admit a unique $\mathcal{F}_t$-adapted solution $(\Psi,\Phi) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, see [76].

Furthermore, we define the $\mathcal{H}-$function corresponding to a given admissible pair (z,v) as follows

$$\mathcal{H}^{(z,v)}(t,x,y,u) = \mathbb{H}(t,x,y,u,\psi_t,\phi_t) + \sigma(t,x,y,u)^\top \Psi_t \sigma(t,z_t,\mathbb{E}z_t,v_t) - \frac{1}{2}\sigma(t,x,y,u)^\top \Psi_t \sigma(t,x,y,u),$$

for $(t,x,y,u) \in [s,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U$, where the processes ψ,ϕ, and Ψ are determined by adjoint equations (4.2.10) and (4.2.11) corresponding to (z,v).

Lemma 4.1.1 *There exists a constant C, independent of (x,u), such that the solutions of* (4.2.9) *and*

(4.2.10) *have the following estimates*

$$\mathbb{E}\left[\sup_{s\leq t\leq T}\left|\psi_t\right|^2+\textstyle\int_s^T\left|\phi_t\right|^2dt\right]\leq C, \tag{4.2.11}$$

$$\mathbb{E}\left[\sup_{s\leq t\leq T}\left|\Psi_t\right|^2+\textstyle\int_s^T\left|\Phi_t\right|^2dt\right]\leq C. \tag{4.2.12}$$

Proof. Squaring both sides of

$$\begin{aligned}&\psi_t+\int_t^T\phi_r dW_r\\&=g_x\left(x_T,\mathbb{E}\left[x_T\right]\right)+\mathbb{E}g_y\left(x_T,\mathbb{E}x_T\right)+\int_t^T\left\{b_x\left(r,x_r,\mathbb{E}x_r,u_r\right)^{\top}\psi_r+\mathbb{E}\left(b_y\left(r,x_r,\mathbb{E}x_r,u_r\right)^{\top}\psi_s\right)\right.\\&\left.+\sigma_x\left(r,x_r,\mathbb{E}x_r,u_r\right)^{\top}\phi_r+\mathbb{E}\left(\sigma_y\left(r,x_r,\mathbb{E}x_r,u_r\right)^{\top}\phi_r\right)+f_x\left(r,x_r,\mathbb{E}x_r,u_r\right)+\mathbb{E}f_y\left(r,x_r,\mathbb{E}x_r,u_r\right)\right\}dr,\end{aligned}$$

and since the derivatives of the coeficients b,σ,f and g are bounded by the constant C, then, by using the fact that $\mathbb{E}\psi_t\int_t^T\phi_r dW_r=0$, we deduce

$$\mathbb{E}\left|\psi_t\right|^2+\mathbb{E}\int_t^T\left|\phi_r\right|^2dr\leq C+C\left(T-t\right)\mathbb{E}\int_t^T\left(\left|\psi_r\right|^2+\left|\mathbb{E}\psi_r\right|^2+\left|\phi_r\right|^2+\left|\mathbb{E}\phi_r\right|^2\right)dr$$

Then by the Jensen inequality, it yields

$$\mathbb{E}\left|\psi_t\right|^2+\mathbb{E}\int_t^T\left|\phi_r\right|^2dt\leq C+CT\mathbb{E}\int_t^T\left|\psi_r\right|^2dr+C\left(T-t\right)\mathbb{E}\int_t^T\left|\phi_r\right|^2dr,$$

for $t\in[T-\delta,T]$ with $\delta=\frac{1}{2C}$. Applying Burkholder-Davis-Gundy inequality and Granwall lemma, we obtain

$$\mathbb{E}\sup_{t\leq r\leq T}\left|\psi_r\right|^2+\frac{1}{2}\mathbb{E}\textstyle\int_t^T\left|\phi_r\right|^2dr\leq C,\ \text{ for } t\in\left[T-\delta,T\right].$$

Similarly we get

$$\mathbb{E}\sup_{t\leq r\leq T-\delta}\left|\psi_r\right|^2+\frac{1}{2}\mathbb{E}\textstyle\int_t^{T-\delta}\left|\phi_r\right|^2dr\leq C,\ \text{ for } t\in\left[T-2\delta,T-\delta\right].$$

Therefore, after a finite number of iterations, we describe the estimate (4.2.11) .■

The objective of the exact-optimality problems, is to minimize the functional $J\left(u,\xi\right)$ over all $\left(u,\xi\right)\in\mathcal{I}$, i.e. we seek $\left(u^*,\xi^*\right)$ such that $J\left(u^*,\xi^*\right)=\inf_{(u,\xi)\in\mathcal{I}}J\left(u,\xi\right)$. Any admissible control $\left(u^*,\xi^*\right)$ that achieves the infimum is called an optimal control, and it implies an associated optimal state evolution x^* from (3.2.1) . The maximum principle then states that, if $\left(u^*,\xi^*\right)$ is an optimal control, then one must have, first from [23]

$$\mathcal{H}^{x^*,u^*}(t,x_t^*,\mathbb{E}x_t^*,u_t^*)=\max_{u\in U}\mathcal{H}^{x^*,u^*}(t,x_t^*,\mathbb{E}x_t^*,u)\,,\ P-a.s.,\ a.e.\ t\in[s,T]\,. \tag{4.2.13}$$

and from [97]

$$0\leq\mathbb{E}\left[\sum\nolimits_{i\geq1}\left(l_\xi\left(\tau_i,\xi_i^*\right)+G_{\tau_i}\psi_{\tau_i}\right)\left(\eta_i-\xi_i^*\right)\right],\ \text{for all }\eta\in\mathcal{V}. \tag{4.2.14}$$

It's worth noting that, for exact optimality, the integral form and the pointwise form of the maximum condition are equivalent, but it is not the case for near-optimality.
Since our objective in this paper is to study near-optimality rather than exact-optimality for a control $(u,\xi)\in\mathcal{I}$, we give the definition of the near-optimality, see for example [114].

Definition 4.1.2 *Let $(u^\varepsilon,\xi^\varepsilon)$ be an admissible control parameterized by $\varepsilon>0$, x^ε is the corresponding trajectory solution to $(3.2.1)$. The control $(u^\varepsilon,\xi^\varepsilon)$ is called near-optimal for the problem $(4.2.2)$ if*

$$\left|J\left(u^\varepsilon,\xi^\varepsilon\right)-\inf_{(u,\xi)\in\mathcal{U}}J\left(u,\xi\right)\right|\leq R\left(\varepsilon\right),$$

holds for sufficiently small ε, where R is a function of ε satisfying $R(\varepsilon)\to0$ as $\varepsilon\to0$.

The estimate $R(\varepsilon)$ is called an error bound. If $R(\varepsilon)=c\varepsilon^\alpha$ for some $\alpha>0$ independent of the constant c, then u^ε is called *near-optimal with order ε^α*.

Remark 4.1.2 *An optimal control is an admissible strategy which achieves the infimum of the cost function $(3.2.2)$, it is usually unrealistic and unnecessary to explore optimal controls that are very sensitive to external perturbations. For an in-depth discussion of the merits of near-optimality we refer to [114]. It should be also noted that, the concepts of the near-optimality and the exact-optimality are coincide if $\varepsilon=0$ in the above definition.*

4.2 Necessary conditions of near-optimality

Let us recall the Ekeland's principle needed in this study

Lemma 4.2.1 (Ekeland's principle [38]) *Let (S,d) be a complet metric space and $\rho:S\to\mathbb{R}$ be lower-semicontinuous and bounded from below. For $\varepsilon\geq0$, suppose $u^\varepsilon\in S$ satisfies $\rho(u^\varepsilon)\leq\inf_{u\in S}\rho(u)+\varepsilon$. Then* *for any* $\lambda > 0$, *there exists* $u^{\lambda} \in S$ *such that*

$$\begin{aligned} &\rho\left(u^{\lambda}\right) \leq \rho\left(u^{\varepsilon}\right), \\ &d\left(u^{\lambda}, u^{\varepsilon}\right) \leq \lambda, \\ &\rho\left(u^{\lambda}\right) \leq \rho\left(u\right) + \frac{\varepsilon}{\lambda} d\left(u, u^{\lambda}\right), \text{ for all } u \in S. \end{aligned}$$

This section is devoted to the presentation of necessary conditions for all near-optimal controls. The main result is stated in the following theorem.

Theorem 4.2.1 (Necessary conditions for all near optimal controls) *For any* $\delta \in \left[0, \frac{1}{3}\right)$, *there exists a constant* $C = C(\delta) > 0$ *such that for any* $\varepsilon > 0$, *and any* $\varepsilon-$*optimal control* $(u^{\varepsilon}, \xi^{\varepsilon})$ *of the problem* (4.2.2), *with the corresponding state* x^{ε} *solution to* (3.2.1), *the following error estimate with respect to* ε *hold*

$$\begin{aligned} -C\varepsilon^{\delta} \leq \mathbb{E}\textstyle\int_s^T &\{\psi_t^{\varepsilon}.\left(b\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) - b\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right)\right) + \phi_t^{\varepsilon}.\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right)\right) \\ &+ \frac{1}{2}\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right)\right)^{\top} \Psi_t^{\varepsilon}\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right)\right) \\ &+ \left(f\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) - f\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right)\right)\} dt, \end{aligned} \tag{4.3.1}$$

and

$$-C\varepsilon^{\delta} \leq \mathbb{E}\left[\textstyle\sum_{i\geq 1}\left(l_{\xi}\left(\tau_i, \xi_i^{\varepsilon}\right) + G_{\tau_i}\psi_{\tau_i}^{\varepsilon}\right)\left(\eta_i - \xi_i^{\varepsilon}\right)\right]. \tag{4.3.2}$$

Here $(\psi^{\varepsilon}, \phi^{\varepsilon})$ *and* $(\Psi^{\varepsilon}, \Phi^{\varepsilon})$ *are the solutions to* (4.2.9) *and* (4.2.10) *respectively, corresponding to* $(x^{\varepsilon}, u^{\varepsilon})$.

Remark 4.2.1 *In particular, the inequality* (4.3.1) *gives the necessary condition of all near-optimal regular controls, in terms of the* $\mathcal{H}$*-function, this inequality can rewritten as follows*

$$-C\varepsilon^{\delta} \leq \mathbb{E}\left[\int_s^T \left\{\mathcal{H}^{x^{\varepsilon}, u^{\varepsilon}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right) - \mathcal{H}^{x^{\varepsilon}, u^{\varepsilon}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right)\right\} dt\right].$$

The proof of the Theorem 4.3.1 is based on some stability results with respect to the control variable of the state and adjoint processes, along with the Ekeland principle. First, we have to endow the set of controls with an appropriate metric

$$d\left(\left(u, v\right), \left(\xi, \eta\right)\right) = d_1\left(u, v\right) + d_2\left(\xi, \eta\right), \tag{4.3.3}$$

where

$$d_1(u,v) = \mathbb{P}\otimes dt\left\{(w,t)\in\Omega\times[0,T], v(\omega,t)\neq u(\omega,t)\right\},$$
$$d_2(\xi,\eta) = \mathbb{E}\left[\sum_{i\geq 1}|\xi_i-\eta_i|^2\right],$$

Here $\mathbb{P}\otimes dt$ is the product measure of $\mathbb{P}$ with the Lebesgue measure dt. It is easy to see that $(\mathcal{V},d_2)$ is a complete metric space. Moreover, it has been shown see e.g. [43] that $(\mathcal{U},d_1)$ is a complete metric space. Hence $\mathcal{I}$ as a product of two complete metric spaces is a complete metric space under d. Further, by assumptions $(4.2.3)-(4.2.6)$ we can prove that $J(.,.)$ is continuous on $\mathcal{I}$ endowed with the metric d.
To arrive at the necessary conditions for all near-optimal controls, we first formulate the necessary conditions only for some near-optimal controls. By applying Ekeland's principle the following intermediate theorem is then obtained

Theorem 4.2.2 (Necessary conditions for some near optimal controls) *For $\varepsilon\geq 0$, there exists $\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)\in\mathcal{I}$ such that for all admissible control (u,ξ), the following error estimate with respect to ε hold*

$$\begin{aligned}-\varepsilon^{\frac{1}{3}} \leq \mathbb{E}\Big[&(f(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},u)-f(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},\widetilde{u}_t^{\varepsilon}))+\widetilde{\psi}_t^{\varepsilon}.(b(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},u)-b(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},\widetilde{u}_t^{\varepsilon}))\\
&+\widetilde{\phi}_t^{\varepsilon}.(\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},u)-\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},\widetilde{u}_t^{\varepsilon}))\\
&+\frac{1}{2}(\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},u)-\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},\widetilde{u}_t^{\varepsilon}))^{\top}\widetilde{\Psi}_t^{\varepsilon}(\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},u)-\sigma(t,\widetilde{x}_t^{\varepsilon},\mathbb{E}\widetilde{x}_t^{\varepsilon},\widetilde{u}_t^{\varepsilon}))\Big],\end{aligned}\tag{4.3.4}$$

and

$$-\varepsilon^{\frac{1}{3}}\leq\mathbb{E}\left[\sum\nolimits_{i\geq 1}\left(l_{\xi}\left(\tau_i,\widetilde{\xi}_i^{\varepsilon}\right)+G_{\tau_i}\widetilde{\psi}_{\tau_i}^{\varepsilon}\right)\left(\eta_i-\widetilde{\xi}_i^{\varepsilon}\right)\right],\tag{4.3.5}$$

where $\left(\widetilde{\psi}^{\varepsilon},\widetilde{\phi}^{\varepsilon}\right)$ and $\left(\widetilde{\Psi}^{\varepsilon},\widetilde{\Phi}^{\varepsilon}\right)$ are the solutions to (4.2.9) *and* (4.2.10) *respectively, corresponding to* $(\widetilde{x}^{\varepsilon},\widetilde{u}^{\varepsilon})$.

Proof. By the Ekeland principle with $\lambda=\varepsilon^{\frac{2}{3}}$, there is an admissible pair $\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)$ such that

$$d\left((u^{\varepsilon},\xi^{\varepsilon}),\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)\right)\leq\varepsilon^{\frac{2}{3}},\ \text{and}\ \widetilde{J}\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)\leq\widetilde{J}(u,\xi),\ \text{for any}\ (u,\xi)\in U,\tag{4.3.6}$$

where $\widetilde{J}(u,\xi)=J(u,\xi)+\varepsilon^{\frac{1}{3}}d\left((u,\xi),\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)\right)$. This means that $\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)$ is an optimal control for the system (4.2.1) with a new cost function $\widetilde{J}(.,.)$. Next, we use a double perturbatios of the control $\left(\widetilde{u}^{\varepsilon},\widetilde{\xi}^{\varepsilon}\right)$. The first perturbation is a spike variation on the absolutly continuous part of the control and the second one is convex, on the impulse control.

Take any Borel measurable set $I^{\theta} \subset [s,T]$, with $\lambda\left(I^{\theta}\right) = \theta$ for any $\theta > 0$, where $\lambda\left(I^{\theta}\right)$ denote the Lebesgue measure of the set I^{θ}. Let $u \in U$ be fixed and consider the first perturbation by

$$\left(\widetilde{u}_t^{\varepsilon,\theta}, \widetilde{\xi}_t^{\varepsilon}\right) = \begin{cases} \left(\widetilde{u}_t^{\varepsilon}, \widetilde{\xi}_t^{\varepsilon}\right) & if \quad t \in [s,T] \setminus I^{\theta}, \\ \left(u, \widetilde{\xi}_t^{\varepsilon}\right) & if \qquad t \in I^{\theta}, \end{cases} \tag{4.3.7}$$

for $\eta \in \mathcal{V}$ we define the second perturbation as follows

$$\left(\widetilde{u}_t^{\varepsilon}, \widetilde{\xi}_t^{\varepsilon,\theta}\right) = \left(\widetilde{u}_t^{\varepsilon}, \widetilde{\xi}_t^{\varepsilon} + \theta\left(\eta_t - \widetilde{\xi}_t^{\varepsilon}\right)\right), \text{ for } t \in [0,T]. \tag{4.3.8}$$

Since $\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right)$ is optimal for the cost $\widetilde{J}(.,.)$, then

$$\widetilde{J}\left(\widetilde{u}^{\varepsilon,\theta}, \widetilde{\xi}^{\varepsilon}\right) \geq \widetilde{J}\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right), \tag{4.3.9}$$

$$\widetilde{J}\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon,\theta}\right) \geq \widetilde{J}\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right). \tag{4.3.10}$$

This imply that

$$J\left(\widetilde{u}^{\varepsilon,\theta}, \widetilde{\xi}^{\varepsilon}\right) - J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right) \geq -\varepsilon^{\frac{1}{3}} d_1\left(\widetilde{u}^{\varepsilon,\theta}, \widetilde{u}^{\varepsilon}\right), \tag{4.3.11}$$

$$J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon,\theta}\right) - J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right) \geq -\varepsilon^{\frac{1}{3}} d_2\left(\widetilde{\xi}^{\varepsilon,\theta}, \widetilde{\xi}^{\varepsilon}\right). \tag{3.3.12}$$

By the fact that $d_1\left(\widetilde{u}^{\varepsilon}, \widetilde{u}^{\varepsilon,\theta}\right) \leq \theta$, we have

$$J\left(\widetilde{u}^{\varepsilon,\theta}, \widetilde{\xi}^{\varepsilon}\right) - J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right) \geq -\varepsilon^{\frac{1}{3}} \theta. \tag{4.3.13}$$

According to Peng's maximum principle [77] and arguing as in [23], we obtain (4.3.4).
Now, the fact that $d_2\left(\widetilde{\xi}^{\varepsilon,\theta}, \widetilde{\xi}^{\varepsilon}\right) \leq \theta$ imply that

$$J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon,\theta}\right) - J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right) \geq -\varepsilon^{\frac{1}{3}} \theta. \tag{4.3.14}$$

The left-hand side of (4.3.14) depends only on the impulse control part, then by the SMP for impulse type control from [97] we get

$$\lim_{\theta \to 0} \frac{1}{\theta} \left\{ J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon,\theta}\right) - J\left(\widetilde{u}^{\varepsilon}, \widetilde{\xi}^{\varepsilon}\right) \right\} = \mathbb{E}\left[\sum_{i \geq 1} \left(l_{\xi}\left(\tau_i, \widetilde{\xi}_i^{\varepsilon}\right) + G_{\tau_i} \widetilde{\psi}_t^{\varepsilon} \right) \left(\eta_i - \widetilde{\xi}_i^{\varepsilon}\right) \right],$$

this guarantee the fulfillment of (4.3.5). ■

The following two lemmas are mainly devoted to investigate some prior estimates which play an important role in proving the main result of this section.

Lemma 4.2.2 *For any $\alpha \in (0,1)$ and $p \in (0,2]$ satisfying $\alpha p < 1$, there is a positive constant $C = C(\alpha, p)$, such that*

$$\mathbb{E}\left[\sup_{s\leq t\leq r} |x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^p\right] \leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.15}$$

Here x^ε (resp. $\widetilde{x}_t^\varepsilon$) is the solutions of the state SDE (4.2.1) *corresponding to* $(u^\varepsilon, \xi^\varepsilon)$ *(resp.* $\left(\widetilde{u}^\varepsilon, \widetilde{\xi}^\varepsilon\right)$*).*

Proof. According to Hölder inequality, it suffices to prove the above estimate for $p = 2$.
First of all, using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leq t\leq r} |x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^2\right] &\leq C\mathbb{E}\int_s^r \Big\{|b(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - b(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2 \\ &\quad + |\sigma(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - \sigma(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2\Big\} dt + C\mathbb{E}\left[\left(\sum_{s\leq\tau_i\leq r}\left|\xi_{\tau_i}^\varepsilon - \widetilde{\xi}_{\tau_i}^\varepsilon\right|\right)^2\right], \\ &\leq C\left(A_1 + A_2 + d_2\left(\xi^\varepsilon, \widetilde{\xi}^\varepsilon\right)\right),\end{aligned}$$

where A_1, A_2 are given by the following

$$A_1 = \mathbb{E}\int_s^r \left(|b(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - b(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2 + |\sigma(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - \sigma(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2\right) 1_{\{u_t \neq u_t'\}}(t)\, dt,$$
$$A_2 = \mathbb{E}\int_s^r \left(|b(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, \widetilde{u}_t^\varepsilon) - b(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2 + |\sigma(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, \widetilde{u}_t^\varepsilon) - \sigma(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)|^2\right) dt.$$

Due to the linear growth of the coefficients and from the Schwarz and Jensen inequalities, it follows

$$\begin{aligned}A_1 &\leq C\mathbb{E}\left[\int_s^r \left(1 + |x_t^\varepsilon|^{\frac{2}{1-\alpha}} + |\mathbb{E}x_t^\varepsilon|^{\frac{2}{1-\alpha}}\right) dt\right]^{1-\alpha} \mathbb{E}\left[\int_s^r 1_{\{u_t^\varepsilon \neq \widetilde{u}_t^\varepsilon\}}(t)\, dt\right]^\alpha, \\ &\leq C\mathbb{E}\left[\int_s^r \left(1 + |x_t^\varepsilon|^{\frac{2}{1-\alpha}}\right) dt\right]^{1-\alpha} d(u^\varepsilon, \widetilde{u}^\varepsilon)^\alpha.\end{aligned}$$

From (3.3.6) this means that $A_1 \leq C\varepsilon^{\frac{2}{3}\alpha}$. Since the coefficients of the SDE (3.2.1) are Lipschitz with respect to the state variable and its expected value, we get $A_2 \leq C\mathbb{E}\int_s^r \left(|x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^2 + |\mathbb{E}(x_t^\varepsilon - \widetilde{x}_t^\varepsilon)|^2\right) dt$. Noting that $\frac{2\alpha}{3} < \frac{2}{3}$ and $\varepsilon < 1$, then we easily check that

$$\begin{aligned}\mathbb{E}\sup_{s\leq t\leq r} |x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^2 &\leq C\left(\int_s^r \mathbb{E}\sup_{s\leq t\leq\theta} |x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^2 d\theta + \int_s^r \sup_{s\leq t\leq\theta} |\mathbb{E}(x_t^\varepsilon - \widetilde{x}_t^\varepsilon)|^2 d\theta + d(u^\varepsilon, \widetilde{u}^\varepsilon)^\alpha + d_2\left(\xi^\varepsilon, \widetilde{\xi}^\varepsilon\right)\right), \\ &\leq C\left(\int_s^r \mathbb{E}\sup_{s\leq t\leq\theta} |x_t^\varepsilon - \widetilde{x}_t^\varepsilon|^2 d\theta + \varepsilon^{\frac{2}{3}\alpha}\right).\end{aligned} \tag{4.3.16}$$

Hence (3.3.15) follows from Gronwall lemma.■

Lemma 4.2.3 *For any $\alpha \in (0,1)$ and $p \in (1,2)$ satisfying $(1+\alpha)p < 2$, there is a positive constant $C = C(\alpha, p)$ such that*

$$\mathbb{E}\int_s^T \left|\psi_t^\varepsilon - \widetilde{\psi}_t^\varepsilon\right|^p dt + \mathbb{E}\int_s^T \left|\phi_t^\varepsilon - \widetilde{\phi}_t^\varepsilon\right|^p dt \leq C\varepsilon^{\frac{\alpha p}{3}}, \tag{4.3.17}$$

$$\mathbb{E}\int_s^T \left|\Psi_t^\varepsilon - \widetilde{\Psi}_t^\varepsilon\right|^p dt + \mathbb{E}\int_s^T \left|\Phi_t^\varepsilon - \widetilde{\Phi}_t^\varepsilon\right|^p dt \leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.18}$$

Where $(\psi^\varepsilon, \phi^\varepsilon)$ and $\left(\widetilde{\psi}^\varepsilon, \widetilde{\phi}^\varepsilon\right)$ (resp. $(\Psi^\varepsilon, \Phi^\varepsilon)$ and $\left(\widetilde{\Psi}^\varepsilon, \widetilde{\Phi}^\varepsilon\right)$) denote the unique solutions to the first-order (resp. second-order) adjoint equation (4.2.9) (*resp.* (4.2.10)), *corresponding to the admissible pair $(x^\varepsilon, u^\varepsilon)$ and $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$.*

Proof. Denote by $\left(\overline{\psi}_t^\varepsilon, \overline{\phi}_t^\varepsilon\right) = \left(\psi_t^\varepsilon - \widetilde{\psi}_t^\varepsilon, \phi_t^\varepsilon - \widetilde{\phi}_t^\varepsilon\right)$ the unique solution of the linear mean-field backward stochastic differential equation for $t \in [s,T]$

$$\begin{cases} d\overline{\psi}_t^\varepsilon = & -\Big\{b_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top \overline{\psi}_t^\varepsilon + \mathbb{E}\left(b_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top \overline{\psi}_t^\varepsilon\right) + \sigma_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top \overline{\phi}_t^\varepsilon \\ & + \mathbb{E}\left(\sigma_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top \overline{\phi}_t^\varepsilon\right) + \overline{f}^\varepsilon(t)\Big\} dt + \overline{\phi}_t^\varepsilon dW_t, \\ \overline{\psi}_T = & g_x(x_T^\varepsilon, \mathbb{E}x_T^\varepsilon) - g_x(\widetilde{x}_T^\varepsilon, \mathbb{E}\widetilde{x}_T^\varepsilon) + \mathbb{E}\left(g_y(x_T^\varepsilon, \mathbb{E}x_T^\varepsilon) - g_y(\widetilde{x}_T^\varepsilon, \mathbb{E}\widetilde{x}_T^\varepsilon)\right), \end{cases} \tag{4.3.19}$$

where we have the following

$$\begin{aligned} \overline{f}^\varepsilon(t) = & \left(b_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top - b_x(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)^\top\right)\widetilde{\psi}_t^\varepsilon + \mathbb{E}\left(\left(b_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top - b_y(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)^\top\right)\widetilde{\psi}_t^\varepsilon\right) \\ & + \left(\sigma_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top - \sigma_x(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)^\top\right)\widetilde{\phi}_t^\varepsilon + \mathbb{E}\left(\left(\sigma_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)^\top - \sigma_y(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)^\top\right)\widetilde{\phi}_t^\varepsilon\right) \\ & + \left(f_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - f_x(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)\right) + \mathbb{E}\left(f_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon) - f_y(t, \widetilde{x}_t^\varepsilon, \mathbb{E}\widetilde{x}_t^\varepsilon, \widetilde{u}_t^\varepsilon)\right). \end{aligned}$$

Now, let Λ be a solution of the following linear stochastic differential equation of mean-field type

$$\begin{cases} d\Lambda_t = & \left\{b_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)\Lambda_t + b_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)\mathbb{E}\Lambda_t + \left|\overline{\psi}_t^\varepsilon\right|^{p-1} sgn\left(\overline{\psi}_t^\varepsilon\right)\right\} dt \\ & + \left\{\sigma_x(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)\Lambda_t + \sigma_y(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)\mathbb{E}\Lambda_t + \left|\overline{\phi}_t^\varepsilon\right|^{p-1} sgn\left(\overline{\phi}_t^\varepsilon\right)\right\} dW_t, \\ \Lambda_s = & 0. \end{cases} \tag{4.3.20}$$

Where $sgn(a) \equiv (sgn(a_1), ..., sgn(a_n))$ for a vector $a = (a_1, ..., a_n)^\top$. In view of the boundness of the coefficients b_x, σ_x, b_y, and σ_y by the constant C, and by the fact that

$$\mathbb{E}\int_s^T \left(\left|\left|\overline{\psi}_t^\varepsilon\right|^{p-1} sgn\left(\overline{\psi}_t^\varepsilon\right)\right|^2 + \left|\left|\overline{\phi}_t^\varepsilon\right|^{p-1} sgn\left(\overline{\phi}_t^\varepsilon\right)\right|^2\right) dt < \infty, \tag{4.3.21}$$

the linear mean-field SDE (4.3.20) satisfies the Itô conditions. Therefore, it has a unique solution. Moreover, we conclude from standard arguments based on the Burkholder-Davis-Gundy inequality, that

$$\mathbb{E}\sup_{s\leq t\leq T}\left|\Lambda_t\right|^q \leq C\left(\int_s^T \mathbb{E}\sup_{s\leq r\leq t}\left|\Lambda_r\right|^q dt+\int_s^T \sup_{s\leq r\leq t}\left|\mathbb{E}\Lambda_r\right|^q dt\right)+C\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{(p-1)q}+\left|\overline{\phi}_t^{\varepsilon}\right|^{(p-1)q}\right)dt.$$

Jensen inequality and Gronwall lemma, gives

$$\mathbb{E}\sup_{s\leq t\leq T}\left|\Lambda_t\right|^q \leq C\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{(p-1)q}+\left|\overline{\phi}_t^{\varepsilon}\right|^{(p-1)q}\right)dt=C\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt, \tag{4.3.22}$$

with $q\in(2,+\infty)$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. In view of (3.2.11), it yields

$$\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt\leq C. \tag{4.3.23}$$

On the other hand, by applying Itô's formula to $\overline{\psi}_t^{\varepsilon}.\Lambda_t$ and taking expectations, we have the representation

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T \overline{f}^{\varepsilon}(t)\,\Lambda_t dt+\left(g_x\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_x\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right)\Lambda_T+\mathbb{E}\left[g_y\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_y\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right]\Lambda_T\right]\\
&=\mathbb{E}\int_s^T\left(\overline{\psi}_t^{\varepsilon}\left|\overline{\psi}_t^{\varepsilon}\right|^{p-1} sgn\left(\overline{\psi}_t^{\varepsilon}\right)+\overline{\phi}_t^{\varepsilon}\left|\overline{\phi}_t^{\varepsilon}\right|^{p-1} sgn\left(\overline{\phi}_t^{\varepsilon}\right)\right)dt,\\
&=\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt.
\end{aligned} \tag{4.3.24}$$

First, it follows from (4.3.22) that

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T \overline{f}^{\varepsilon}(t)\,\Lambda_t dt+\left(g_x\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_x\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right)\Lambda_T+\mathbb{E}\left[g_y\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_y\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right]\Lambda_T\right]\\
&\leq C\mathbb{E}\left[\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt\right]^{\frac{1}{q}}\left(\mathbb{E}\left[\int_s^T\left|\overline{f}^{\varepsilon}(t)\right|^{p}dt\right]^{\frac{1}{p}}+\mathbb{E}\left[\left|g_x\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_x\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right|^{p}\right]^{\frac{1}{p}}\right.\\
&\left.+\mathbb{E}\left[\left|g_y\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_y\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right|^{p}\right]^{\frac{1}{p}}\right),
\end{aligned}$$

according to (4.3.24), we get

$$\begin{aligned}
&\mathbb{E}\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt\\
&\leq C\mathbb{E}\left[\int_s^T\left(\left|\overline{\psi}_t^{\varepsilon}\right|^{p}+\left|\overline{\phi}_t^{\varepsilon}\right|^{p}\right)dt\right]^{\frac{1}{q}}\left(\mathbb{E}\left[\int_s^T\left|\overline{f}^{\varepsilon}(t)\right|^{p}dt\right]^{\frac{1}{p}}+\mathbb{E}\left[\left|g_x\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_x\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right|^{p}\right]^{\frac{1}{p}}\right.\\
&\left.+\mathbb{E}\left[\left|g_y\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)-g_y\left(\widetilde{x}_T^{\varepsilon},\mathbb{E}\widetilde{x}_T^{\varepsilon}\right)\right|^{p}\right]^{\frac{1}{p}}\right),
\end{aligned}$$

since $\dfrac{1}{p}+\dfrac{1}{q}=1$, then

$$\mathbb{E}\left[\int_s^T\left(\left|\overline{\psi}_t^\varepsilon\right|^p+\left|\overline{\phi}_t^\varepsilon\right|^p\right)dt\right]\leq C\left(\mathbb{E}\int_s^T\left|\overline{f}^\varepsilon(t)\right|^p dt+\mathbb{E}\left[\left|g_x\left(x_T^\varepsilon,\mathbb{E}x_T^\varepsilon\right)-g_x\left(\widetilde{x}_T^\varepsilon,\mathbb{E}\widetilde{x}_T^\varepsilon\right)\right|^p\right]\right.$$
$$\left.+\mathbb{E}\left[\left|g_y\left(x_T^\varepsilon,\mathbb{E}x_T^\varepsilon\right)-g_y\left(\widetilde{x}_T^\varepsilon,\mathbb{E}\widetilde{x}_T^\varepsilon\right)\right|^p\right]\right).$$

To derive the inequality $(3.3.17)$, it is sufficient to prove the following assertions

$$\mathbb{E}\left[\int_s^T\left|\overline{f}^\varepsilon(t)\right|^p dt\right]\leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.25}$$

$$\mathbb{E}\left[\left|g_x\left(x_T^\varepsilon,\mathbb{E}x_T^\varepsilon\right)-g_x\left(\widetilde{x}_T^\varepsilon,\mathbb{E}\widetilde{x}_T^\varepsilon\right)\right|^p\right]\leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.26}$$

$$\mathbb{E}\left[\left|g_y\left(x_T^\varepsilon,\mathbb{E}x_T^\varepsilon\right)-g_y\left(\widetilde{x}_T^\varepsilon,\mathbb{E}\widetilde{x}_T^\varepsilon\right)\right|^p\right]\leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.27}$$

Let us prove the inequalities (4.3.26) and (4.3.27), since g_x and g_y are Lipschitz with respect to (x,y), and from Jensen inequality and by the fact that $\dfrac{\alpha p}{2}<1-\dfrac{p}{2}<1$, which combined with Lemma 3.2.2, leads to (4.3.26) and $(4.3.27)$.

Next, by repeatedly applying the Schwarz inequality, we can estimate

$$\mathbb{E}\left[\int_s^T\left|\left(b_x\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)^\top-b_x\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)^\top\right)\widetilde{\psi}_t^\varepsilon\right|^p dt\right]\leq B_1+B_2.$$

where, the following hold

$$B_1=\mathbb{E}\left[\int_s^T\left|\left(b_x\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)^\top-b_x\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)^\top\right)\widetilde{\psi}_t^\varepsilon\right|^p 1_{\{u_t^\varepsilon\neq\widetilde{u}_t^\varepsilon\}}(t)\,dt\right],$$
$$B_2=\mathbb{E}\left[\int_s^T\left|\left(b_x\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)^\top-b_x\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)^\top\right)\widetilde{\psi}_t^\varepsilon\right|^p dt\right].$$

Noting that $1-\dfrac{p}{2}>\dfrac{\alpha p}{2}$, and $d\left(u^\varepsilon,\widetilde{u}^\varepsilon\right)\leq\varepsilon^{\frac{2}{3}}<1$, then by the fact that b_x is bounded together with (3.2.11), it follows that

$$B_1\leq C\mathbb{E}\left[\int_s^T\left|\widetilde{\psi}_t^\varepsilon\right|^2 dt\right]^{\frac{p}{2}}d\left(u^\varepsilon,\widetilde{u}^\varepsilon\right)^{1-\frac{p}{2}}\leq C\varepsilon^{\frac{\alpha p}{3}}.$$

From Lipschitz condition on the coefficients, and by the fact that $\dfrac{\alpha p}{2-p}<1$, we conclude from Lemma 3.2.2 and estimate (4.2.11) that

$$\begin{aligned}B_2 &\leq \mathbb{E}\int_s^T\left(\left|x_t^\varepsilon-\widetilde{x}_t^\varepsilon\right|^p\left|\widetilde{\psi}_t^\varepsilon\right|^p+\left|\mathbb{E}x_t^\varepsilon-\mathbb{E}\widetilde{x}_t^\varepsilon\right|^p\left|\widetilde{\psi}_t^\varepsilon\right|^p\right)dt,\\ &\leq C\mathbb{E}\left[\int_s^T\left|\widetilde{\psi}_t^\varepsilon\right|^2dt\right]^{\frac{p}{2}}\mathbb{E}\left[\int_s^T\left|x_t^\varepsilon-\widetilde{x}_t^\varepsilon\right|^{\frac{2p}{2-p}}dt\right]^{1-\frac{p}{2}}+\left[\int_s^T\mathbb{E}\left[\left|\widetilde{\psi}_t^\varepsilon\right|^p\right]^{\frac{2}{p}}dt\right]^{\frac{p}{2}}\left[\int_s^T\mathbb{E}\left[\left|x_t^\varepsilon-\widetilde{x}_t^\varepsilon\right|^p\right]^{\frac{2}{2-p}}dt\right]^{1-\frac{p}{2}},\\ &\leq C\mathbb{E}\left[\int_s^T\left|\widetilde{\psi}_t^\varepsilon\right|^2dt\right]^{\frac{p}{2}}\left(d\left(u^\varepsilon,\widetilde{u}^\varepsilon\right)^{\frac{\alpha p}{2-p}}\right)^{1-\frac{p}{2}}\leq C\varepsilon^{\frac{\alpha p}{3}}.\end{aligned}$$

This proves

$$\mathbb{E}\left[\int_s^T\left|\left(b_x\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)^\top-b_x\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)^\top\right)\widetilde{\psi}_t^\varepsilon\right|^pdt\right]\leq C\varepsilon^{\frac{\alpha p}{3}}. \tag{4.3.28}$$

and we have the bound $\mathbb{E}\left[\int_s^T\left|\overline{f}^\varepsilon(t)\right|^pdt\right]\leq C\varepsilon^{\frac{\alpha p}{3}}$.■

Prof of Theorem 4.3.1. To arrive at the necessary conditions expressed for all near-optimal controls $(u^\varepsilon,\xi^\varepsilon)$, it is sufficient to derive an estimate for the term similar to the right sides of the inequalities (3.3.4) and (3.3.5) with all the $\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon$, etc. replaced by $x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon$, etc. To this end, we first estimate the following difference

$$\begin{aligned}&\mathbb{E}\left[\int_s^T\left\{\widetilde{\phi}_t^\varepsilon.(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u)-\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon))-\phi_t^\varepsilon.\left(\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right)\right\}dt\right],\\ &\leq I_1+I_2-I_3,\end{aligned}$$

where the I_1,I_2, and I_3 are defined by

$$\begin{aligned}I_1&=\mathbb{E}\int_s^T\left(\widetilde{\phi}_t^\varepsilon-\phi_t^\varepsilon\right).(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u)-\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon))dt,\\ I_2&=\mathbb{E}\int_s^T\phi_t^\varepsilon.(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u)-\sigma(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u))dt,\\ I_3&=\mathbb{E}\int_s^T\phi_t^\varepsilon.(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon)-\sigma(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon))dt.\end{aligned}$$

With this notation we have writing: for any $\delta\in\left[0,\frac{1}{3}\right)$, let $\alpha=3\delta\in[0,1)$, and fix a $p\in(1,2)$ so that $(1+\alpha)p<2$. Take $q\in(2,+\infty)$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$, it holds by using Lemma 4.3.3, that

$$\begin{aligned}I_1&\leq\mathbb{E}\left[\int_s^T\left|\widetilde{\phi}_t^\varepsilon-\phi_t^\varepsilon\right|^pdt\right]^{\frac{1}{p}}\mathbb{E}\left[\int_s^T(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u)-\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon))^q\,dt\right]^{\frac{1}{q}},\\ &\leq\left(C\varepsilon^{\frac{\alpha p}{3}}\right)^{\frac{1}{p}}\left(C\mathbb{E}\left[\int_s^T\left(1+\left|\widetilde{x}_t^\varepsilon\right|^q+\left|\mathbb{E}\widetilde{x}_t^\varepsilon\right|^q\right)dt\right]\right)^{\frac{1}{q}}.\\ &\leq C\varepsilon^{\frac{\alpha}{3}}=C\varepsilon^\delta.\end{aligned}$$

In vieu of the Lipschitz condition on σ, together with the estimates (4.2.11) and (4.3.15), we get from Schwarz inequality

$$\begin{aligned}
I_2 &\leq \mathbb{E}\left[\int_s^T |\phi_t^\varepsilon|^2\, dt\right]^{\frac{1}{2}} \mathbb{E}\left[\int_s^T |\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u)-\sigma(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u)|^2\, dt\right]^{\frac{1}{2}},\\
&\leq C\mathbb{E}\left[\int_s^T \left(|\widetilde{x}_t^\varepsilon - x_t^\varepsilon|^2 + |\mathbb{E}\left(\widetilde{x}_t^\varepsilon - x_t^\varepsilon\right)|^2\right) dt\right]^{\frac{1}{2}},\\
&\leq C\left(\varepsilon^{\frac{2\alpha}{3}}\right)^{\frac{1}{2}} = C\varepsilon^\delta.
\end{aligned}$$

Further, by the Schwarz and Jensen inequalities, one has

$$\begin{aligned}
I_3 &= \mathbb{E}\int_s^T \phi_t^\varepsilon.\left(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon)-\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u_t^\varepsilon)\right)1_{\{\widetilde{u}_t^\varepsilon\neq u_t^\varepsilon\}}(t)\, dt\\
&\quad+\mathbb{E}\int_s^T \phi_t^\varepsilon.\left(\sigma(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u_t^\varepsilon)-\sigma(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon)\right) dt,\\
&\leq C\mathbb{E}\left[\int_s^T |\phi_t^\varepsilon|^2\, dt\right]^{\frac{1}{2}} \mathbb{E}\left[\int_s^T \left(1+|\widetilde{x}_t^\varepsilon|^4+|\mathbb{E}\widetilde{x}_t^\varepsilon|^4\right) dt\right]^{\frac{1}{4}} d\left(\widetilde{u}^\varepsilon,u^\varepsilon\right)^{\frac{1}{4}},\\
&\quad+C\mathbb{E}\left[\int_s^T |\phi_t^\varepsilon|^2\, dt\right]^{\frac{1}{2}} \mathbb{E}\left[\int_s^T \left(1+|\widetilde{x}_t^\varepsilon - x_t^\varepsilon|^2+|\mathbb{E}\left(\widetilde{x}_t^\varepsilon - x_t^\varepsilon\right)|^2\right) dt\right]^{\frac{1}{2}},
\end{aligned}$$

Thus from the first inequality in (3.3.6) , it yields $I_3 \leq C\varepsilon^\delta$. Analogously we have

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T \left(\left(f\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u\right)-f\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)\right)+\widetilde{\psi}_t^\varepsilon.\left(b\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u\right)-b\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)\right)\right.\right.\\
&+\widetilde{\phi}_t^\varepsilon.\left(\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u\right)-\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)\right)\\
&\left.+\frac{1}{2}\left(\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u\right)-\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)\right)^\top \widetilde{\Psi}_t^\varepsilon\left(\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,u\right)-\sigma\left(t,\widetilde{x}_t^\varepsilon,\mathbb{E}\widetilde{x}_t^\varepsilon,\widetilde{u}_t^\varepsilon\right)\right) dt\right]\\
&-\mathbb{E}\left[\int_s^T \left(\left(f\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-f\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right)+\psi_t^\varepsilon.\left(b\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-b\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right)\right.\right.\\
&+\phi_t^\varepsilon.\left(\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right)\\
&\left.+\frac{1}{2}\left(\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right)^\top \Psi_t^\varepsilon\left(\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u\right)-\sigma\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)\right) dt\right]\\
&\leq C\varepsilon^\delta,
\end{aligned}$$

and directly deduce (3.3.1) .

By using similar arguments developed above, we can obtain the second variational inequality (3.3.2). We first estimate the following difference

$$\mathbb{E}\left[\sum_{i\geq 1}\left\{\left(l_\xi\left(\tau_i,\widetilde{\xi}_i^\varepsilon\right)+G_{\tau_i}\widetilde{\psi}_{\tau_i}^\varepsilon\right)\left(\eta_i-\widetilde{\xi}_i^\varepsilon\right)-\left(l_\xi\left(\tau_i,\xi_i^\varepsilon\right)+G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\left(\eta_i-\xi_i^\varepsilon\right)\right\}\right]=L_1+L_2,$$

where

$$L_1 = \mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i^\varepsilon\right)+G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\left(\xi_i^\varepsilon-\widetilde{\xi}_i^\varepsilon\right)\right],$$
$$L_2 = \mathbb{E}\left[\sum_{i\geq 1}\left\{\left(\left(l_\xi\left(\tau_i,\xi_i^\varepsilon\right)-l_\xi\left(\tau_i,\widetilde{\xi}_i^\varepsilon\right)\right)+G_{\tau_i}\left(\psi_{\tau_i}^\varepsilon-\widetilde{\psi}_{\tau_i}^\varepsilon\right)\right)\left(\widetilde{\xi}_i^\varepsilon-\eta_i\right)\right\}\right].$$

By the fact that l_ξ and G are bounded, together with the estimate (3.2.11) and Schwarz inequality, one has $L_1 \leq C\varepsilon^\delta$. Since l_ξ is Lipschitz with respect to the control variable it follows from (3.3.17) and Schwarz inequality that $L_2 \leq C\varepsilon^\delta$.

This completes the proof of Theorem 4.3.1■

Corollary 4.2.1 *Under the conditions of Theorem 4.3.1, it hold that*

$$\mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon,u^\varepsilon}\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon\right)dt\right] \geq \sup_{u\in\mathcal{U}}\mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon,u^\varepsilon}\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t\right)dt\right]-C\varepsilon^\delta, \tag{4.3.29}$$

and

$$\mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i^\varepsilon\right)+G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\eta_i\right] \geq \mathbb{E}\left[\sum_{i\geq 1}\left(l_\xi\left(\tau_i,\xi_i^\varepsilon\right)+G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\xi_i^\varepsilon\right]-C\varepsilon^\delta. \tag{4.3.30}$$

Proof. In the spike variations technique, we can replace the point $u\in U$ by any control $u\in\mathcal{U}$, and the subsequent argument still goes through. So the inequality in the estimate (4.3.1) holds with $u\in U$ replaced by $u\in\mathcal{U}$. The inequality (4.3.30) is an immediate consequence of (4.3.2) .■

4.3 Sufficient conditions of near-optimality

In this section, we will show that, under certain concavity conditions, the necessary conditions given by (4.3.29) and (4.3.30) are in fact sufficient for near-optimality. The classical maximum principle is effectively based on the fact that a maximum point of a function implies zero derivative at this point, while this is no longer the case for near-optimality. The key step here is to show that $H_u\left(t,x_t^\varepsilon,\mathbb{E}x_t^\varepsilon,u_t^\varepsilon,\psi_t^\varepsilon,\phi_t^\varepsilon\right)$ is very small and to estimate it in terms of ε.

We make the following assumptions

(H**4**) b,σ, and f are differentiable in u, and there is a constant $C>0$, such that for $\rho=b,\sigma$, and f

$$\left|\rho\left(t,x,y,u\right)-\rho\left(t,x,y,u'\right)\right|+\left|\rho_u\left(t,x,y,u\right)-\rho_u\left(t,x,y,u'\right)\right|\leq C\left|u-u'\right|. \tag{4.4.1}$$

Definition 4.3.1 (Clarke[32]) *Let Q be a convex set in $\mathbb{R}^d$ and let $h:Q\to\mathbb{R}$ be a locally Lipschitz* *function. The generalized gradient of h at $\hat{x} \in Q$, denoted by $\partial_x h$, is a set defined by*

$$\partial_x h(\hat{x}) = \left\{ p \in \mathbb{R}^d \diagup p.\xi \leq \limsup_{x \to \hat{x}, \theta \to 0+} \frac{h(x+\theta\xi) - h(x)}{\theta};\ \text{for any } \xi \in \mathbb{R}^d,\ \text{and } x, x+\theta\xi \in Q \right\}. \quad (4.4.2)$$

We can now state and prove the main result of this section.

Theorem 4.3.1 *Let $(u^\varepsilon, \xi^\varepsilon)$ be an admissible control, and $(\psi^\varepsilon, \phi^\varepsilon)$ be the solution to the corresponding BSDE (4.2.9). Assume that $H(t, ., ., ., \psi_t^\varepsilon, \phi_t^\varepsilon)$ is concave for a.e. $t \in [s, T]$, $P-a.s.$ The functions $g(., .)$ and $l(\tau_i, .)$ are convex. If for some $\varepsilon > 0$*

$$\mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon, u^\varepsilon}(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t^\varepsilon)\, dt\right] \geq \sup_{u \in \mathcal{U}} \mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon, u^\varepsilon}(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t)\, dt\right] - \varepsilon, \quad (4.4.3)$$

and

$$\mathbb{E}\left[\sum_{i \geq 1}\left(l_\xi(\tau_i, \xi_i^\varepsilon) + G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\eta_i\right] \geq \mathbb{E}\left[\sum_{i \geq 1}\left(l_\xi(\tau_i, \xi_i^\varepsilon) + G_{\tau_i}\psi_{\tau_i}^\varepsilon\right)\xi_i^\varepsilon\right] - \varepsilon. \quad (4.4.4)$$

Then $(u^\varepsilon, \xi^\varepsilon)$ is a near-optimal control with an error bound $\varepsilon + C\varepsilon^{\frac{1}{2}}$, i.e.

$$J(u^\varepsilon, \xi^\varepsilon) \leq \inf_{(u,\xi) \in \mathcal{I}} J(u, \xi) + \varepsilon + C\varepsilon^{\frac{1}{2}},$$

where $C > 0$ is a constant independent of ε.

Proof. We first fix an $\varepsilon > 0$, and define a new metric $\widehat{d}$ on $\mathcal{U}$, by setting

$$\widehat{d}(u, u') = \mathbb{E}\left[\int_s^T \mathcal{L}^\varepsilon(t)\, |u_t - u_t'|\, dt\right], \quad (4.4.5)$$

where $\mathcal{L}^\varepsilon(t) = 1 + |\psi_t^\varepsilon| + |\phi_t^\varepsilon| + 2|\Psi_t^\varepsilon|(1 + |x_t^\varepsilon| + |\mathbb{E}x_t^\varepsilon|) \geq 1$. Obviously $\widehat{d}$ is a metric, and it is a complete metric as a weighted L^1 norm. A simple computation shows that

$$\left|\mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon, u^\varepsilon}(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t)\, dt\right] - \mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon, u^\varepsilon}(t, x_t^\varepsilon, \mathbb{E}x_t^\varepsilon, u_t')\, dt\right]\right| \leq C\widehat{d}(u, u').$$

Therefore, $\mathbb{E}\left[\int_s^T \mathcal{H}^{x^\varepsilon, u^\varepsilon}(t, x_t^\varepsilon, .)\, dt\right]$ is continuous on $\mathcal{U}$ with respect to $\widehat{d}$. It follows from (4.4.3) and the Ekeland principle that, there exists a $\widetilde{u}^\varepsilon \in \mathcal{U}$ such that

$$\widehat{d}(\widetilde{u}^\varepsilon, u^\varepsilon) \leq \varepsilon^{\frac{1}{2}}, \quad (4.4.6)$$

and the following maximum condition holds

$$\mathbb{E}\left[\int_s^T \overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right) dt\right] = \max_{u \in U} \mathbb{E}\left[\int_s^T \overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right) dt\right], \tag{4.4.7}$$

where the $\overline{\mathcal{H}}$-function associated with random variables $x \in L^1\left(\Omega, \mathcal{F}, P\right)$, has the representation

$$\overline{\mathcal{H}}\left(t, x, \mathbb{E}x, u\right) = \mathcal{H}^{x^{\varepsilon}, u^{\varepsilon}}\left(t, x, \mathbb{E}x, u\right) - \varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right) \left|u - \widetilde{u}_t^{\varepsilon}\right|. \tag{4.4.8}$$

The integral-form maximum condition (4.4.7) implies a pointwise maximum condition, namely, for *a.e.* $t \in [s, T]$ and $P - a.s.$, $\overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right) = \max_{u \in U} \overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u\right)$. Recall from Proposition 2.3.2 in [32]

$$0 \in \partial_u \overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right). \tag{4.4.9}$$

By (4.4.8) and the fact that the generalized gradient of the sum of two functions is contained in the sum of the generalized gradients of the two functions, it follows from Proposition 2.3.3 in [32]

$$\begin{aligned}
\partial_u \overline{\mathcal{H}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right) \subset{}& \partial_u \mathcal{H}^{x^{\varepsilon}, u^{\varepsilon}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right) + \left[-\varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right), \varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right)\right] \\
&+ \sigma_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)^{\mathrm{T}} \Psi_t^{\varepsilon}\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)\right).
\end{aligned}$$

Since the Hamiltonian H is differentiable in u, we deduce from the inclusion (4.4.9) that, there is

$$K^{\varepsilon}\left(t\right) \in \left[-\varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right), \varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right)\right],$$

such that

$$\mathbb{H}_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}, \psi_t^{\varepsilon}, \phi_t^{\varepsilon}\right) = -K^{\varepsilon}\left(t\right) - \sigma_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)^{\mathrm{T}} \Psi_t^{\varepsilon}\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)\right). \tag{4.4.10}$$

Therefore, by assumption (4.4.1), we get

$$\begin{aligned}
\left|\mathbb{H}_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}, \psi_t^{\varepsilon}, \phi_t^{\varepsilon}\right)\right| \leq{}& \left|H_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}, \psi_t^{\varepsilon}, \phi_t^{\varepsilon}\right) - \mathbb{H}_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}, \psi_t^{\varepsilon}, \phi_t^{\varepsilon}\right)\right| \\
&+ \left|K^{\varepsilon}\left(t\right)\right| + \left|\sigma_u\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)^{\mathrm{T}} \Psi^{\varepsilon}\left(t\right)\left(\sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right) - \sigma\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, \widetilde{u}_t^{\varepsilon}\right)\right)\right| \\
\leq{}& C \mathcal{L}^{\varepsilon}\left(t\right) \left|u_t^{\varepsilon} - \widetilde{u}_t^{\varepsilon}\right| + \varepsilon^{\frac{1}{2}} \mathcal{L}^{\varepsilon}\left(t\right).
\end{aligned} \tag{4.4.11}$$

By the concavity of $\mathbb{H}\left(t,.,.,.,\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)$, we have

$$\mathbb{H}\left(t,x_t,\mathbb{E}x_t,u_t,\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)-\mathbb{H}\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u^{\varepsilon}\left(t\right),\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\leq H_x\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\left(x_t-x_t^{\varepsilon}\right)$$
$$+\mathbb{H}_y\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\mathbb{E}\left(x_t-x_t^{\varepsilon}\right)+\mathbb{H}_u\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\left(u_t-u_t^{\varepsilon}\right),$$

for any admissible pair (x,u), integrating this inequality with respect to t and taking expectations we obtain from (3.4.5), (3.4.6) and (3.4.11)

$$\begin{aligned}&\mathbb{E}\left[\int_s^T\left(\mathbb{H}\left(t,x_t,\mathbb{E}x_t,u_t,\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)-\mathbb{H}\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\right)dt\right]\\&\leq\mathbb{E}\left[\int_s^T\left\{\mathbb{H}_x\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)+\mathbb{E}\mathbb{H}_y\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\right\}\left(x_t-x_t^{\varepsilon}\right)dt\right]+C\varepsilon^{\frac{1}{2}}.\end{aligned}\tag{4.4.12}$$

On the other hand, by the convexity of g, it yields

$$\begin{aligned}\mathbb{E}\left[g\left(x_T,\mathbb{E}x_T\right)-g\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)\right]&\geq\mathbb{E}\left[\left\{g_x\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)+\mathbb{E}g_y\left(x_T^{\varepsilon},\mathbb{E}x_T^{\varepsilon}\right)\right\}\left(x_T-x_T^{\varepsilon}\right)\right]\\&=\mathbb{E}\left[\psi_T^{\varepsilon}\left(x_T-x_T^{\varepsilon}\right)\right].\end{aligned}\tag{4.4.13}$$

Thus it follows by the Itô formula applied to $\psi_t^{\varepsilon}\left(x_t-x_t^{\varepsilon}\right)$, together with $(4.4.4)$, (4.4.12) and (4.4.13)

$$\begin{aligned}&\mathbb{E}\left[\psi_T^{\varepsilon}\left(x_T-x_T^{\varepsilon}\right)\right]\\&=\mathbb{E}\Big[\int_s^T\left\{\mathbb{H}_x\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)+\mathbb{E}\mathbb{H}_y\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\right\}\left(x_t-x_t^{\varepsilon}\right)\\&\quad+\psi_t^{\varepsilon}.\left(b\left(t,x_t,\mathbb{E}x_t,u_t\right)-b\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon}\right)\right)+\phi_t^{\varepsilon}.\left(\sigma\left(t,x_t,\mathbb{E}x_t,u_t\right)-\sigma\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon}\right)\right)dt\\&\quad+\sum_{i\geq1}\psi_{\tau_i}^{\varepsilon}G_{\tau_i}\left(\eta_i-\xi_i^{\varepsilon}\right)\Big],\\&\geq\mathbb{E}\Big[\int_s^T\left(\mathbb{H}\left(t,x_t,\mathbb{E}x_t,u_t,\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)-\mathbb{H}\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon},\psi_t^{\varepsilon},\phi_t^{\varepsilon}\right)\right.\\&\quad+\psi_t^{\varepsilon}.\left(b\left(t,x_t,\mathbb{E}x_t,u_t\right)-b\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon}\right)\right)+\phi_t^{\varepsilon}.\left(\sigma\left(t,x_t,\mathbb{E}x_t,u_t\right)-\sigma\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon}\right)\right)dt\\&\quad-\sum_{i\geq1}l_{\xi}\left(\tau_i,\xi_i^{\varepsilon}\right)\left(\eta_i-\xi_i^{\varepsilon}\right)\Big]-\varepsilon-C\varepsilon^{\frac{1}{2}},\\&=\mathbb{E}\left[\int_s^T\left(f\left(t,x_t^{\varepsilon},\mathbb{E}x_t^{\varepsilon},u_t^{\varepsilon}\right)-f\left(t,x_t,\mathbb{E}x_t,u_t\right)\right)dt-\sum_{i\geq1}\left\{l\left(\tau_i,\eta_i\right)-l\left(\tau_i,\xi_i^{\varepsilon}\right)\right\}\right]-\varepsilon-C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

where the last inequality above is due to the convexity of $l\left(\tau_i,.\right)$. This shows that $J\left(u,\xi\right)\geq J\left(u^{\varepsilon},\xi^{\varepsilon}\right)-\varepsilon-C\varepsilon^{\frac{1}{2}}$.■

Corollary 4.3.1 *Under the assumptions of the Theorem* 4.2*, a sufficient condition for an admissible pair*

$(u^{\varepsilon}, \xi^{\varepsilon})$ *to be* $\varepsilon-$*optimal is*

$$\mathbb{E}\left[\int_s^T \mathcal{H}^{x^{\varepsilon},u^{\varepsilon}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t^{\varepsilon}\right) dt\right] \geq \sup_{u \in \mathcal{U}} \mathbb{E}\left[\int_s^T \mathcal{H}^{x^{\varepsilon},u^{\varepsilon}}\left(t, x_t^{\varepsilon}, \mathbb{E}x_t^{\varepsilon}, u_t\right) dt\right] - \left(\frac{\varepsilon}{2C}\right)^2. \tag{4.4.14}$$

and

$$\mathbb{E}\left[\sum_{i\geq 1}\left(l_{\xi}\left(\tau_i, \xi_i^{\varepsilon}\right) + G_{\tau_i}\psi_{\tau_i}^{\varepsilon}\right)\eta_i\right] \geq \mathbb{E}\left[\sum_{i\geq 1}\left(l_{\xi}\left(\tau_i, \xi_i^{\varepsilon}\right) + G_{\tau_i}\psi_{\tau_i}^{\varepsilon}\right)\xi_i^{\varepsilon}\right] - \frac{\varepsilon}{2}.$$

Remark 4.3.1 *If we assume that* $\varepsilon = 0$. *Theorems 4.3.1 and 4.4.1 reduces to the maximum principle for exact optimal controls.*

Conclusion

Classical optimal control problems are time-consistent, which means that an optimal control constructed for a given initial pair of time and state will remain optimal thereafter. It seems to be not so ideal in real world. In fact, more than often, an optimal control selected/designed at a given moment will hardly be optimal at later time moments. This is called time-inconsistency of the problem. Among many possible reasons causing the time-inconsistency, there are two playing some essential roles:

(i) People usually over-discount on the immediate future utility/disutility than on farther future ones;

(ii) and people's attitude towards risks are subjective rather than objective.

Mathematically, the former can be described by the hyperbolic discounting, and an important special case of the later can be described by certain nonlinear appearance of conditional expectations for the state process and/or control process in the cost functional.

In this Thesis we have investigated about 4 stochastic optimal control problems which, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. *In chapter 1, we have studied a class a general time-inconsistent stochastic linear–quadratic* (LQ) control problem*. We have used the game theoretic approach to handle the time inconsistency. During this study open-loop Nash equilibrium controls are constructed as an alternative of optimal controls. This has been accomplished through stochastic maximum principle that includes a flow of forward-backward stochastic differential equations under maximum condition. The inclusion of concrete examples confirms the validity of our proposed study. The work can be extended in several ways. For example, this approach can be extended to a* theory for general random coefficients case*. The research on this topic is in progress and will appear in our forthcoming paper.*

In Chapter 2 , we have studied optimal investment and reinsurance problem, incorporating jumps, for mean–variance insurers*. We have used the game theoretic approach to handle the time inconsistency.* We have provided a necessary and sufficient conditions for equilibriums and derived the equilibrium investment and reinsurance strategies and the corresponding value function explicitly. In addition, some special cases of our model have been discussed and the explicit expressions of the corresponding solutions have been derived. Moreover, this paper extends the models and results of Zheng et al [111]. *The work can be extended in several ways. For example, as the insurer updates its policy continuously in the time inconsistent problems, it may be interesting to consider a time and state dependent coefficient of risk aversion, instead of a constant one, in order to analyze how time and state dependent risk aversion modify the equilibrium strategies.*

In Chapter 3, we investigated equilibrium consumption-investment problem with a general discount function and a general utility function in a non-Markovian framework. Using the variational method, we characterized open loop Nash equilibrium strategies. We obtained the equilibrium solution in explicit form under some special cases of the utility function. Possible extensions of the results in the paper include several financial and actuarial applications, such as contribution and portfolio selection in pension funding (see, e.g., Josa-Fombellida and Rincón-Zapatero *[54]* and references therein).

In chapter 4, we have studied a class of dynamic decision problems of a general time-inconsistent type in the sense that the coefficients in the state equation and the cost functional involve the expected value of the solution. The control variable has two components, the first being absolutely continuous and the second is a piecewise impulse process which is not necessarily increasing. During this study near-optimal controls are constructed as an alternative of optimal controls. This has been accomplished through Ekeland's variational principle and a double perturbations approach, then a stochastic maximum principle is proved for all near-optimal controls. This result includes two approximate variational inequalities in integral form, under two adjoint processes which are backward stochastic differential equations. The work can be extended in several ways. For example,

1) The paper [19] considers a sector-wise allocation in a portfolio consisting of a very large number of stocks. Their interdependence is captured by the dependence of the drift coefficient of each stock on an averaged effect of the sectors. This leads to decoupled dynamics in the limit of large numbers. In this case, the so-called consistent mean-field approximation which represents in some sense the average behavior of the infinite number of players. Note that, the general solvability for this type of problems optimally or near-optimally, remains an outstanding open problem.

2) This approach can be extended to a mean-field game, see e.g. [37], to construct decentralized strategies and obtain an estimate of their performance.

We hope to study these problems in forthcoming papers.

Bibliography

[1] N.U. Ahmed, *Nonlinear diffusion governed by McKean–Vlasov equation on Hilbert space and optimal control*. SIAM J. Control Optim., 46, (2007), 356–378.

[2] N.U. Ahmed, X. Ding, *Controlled McKean-Vlasov equations*. Communications in Applied Analysis, 5(2), (2001), 183-206.

[3] G. Ainslie, *Special reward: A behavioral theory of impulsiveness and impulse control*. Psychological Bulletin, 82, (1975), 463–496

[4] I. Alia, F. Chighoub, A. Sohail, *The Maximum Principle in Time-Inconsistent LQ Equilibrium*. arXiv:1505.04674v1, (2015), (submitted).

[5] I. Alia, F. Chighoub, A. Sohail, *A Characterization of Equilibrium Strategies in Continuous-Time Mean-Variance Problems for Insurers*. (submitted).

[6] I. Alia, F. Chighoub, A. Sohail, *Open Loop Equilibrium Strategies in General Discounting Merton Portfolio Problem*. (Preprint).

[7] D. Andersson, B. Djehiche, *A maximum principle for SDE's of mean-field type*. Applied Math. and Optimization, 63(3), (2010), 341-356.

[8] K. Bahlali, F. Chighoub, B. Djehiche, B. Mezerdi, *Optimality necessary conditions in singular stochastic control problems with non smooth data*. J. Math. Anal. Appl., 355, (2009), 479-494.

[9] L.H. Bai, H.Y. Zhang, *Dynamic mean-variance problem with constrained risk control for the insurers*. Mathematical Methods of Operations Research, 68, (2008),181–205.

[10] I. Bajeux-Besnainou, R. Portait, *Dynamic asset allocation in a mean-variance framework*. Management Science 44, 11, (1998), 79–95.

[11] S. Basak and G. Chabakauri, *Dynamic mean-variance asset allocation.* Review of Financial Studies, 23, (2010), 2970-3016.

[12] N. Bäuerle, Benchmark, *mean-variance problems for insurers.* Mathematical Methods of Operations Research. 62, (2005), 159-165.

[13] A. Bensoussan, *Lectures on stochastic control.* In S.K. Mitter and A. Moro, editors, Lecture Notes in Mathematics: Nonlinear Filtering and Stochastic Control, Springer-Verlag, 972, (1982),1-62.

[14] A. Bensoussan , K.C.J Sung, , and S.C.P. Yam, *Linear-quadratic time-inconsistent mean field games.* Dynamic Games Appl, 3(4), (2013), 537-552.

[15] T.R. Bielecki, H. Jin, S. Pliska, X. Zhou, *Continuous-time mean-variance portfolio selection with bankruptcy prohibition.* Mathematical Finance 15(2), (2005), 213–244.

[16] J.M. Bismut, *An Introductory Approach to Duality in Optimal Stochastic Control.* SIAM Rev, 20(1), (2006), 62–78.

[17] T. Björk and A. Murgoci, *A general theory of Markovian time-inconsistent stochastic control problems*, SSRN:1694759, (2008).

[18] T. Björk, A. Murgoci, X.Y. Zhou, *Mean-variance portfolio optimization with state-dependent risk aversion.* Mathematical Finance, 24(1), (2014), 1-24.

[19] V.S. Borkar, K.S. Kumar, *McKean Vlasov limit in portfolio optimization.* Stochastic Analysis and Applications, 28(5), (2010), 884-906.

[20] M. Bossy, D. Talay, *A stochastic particle method for the McKean-Vlasov and the Burgers equation.* Mathematics of Computation, 66(217), (1997), 157-192.

[21] D.T. Breeden, *An intertemporal asset pricing model with stochastic consumption and investment opportunities.* J. Finan. Econ., 7 (1979), 265-296.

[22] S. Browne, *Optimal investment policies for a firm with random risk process: exponential utility and minimizing the probability of ruin.* Math. Metho. Oper. Res, 20(4), (1995), 937-958.

[23] R. Buckdahn, B. Djehiche, J. Li, *A General Stochastic Maximum Principle for SDEs of Mean-Field Type.* Applied Math. and Optimization, 64(2), (2011), 197-216.

[24] R. Buckdahn, B. Djehiche, J. Li, S. Peng, *Mean-Field backward stochastic differential equations: a limit approach.* The Annals of Probability, 37(4), (2009), 1524-1565.

[25] R. Buckdahn, J. Li, S. Peng, *Mean-Field backward stochastic differential equations and related partial diferential equations*. Stochastic Processes and their Applications, 119, (2009), 3133-3154.

[26] A. Cadenillas, I. Karatzas, *The Stochastic Maximum Principle for Linear Convex Optimal Control with Random Coefficients*. SIAM J. Control Optim, 33(2), (1995), 590–624.

[27] A. Cadenillas, F. Zapatero, *Classical and impulse stochastic control of the exchange rate using interest rates and reserves.* Math. Finance, 10, (2000), 141-156.

[28] Y. Cao, N. Wan, *Optimal proportional reinsurance and investment based on Hamilton–Jacobi–Bellman equation*. Insurance: Mathematics and Economics, 45, (2009), 157-162.

[29] F. Chighoub, A. Sohail, I. Alia, *Near-optimality conditions in mean-field control models involving continuous and impulse controls.* Nonlinear Studies, 22(4), (2015).

[30] F. Chighoub, B. Mezerdi, *near-optimality Conditions in Stochastic Control of Jump Diffusion Processes*. Systems & Control Letters, 60, (2011), 907-916.

[31] F. Chighoub, B. Mezerdi, *A stochastic maximum principle in mean-field optimal control problems for jump diffusions.* Arab Journal of Mathematical Sciences, 19(2), (2013), 223–241.

[32] F.H. Clarke, *Optimization and Nonsmooth Analysis*. Wiley, New York, (1983).

[33] C. Czichowsky, *Time-consistent mean-variance porftolio selection in discrete and continuous time*, arXiv:1205.4748v1, (2012).

[34] M. Dai, Z. Xu, X. Zhou, *Continuous-time Markowitz's model with transaction costs*. SIAM Journal on Financial Mathematics, 1, (2010), 96–125.

[35] M.H.A. Davis, A. Norman, *Portfolio selection with transaction costs*. Math. Oper. Res, 15, (1990), 676-713.

[36] Ł. Delong, R. Gerrard, *Mean-variance portfolio selection for a nonlife insurance company.* Mathematical Methods of Operations Research, 66, (2007), 339–367..

[37] B. Djehiche, M. Huang, *A characterization of sub-game perfect Nash equilibria for SDEs of mean-field type*. Dynamic Games and Applications, DOI:10.1007/s13235-015-0140-8, (2015).

[38] I. Ekeland, *Non convex minimization problems*. Bull. Am. Math. Soc, (N.S.) 1, (1979), 443-474.

[39] I. Ekeland, A. Lazrak, *Equilibrium policies when preferences are time-inconsistent.* arXiv:0808.3790v1, (2008).

[40] I. Ekeland, T.A. Pirvu, *Investment and consumption without commitment.* Mathematics and Financial Economics, 2, (2008), 57-86.

[41] I. Ekeland, O.Mbodji , TA.Pirvu . Time-consistent portfolio management. SIAM J Financ Math, 3, (2012), 1–32

[42] Z.F. Li, Y. Zeng, Y.Z. Lai, *Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model.* Insurance: Mathematics and Economics, 51, (2012) 191–203.

[43] R.J. Elliott, M. Kohlmann, *The variational principle and stochastic optimal control.* Stochastics, 3, (1980), 229-241.

[44] W.H. Fleming, T. Zariphopoulou, *An optimal investment/consumption model with borrowing constraints.* Math. Oper. Res., 16, (1991), 802-822

[45] S. M. Goldman, *Consistent plans.* Rev. Financial Stud, 47, (1980), 533-537.

[46] J. Grandll, *Aspects of Risk Theory.* Springer-Verlag, New York, (1991).

[47] M.D. Gu, Y.P Yang, S.D. Li, J.Y. Zhang, *Constant elasticity of variance model for proportional reinsurance and investment strategies.* Insurance: Mathematics and Economics, 46(3) (2010), 580-587.

[48] M. Hafayed, S. Abbas, P. Veverka, *On necessary and sufficient conditions for near-optimal singular stochastic controls.* Optim. Lett, 7(5), (2013), 949–966.

[49] J. Huang, X. Li, G. Wang, *Near-optimal control problems for linear forward-backward stochastic systems.* Automatica, 46(2), (2010), 397–404.

[50] E. Hui, J. Huang, X. Li, G. Wang, *Near-optimal control for stochastic recursive problems.* Syst. Control Lett, 60, (2011), 161–168.

[51] Y. Hu, H. Jin, X.Y. Zhou, *Time-inconsistent stochastic linear quadratic control.* SIAM Journal on Control and Optimization, 50(3), (2012), 1548-1572.

[52] Y. Hu, H. Jin, X.Y. Zhou,*The Uniqueness of Equilibrium for Time-Inconsistent Stochastic Linear–Quadratic Control.* arXiv:1504.01152v1, (2015).

[53] Y. Hu, B. Økendal, *Partial information linear quadratic control for jump diffusions.* SIAM Journal on Control and Optimization, 47, (2008), 1744–1761.

[54] R. Josa-Fombellida, J.P. Rincón-Zapatero, *Mean-variance portfolio and contribution selection in stochastic pension funding*. European Journal of Operational Research ,187, (2008), 120-137.

[55] I. Karatzas, J. Lehoczky, S.E. Shreve, *Optimal portfolio and consumption decisions for a "small investor" on a finite horizon*. SIAM J. Control & Optim., 25, (1987), 1157-1186.

[56] L. Karp, *Non-constant discounting in continuous time*. Journal of Economic Theory 132, (2007), 557-568.

[57] R. Korn, *Some appliations of impulse control in mathematical finance*. Math. Meth. Oper. Res, 50, (1999), 493-518.

[58] P. Krusell, A. Smith, *Consumption and savings decisions with quasi-geometric discounting*. Econometrica, 71, (2003), 366–375.

[59] F.E. Kydland, E. Prescott, *Rules rather than discretion: The inconsistency of optimal plans*. Journal of Political Economy, 85, (1997), 473–492.

[60] J.M. Lasry, P.L. Lions, *Mean-field games*. Japanese Journal of Mathematics, 2, (2007), 229–260.

[61] E. B. A. Lim, *Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market*. Mathematics of Opera- tions Research, 29, (2004), 132–161.

[62] A. E. B. Lim, X. Zhou, *Quadratic hedging and mean-variance portfolio selection with random parameters in a complete market*. Mathematics of Operations Research 27, 1 (2002), 101–120.

[63] J. Li, *Stochastic maximum principle in the mean-field controls*. Automatica, 48, (2012), 366–373.

[64] D. Li, and W.Ng, *Optimal dynamic portfolio selection: Multi-period mean-variance formulation*. Mathematical Finance, 10, (2000), 387–406.

[65] Y. Li , and Z. Li, *Optimal time-consistent investment and reinsurance strategies for mean–variance insurers with state dependent risk aversion*. Insurance: Mathematics and Economics, 53(1), (2013), 86-97.

[66] G. Loewenstein, D. Prelec, *Anomalies in intertemporal choice: Evidence and an interpretation*. The Quarterly Journal of Economics, 107(2), (1992), 573–597.

[67] H.M. Markowitz, *Portfolio selection*. Jornal of Finance, 7, (1952), 77–91.

[68] Q. Meng, *General linear quadratic optimal stochastic control problem driven by a Brownian motion and a Poisson random martingale measure with random coefficients*. Stochastic Analysis and Applications, DOI: 10.1080/07362994.2013.845106, (2013).

[69] R.C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time case*. Rev. Econ. Statist, 51, (1969),247–257.

[70] R.C. Merton, *Optimum consumption and portfolio rules in a continuous-time model*. J. Econ. Theory, 3, (1971),373–413.

[71] T. Meyer-Brandis, B. Øksendal, X.Y. Zhou, *A mean-field stochastic maximum principle via Malliavin calculus*. Stochastics (A Special Issue for Mark Davis' Festschrift), 84, (2012), 643–666.

[72] B.M. Miller, E.Y. Rubinovich, *Impulsive Control in Continuous and Discrete-Continuous Systems.* Kluwer Academic/Plenum Publishers, New York, (2003).

[73] B. ∅ksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions, Second Edition*. Springer, (2007).

[74] B. ∅ksendal, A. Sulem, *Optimal consumption and portfolio with both fixed and proportional transaction costs*. SIAM J. Control Optim, 40, (2002), 1765-1790.

[75] C. Orrieri, *A stochastic maximum principle with dissipativity conditions*. Disc. Cont. Dyn. Sist. A, 35, (2015), 5499-5519.

[76] E. Pardoux, S. Peng, *Adapted solution of a backward stochastic differential equation*. Systems Control Lett, 14(1-2), (1990), 61-74.

[77] S. Peng, *A general stochastic maximum principle for optimal control problems*. SIAM J Control Optim, 28, (1990), 966-979.

[78] E.S. Phelps and R.A. Pollak, *On·second-best national saving and game-equilibrium growth.* Review of Economic Studies, 35**,** (1968), 185-199.

[79] S. Pliska, *A stochastic calculus model of continuous trading: Optimal portfolios*, Math. Oper. Res., 11, (1986), 371-382.

[80] R. Pollak, *Consistent planning*, Rev. Financial Stud, 35, (1968), 185–199.

[81] F.P. Ramsey, *A Mathematical Theory of Saving*. Economic Journal, 38, (1928),543-559.

[82] S. Richard, *Optimal consumption, portfolio, and life insurance rules for an uncertain lived individual in a continuous time model*. J. Finance Econ., 2, (1975), 187-203.

[83] H.R. Richardson, *A minimum variance result in continuous trading portfolio optimization*. Management Science, 35, 9, (1989), 1045–1055.

[84] P. A. Samuelson, *An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money*. Journal of Political Economy, 66, (1958), 467-482.

[85] S. Sethi, X.Y. Zhou, *Asymptotic optimal feedback controls in stochastic dynamic two-machine flowshops*. Lecture Notes in Control and Infor. Sci. 214, G. Yin and Q. Zhang, eds., Springer-Verlag, New York, (1996), 147-180.

[86] Y. Shen, T. Siu, *The maximum principle for a jump-diffusion mean-field model and its application to the mean-variance problem*. Nonlinear Analysis: Theory, Methods & Applications, 86, (2013), 58-73.

[87] J.M. Solano, J. Navas, Consumption and portfolio rules for time-inconsistent investors, European J. Operational Research, 201, (2010), 860-872.

[88] R. Strotz, *Myopia and inconsistency in dynamic utility maximization*. Review of Economic Studies, 23, (1955), 165–180.

[89] A.S. Sznitman, *Topics in propagation of chaos*, Ecole de Probabilités de Saint Flour. Lecture Notes in Math. Springer-Verlag, Berlin, 1464, (1989), 165-251.

[90] S. Tang and X. Li, *Necessary conditions for optimal control for stochastic systems with random jumps*. SIAM J. Control Optim, 32**,** (1994), 1447-1475.

[91] H.L. Tsai and J.M. Lin, *Optimal control of fuel processing system using generalized linear quadratic Gaussian and loop transfer recovery method.* Asian J. Control, 12(5), (2010) 597–606.

[92] Z. Wang, J. Xia, L. Zhang, *Optimal investment for an insurer: the martingale approach.* Insur. Math. Econ, 40**,** (2007), 322–334.

[93] N. Wang , *Optimal investment for an insurer with utility preference.* Insur. Math. Econ, 40**,** (2007), 77–84.

[94] H. Wang, Z. Wu, *Partially Observed Time-Inconsistency Recursive Optimization Problem and Application*. Journal of Optimization Theory and Applications, 161(2), (2014) ,664-687.

[95] H. Wang, Z. Wu, *Time-inconsistent optimal control problem with random coefficients and stochastic equilibrium HJB equation*. Mathematical Control & Related Fields, 5(3), (2015), 651-678.

[96] Z. Wu, X. Wang, *FBSDE with Poisson process and its application to linear quadratic stochastic optimal control problem with random jumps*. Acta Automatica Sinica, 29, (2003), 821–826.

[97] Z. Wu, F. Zhang, *Stochastic maximum principle for optimal control problems of forward-backward systems involving impulse controls*. IEEE Transactions on Automatic Control, 56, (2011), 1401-1406.

[98] J. M. Xia, *Mean-variance portfolio choice: Quadratic partial hedging*. Mathematical Finance 15(3), (2005), 533–538.

[99] L. Xu, R.M. Wang, D.J. Yao, *On maximizing the expected terminal utility by investment and reinsurance*. Journal of Industrial and Management Optimization, 4(4), (2008), 801–815.

[100] H. Yang, L. Zhang, *Optimal investment for insurer with jump-diffusion risk process*. Insurance: Mathematics and Economics, 37, (2005), 615–634.

[101] J. Yong, *A deterministic linear quadratic time-inconsistent optimal control problem.* Math. Control Related Fields, 1, (2011), 83-118.

[102] J. Yong, *Linear quadratic optimal control problems for mean-field stochastic differential equations: Time-consistent solutions.* arXiv:1304.3964, (2013).

[103] J. Yong, *Time-inconsistent optimal control problems and the equilibrium HJB equation,* Math. Control Related Fields, 2(3), (2012), 271-329.

[104] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*. SIAM J. Control Optim., 51(4) (2013),2809-2838.

[105] J. Yong, X.Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York. (1999)

[106] Z. Yu, *The stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls*. Automatica, 48(10), (2012), 2420-2432.

[107] Q. Zhaoa, Y. Shenc, J. Weib, Consumption-investment strategies with non-exponential discounting and logarithmic utility. European Journal of Operational Research, 283(3), (2014), 824-835.

[108] Y. Zeng, Z.F. Li, *Optimal reinsurance–investment strategies for insurers under mean-CaR criteria.* Journal of Industrial and Management Optimization, 8(3), (2012), 673–690.

[109] Y. Zeng, Z.F. Li, J.J. Liu, *Optimal strategies of benchmark and mean–variance portfolio selection problems for insurers.* Journal of Industrial and Management Optimization, 6, (2011), 483–496.

[110] Y. Zeng, Z.F. Li, *Optimal time-consistent investment and reinsurance policies for mean–variance insurers.* Insurance: Mathematics and Economics, 49, (2011), 145–154.

[111] Y. Zeng, Z.F. Li, Y.Z. Lai, *Time-consistent investment and reinsurance strategies for mean–variance insurers with jumps.* Insurance: Mathematics and Economics, 52, (2013), 498-507.

[112] X.Y. Zhou, D. Li, *Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework.* Appl. Math. Optim, 42, (2000), 19-33.

[113] X.Y. Zhou, S. Sethi, *A sufficient condition for near-optimal stochastic controls and its applications to manufacturing systems.* Appl. Math. Optim., 29, (1994), 67-92.

[114] X.Y. Zhou, *Stochastic near-optimal controls: necessary and sufficient conditions for near-optimality.* SIAM J. Control Optim., 36(3), (1998), 929-947.

[115] X.Y. Zhou, *Deterministic near-optimal controls, part I: Necessary and suffcient conditions for near-optimality.* J. Optim. Theory Appl., 85, (1995), 473-488