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Title

Existence and asymptotic behavior for some hyperbolic systems

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Abstract

This thesis is devoted to study the Existence and asymptotic behavior for some hyperbolic systems . The first part of the thesis is composed of two chapters 2 and 3. We studied a one-dimensional linear thermoelastic system of Timoshenko type, where the heat flux is given by Cattaneo's law, noting that in the chapter 3 we have introduced a delay term in the feedback and forcing term.

We established several exponential decay results for classical and weak solutions in one-dimensional. Our technics of proof is based on the construction of the appropriate Lyapunov function equivalent to the energy of the considered solution, and which satisfies a differential inequality leading to the desired decay.

In chapter 4, we consider a system of nonlinear wave equation with degenerate damping and strong nonlinear source terms. We prove that the solution blows up in time.

Keywords: Nonlinear damping, Strong damping, Viscoelasticity, Nonlinear source, Locale solutions, Global solution, Exponential decay, Polynomial decay, Blow up.

Résumé

Cette thèse est consacrée á l'étude de l'existence et le comportement asymptotique pour certains systèmes hyperboliques. La première partie de la thèse est composée de deux chapitre 2 et 3. Les deux sont consacrés en premier lieu á l'étude du systme thermo-élastique linéaire en dimension un de type Timoshenko, dans lequel le flux de chaleur est donné par la loi de Cattanéo. noton ici que l'introduction du term de retard et le terme de la force extrieure ne concerne que le chapitre 3 .

Et on montre l'existence, ainci que la stabilité exponentielle de la solution. La preuve que nous avons établie est basées sur la construction d'une fonction de Lyapunov appropriée et équivalente á l'énergie de la solution considérée. Cette fonction vérifie une inequation différentielle menant au résultat de la décroisence désirée

Ensuite, et dans le chapitre 4, on considére un systme d'équations des ondes avec termes dissipatifs afin de prouver que la solution de ce système explose en temps fini.

Mots clés : Dissipation nonlinéaire, dissipation forte, viscoélasticité, source non linéaire, solution locales, solution globales, décroissance exponentielle, décroissance polynômiale, explosion en temps fini.

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Contents

Introduction

Our work, in this thesis, lies in the study some hyperbolic systems with the presence of different mechanisms of damping, and under assumptions on initial data and boundary conditions. Our goal is to establish the existence of the solutions and a general decay estimate using the energy methode. In fact, we prove that under some assumptions on the parameters in the systems and on the size of the initial data, the solutions can be proved to be either *global* in time or may *blow up* in finite time. If the solution are global in time, then the natural question is about their convergence to the steady state and the rate of the convergence. The system that we treated here are the following:

The Timoshenko systems

In 1921, Timoshenko [92] gave the following system of coupled hyperbolic equations

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{1}$$

wich describe the transverse vibration of a beam of length L in its equilibrium configuration. where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, φ is the transverse displacement of the beam and ψ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.
System (1), together with boundary conditions of the form

$$EI\varphi_x \mid_{x=0}^{x=L} = 0, \qquad K(u_x - \varphi) \mid_{x=0}^{x=L} = 0$$

is conservative, and thus the total energy is preserved, as time goes to infinity. Several authors introduced different types of dissipative mechanisms to stabilize system

(1), and several results concerning uniform and asymptotic decay of energy have been established.

Kim and Renardy [39] considered (1) together with two boundary controls of the form

$$\begin{cases} K\psi(L,t) - K\varphi_x(L,t) = \alpha\varphi_t(L,t), & \forall t \geq 0 \\ EI\psi_x(L,t) = -\beta\varphi_t(L,t), & \forall t \geq 0 \end{cases}$$

and used the multiplier techniques to establish an exponential decay result for the total energy of (1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1). Raposo *et al.* [81] treated the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x - \psi)_x + \varphi_t = 0, & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x - \psi) + \psi_t = 0, & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{2}$$

with homogeneous Dirichlet boundary conditions and two linear frictional dampings, and proved that the associated energy decays exponentially. Soufyane and Wehbe [89] showed that it is possible to stabilize uniformly (1) by using a unique locally distributed feedback. They considered

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi) - b\psi_t, & \text{in } (0,L)\times(0,+\infty) \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \forall t > 0, \end{cases} \tag{3}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (3) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. Also, Muñoz Rivera and Racke [65] studied a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Concerning the Timoshenko system with viscoelastic damping, Ammar-Khodja et *al.* [5] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{4}$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate.

Messaoudi and Mustafa [50] improved the results of [5] and [27] by allowing more general decaying relaxation functions and showed that the rate of decay of the solution energy is exactly the rate of decay of the relaxation function. Also, Muñoz Rivera and Fernndez Sare [69], considered Timoshenko type system with past history acting only in one equation. More precisely they studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^\infty g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0, \end{cases} \tag{5}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work was improved recently by Messaoudi and Said-Houari [48], where the authors considered system (5) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function weaker than those in [69]. The case of g having a general decay has been studied in [30–32] for Timoshenko-type and [29, 33] for abstract systems, where a general relation between the growth of g at infinity and the decay rate of solutions is explicitly found in terms of the growths at infinity.

Messaoudi *et al.* [53] studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases}$$

where $(x, t) \in (0, L) \times (0, \infty)$ and $\varphi = \varphi(t, x)$ is the displacement vector, $\psi = \psi(t, x)$ is the rotation angle of the filament, $\theta = \theta(t, x)$ is the temperature difference, $q = q(t, x)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = k$$

and

$$\sigma_{\varphi_x\varphi_x}(0, 0) = \sigma_{\varphi_x\psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

Concerning the Timoshenko system with delay, the investigation started with the paper [82] where the authors studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt}(x, t) - K(\varphi_x + \psi)_x(x, t) = 0, \\ \rho_2\psi_{tt}(x, t) - b\psi_{xx}(x, t) + K(\varphi_x + \psi)(x, t) + \mu_1\psi_t(x, t) + \mu_2\psi_t(x, t - \tau) = 0. \end{cases} \tag{6}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{7}$$

Subsequently, the work in [82] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari and Anwar [40]. The case where the damping $\mu_1\psi_t$ is replaced by $(\int_0^\infty g(s)\psi_{xx}(t - s)\,ds)$ (with either discrete delay $\mu_2\psi_t(t - \tau)$ or distributed one $\int_0^\infty f(s)\psi_t(t - s)\,ds$) has been treated in [32] (in case (7) and the opposite one), where several general decay estimates have been proved.

Our main results in this part can be summarized as follows:

Chapter 2. In this chapter we studied a one-dimensional linear thermoelastic system of Timoshenko type, where the heat flux is given by Cattaneo's law, see for example [53]. We consider damping terms acting on the second equation and we establish a general decay estimate without the usual assumption of the wave speeds,with theintroductio of damping term $\mu\varphi_t$ in the first equation see [74].Also, the results obtained in [74] has been improved without $\mu\varphi_t$. Our method of proof uses the energy method together with some properties of convex functions. The advantage here is that from our general estimates we can derive the exponential, polynomial or logarithmic decay rate. We also give some examples to illustrate our result.This work has been recently published in [74].

Chapter 3. In this chapter we consider a one-dimonsional linear thermoelastic system of Timoshenko type with delay term in the feedback. The heat conduction in given by Cattaneo's law. Under an appropriate assumption between the weight of the delay and the weight of the damping, we proved a well-posedness result. Furthermore an exponential stability result has been shown without the usual assumption on the wave speeds. To achieve our goals, we made use of the semigroup method and the energy method.

The damped wave equation (Blow up)

The study of the interaction between the source term and the damping term in the wave equation

$$u_{tt} - \Delta u + a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u, \text{ in } \Omega \times (0, T), \tag{8}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history.

It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters m, p and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $\left|u\right|^{p-2} u$ then a uniform estimate of the form

$$\|u_t(t)\|_2 + \|\nabla u(t)\|_2 \leq C, \tag{9}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1(\Omega) \times L^2(\Omega)$, where C is a positive constant independent of t. The estimate (9) shows that any local solution u of problem (8) can be continued in time as long as (9) is verified. This result has been proved by several authors. See for example [34, 38]. On the other hand in the absence of the damping term $|u_t|^{m-2} u_t$, the solution of (8) ceases to exist and there exists a finite value T^* such that

$$\lim_{t\to T^*} \|u(t)\|_p = +\infty, \tag{10}$$

the reader is refereed to Ball [8] and Kalantarov & Ladyzhenskaya [38] for more details.

When both terms are present in equation (8), the situation is more delicate. This case has been considered by Levine in [43, 44], where he investigated problem (8) in the linear damping case ($m = 2$) and showed that any local solution u of (8) cannot be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [43] and [44] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim_{t\to T^*} \theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [22] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (4.1) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function L which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{11}$$

In $[0, \infty)$, where $\nu > 0$. Inequality (11) leads to a blow up of the solutions in finite tim $t \geq L(0)^{-\nu} \xi^{-1} \nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [22] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [22] (see for example [42, 45, 62, 93]. In particular, Vitillaro in [93] combined the arguments

in [22] and [42] to extend the result in [22] to situations where the damping is nonlinear and the solution has positive initial energy.
In [95], Young, studied the problem

$$u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) + a\,|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (12)$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [95][Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$([95][Remark 2]).

Messaoudi and Said-Houari [60] improved the result in [95] and showed that the blow up of solutions of problem (12) takes place for negative initial data only regardless of the size of Ω.

To the best of our knowledge, the system of wave equations is not well studied, and only few results are available in literature. Let us mention some of them. Milla Miranda and Medeiros [63] considered the following system

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{\rho+2}|u|^{\rho}u = f_1(x) \\ v_{tt} - \Delta v + v - |u|^{\rho+2}|v|^{\rho}v = f_2(x), \end{cases} \quad (13)$$

in $\Omega\times(0,T)$. By using the method of potential well, the authors determined the existence of weak solutions of system (13). Some special cases of system (13) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field. See [87] and [36]. Agre and Rammaha [3] studied the system

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v), \end{cases} \quad (14)$$

in $\Omega\times(0,T)$ with initial and boundary conditions of Dirichlet type and the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ satisfying

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \quad (15)$$

They proved, under some appropriate conditions on $f_1(u;v)$, $f_1(u;v)$ and the initial data, several results on local and global existence, but no rate of decay has been discussed. They also showed that any weak solution with negative initial energy blows up in finite time, using the same techniques as in [22]. Recently, the blow up result in [3] has been improved by Said-Houari [83] by considering certain class of initial data with positive initial energy. Subsequently, the paper [83] has been followed by [85], where the author proved that if the initial data are small enough, then the solution of (14) is global and decays with an exponential rate if $m = r = 1$ and with a polynomial rate like $t^{-2/(\max(m,r)-1)}$ if $\max(m, r) > 1$. Several authors and many results appeared in the literature see for example [[9],[75]]

Chapter 4. In this chapter, we consider the following system of wave equations

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2} \nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2} \nabla u_t\right) + a_1 |u_t|^{m-2} u_t & = f_1(u, v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2} \nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2} \nabla v_t\right) + a_2 |v_t|^{r-2} v_t & = f_2(u, v), \end{cases} \tag{16}$$

where the functions $f_1(u, v)$ and $f_2(u, v)$ satisfying (15). In (16), $u = u(t, x)$, $v = v(t, x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and $a_1, a_2, b_1, b_2 > 0$ and $\beta_1, \beta_2, m, r \geq 2$, $\alpha > 2$. System (16) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), & x \in \Omega \\ u(x) = v(x) = 0 & x \in \partial\Omega, \end{cases} \tag{17}$$

Our main interest in this chapter is to prove a global nonexistence result of solutions of system (16) - (17) for large initial data. We use the method in [83] with the necessary modification imposed by the nature of our problem. The core of this method relies on the use of an auxiliary function L in order to obtain a differential inequality of the form (11) which leads to the desired result. This work has been recently published in [80].

Chapter 1

Preliminaries

In this chapter, we recall some basic knowledge in fonctional analysis, moste of wich will be used in the subseqent chapter. The reader can easily find the detailed in the related literature, see, e.g.,[2], [12], [77], [96]

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is bonded smoth domain,$C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω

Definition 1.1. Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $||.|| : X \longrightarrow \mathbb{R}$, such that :
a) $||x|| \geqslant 0$ for all $x \in X$,
b) $||\alpha x|| = |\alpha|||x||$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
c) $||x + y|| \leqslant ||x|| + ||y||$ for all $x, y \in X$.
A norm on X is a semi-norm wich also satisfies :
d) $||x|| = 0 \Rightarrow x = 0$. A vector space X toghether with a norm $||.||$ is called a normed linear space, a normed vector space or simply, a normed space.

Definition 1.2. (Convergent and Cauchy sequences). Let X be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

a) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} ||x_n - x|| = 0,$$

i.e. if

$$\forall \varepsilon > 0; \exists N > 0, \forall n \geq N, ||x_n - x|| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall \varepsilon > 0; \exists N > 0, \forall m, n \geq N, ||x_m - x_n|| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

Definition 1.3. (Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space X' is the linear space of all contnuous linear functional $f : X \to \mathbb{R}$.

Proposition 1.4. *X' equipped with the norm $||.||_{X'}$ defined by*

$$||f||_{X'} = \sup\{|f(u)| : ||u|| \leq 1\}$$

is olso a Banach space.

Remark 1.5. From X' we construct the bidual or secend dual $X'' = (X')'$. Futhermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$, this satisfies clearly $||\varphi(x)|| \leq ||u||$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = ||u||$ and $||f|| = 1$, so it follows that $||\varphi(x)|| = ||u||$.

Definition 1.6. Since φ is linear we see that

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.7. if φ (in the above definition) is onto X'' we say X is reflexive, $X \cong X''$

1.1.1 The weak and weak star topologies:

Let X be a Banach space and $f \in X'$. Denot by

$$\begin{aligned} \varphi_f : & X \to \mathbb{R} \\ & x \mapsto \varphi_f \end{aligned}$$

When f cover X', we obtain a family $(\varphi_f)_{f \in X'}$ of appmications to X in $\mathbb{R}$.

Definition 1.8. The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f \in X'}$ is continuous.

We will define the topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : & X' \to \mathbb{R} \\ & f \mapsto \varphi_x(f) = \langle f, x \rangle_{X', X} \end{aligned}$$

Definition 1.9. The weak star topology on X' is the weakest topology on X' for wich every $(\varphi_x)_{x \in X'}$ is continuous.

Remark 1.10. Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.

Definition 1.11. A sequence (x_n) in X is weakly convergent to x if and only if

$$\lim_{n \to \infty} f(x_n) = f(x)$$

for evry $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

Remark 1.12. :
1.If the weak limit exist, it is unique.
2.If $x_n \to x \in X$(strongly), then $x_n \rightharpoonup x$ (weakly).
3.If $\dim X < \infty$, then the weak convergent implise the strong convergent.

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyse, known as Hilbert spaces. Then, we most give some impotant result on these spaces here.

Definition 1.13. A Hilbert space $H$is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $||u|| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.

Theorem 1.14. *Let $(x_n)_{n\in\mathbb{N}}$ is a bounded squence in the Hilbrt sce H, then it possess a subsequence which converges in the weak topology of H.*

Theorem 1.15. *In the Hilbrt space, all sequence which converges in the weak topology is bounded.*

Theorem 1.16. *Let $(x_n)_{n\in\mathbb{N}}$ be sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other squence which converge weakly to y, then*

$$\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 1.17. *Let X and Y be tow Hilbert space, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which conveges weakly to $x \in X$, let $A \in \mathcal{L}(X, Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

Theorem 1.18. *(The Lax-Milgram Theorem)*
Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Asume that there existe tow constants $C < \infty, \alpha > 0$ such that:
(i) $|a(u, v)| \leq C||u||.||v||$ *for all* $(u, v) \in X \times X$ *(continuity);*
(ii) $a(u, u) \geq \alpha||u||^2$ *for all* $u \in X$ *(coerciveness).*
Then, for evry $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in X$.

1.1.3 The $L^p(\Omega)$ spaces

Definition 1.19. Let $1 \leq p \leq \infty$, and let Ω be on open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \;\; is \;\; measurable \;\; and \int_\Omega |f(x)|^p dx < \infty \right\}$$

Notation 1 : for $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

. If $p = \infty$, we have

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : f \ \ is \ \ measurable \ \ and \ \ there \ \ exists \ \ C \ \ suche \ \ that, |f(x)| \leq C \ \ in \ \ \Omega\}$$

Notation 2 : Let $1 \leq p \leq \infty$,we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

Theorem 1.20. *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all* $1 \leq p \leq \infty$

Remark 1.21. In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) dx,$$

is a Hilbert space .

Theorem 1.22. *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

1.1.4 The Sobolev space $W^{m,p}(\Omega)$

Definition 1.23.

i) Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega), such \ \ that \ \ \partial^\alpha f \in L^p(\Omega) for \ \ all \ \ \alpha \in \mathbb{N}^m\}$$

such that $|\alpha| = \sum_{j=1}^n \alpha_j \leq m \ \ where, \ \ \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .... \partial_n^{\alpha_n}.$
ii) if $f \in W^{m,p}(\Omega)$, we define its norm to be

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} (\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p dx)^{\frac{1}{p}} & ; (1 \leq p < \infty), \\ \sum_{|\alpha|<k} ess \sup |D^\alpha f| & ; (p = \infty) \end{cases}$$

Definition 1.24. We denote by

$$W_0^{m,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$

Remark 1.25. i) if $p = 2$ we usully write

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \le m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}$$

The letter H is used, since - as we will see - $H^m(\Omega)$ is a Hilber space.
with usual scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Note that $H^0(\Omega) = L^2(\Omega)$

Theorem 1.26. .
1. $H^m(\Omega)$ supplied with inner product $\langle ., . \rangle_{H^m(\Omega)}$ is Hilbert space.
2. If $m \ge m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.

Theorem 1.27. *Assume that Ω is an open domain in $\mathbb{R}^n, n \ge 1$, with smooth boundary Γ. Then,*
i) if $1 \le p \le n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p^*]$, where $p^* = \dfrac{np}{n-p}$.
ii) if $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.
iii) if $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \dfrac{p-n}{p}$.

1.1.5 The $L^p(0, T, X)$ space

Definition 1.28. Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions

$$f :]0, T[\to X$$
$$t \mapsto f(t)$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \le p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \|f(t)\|_X$$

Theorem 1.29. *$L^p(0,T,X)$ equipped with the norm $||.||_{L^p(0,T,X)}$ is a Banach space .*

Proposition 1.30. *Let X be a reflexive Banach space, X' it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$*

1.2 Some useful inequalities

In this section, we shall recall some inqualities which will be used in the supsequent chapters.

1.2.1 Young inequalities

Theorem 1.31. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b > 0$$

Theorem 1.32. *(Young inequality with ε) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q}, a, b > 0$$

The Young inequality has several variants in the following.

Corollary 1.33. *Let $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then*
i) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \quad , \forall \varepsilon > 0.$
iii) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \quad , 0 < \alpha < 1.$

1.2.2 The Holder inequalities

Theorem 1.34. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,*
then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$||fg||_{L^1(\Omega)} \leq ||f||_{L^p(\Omega)} . ||g||_{L^q(\Omega)}$$

Theorem 1.35. *(Generalized Holder inequality) Let* $1 \le p_1, ..., p_m \le \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$, *we have*

$$\int_\Omega |f_1....f_m| dx \le \prod_{k=1}^{m} ||f_k||_{L^{p_k}(\Omega)}$$

Remark 1.36. We have the corresponding weighted Holder inequality of the integral form. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$,$\omega(x) > 0$ on Ω. Then

$$\int_\Omega |fg|\omega(x)dx \le \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \omega(x)dx\right)^{\frac{1}{q}}.$$

1.2.3 The Minkowski inequality

Theorem 1.37. *Assume* $1 \le p \le \infty$, $f, g \in L^p(\Omega)$, *then*

$$||f + g||_{L^p(\Omega)} \le ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}.$$

If $0 \ne p < 1$, *then*

$$||f + g||_{L^p(\Omega)} \ge ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}.$$

In the applications, the integral form from the Minkowski inequality is used frequentely.

1.2.4 The Poincar inequality

In this subsection, we shall recall the Poincar inequality in different forms.

Theorem 1.38. . *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant C such that*

$$||f||_{L^2(\Omega)} \le C||\nabla f||_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega)$$

Theorem 1.39. *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant C , such that for any* $f \in H^1(\Omega)$.

$$||f - \tilde{f}||_{L^2(\Omega)} \le C||\nabla f||_{L^2(\Omega)}$$

Where $\tilde{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ *is the integral average of f over* Ω, *and* $|\Omega|$ *is the volume fo* Ω.

Theorem 1.40. *Under assumption of Theorem* (1.39) *for any* $f \in H^1(\Omega)$*, we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + |\int_\Omega f dx|\right).$$

1.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. Ageneral rference to this topic is [12], [77],

1.3.1 C_0−Semigroups of Linear Operators

Definition 1.41. (Semigroups)
Let X be a Banach space, the one-parametre family $S(t), 0 \leq t < \infty$ from X to X is called a Semigroups if
(i) $S(0) = I$ (I is the identity operateur on X), (ii) $S(t + s) = S(t) + S(s)$ for every $t, s \geq 0$ (the Semigroup property).

Definition 1.42. The linear operator A defined by

$$D(A) = \left\{x \in X : \lim_{t\to 0^+}(S(t)x - x)/t \quad exists\right\}$$

and

$$Ax = \lim_{t\to 0^+}(S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \quad all \quad x \in D(A)$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(A)$ is called the domain of A.

Definition 1.43. (C_0−Semigroups).
A Semigroup $S(t)$,$0 \leq t < \infty$,from X to X is called a strong continuous Semigroup of bounded lineaar operators if

$$\lim_{t\to 0^+} S(t)x = x \quad for \quad all \quad x \in X,$$

or

$$\lim_{t\to 0^+} \|S(t)x - x\| = 0 \quad for \quad all \quad x \in X.$$

i.e $S(t)$ C_0−Semigroup.

Definition 1.44. A semigroup $S(t)$,$0 \leq t < \infty$ is called a semigroup of contraction if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t > 0$,

$$||S(t)x - S(t)y|| \leq \alpha||x - y||, \quad for \ \ all \quad x, y \in X.$$

1.3.2 Hille-Yoshida Theorem

Definition 1.45. An unbonded linear operator $A : D(A) \subset H \to H$[1] is said to be monotone[2] if it satisfies

$$\langle Av, v\rangle \geq 0 \qquad \forall v \in D(A).$$

It is called maximal monotone if, in addition; $R(I + A) = H$ i.e

$$\forall f \in H \ \ \exists u \in D(A) \quad such \ \ that \ \ u + Au = f.$$

Proposition 1.46. *Let A be a maximal monotone operator. Then*
1. $D(A)$ is dense in H.
2. A is closed operator.
3. For evry $\lambda > 0, (I + \lambda A)$ is bijective from $D(A)$ onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and $||(I + \lambda A)^{-1}||_{\mathcal{L}(H)} \leq 1$.

Theorem 1.47. *(Hille-Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad on[0, +\infty) \\ u(0) = u_0. \end{cases}$$

[1] H denotes a Hilbert space
[2] Some authors say that A is accretive or $-A$ is dissipative.

Chapter 2

Stability of a thermo-elastic Timoshenko Beam system of second sound

2.1 Introduction

Here, the long-term behavior of solutions to the following system is investigating:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t = 0 \\ \rho_2\psi_{tt} - \bar{b}\psi_{xx} + \int_0^t g(t-s)(a(x)\psi_x(s))_x\, ds + k(\varphi_x + \psi) + b(x)h(\psi_t) + \gamma\theta_x = 0 \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0. \end{cases} \tag{2.1}$$

where $t \in (0,\infty), x \in (0,1)$, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, the function θ is the temperature difference, $q = q(t,x) \in \mathbb{R}$ is the heat flux, and $\rho_1, \rho_2, \rho_3, \gamma, \tau_0, \delta, \kappa, \bar{b}$ and k are a positive constants and the following are initial conditions:

$$\begin{aligned} \varphi(.,0) &= \varphi_0(x),\ \varphi_t(.,0) = \varphi_1(x),\ \psi(.,0) = \psi_0(x) \\ \psi_t(.,0) &= \psi_1(x),\ \theta(.,0) = \theta_0(x),\ q(.,0) = q_0(x), \end{aligned} \tag{2.2}$$

and boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \ \forall t \geq 0. \tag{2.3}$$

Timoshenko in 1921, proposed the following system of coupled hyperbolic equations

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0,L) \times (0,+\infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in } (0,L) \times (0,+\infty), \end{cases} \tag{2.4}$$

which describes the transverse vibration of a beam of length L in its equilibrium configuration. Where t denotes the time variable, x is the space variable along the beam, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

System (2.4), together with boundary conditions of the form

$$EI\varphi_x \mid_{x=0}^{x=L} = 0, \qquad K(u_x - \varphi) \mid_{x=0}^{x=L} = 0$$

is conservative, and so the total energy of the beam remains constant along the time.

The subject of stability of Timoshenko-type systems has received a lot of attention in the last years and several outstanding results have been proved by some of the major experts in the fields of partial deferential equations, and several results concerning uniform and asymptotic decay of energy have been established.

An important issue of research is to look for a minimum dissipation by which solutions of system (2.4) decay uniformly to the stable state as time goes to infinity.

Kim and Renardy [39] considered (2.4) together with two boundary controls of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t), \quad \forall t \geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t), \quad \forall t \geq 0 \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (2.4). They also provided numerical estimates to the eigenvalues of the operator associated with system (2.4)). Raposo *et al.* [81] studied the following system

$$\begin{cases} \rho_1 u_{tt} - K(u_x - \varphi)_x + u_t = 0, & \text{in } (0, L) \times (0, +\infty) \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t = 0, & \text{in } (0, L) \times (0, +\infty) \end{cases} \tag{2.5}$$

with homogeneous Dirichlet boundary conditions, and proved that the associated energy decays exponentially. Soufyane and Wehbe [89] showed that it is possible to stabilize uniformly (2.4) by using a unique locally distributed feedback. They considered

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0, L) \times (0, +\infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, & \text{in } (0, L) \times (0, +\infty) \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0, \quad \forall t > 0, \end{cases} \tag{2.6}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\ x\ \in\ [a_0, a_1] \subset [0, L]$$

and proved that the uniform stability of (2.6) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. Recently, Muñoz Rivera and Racke [68] obtained a similar result in a work where the damping function $b = b(x)$ is allowed to change its sign. Also, Muñoz Rivera and Racke [65] treated a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2 \psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Ammar-Khodja *et al.* [5] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{2.7}$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system (2.7) is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [27] proved the same result without imposing the extra technical conditions of [5]. Recently, Messaoudi and Mustafa [50] have improved the results of [5] and [27] by allowing more general decaying relaxation functions and showed that the rate of decay of the solution energy is exactly the rate of decay of the relaxation function.

Also, Muñoz Rivera and Fernández Sare [69], considered Timoshenko type system with past history acting only in one equation. More precisely they looked into the following problem

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^{\infty} g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{2.8}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work was improved recently by Messaoudi and Said-Houari [48], where the authors considered system (2.8) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function weaker than those in [69].

Concerning the Timoshenko systems in thermo-elasticity, Rivera and Racke [64] considered

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 & \text{in } (0, L) \times (0, +\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 & \text{in } (0, L) \times (0, +\infty) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 & \text{in } (0, L) \times (0, +\infty) \end{cases} \tag{2.9}$$

where φ, ψ and θ are functions of (x, t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

Modeling heat conduction with the so-called Fourier law (as in (2.9)), which assumes the flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q + \kappa\nabla\theta, \qquad (\kappa > 0),$$

leads to the phenomenon of infinite heat propagation speed. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. To overcome this physical problem, a number of modification of the basic assumption on the relation between the heat flux and the temperature have been made. The common feature of these theories is that all lead to hyperbolic differential equation and the speed of propagation becomes limited. See [16] for more details. Among them Cattaneo's law,

$$\tau q_t + q + \kappa\nabla\theta = 0, \qquad (\tau > 0, \text{ relatively small}),$$

leading to the system with *second sound*, ([90], [78], [52]) and a suggestion by Green and Naghdi [24], [26], for thermo-elastic systems introducing what is called thermo-elasticity of type III, where the constitutive equations for the heat flux is characterized by

$$q + \kappa^* p_x + \tilde{\kappa}\nabla\theta = 0, \qquad (\tilde{\kappa} > \kappa > 0, \ p_t = \theta).$$

In the present work we are concerned with system (2.1) - (2.3) where the heat conduction is given by Cattaneo's law instead of usual Fourier's one. We should note here that the dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law. This justifies the presence of the extra damping terms in system (2.1). In fact if $a = b = 0$, Fernandez Sare and Racke [20] have proved recently that (2.1) - (2.3) is no longer exponentially stable even in the case of equal propagation speed $\rho_1/\rho_2 = K/\bar{b}$.

2.2 Preliminaries

In this section, we introduce some notations and some techenical lemmas to be used throughout this chapter. Also, we give a local existence theorem. In order to state and prove our result, we formulate the following assumptions:

- **(H1)** a, b: $[0,1] \to \mathbb{R}^+$ are such that

$$\begin{aligned} a &\in C^1([0,1]), \qquad b \in L^\infty([0,1]) \\ a &= 0 \text{ or } a(0)+a(1)>0, \qquad \inf_{x\in[0,1]}\{a(x)+b(x)\}>0. \end{aligned}$$

- **(H2)** $h : \mathbb{R} \to \mathbb{R}$ is a differentiable nondecreasing function such that there exist constants $\varepsilon', c_1, c_2 > 0$ and a convex and increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2((0,\infty))$ satisfying $H(0) = 0$ and H is linear on $[0,\varepsilon']$ or $H'(0) = 0$ and $H'' > 0$ on $(0,\varepsilon']$ such that

$$\begin{cases} c_1|s| \le h(s) \le c_2|s| & \text{if } |s| \ge \varepsilon' \\ s^2 + h^2(s) \le H^{-1}(sh(s)) & \text{if } |s| \le \varepsilon'. \end{cases}$$

- **(H3)** $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \|a\|_\infty \int_0^\infty g(s)\,ds = l > 0.$$

- **(H4)** There exists a non-increasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g'(s) \le -\xi(s)g(s), \qquad \forall s \ge 0.$$

Throughout this chapter, we use the following notations

$$\begin{aligned} (\phi * \psi)(t) &:= \int_0^t \phi(t-\tau)\psi(\tau)\,d\tau \\ (\phi \diamond \psi)(t) &:= \int_0^t \phi(t-\tau)|\psi(t)-\psi(\tau)|\,d\tau \\ (\phi \circ \psi)(t) &:= \int_0^t \phi(t-\tau)\int_\Omega |\psi(t)-\psi(\tau)|^2\,dxd\tau. \end{aligned}$$

The following lemma was introduced in [21].

Lemma 2.1. *For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,1)$, we have*

$$\begin{aligned} (\phi * \psi)(t)\psi_t(t) &= -\frac{1}{2}\phi(t)|\psi(t)|^2 + \frac{1}{2}(\phi' \diamond \psi)(t) \\ &\quad -\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,d\tau\right)|\psi(t)|^2\right\}. \end{aligned}$$

Now, we are going to prepare some materials in order to state two lemmas due to Cavalcanti and Oquendo [15]. See also [28] for the proof.

By using the fact that $a(0) > 0$ and since a is continuous, then there exists $\varepsilon > 0$ such that $\inf_{x\in[0,\varepsilon]} a(x) \geq \varepsilon$. Let us denote

$$d = \min\left\{\varepsilon, \inf_{x\in[0,1]} \{a(x) + b(x)\}\right\} > 0$$

and let $\alpha \in C^1([0,1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{cases} \alpha(x) = 0 & \text{if } a(x) \leq \frac{d}{4} \\ \alpha(x) = a(x) & \text{if } a(x) \geq \frac{d}{2} \end{cases} \tag{2.10}$$

To simplify the notations we introduce the following

$$g \odot v = \int_0^1 \alpha(x) \int_0^t g(s)(v(t) - v(s))\, ds dx$$

for all $v \in L^2(0,1)$. Here and in the sequel, we denote various generic positive constants by C or c.

Lemma 2.2. *The function α is not identically zero and satisfies*

$$\inf_{x\in[0,1]} \{\alpha(x) + b(x)\} \geq \frac{d}{2}.$$

Lemma 2.3. *There exists a positive constant c such that*

$$(g \odot v)^2 \leq c g \circ v_x,$$

for all $v \in H_0^1(0,1)$.

In order to make this chapter self contained we state, without proof, a local existence result. The proof can be established by the classical Galerkin method.

Theorem 2.4. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ and $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ be given. Assume that **(H1)**–**(H4)** are satisfied, then problem (2.1)–(2.3) has a unique global (weak) solution satisfying*

$$\begin{aligned} \varphi, \psi &\in C\left(\mathbb{R}_+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0,1)\right) \\ \theta, q &\in C\left(\mathbb{R}_+; L^2(0,1)\right). \end{aligned}$$

2.3 Stability result for $\mu > 0$

In this section, we show the uniform decay property of the solution of the system (2.1)−(2.3). In order to use the Poincaré inequality for θ, we introduce, as in [26],

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

Then, by the third equation in (2.1) we easily verify that

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\bar{\theta}$. On the other hand, $(\varphi, \psi, \bar{\theta}, q)$ satisfies the same system (2.1) and the boundary conditions (2.3). So, in the sequel, we shall work with $\bar{\theta}$ but we write θ for simplicity. The first-order energy, associated to (2.1)−(2.3), is then given by

$$\begin{aligned} E\left(t,\varphi,\psi,\bar{\theta},q\right) &= \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(\bar{b} - a(x)\int_0^t g(s)\,ds\right)\psi_x^2\right\}dx \\ &+ \frac{1}{2}\int_0^1 \left\{K(\varphi_x+\psi)^2 + \rho_3\theta^2 + \tau_0 q^2\right\}dx + \frac{1}{2}(g\circ\psi_x). \end{aligned} \tag{2.11}$$

In what follows, we denote $E(t) = E\left(t,\varphi,\psi,\bar{\theta},q\right)$ and $E(0) = E\left(0,\varphi_0,\psi_0,\bar{\theta}_0,q_0\right)$ for simplicity. The main result of this chapter is given by the following theorem:

Theorem 2.5. *Let $(\varphi_0,\varphi_1), (\psi_0,\psi_1), \in H_0^1(0,1)\times L^2(0,1)$ and $(\theta_0,q_0) \in L^2(0,1)\times L^2(0,1)$ be given. Assume that* ***(H1)*****−*****(H4)*** *are satisfied, then there exist positive constants c', c'' and ε_0 for which the (weak) solution of problem (2.1)− (2.3) satisfies*

$$E(t) \leq c'' H_1^{-1}\left(c'\int_0^t \xi(s)\,ds\right), \qquad \forall t \geq 0, \tag{2.12}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2}(s)\,ds$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } \left[0,\varepsilon'\right] \\ tH'(\varepsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on } \left(0,\varepsilon'\right] \end{cases} \tag{2.13}$$

and $\xi = 1$ if $a = 0$.

To prove Theorem (2.5), we will use the energy method to produce a suitable Lyapunov functional. This will be established through several lemmas. A starting point is, as usual, the dissipativity inequality which states that the energy E of the entire system (2.1)-(2.3) is a non-increasing function. Of course this fact is a necessary preliminary step of stability analysis. More precisely, we have the following result:

Lemma 2.6. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3), then the energy E is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\delta \int_0^1 q^2 dx - \frac{1}{2} g(t) \int_0^1 a(x) \psi_x^2 dx - \int_0^1 b(x) \psi_t h(\psi_t) dx \\ &\quad + \frac{1}{2} (g' \circ \psi_x) - \mu \int_0^1 \varphi_t^2 dx, \\ &\leq -\delta \int_0^1 q^2 dx - \int_0^1 b(x) \psi_t h(\psi_t) dx + \frac{1}{2} (g' \circ \psi_x) - \mu \int_0^1 \varphi_t^2 dx \leq 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.1) by φ_t, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \rho_1 \varphi_t^2 dx + K \int_0^1 \varphi_{tx} \varphi_x dx + K \int_0^1 \varphi_{tx} \psi dx = -\mu \int_0^1 \varphi_t^2 dx. \tag{2.15}$$

Similarly, multiplying the second equation in (2.1) by ψ_t, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 \rho_2 \psi_t^2 dx + \bar{b} \int_0^1 \psi_x \psi_{tx} dx + \int_0^1 \psi_t \int_0^t g(t-s) (a(x) \psi_x(s))_x \, ds dx \\ &+ K \int_0^1 \psi_t \varphi_x dx + K \int_0^1 \psi_t \psi dx - \gamma \int_0^1 \psi_{tx} \theta dx \\ = &- \int_0^1 b(x) \psi_t h(\psi_t) dx. \end{aligned} \tag{2.16}$$

Also, multiplying the third equation in (2.1) by θ, we find

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \rho_3 \theta^2 dx + \kappa \int_0^1 q_x \theta dx + \gamma \int_0^1 \psi_{tx} \theta dx = 0. \tag{2.17}$$

Finally, multiplying the fourth equation in (2.1) by q, we deduce

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \tau_0 q^2 dx - \kappa \int_0^1 \theta q_x dx = -\delta \int_0^1 q^2 dx. \tag{2.18}$$

Now, using Lemma 2.1, to handle the last term in first line of (2.16) and summing up (2.15)–(2.18), then (2.14) holds. □

Let us now define the functional I_1 as follows:

$$I_1(t) := -\int_0^1 \rho_2 \alpha(x)\psi_t \int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx + \frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\,q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx,$$

for simplicity we write

$$I_1(t) := \chi_1(t) + \chi_2(t). \tag{2.19}$$

Then, we have the following result:

Lemma 2.7. *Let (φ,ψ,θ,q) be the solution of (2.1)–(2.3). Assume that **(H1)–(H4)** hold. Then we have, for any ε_1, $\varepsilon_1' > 0$,*

$$\begin{aligned}\frac{dI_1}{dt} \le\ & -\left(\rho_2\int_0^t g(s)\,ds - \varepsilon_1\left(\rho_2^2 + \int_0^t g(s)\,ds\right)\right)\int_0^1 \alpha(x)\psi_t^2dx \\ & +\varepsilon_1' K^2\int_0^1 (\varphi_x+\psi)^2\,dx + \varepsilon_1\int_0^1 b(x)h^2(\psi_t)dx \\ & +\varepsilon_1'\left(2\bar{b}^2+1\right)\int_0^1 \psi_x^2dx + \left(c\varepsilon_1 + \frac{1}{\varepsilon_1}\int_0^t g(s)\,ds\right)\int_0^1 q^2dx \\ & +c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right)g\circ\psi_x + c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)g\circ\psi_x - \frac{c}{\varepsilon_1}g'\circ\psi_x\end{aligned} \tag{2.20}$$

Proof. Differentiating χ_1 with respect to t to obtain

$$\begin{aligned}\chi_1'(t) =\ & -\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\ & -\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx \\ & -\int_0^1 \rho_2\alpha(x)\psi_t^2\int_0^t g(s)\,dsdx.\end{aligned} \tag{2.21}$$

Now, using the second equation in (2.1), we get

$$
\begin{aligned}
&-\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&=\int_0^1 \bar{b}\alpha(x)\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\\
&+\int_0^t K\alpha(x)(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1 \alpha(x)a(x)\left(\int_0^t g(t-s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx\\
&+\int_0^1 b(x)h(\psi_t)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\int_0^1 \alpha(x)\gamma\theta_x\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\int_0^1 \alpha'(x)\left(\bar{b}\psi_x-a(x)\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx.
\end{aligned}
\tag{2.22}
$$

Next, we will estimate the second term in the right-hand side of (2.21). So, by using Lemma 2.3, we have, for any $\varepsilon_1>0$

$$
\begin{aligned}
&-\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
&\le \varepsilon_1\rho_2^2\int_0^1 \alpha(x)\psi_t^2dx-\frac{c}{\varepsilon_1}g'\circ\psi_x.
\end{aligned}
\tag{2.23}
$$

Also, as above we have

$$
\begin{aligned}
\chi_2'(t) \;=\; &\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)q_t\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)q\psi_t\int_0^t g(s)\,ds.
\end{aligned}
$$

Using the fourth equation in (2.1), we get

$$
\begin{aligned}
\chi_2'(t) \;=\; &-\frac{\gamma\delta}{\kappa}\int_0^1 \alpha(x)q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1 \alpha(x)\gamma\theta_x\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\tau_0}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)q\psi_t dx.
\end{aligned}
\tag{2.24}
$$

Similarly to (2.23), by exploiting Young's inequality, we estimate the terms in the right-hand side of (2.22) as follows:

$$\begin{aligned}&\int_0^1 \bar{b}\alpha(x)\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\\&\leq \varepsilon_1'\bar{b}^2\int_0^1\psi_x^2dx+\frac{c}{\varepsilon_1'}g\circ\psi_x.\end{aligned}\tag{2.25}$$

Similarly,

$$\begin{aligned}&\int_0^t K\alpha(x)(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq \varepsilon_1'K^2\int_0^1(\varphi_x+\psi)^2\,dx+\frac{c}{\varepsilon_1'}g\circ\psi_x.\end{aligned}\tag{2.26}$$

By the same method used in [28], we have the following estimates:

$$\begin{aligned}&-\int_0^1\alpha(x)a(x)\left(\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx\\&\leq \varepsilon_1'\int_0^1\psi_x^2dx+c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)g\circ\psi_x\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}&\int_0^1 b(x)h(\psi_t)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\&\leq \varepsilon_1\int_0^1 b(x)h^2(\psi_t)dx+c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)g\circ\psi_x.\end{aligned}\tag{2.28}$$

Finally,

$$\begin{aligned}&\int_0^1\alpha'(x)\left(\bar{b}\psi_x-a(x)\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\&\leq \varepsilon_1'\bar{b}^2\int_0^1\psi_x^2dx+c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)g\circ\psi_x.\end{aligned}\tag{2.29}$$

As in (2.23), it is easy to prove

$$\begin{aligned}&\frac{\gamma\tau_0}{\kappa}\int_0^1\alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq \varepsilon_1\int_0^1 q^2dx-\frac{c}{\varepsilon_1}g'\circ\psi_x.\end{aligned}\tag{2.30}$$

Also, we estimate the first term in the right-hand side of (2.24) as follows:

$$\begin{aligned}&-\frac{\gamma\delta}{\kappa}\int_0^1\alpha(x)q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq\left(\frac{\gamma\delta}{\kappa}\right)^2\varepsilon_1\int_0^1 q^2dx+\frac{c}{\varepsilon_1}g\circ\psi_x\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}&\frac{\gamma\tau_0}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)\,q\psi_t dx\\ \leq\ &\left(\int_0^t g(s)\,ds\right)\frac{1}{\varepsilon_1}\int_0^1 q^2 dx+\left(\int_0^t g(s)\,ds\right)c\varepsilon_1\int_0^1 \psi_t^2 dx. \end{aligned} \tag{2.32}$$

Consequently, by combining all the above estimates (2.21)–(2.32), the assertion of Lemma (2.7) is fulfilled. □

Now, as in [64], let w be the solution of

$$\begin{cases} -w_{xx}=\psi_x,\\ w(0)=w(1)=0. \end{cases} \tag{2.33}$$

Then, we have the following inequalities:

Lemma 2.8. *The solution of* (2.33) *satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply Equation (2.33) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (2.33) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 2.8. □

Let w be the solution of (2.33). We introduce the following functional:

$$I_2(t):=\int_0^1\left(\rho_2\psi_t\psi+\rho_1\varphi_t w-\frac{\gamma\tau_0}{\kappa}\psi q\right)dx. \tag{2.34}$$

Then, we have the following estimate:

Lemma 2.9. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3). Assume that **(H1)–(H4)** hold. Then we have, for any $\varepsilon_2 > 0$*

$$\begin{aligned} \frac{dI_2}{dt} \leq & -\left(\bar{b} + \frac{c\mu\varepsilon_2}{2} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right)\int_0^1 \psi_x^2 dx + \left(\frac{\rho_1}{2\varepsilon_2} + \frac{\mu}{2\varepsilon_2}\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2} g \circ \psi_x \\ & + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\int_0^1 q^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x) h^2(\psi_t)\, dx. \end{aligned} \tag{2.35}$$

Proof. By taking the derivative of I_2 with respect to t we get

$$\begin{aligned} I_2'(t) = & \int_0^1 \left(\rho_2 \psi_{tt}\psi + \rho_2 \psi_t^2\right) dx + \int_0^1 (\rho_1 \varphi_{tt} w + \rho_1 \varphi_t w_t)\, dx \\ & -\frac{\gamma\tau_0}{\kappa}\int_0^1 (\psi_t q + \psi q_t)\, dx \\ := & J_1 + J_2 + J_3. \end{aligned} \tag{2.36}$$

Next, using the first and the fourth equations in (2.1) we get

$$\begin{aligned} J_2 + J_3 = & -K\int_0^1 \varphi\psi_x dx + K\int_0^1 w_x^2 dx + \rho_1 \int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx + \gamma\int_0^1 \psi\theta_x dx. \end{aligned} \tag{2.37}$$

Next, using the second equation in (2.1), we get

$$\begin{aligned} J_1 = & -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 \psi_x \int_0^t g(t-s)\, a(x)\, \psi_x(s)\, ds dx \\ & -K\int_0^1 \psi^2 dx - K\int_0^1 \varphi_x \psi dx - \int_0^1 b(x)\,\psi h(\psi_t)\, dx - \int_0^1 \gamma\psi\theta_x dx. \end{aligned} \tag{2.38}$$

From (2.37), (2.38) and by using Lemma 2.8, we deduce

$$\begin{aligned} I_2'(t) \leq & -\mu\int_0^1 \varphi_t w dx + \rho_1 \int_0^1 \varphi_t w_t dx - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx \\ & -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx - \int_0^1 b(x)\,\psi h(\psi_t)\, dx \\ & + \int_0^1 a(x)\,\psi_x \int_0^t g(t-s)\,\psi_x(s)\, ds dx. \end{aligned} \tag{2.39}$$

By exploiting the inequality

$$|ab| \leq \frac{\nu}{2}a^2 + \frac{1}{2\nu}b^2, \qquad a, b \in \mathbb{R},\ \nu > 0,$$

we easily find, for any $\varepsilon_2 > 0$,

$$\begin{aligned} I_2'(t) \leq & -\bar{b}\int_0^1 \psi_x^2 dx + \frac{\mu}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w^2\right) + \frac{\rho_1}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w_t^2\right) dx \\ & + \frac{\gamma\tau_0}{2\kappa}\int_0^1 \left(\varepsilon_2\psi_t^2 + \frac{1}{\varepsilon_2}q^2\right) dx + \frac{\delta\gamma}{2\kappa}\int_0^1 \left(\varepsilon_2\psi^2 + \frac{1}{\varepsilon_2}q^2\right) dx \\ & + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\,\psi h(\psi_t)\,dx \\ & + \int_0^1 a(x)\,\psi_x \int_0^t g(t-s)\,\psi_x(s)\,ds dx. \end{aligned} \tag{2.40}$$

We now proceed to the evaluation of the last two terms in the right-hand side of (2.40). First, by Young's and Poicaré's inequalities we have

$$\left|\int_0^1 b(x)\,\psi h(\psi_t)\,dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x)\,h^2(\psi_t)\,dx. \tag{2.41}$$

Furthermore, by Young's and Cauchy-Schwartz inequalities we have

$$\left|\int_0^1 a(x)\,\psi_x \int_0^t g(t-s)\,\psi_x(s)\,ds dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_2} g \circ \psi_x. \tag{2.42}$$

Then, plugging (2.41) and (2.42) into (2.40) and using the second inequality in Lemma 2.8, there fore the assertion of Lemma 2.9 holds. □

Now, following [28], we define the functional I_3 as follows:

$$I_3(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu}{2}\int_0^1 \varphi^2 dx. \tag{2.43}$$

Then, we have the following estimate:

Lemma 2.10. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of* (2.1)–(2.3)*. Then, for any* $\varepsilon_3 > 0$*, we have*

$$I_3'(t) \leq \left(\frac{K\varepsilon_3}{2} - K\right)\int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_3}\int_0^1 \psi_x^2 dx + \rho_1\int_0^1 \varphi_t^2 dx. \tag{2.44}$$

Proof. By exploiting the first equation in (2.1) and using Young's inequality, we get

$$
\begin{aligned}
I_3'(t) &= \int_0^1 \rho_1 \varphi_{tt}\varphi dx + \rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_t \varphi dx \\
&= \int_0^1 K\varphi\left(\varphi_{xx} + \psi_x\right) dx + \rho_1 \int_0^1 \varphi_t^2 dx \\
&= -K \int_0^1 \varphi_x^2 dx + K \int_0^1 \varphi \psi_x dx + \rho_1 \int_0^1 \varphi_t^2 dx \\
&\leq -K \int_0^1 \varphi_x^2 dx + \frac{K}{2} \int_0^1 \left(\varepsilon_3 \varphi^2 + \frac{1}{\varepsilon_3}\psi_x^2\right) dx + \rho_1 \int_0^1 \varphi_t^2 dx.
\end{aligned}
$$

A simple use of Poincaré's inequality completes the proof of Lemma 2.10. □

Now, in order to obtain negative terms of $\int_0^1 \theta^2 dx$ we introduce the following functional:

$$
I_4(t) := -\tau_0 \rho_3 \int_0^1 q(t,x)\left(\int_0^x \theta(t,y) dy\right) dx. \tag{2.45}
$$

Then we have the following estimate:

Lemma 2.11. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of* (2.1)–(2.3)*. Then, for any* $\varepsilon_4 > 0$*, we have*

$$
\begin{aligned}
I_4'(t) \leq\ & \left(-\rho_3 \kappa + \frac{\varepsilon_4 \rho_3 \delta c}{2}\right) \int_0^1 \theta^2 dx + \frac{\varepsilon_4 \tau_0 \gamma}{2} \int_0^1 \psi_t^2 dx \\
& + \left(\tau_0 \kappa + \frac{\rho_3 \delta}{2\varepsilon_4} + \frac{\tau_0 \gamma}{2\varepsilon_4}\right) \int_0^1 q^2 dx.
\end{aligned} \tag{2.46}
$$

Proof. By using the fourth equation in (2.1), we get

$$
\begin{aligned}
I_4'(t) &= -\rho_3 \int_0^1 \tau_0 q_t \left(\int_0^x \theta dy\right) dx - \tau_0 \int_0^1 q\left(\int_0^x \rho_3 \theta_t dy\right) dx \\
&= -\rho_3 \int_0^1 \left(-\delta q - \kappa \theta_x\right)\left(\int_0^x \theta dy\right) dx - \tau_0 \int_0^1 q\left(\int_0^x \left(-\kappa q_x - \gamma \psi_{tx}\right) dy\right) dx \\
&= \rho_3 \delta \int_0^1 q\left(\int_0^x \theta dy\right) dx + \rho_3 \kappa \int_0^1 \theta_x \left(\int_0^x \theta dy\right) dx \\
&\quad + \tau_0 \kappa \int_0^1 q\left(\int_0^x q_x dy\right) dx + \tau_0 \gamma \int_0^1 q\left(\int_0^x \psi_{tx} dy\right) dx.
\end{aligned}
$$

That is

$$I_4'(t) \leq \frac{\rho_3\delta}{2}\int_0^1\left(\varepsilon_4\left(\int_0^x\theta^2dy\right)^2+\frac{1}{\varepsilon_4}q^2\right)dx-\rho_3\kappa\int_0^1\theta^2dx$$
$$+\tau_0\kappa\int_0^1 q^2dx+\frac{\tau_0\gamma}{2}\int_0^1\left(\varepsilon_4\psi_t^2+\frac{1}{\varepsilon_4}q^2\right)dx. \tag{2.47}$$

Consequently, the assertion of Lemma 2.11 immediately follows. □

Proof of Theorem 2.5. For $N, N_1, N_2 > 0$, we can define an auxiliary functional $\mathcal{F}$ by

$$\mathcal{F}(t) := NE(t) + N_1I_1 + N_2I_2 + I_3 + I_4 \tag{2.48}$$

and let $t_0 > 0$, and $g_0(t) = \int_0^t g(s)ds > 0$. By combining (2.14), (2.20), (2.35), (2.44) and (2.47), and by using the inequality

$$(\varphi_x+\psi)^2 \leq 2\varphi_x^2 + 2\psi^2$$

and Poincaré's inequality, we arrive at

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} \leq\ & -N_1\left(\rho_2g_0-\varepsilon_1\left(\rho_2^2+g_0\right)\right)\int_0^1(\alpha(x)+b(x))\psi_t^2dx\\
&+\left(N_2\left(\rho_2+\frac{\gamma\tau_0\varepsilon_2}{2\kappa}+\frac{\rho_1\varepsilon_2}{2}\right)+\frac{\tau_0\gamma\varepsilon_4}{2}\right)\int_0^1\psi_t^2dx-N\int_0^1 b(x)\psi_th(\psi_t)dx\\
&+\left(N_2\left(\frac{\rho_1}{2\varepsilon_2}+\frac{\mu}{2\varepsilon_2}\right)+\rho_1-N\mu\right)\int_0^1\varphi_t^2dx+\left(N_1\varepsilon_1+\frac{N_2}{2\varepsilon_2}\right)\int_0^1 b(x)h^2(\psi_t)dx\\
&+N_1\left(\rho_2g_0-\varepsilon_1\left(\rho_2^2+g_0\right)\right)\int_0^1 b(x)\psi_t^2dx\\
&+\left\{N_1\varepsilon_1'\left(2\bar{b}^2+1+2K^2\right)-N_2\left(\bar{b}-2c\varepsilon_2-\frac{\delta\gamma\varepsilon_2}{2\kappa}\right)+\frac{K}{2\varepsilon_3}\right\}\int_0^1\psi_x^2dx\\
&+\left(2N_1\varepsilon_1'K^2+\frac{K\varepsilon_3}{2}-K\right)\int_0^1\varphi_x^2dx+\left(-\rho_3\kappa+\frac{\varepsilon_4\rho_3\delta c}{2}\right)\int_0^1\theta^2dx\\
&+\left\{cN_1\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)+cN_1\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)+\frac{N_2c}{\varepsilon_2}\right\}g\circ\psi_x+\left(\frac{N}{2}-\frac{cN_1}{\varepsilon_1}\right)g'\circ\psi_x\\
&+\left\{N_1\left(c\varepsilon_1+\frac{g_0}{\varepsilon_1}\right)+N_2\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2}+\frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\right.\\
&\left.+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_4}+\frac{\tau_0\gamma}{2\varepsilon_4}\right)-\delta N\right\}\int_0^1 q^2dx\end{aligned}$$

for all $t \geq t_0$. At this point, we have to choose our constants very carefully. First, let us take $\varepsilon_3 < 1$, $\varepsilon_1, \varepsilon_2$ and ε_4 small enough such that

$$\begin{aligned}
\varepsilon_1 &\leq \min\left\{\left(\frac{\rho_2 g_0}{2}\right)/\left(\rho_2^2+g_0\right), \frac{1}{4K}\right\}, \\
\varepsilon_2 &\leq \left(\frac{\bar{b}}{2}\right)/\left(2c+\frac{\delta\gamma}{2\kappa}\right)
\end{aligned}$$

and

$$\varepsilon_4 \leq \frac{\kappa}{\delta c}.$$

After that, we pick N_2 large enough so that

$$N_2 \geq \frac{2K\bar{b}}{\varepsilon_3}.$$

Now, by using Lemma 2.2, and choosing N_1 large enough such that

$$\frac{N_1\rho_2 g_0}{2} > \left(N_2\left(\rho_2+\frac{\gamma\tau_0\varepsilon_2}{2k}+\frac{\rho_1\varepsilon_2}{2}\right)+\frac{\tau_0\gamma\varepsilon_4}{2}\right)\frac{2}{d}$$

then, we can select $\varepsilon_1^{'}$ small enough such that

$$\varepsilon_1^{'} \leq \min\left\{\frac{1}{4N_1K}, \left(\frac{N_2\bar{b}}{4}\right)/N_1\left(2\bar{b}^2+1+2K^2\right)\right\}. \tag{2.49}$$

Finally, we choose N large enough so that, there exist positive constants η, η_1, and η_2 such that, for $t \geq t_0$,

$$\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} &\leq -\eta\left\{\int_0^1 (\alpha(x)+b(x))\psi_t^2dx+\int_0^1 \varphi_t^2dx\right. \\
&\quad \left.+\int_0^1 \theta^2dx+\int_0^1 q^2dx\right\}-\eta_1\int_0^1 \psi_x^2dx-\eta_2\int_0^1 \varphi_x^2dx \\
&\quad +cg\circ\psi_x+c\int_0^1 b(x)\left(\psi_t^2+h^2(\psi_t)\right)dx.
\end{aligned}$$

By the same method as in [55] (see inequality (25) in [55]), we can find $\eta_3 > 0$ such that, for $t \geq t_0$,

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} \leq & -\eta_3\left\{\int_0^1 (\alpha(x)+b(x))\,\psi_t^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_x^2 dx \right.\\ & \left. + \int_0^1 (\varphi_x+\psi)^2\,dx + \int_0^1 \theta^2 dx + \int_0^1 q^2 dx\right\} \\ & + cg\circ\psi_x + c\int_0^1 b(x)\left(\psi_t^2 + h^2(\psi_t)\right)dx. \end{aligned} \tag{2.50}$$

Moreover, we have the following: there exist two positive constants β_1 and β_2 depending on N, N_1, N_2, such that

$$\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{2.51}$$

This can be seen simply from estimate (2.14), (2.19), (2.34), (2.43), (2.45), (2.48), Young's and Poincaré's inequalities, that

$$|\mathcal{F}(t) - NE(t)| \leq CE(t), \qquad \forall t \geq 0.$$

Consequently, we can choose N large enough such that $\beta_1 = N - C > 0$ and (2.50) therefore (2.51) holds true. Our goal now is to estimate the last term in the right-hand side of (2.50). Following the method presented in [28], we consider the following partition of the interval $(0,1)$:

$$\Omega^+ = \left\{x \in (0,1) : |\psi_t| > \varepsilon'\right\} \text{ and } \Omega^- = \left\{x \in (0,1) : |\psi_t| \leq \varepsilon'\right\} \tag{2.52}$$

where ε' is defined in (H2). By using the hypothesis (H2) , we have $|\psi_t| \leq c_1^{-1}\psi_t h(\psi_t)$ on Ω^+ and therefore taking into account the estimate (2.14), we arrive at

$$\begin{aligned}\int_{\Omega^+} b(x)\left(\psi_t^2 + h^2(\psi_t)\right)dx &\leq c\int_{\Omega^+} b(x)\,\psi_t h(\psi_t)\,dx \\ &\leq c\int_0^1 b(x)\,\psi_t h(\psi_t)\,dx \\ &\leq -cE'(t). \end{aligned} \tag{2.53}$$

According to (H2) , we distinguish two cases:

Case 1: H is linear on $\left[0,\varepsilon'\right]$. Consequently, there exist two positive constants c_1' and c_2' such that $c_1'|s| \leq |h(s)| \leq c_2'|s|$, for all $s \in \mathbb{R}_+$, therefore the above inequality (2.53) holds on $(0,1)$. Now, from (2.50) and (2.53), we arrive at

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}(t)+cE(t)) &\le -cE(t)+cg\circ\psi_x\\ &= -cH_2(E(t))+cg\circ\psi_x, \qquad \forall t\ge t_0,\end{aligned} \tag{2.54}$$

where the function H_2 is defined by (2.13).

Case 2: $H^{'}(0)=0$ and $H^{''}(0)>0$ on $\left[0,\varepsilon^{'}\right]$. Let H^{*} denote the dual of H in the sense of Young, then we have (see [39] for more details)

$$H^{*}(s)=s\left(H^{'}\right)^{-1}(s)-H\left[\left(H^{'}\right)^{-1}(s)\right], \qquad \forall s\in\mathbb{R}_{+}.$$

By using Jensen's inequality, we deduce

$$\begin{aligned}\int_{\Omega^{-}} b(x)\left(\psi_t^2+h^2(\psi_t)\right)dx &\le c\int_{\Omega^{-}} b(x)H^{-1}(\psi_t h(\psi_t))\,dx\\ &\le c\int_{\Omega^{-}} H^{-1}(b(x)\psi_t h(\psi_t))\,dx\\ &\le cH^{-1}\left(\int_{\Omega^{-}} b(x)\psi_t h(\psi_t)\,dx\right)\\ &\le cH^{-1}\left(-cE^{'}(t)\right).\end{aligned} \tag{2.55}$$

Thus, it follows from (2.50), (2.53) and (2.55) that

$$\mathcal{F}^{'}(t)\le -cE(t)+cH^{-1}\left(-cE^{'}(t)\right)-cE^{'}(t)+cg\circ\psi_x, \qquad \forall t\ge t_0.$$

By using Young's inequality and the fact that

$$H^{*}(s)\le s(H')(s),\ E^{'}(t)\le 0,\ H^{''}\ge 0,$$

we obtain by the same method as in [28] (we omit the details)

$$H'(\varepsilon_0 E(t))(\mathcal{F}'(t)+cE'(t)+c_0E'(t))\le -cH_2(E(t))+cg\circ\psi_x \tag{2.56}$$

where ε_0 is a small positive constant and c_0 is a large positive constant. Now, let us define the following functional:

$$\mathcal{L}(t)=\begin{cases}\mathcal{F}(t)+cE(t) & \text{if } H \text{ is linear on } \left[0,\varepsilon^{'}\right]\\ H^{'}(\varepsilon_0E(t))(\mathcal{F}(t)+cE(t))+c_0E(t) & \text{if } H^{'}(0)=0 \text{ and } H^{''}>0 \text{ on } (0,\varepsilon^{'}].\end{cases}$$

We can easily show that

$$\mathcal{L} \sim E.$$

On the other hand, by making use of (2.54) and (2.56), we easily deduce that the following inequality

$$\mathcal{L}^{'}(t) \leq -cH_2(E(t)) + cg \circ \psi_x$$

holds for all $t \geq t_0$. By using (2.14) and (H4) , we obtain

$$\begin{aligned}(\xi(t)\mathcal{L}(t))^{'} &= \xi^{'}(t)\mathcal{L}(t) + \xi(t)\mathcal{L}^{'}(t) \\ &\leq -c\xi(t)H_2(E(t)) - cE^{'}(t).\end{aligned}$$

Next, let $\mathcal{K}(t) = \varepsilon(\xi(t)\mathcal{L}(t) + cE(t))$, where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\xi(t)\mathcal{L}(t) + cE(t) \leq \frac{1}{\bar{\varepsilon}}E(t), \quad \forall t \geq 0.$$

We can also show that

$$\mathcal{K} \sim E$$

and, for $t \geq t_0$,

$$\mathcal{K}^{'}(t) \leq -c\varepsilon\xi(t)H_2(\mathcal{K}(t)).$$

A simple integration of the above inequality over (t_0, t) yields

$$\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds + H_1(\mathcal{K}(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds\right), \forall t \geq t_0,$$

where $H_1(t) = \int_t^1 \left(\frac{1}{H_2(t)}\right)ds$. Since $\lim_{t\to 0^+} H_1(t) = \infty$ and

$$0 \leq \mathcal{K}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(0).$$

We may choose ε small enough such that

$$H_1(F(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds \geq 0.$$

Therefore, $\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds\right)$, for $t \geq t_0$. Consequently, there exist two positive constants $c^{'}$, and $c^{''}$ for which

$$\mathcal{K}(t) \leq c^{''}H_1^{-1}\left(c^{'}\int_0^t \xi(s)\,ds\right), \ \forall t \geq 0,$$

since $\mathcal{K}$ is bounded, which gives (2.12).

This completes the proof of the Theorem 2.5 □

Remark 2.12. We can also prove the same decay results for the following boundary conditions:

$$\varphi_x(0,\ t) = \varphi_x(1,\ t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0.$$

2.4 Stability results for $\mu = 0$

This is the main section, where we show the uniform decay property of the solution of the system (2.1)−(2.3). As in [78], and in order to use the Poincaré inequality for θ, we introduce

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

Then, by the third equation in (2.1) we easily verify that

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\bar{\theta}$. On the other hand, $(\varphi, \psi, \bar{\theta}, q)$ satisfies the same system (2.1) and the boundary conditions (2.3). So, in the sequel, we shall work with $\bar{\theta}$ but we write θ for simplicity. The first-order energy, associated to (2.1)−(2.3), is then given by

$$\begin{aligned} E\left(t,\varphi,\psi,\bar{\theta},q\right) &= \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(\bar{b} - a(x)\int_0^t h(s)\,ds\right)\psi_x^2\right\}dx \\ &+\frac{1}{2}\int_0^1 \left\{k(\varphi_x+\psi)^2 + \rho_3\theta^2 + \tau_0 q^2\right\}dx + \frac{1}{2}(h\circ\psi_x). \end{aligned} \tag{2.57}$$

Theorem 2.13. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1), \in H_0^1(0,1) \times L^2(0,1)$ *and* $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ *be given. Assume that* ***(H1)*–*(H4)*** *are satisfied, then there exist positive constants* c', c'' *and* ε_0 *for which the weak solution of problem* (2.1)−(2.3) *satisfies*

$$E(t) \leq c'' H_1^{-1}\left(c'\int_0^t \xi(s)\,ds\right), \qquad \forall t \geq 0, \tag{2.58}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2}(s)\,ds$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } \left[0, \varepsilon'\right] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } \left(0, \varepsilon'\right] \end{cases} \tag{2.59}$$

and $\xi = 1$ *if* $a = 0$.

To prove Theorem 2.13, we will use the energy method to produce a suitable Lyapunov functional. This will be established through several lemmas.

Lemma 2.14. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of* (2.1) − (2.3), *then the energy* $E(t)$ *is no-increasing function and satisfies, for all* $t \geq 0$,

$$\begin{aligned} \frac{dE(t)}{dt} &= -\delta \int_0^1 q^2 dx - \frac{1}{2}h(t)\int_0^1 a(x)\psi_x^2 dx - \int_0^1 b(x)\psi_t g(\psi_t)\,dx \\ &\quad + \frac{1}{2}(h' \circ \psi_x) \\ &\leq -\delta \int_0^1 q^2 dx - \int_0^1 b(x)\psi_t g(\psi_t)\,dx + \frac{1}{2}(h' \circ \psi_x) \leq 0. \end{aligned} \tag{2.60}$$

Proof. By multiplying the first equation in (2.1) by φ_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_1 \varphi_t^2 dx + k\int_0^1 \varphi_{tx}\varphi_x dx + k\int_0^1 \varphi_{tx}\psi dx = 0 \tag{2.61}$$

And the second equation in (2.1) by ψ_t, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_2 \psi_t^2 dx + \bar{b}\int_0^1 \psi_x \psi_{tx} dx + \int_0^1 \psi_t \int_0^t h(t-s)\left(a(x)\psi_x(s)\right)_x ds dx \\ &+k\int_0^1 \psi_t \varphi_x dx + k\int_0^1 \psi_t \psi dx - \gamma\int_0^1 \psi_{tx}\theta dx \\ = &-\int_0^1 b(x)\psi_t g(\psi_t)\,dx. \end{aligned} \tag{2.62}$$

Multiplying the third equation in (2.1) by θ, we find

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3 \theta^2 dx + \kappa\int_0^1 q_x \theta dx + \gamma\int_0^1 \psi_{tx}\theta dx = 0. \tag{2.63}$$

Finally, multiplying the fourth equation in (2.1) by q, we deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tau_0 q^2 dx - \kappa\int_0^1 \theta q_x dx = -\delta\int_0^1 q^2 dx \tag{2.64}$$

To handle the last term in first line of (2.62), using Lemma (2.1) and summing up (2.61)– (2.64), then (2.60) holds. □

We define the functional I_1 as follows:

$$\begin{aligned} I_1(t) &:= -\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx \\ &\quad +\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)q\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx \\ &:= \chi_1(t)+\chi_2(t). \end{aligned} \tag{2.65}$$

Then, we have the following Lemma.

Lemma 2.15. *Let (φ,ψ,θ,q) be the solution of (2.1)–(2.3). Assume that **(H1)–(H4)** hold. Then we have, for any ε_1, $\varepsilon_1' > 0$,*

$$\begin{aligned} \frac{dI_1}{dt} &\le -\left(\rho_2\int_0^t h(s)\,ds - \varepsilon_1\left(\rho_2^2+\int_0^t h(s)\,ds\right)\right)\int_0^1 \alpha(x)\psi_t^2 dx \\ &\quad +\varepsilon_1' k^2\int_0^1 (\varphi_x+\psi)^2\,dx + \varepsilon_1\int_0^1 b(x)g^2(\psi_t)dx \\ &\quad +\varepsilon_1'\left(2\bar{b}^2+1\right)\int_0^1 \psi_x^2 dx + \left(c\varepsilon_1+\frac{1}{\varepsilon_1}\int_0^t h(s)\,ds\right)\int_0^1 q^2 dx \\ &\quad +c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)(h\circ\psi_x) + c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)(h\circ\psi_x) - \frac{c}{\varepsilon_1}(h'\circ\psi_x) \end{aligned} \tag{2.66}$$

Proof. Differentiating χ_1 with respect to t to obtain

$$\begin{aligned} \chi_1'(t) &= -\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx \\ &\quad -\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t h'(t-s)(\psi(t)-\psi(s))\,dsdx \\ &\quad -\int_0^1 \rho_2\alpha(x)\psi_t^2\int_0^t h(s)\,dsdx. \end{aligned} \tag{2.67}$$

By using Lemma (2.3), we have, for any $\varepsilon_1 > 0$

$$\begin{aligned} & -\int_0^1 \rho_2 \alpha(x) \psi_t \int_0^t h'(t-s)(\psi(t)-\psi(s))\, ds dx \\ & \leq \varepsilon_1 \rho_2^2 \int_0^1 \alpha(x) \psi_t^2 dx - \frac{c}{\varepsilon_1}(h' \circ \psi_x). \end{aligned} \tag{2.68}$$

Next, using the second equation in (2.1), we get

$$\begin{aligned} & -\int_0^1 \rho_2 \alpha(x) \psi_{tt} \int_0^t h(t-s)(\psi(t)-\psi(s))\, ds dx \\ & = \int_0^1 \bar{b} \alpha(x) \psi_x \int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\, ds dx \\ & + \int_0^t k\alpha(x)(\varphi_x+\psi) \int_0^t h(t-s)(\psi(t)-\psi(s))\, ds dx \\ & - \int_0^1 \alpha(x) a(x) \left(\int_0^t h(t-s)\psi_x(s)\, ds\right)\left(\int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\, ds\right) dx \\ & + \int_0^1 b(x) g(\psi_t) \left(\int_0^t h(t-s)(\psi(t)-\psi(s))\, ds\right) dx \\ & + \int_0^1 \alpha(x) \gamma \theta_x \left(\int_0^t h(t-s)(\psi(t)-\psi(s))\, ds\right) dx \\ & + \int_0^1 \alpha'(x) \left(\bar{b}\psi_x - a(x) \int_0^t h(s)\psi_x(s)\, ds\right)\left(\int_0^t h(t-s)(\psi(t)-\psi(s))\, ds\right) dx. \end{aligned} \tag{2.69}$$

Also, as above we have

$$\begin{aligned} \chi_2'(t) &= \frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q_t \int_0^t h(t-s)(\psi(t)-\psi(s))\, ds dx \\ & +\frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q \int_0^t h'(t-s)(\psi(t)-\psi(s))\, ds dx \\ & +\frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q \psi_t \int_0^t h(s)\, ds. \end{aligned}$$

Using the fourth equation in (2.1), we get

$$\begin{aligned} \chi_2'(t) &= -\frac{\gamma\delta}{\kappa} \int_0^1 \alpha(x) q \int_0^t h(t-s)(\psi(t)-\psi(s))\, ds dx \\ & - \int_0^1 \alpha(x) \gamma \theta_x \left(\int_0^t h(t-s)(\psi(t)-\psi(s))\, ds\right) dx \\ & +\frac{\gamma\tau_0}{\kappa} \int_0^1 \alpha(x) q \int_0^t h'(t-s)(\psi(t)-\psi(s))\, ds dx \\ & +\frac{\gamma\tau_0}{\kappa} \left(\int_0^t h(s)\, ds\right) \int_0^1 \alpha(x) q \psi_t dx. \end{aligned} \tag{2.70}$$

Similarly to (2.68), by exploiting Young's inequality, we estimate the terms in the right-hand side of (2.69) as follows:

$$\begin{aligned}&\int_0^1 \bar{b}\alpha(x)\psi_x \int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\\&\leq \varepsilon_1'\bar{b}^2\int_0^1 \psi_x^2 dx+\frac{c}{\varepsilon_1'}(h\circ\psi_x).\end{aligned}\tag{2.71}$$

Similarly,

$$\begin{aligned}&\int_0^t k\alpha(x)(\varphi_x+\psi)\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq \varepsilon_1' k^2\int_0^1(\varphi_x+\psi)^2\,dx+\frac{c}{\varepsilon_1'}(h\circ\psi_x).\end{aligned}\tag{2.72}$$

By the same method used in [28], we have the following estimates:

$$\begin{aligned}&-\int_0^1 \alpha(x)a(x)\left(\int_0^t h(s)\psi_x(s)\,ds\right)\left(\int_0^t h(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx\\&\leq \varepsilon_1'\int_0^1\psi_x^2dx+c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)(h\circ\psi_x)\end{aligned}\tag{2.73}$$

and

$$\begin{aligned}&\int_0^1 b(x)g(\psi_t)\left(\int_0^t h(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\&\leq \varepsilon_1\int_0^1 b(x)g^2(\psi_t)dx+c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)(h\circ\psi_x).\end{aligned}\tag{2.74}$$

Finally,

$$\begin{aligned}&\int_0^1 \alpha'(x)\left(\bar{b}\psi_x-a(x)\int_0^t h(s)\psi_x(s)\,ds\right)\left(\int_0^t h(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\&\leq \varepsilon_1'\bar{b}^2\int_0^1\psi_x^2dx+c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right)(h\circ\psi_x).\end{aligned}\tag{2.75}$$

As in (2.68), it is obvious that

$$\begin{aligned}&\frac{\gamma\tau_0}{\kappa}\int_0^1\alpha(x)q\int_0^t h'(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq \varepsilon_1\int_0^1 q^2dx-\frac{c}{\varepsilon_1}(h'\circ\psi_x).\end{aligned}\tag{2.76}$$

Also, we estimate the first term in the right-hand side of (2.70) as follows:

$$\begin{aligned}&-\frac{\gamma\delta}{\kappa}\int_0^1\alpha(x)q\int_0^t h(t-s)(\psi(t)-\psi(s))\,dsdx\\&\leq\left(\frac{\gamma\delta}{\kappa}\right)^2\varepsilon_1\int_0^1 q^2dx+\frac{c}{\varepsilon_1}(h\circ\psi_x)\end{aligned}\tag{2.77}$$

and

$$\frac{\gamma\tau_0}{\kappa}\left(\int_0^t h(s)\,ds\right)\int_0^1 \alpha(x)\,q\psi_t dx$$
$$\leq \left(\int_0^t h(s)\,ds\right)\frac{1}{\varepsilon_1}\int_0^1 q^2 dx+\left(\int_0^t h(s)\,ds\right)c\varepsilon_1\int_0^1 \psi_t^2 dx. \tag{2.78}$$

Consequently, by combining all the above estimates (2.67)–(2.78), the assertion of Lemma 2.15 is fulfilled. □

Now, as in [64], let w be the solution of

$$\begin{cases} -w_{xx}=\psi_x, \\ w(0)=w(1)=0. \end{cases} \tag{2.79}$$

Then, we have

Lemma 2.16. *The solution of* (2.79) *satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply Equation (2.79) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (2.79) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma (2.16). □

Let w be the solution of (2.79). We introduce the following functional:

$$I_2(t):=\int_0^1\left(\rho_2\psi_t\psi+\rho_1\varphi_t w-\frac{\gamma\tau_0}{\kappa}\psi q\right)dx. \tag{2.80}$$

Then, we have

Lemma 2.17. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3). Assume that **(H1)–(H4)** hold. Then we have, for any $\varepsilon_2 > 0$*

$$\begin{aligned}\frac{dI_2}{dt} \leq & -\left(\bar{b} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right)\int_0^1 \psi_x^2 dx + \frac{\rho_1}{2\varepsilon_2}\int_0^1 \varphi_t^2 dx \\ & +\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2}(h \circ \psi_x) \\ & +\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\int_0^1 q^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x) g^2(\psi_t) dx.\end{aligned} \tag{2.81}$$

Proof. By taking the derivative of I_2 with respect to t we get

$$\begin{aligned} I_2'(t) = & \int_0^1 \left(\rho_2\psi_{tt}\psi + \rho_2\psi_t^2\right) dx + \int_0^1 \left(\rho_1\varphi_{tt}w + \rho_1\varphi_t w_t\right) dx \\ & -\frac{\gamma\tau_0}{\kappa}\int_0^1 (\psi_t q + \psi q_t)\, dx \\ := & J_1 + J_2 + J_3. \end{aligned} \tag{2.82}$$

Next, using the first and the fourth equations in (2.1) we get

$$\begin{aligned} J_2 + J_3 = & -k\int_0^1 \varphi\psi_x dx + k\int_0^1 w_x^2 dx + \rho_1\int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx + \gamma\int_0^1 \psi\theta_x dx. \end{aligned} \tag{2.83}$$

Next, using the second equation in (2.1), we get

$$\begin{aligned} J_1 = & -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x \int_0^t h(t-s)\, a(x)\, \psi_x(s)\, ds dx \\ & -k\int_0^1 \psi^2 dx - k\int_0^1 \varphi_x\psi dx - \int_0^1 b(x)\,\psi g(\psi_t)\, dx - \int_0^1 \gamma\psi\theta_x dx. \end{aligned} \tag{2.84}$$

From (2.83), (2.84) and by using Lemma (2.16), we deduce

$$\begin{aligned} I_2'(t) \leq & \rho_1\int_0^1 \varphi_t w_t dx - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx \\ & -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\,\psi g(\psi_t)\, dx \\ & +\int_0^1 a(x)\,\psi_x \int_0^t h(t-s)\,\psi_x(s)\, ds dx. \end{aligned} \tag{2.85}$$

By exploiting the inequality

$$|ab| \leq \frac{\nu}{2}a^2 + \frac{1}{2\nu}b^2, \qquad a, b \in \mathbb{R},\ \nu > 0,$$

we easily find, for any $\varepsilon_2 > 0$,

$$\begin{aligned} I_2'(t) \leq\ & -\bar{b}\int_0^1 \psi_x^2 dx + \frac{\rho_1}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w_t^2\right)dx \\ & + \frac{\gamma\tau_0}{2\kappa}\int_0^1 \left(\varepsilon_2\psi_t^2 + \frac{1}{\varepsilon_2}q^2\right)dx + \frac{\delta\gamma}{2\kappa}\int_0^1 \left(\varepsilon_2\psi^2 + \frac{1}{\varepsilon_2}q^2\right)dx \\ & + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\psi g(\psi_t)\,dx \\ & + \int_0^1 a(x)\psi_x \int_0^t h(t-s)\psi_x(s)\,dsdx. \end{aligned} \tag{2.86}$$

We now proceed to the evaluation of the last two terms in the right-hand side of (2.86). First, by Young's and Poicaré's inequalities we have

$$\left|\int_0^1 b(x)\psi g(\psi_t)\,dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x)\,g^2(\psi_t)\,dx. \tag{2.87}$$

Furthermore, we have the following inequality

$$\left|\int_0^1 a(x)\psi_x \int_0^t h(t-s)\psi_x(s)\,dsdx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_2}(h \circ \psi_x). \tag{2.88}$$

Then, plugging (2.87) and (2.88) into (2.86) and using the second inequality in Lemma (2.16), there fore the assertion of Lemma (2.17) holds. □

Now, following [28], we define the functional I_3 as follows:

$$I_3(t) := -\int_0^1 (\rho_2\psi\psi_t + \rho_1\varphi\varphi_t)dx \tag{2.89}$$

Then, we have the following estimate:

Lemma 2.18. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3) Assume that **(H1)–(H4)** hold. Then, for any $\varepsilon_3 > 0$, we have*

$$\begin{aligned} I_3'(t) \leq & -\int_0^1 (\rho_2\psi_t^2 + \rho_1\varphi_t^2)dx + k\int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{1}{2\varepsilon_3}\int_0^1 b(x)h^2(\psi_t)dx + \left(\frac{\gamma}{2\varepsilon_3}\right)\int_0^1 \theta^2 dx \\ & + \left(\gamma\varepsilon_3 + \bar{b} + c\varepsilon_3\right)\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_3}(h \circ \psi_x) \end{aligned} \tag{2.90}$$

Proof. By exploiting the first equation in (2.1)

$$\begin{aligned} I_3'(t) = & -\int_0^1 (\rho_2\psi_t^2 + \rho_1\varphi_t^2)dx - k\int_0^1 \varphi(\varphi_x + \psi)_x dx \\ & -\int_0^1 \psi[\bar{b}\psi_{xx} - \int_0^t h(t-s)(a(x)\psi_x(s))_x ds - k(\varphi_x + \psi) - b(x)g(\psi_t) - \gamma\theta_x]dx \\ = & -\int_0^1 (\rho_2\psi_t^2 + \rho_1\varphi_t^2)dx + \bar{b}\int_0^1 \psi_x^2 dx - \int_0^1 a(x)\psi_x\left(\int_0^t h(t-s)\psi_s ds\right)dx \\ & + k\int_0^1 (\varphi_x + \psi)^2 dx + \int_0^1 \psi b(x)g(\psi_t)dx + \gamma\int_0^1 \psi\theta_x dx \\ \leq & -\int_0^1 (\rho_2\psi_t^2 + \rho_1\varphi_t^2)dx + k\int_0^1 (\varphi_x + \psi)^2 dx + \left(\gamma\varepsilon_3 + \bar{b} + c\varepsilon_3\right)\int_0^1 \psi_x^2 dx \\ & + \frac{c}{\varepsilon_3}(h \circ \psi_x) + \frac{1}{2\varepsilon_3}\int_0^1 b(x)g^2(\psi_t)dx + \left(\frac{\gamma}{2\varepsilon_3}\right)\int_0^1 \theta^2 dx. \end{aligned} \tag{2.91}$$

□

Now, we define the functional I_4 as follows :

$$I_4(t) := \rho_2\int_0^1 \psi_t(\varphi_x + \psi)dx + \rho_2\int_0^1 \psi_x\varphi_t dx - \frac{\rho_1}{k}\int_0^1 a(x)\varphi_t\int_0^t h(t-s)\psi_x(s)ds dx. \tag{2.92}$$

Lemma 2.19. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3) Assume that **(H1)–(H4)** and*

$$\frac{\rho_2}{\bar{b}} = \frac{\rho_1}{k} \tag{2.93}$$

hold. Then, for any $0 < \varepsilon \leq \frac{k}{1+\gamma}$, *we have*

$$
\begin{aligned}
I_4'(t) \leq & \left[\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=1} - (k-\varepsilon-\gamma\varepsilon)\int_0^1 (\varphi_x+\psi)^2 dx \\
& + \frac{\varepsilon\rho_1}{k}\int_0^1 \varphi_t^2 dx - \frac{c\rho_1}{k\varepsilon} h' \circ \psi_x + \frac{c\rho_1}{\varepsilon}\int_0^1 \psi_x^2 dx \\
& + \rho_2 \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 b(x) g^2(\psi_t) dx \\
& + \frac{c\gamma}{\varepsilon}\int_0^1 \theta_x^2 dx
\end{aligned}
\tag{2.94}
$$

Proof. By exploiting the first and second equation in (2.1) and (2.93); we have

$$
\begin{aligned}
I_4'(t) = & \int_0^1 (\varphi_x+\psi)\left[\bar{b}\psi_{xx} - \int_0^t h(t-s)(a(x)\psi_x(s))_x dx - k(\varphi_x+\psi) - b(x)g(\psi_t) - \gamma\theta_x\right] \\
& + \rho_2 \int_0^1 (\varphi_{xt}+\psi_t)\psi_t dx + \rho_2 \int_0^1 \psi_{xt}\varphi_t dx + \frac{\rho_2 k}{\rho_1}\int_0^1 \psi_x(\varphi_x+\psi)_x dx \\
& - \frac{\rho_1}{k}\int_0^1 a(x)\varphi_t\left(h(0)\psi_x + \int_0^t h'(t-s)\psi_x(s)ds\right)dx \\
& - \int_0^1 a(x)(\varphi_x+\psi)_x \int_0^t h(t-s)\psi_x(s)dsdx \\
= & \ \bar{b}\int_0^1 (\varphi_x+\psi)\psi_{xx} - \int_0^1 (\varphi_x+\psi)\int_0^t h(t-s)(a(x)\psi_x(s))_x dsdx - k\int_0^1 (\varphi_x+\psi)^2 \\
& - \int_0^1 (\varphi_x+\psi) b(x) h(\psi_t) - \gamma \int_0^1 (\varphi_x+\psi)\theta_x + \rho_2 \int_0^1 \psi_t^2 + \frac{\rho_2 k}{\rho_1}\int_0^1 \psi_x(\varphi_x+\psi)_x \\
& - \frac{\rho_1}{k}\int_0^1 (\varphi_x+\psi)_x a(x)\int_0^t h(t-s)\psi_x(s)dsdx \\
& - \frac{\rho_1}{k}\int_0^1 a(x)\varphi_t\left(h(0)\psi_x + \int_0^t h'(t-s)\psi_x(s)ds\right)dx
\end{aligned}
\tag{2.95}
$$

by Young's inequality, (2.94) is established. □

Now, following [28], we define the functionals I_5 and I_6, let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$.

$$
I_5(t) := \int_0^1 \rho_2 m(x)\psi_t\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)dx. \tag{2.96}
$$

$$
I_6(t) := \frac{1}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x dx. \tag{2.97}
$$

Lemma 2.20. *Let $(\varphi, \psi, \theta, q)$ be the solution of (2.1)–(2.3) Assume that* ***(H1)–(H4)*** *hold. Then, for any $\varepsilon > 0$, we have*

$$\begin{aligned} I_5'(t) \leq & -\left(\left(\bar{b}\psi_x(1,t) - a(1)\int_0^t h(t-s)\psi_x(1,s)ds\right)^2 + \left(\bar{b}\psi_x(0,t) - a(0)\int_0^t h(t-s)\psi_x(0,s)ds\right)^2\right) \\ & +\varepsilon k\int_0^1 (\varphi_x+\psi)^2 dx + \frac{c\rho_2}{\varepsilon}\left(\int_0^1 \psi_x^2 dx + (h\circ\psi_x)\right) \\ & +\bar{b}\rho_2 c\left(\int_0^1 (\psi_t^2 + b(x)g^2(\psi_t))dx - (h'\circ\psi_x)\right) + c\gamma\int_0^1 \theta_x^2 dx \end{aligned} \tag{2.98}$$

and

$$I_6'(t) \leq -(\varphi_x^2(1,t) - \varphi_x^2(0,t)) + \frac{c\rho_1}{k}\int_0^1 \varphi_t^2 + \frac{ck}{2}\int_0^1 (\varphi_x^2 + \psi_x^2)dx \tag{2.99}$$

Proof. By exploiting the first and second equation in (2.1) and using Young's inequality and lemma (2.3) , we have

$$\begin{aligned} I_5'(t) = & \int_0^1 \rho_2 m(x)\psi_{tt}\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_s ds\right)dx \\ & + \int_0^1 m(x)\psi_t\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_s ds\right)' dx \\ = & \int_0^1 m(x)\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)_x\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_s ds\right)dx \\ & - \int_0^1 m(x)\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)(\varphi_x + \psi + b(x)g(\psi_t) + \gamma\theta_x)dx \\ & + \int_0^1 m(x)\psi_t\left(\bar{b}\psi_{xt} - a(x)h(0)\psi_x - \int_0^t h(t-s)\psi_x(s)ds\right)dx \\ = & -\left(\left(\bar{b}\psi_x(1,t) - a(1)\int_0^t h(t-s)\psi_x(1,s)ds\right)^2 + \left(\bar{b}\psi_x(0,t) - a(0)\int_0^t h(t-s)\psi_x(0,s)ds\right)^2\right) \\ & -\frac{1}{2}\int_0^1 m'(x)\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)^2 dx \\ & - \int_0^1 m(x)\left(\bar{b}\psi_x - a(x)\int_0^t h(t-s)\psi_x(s)ds\right)(k(\varphi_x + \psi) + b(x)g(\psi_t) + \gamma\theta_x)dx \\ & -\frac{\bar{b}\rho_2}{2}\int_0^1 m'(x)\psi_t^2 dx \\ & +\rho_2\int_0^1 m(x)a(x)\psi_t\left(\int_0^t h'(t-s)(\psi_x(t) - \psi_x(s))ds\right)dx + h(t)\int_0^1 m(x)a(x)\psi_x\psi_t dx, \end{aligned} \tag{2.100}$$

Applying Young's inequality, we obtain (2.98). Similarly, we can prove the second estimate of Lemma (2.20) □

Now, we introduce the followige functionel I_7

$$I_7(t) := \varepsilon I_4(t) + \frac{1}{4\varepsilon} I_5(t) + \varepsilon I_6(t). \tag{2.101}$$

Lemma 2.21. *Let* $(\varphi, \psi, \theta, q)$ *be the solution of* (2.1)−(2.3) *Assume that* ***(H1)−(H4)*** *hold. Then, for any* $0 < \varepsilon \leq \frac{3k^2}{4c}$ *, we have*

$$\begin{aligned} I_7'(t) \leq & -(\frac{3k}{4} - \frac{c\varepsilon}{k}) \int_0^1 (\varphi_x + \psi)^2 dx + c\varepsilon\rho_1 \int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon} \int_0^1 \psi_t^2 dx \\ & + \frac{c}{\varepsilon} \int_0^1 b(x) g^2(\psi_t) dx + \frac{c}{\varepsilon} \int_0^1 \psi_x^2 dx + \frac{c\gamma}{\varepsilon} \int_0^1 \theta_x^2 dx \\ & - \frac{c}{\varepsilon}(h' \circ \psi_x) + \frac{c}{\varepsilon}(h \circ \psi_x) \end{aligned} \tag{2.102}$$

Proof. By using Lemmas (2.19) and (2.20) ,Young's and Poincare's inequalities and the fact that

$$\varphi_x^2 \leq 2(\psi + \varphi_x)^2 + 2\psi^2$$

and

$$\left(\psi_x - a(x) \int_0^t h(t-s)\psi_x(s) ds\right) \varphi_x \leq \varepsilon \varphi_x^2 + \frac{1}{4\varepsilon} \left(\psi_x - a(x) \int_0^t h(t-s)\psi_x(s) ds\right)^2$$

we obten (2.102) □

Finally, we set

$$I_8(t) := -\tau_0 \rho_3 \int_0^1 q(t,x) \left(\int_0^x \theta(t,y) dy\right) dx. \tag{2.103}$$

Lemma 2.22. *[53] Let* $(\varphi, \psi, \theta, q)$ *be the solution of* (2.1)−(2.3) *Assume that* ***(H1)−(H4)*** *hold. Then, we have for any* $\varepsilon_8 > 0$,

$$\begin{aligned} I_8'(t) \leq & \left(-\rho_3 \kappa + \frac{\varepsilon_8 \rho_3 \delta c}{2}\right) \int_0^1 \theta^2 dx + \frac{\varepsilon_8 \tau_0 \gamma}{2} \int_0^1 \psi_t^2 dx \\ & + \left(\tau_0 \kappa + \frac{\rho_3 \gamma}{2\varepsilon_8} + \frac{\tau_0 \gamma}{2\varepsilon_8}\right) \int_0^1 q^2 dx \end{aligned} \tag{2.104}$$

proof of Theorem 2.13. For $N, N_1, N_2, N_3, N_7 > 0$, we can define an auxiliary functional $\mathcal{F}$ by

$$\mathcal{F}(t) := NE(t) + N_1 I_1 + N_2 I_2 + N_3 I_3 + N_7 I_7 + N_8 I_8 \tag{2.105}$$

and let $t_0 > 0$, and $g_0 = \int_0^t g(s)ds > 0$. By combining (2.60), (2.66), (2.81), (2.90) and (2.102),(2.104) and by using the inequality

$$(\varphi_x + \psi)^2 \leq 2\varphi_x^2 + 2\psi^2$$

and Poincaré's inequality, we arrive at

$$\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} \leq\; & -N_1\left(\rho_2 g_0 - \varepsilon_1\left(\rho_2^2 + g_0\right)\right)\int_0^1 (\alpha(x) + b(x))\,\psi_t^2 dx \\
& + \left(N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) - N_3\rho_2 + \frac{N_7 c}{\varepsilon} + N_8\frac{\varepsilon_8\tau_0\gamma}{2}\right)\int_0^1 \psi_t^2 dx - N\int_0^1 b(x)\,\psi_t g(\psi_t)\,dx \\
& + \left(\frac{N_2\rho_1}{2\varepsilon_2} - N_3\rho_1 + N_7 c\varepsilon\rho_1\right)\int_0^1 \varphi_t^2 dx + \left(N_1\varepsilon_1 + \frac{N_2}{2\varepsilon_2} + \frac{N_3}{2\varepsilon_3} + \frac{cN_7}{\varepsilon}\right)\int_0^1 b(x)g^2(\psi_t)dx \\
& + N_1\left(\rho_2 h_0 - \varepsilon_1\left(\rho_2^2 + h_0\right)\right)\int_0^1 b(x)\,\psi_t^2 dx \\
& + \left\{N_1\varepsilon_1'\left(2\bar{b}^2 + 1 + 2ck^2\right) - N_2\left(\bar{b} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right) + N_3(\gamma\varepsilon_3 + \bar{b} + c\varepsilon_3)\right. \\
& \left. + 2kcN_3 + \frac{N_7 c}{\varepsilon}\left(\frac{3k}{4} - \frac{c\varepsilon}{k}\right)\right\}\int_0^1 \psi_x^2 dx \\
& + \left\{2N_1\varepsilon_1' k^2 + 2N_3 k - N_7\left(\frac{3k}{4} - \frac{c\varepsilon}{k}\right) + 2N_7\varepsilon\left(\frac{3k}{4} - \frac{c\varepsilon}{k}\right)\right\}\int_0^1 \varphi_x^2 dx \\
& + \left(\frac{N_3\gamma}{2\varepsilon_3} + \frac{N_7 C\gamma}{c\varepsilon} + N_8\left(-\rho_3\kappa + \frac{\varepsilon_8\rho_3\delta c}{2}\right)\right)\int_0^1 \theta^2 dx \\
& + \left\{cN_1\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) + cN_1\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) + \frac{N_2 c}{\varepsilon_2} + \frac{N_3 c}{\varepsilon_3} + \frac{cN_7}{\varepsilon}\right\}(h \circ \psi_x) \\
& + \left(\frac{N}{2} - \frac{cN_1}{\varepsilon_1} - \frac{cN_7}{\varepsilon}\right)(h' \circ \psi_x) \\
& + \left\{N_1\left(c\varepsilon_1 + \frac{g_0}{\varepsilon_1}\right) + N_2\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\right. \\
& \left. + N_8\left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_8} + \frac{\tau_0\gamma}{2\varepsilon_8}\right) - \delta N\right\}\int_0^1 q^2 dx
\end{aligned}$$

for all $t \geq t_0$. At this point, we have to choose our constants very carefully. First, let us take $\varepsilon_3 < 1$, $\varepsilon_1, \varepsilon_2, \varepsilon$ and ε_8 small enough such that

$$\varepsilon_1 \leq \min\left\{\left(\frac{\rho_2 g_0}{2}\right) / \left(\rho_2^2 + h_0\right), \frac{1}{4k}\right\},$$
$$\varepsilon_2 \leq \left(\frac{\bar{b}}{2}\right) / \left(2c + \frac{\delta\gamma}{2\kappa}\right),$$

$$\varepsilon \leq \min\left\{\frac{k}{1+\gamma}, \frac{3k^2}{4c}\right\},$$

and

$$\varepsilon_8 \leq \frac{2k}{\delta c}.$$

After that, we pick N_2 large enough so that

$$N_2 \geq \frac{2k\bar{b}}{\varepsilon_3}.$$

Now, by using Lemma (2.2), and choosing N_1 large enough such that

$$\frac{N_1\rho_2 h_0}{2} > \left(N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) - N_3\rho_2 + \frac{N_7 C}{\varepsilon} + N_8\frac{\varepsilon_8\tau_0\gamma}{2}\right)\frac{2}{d}$$

then, we can select $\varepsilon_1^{'}$ small enough such that

$$\varepsilon_1^{'} \leq \min\left\{\frac{1}{4N_1 k}, \left(\frac{N_2\bar{b}}{4}\right)/N_1\left(2\bar{b}^2 + 1 + 2ck^2\right)\right\}. \tag{2.106}$$

Then we take N_3 so large that

$$N_3 > \frac{N_2}{2\varepsilon_2}$$

After that, we pick N_7 so large that

$$N_7 > N_3 k$$

Next, let N_8 be large enough so that

$$N_8 > \frac{N_7 c\gamma}{\rho_3\kappa c\varepsilon}$$

Finally, we choose N large enough so that, there exist positive constants η, η_1, and η_2 such that, for $t \geq t_0$,

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} &\leq -\eta\left\{\int_0^1 (\alpha(x)+b(x))\psi_t^2dx + \int_0^1 \varphi_t^2dx\right.\\&\left.+\int_0^1 \theta^2dx + \int_0^1 q^2dx\right\} - \eta_1\int_0^1 \psi_x^2dx - \eta_2\int_0^1 \varphi_x^2dx\\&+c(h\circ\psi_x) + c\int_0^1 b(x)\left(\psi_t^2 + g^2(\psi_t)\right)dx.\end{aligned}$$

As in [28], we can find $\eta_3 > 0$ such that, for $t \geq t_0$,

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt} &\leq -\eta_3\left\{\int_0^1 (\alpha(x)+b(x))\psi_t^2dx + \int_0^1 \varphi_t^2dx + \int_0^1 \psi_x^2dx\right.\\&\left.+\int_0^1 (\varphi_x+\psi)^2dx + \int_0^1 \theta^2dx + \int_0^1 q^2dx\right\}\\&+c(h\circ\psi_x) + c\int_0^1 b(x)\left(\psi_t^2 + g^2(\psi_t)\right)dx. \qquad (2.107)\end{aligned}$$

Moreover, we have the following:

Lemma 2.23. *There exist two positive constants β_1 and β_2 depending on N, N_1, N_2, such that*

$$\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \qquad (2.108)$$

Proof. As in [28],it is clear, for (2.60), (2.65), (2.80), (2.89), (2.101), (2.105), Young's and Poincaré's inequalities, that

$$|\mathcal{F}(t) - NE(t)| \leq CE(t), \qquad \forall t \geq 0.$$

Consequently, we can choose N large enough such that $\beta_1 = N - C > 0$ and (2.107) and therefore (2.108) holds true.

□

Our goal now is to estimate the last term in the right-hand side of (2.107). Following the method presented in [28], we consider the following partition of the interval $(0, 1)$:

$$\Omega^+ = \left\{x \in (0,1) : |\psi_t| > \varepsilon'\right\} \text{ and } \Omega^- = \left\{x \in (0,1) : |\psi_t| \leq \varepsilon'\right\} \qquad (2.109)$$

where ε' is defined in (H2). By using the hypothesis (H2) , we have $|\psi_t| \leq c_1^{-1}\psi_t g(\psi_t)$ on Ω^+ and therefore taking into account the estimate (2.60) , we arrive at

$$\begin{aligned}\int_{\Omega^+} b(x)\left(\psi_t^2 + g^2(\psi_t)\right) dx &\leq c\int_{\Omega^+} b(x)\psi_t g(\psi_t)\, dx \\ &\leq c\int_0^1 b(x)\psi_t h(\psi_t)\, dx \\ &\leq -cE'(t). \end{aligned} \tag{2.110}$$

According to (H2) , we distinguish two cases:

Case 1: H is linear on $\left[0, \varepsilon'\right]$. Consequently, there exist two positive constants c_1' and c_2' such that $c_1'|s| \leq |g(s)| \leq c_2'|s|$, for all $s \in (0,1)$. Now, from (2.107) and (2.110), we arrive at

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}(t) + cE(t)) &\leq -cE(t) + c(h\circ\psi_x) \\ &= -cH_2(E(t)) + c(h\circ\psi_x), \forall t \geq t_0, \end{aligned} \tag{2.111}$$

where the function H_2 is defined by (2.59) .

Case 2: $H'(0) = 0$ and $H''(0) > 0$ on $\left[0, \varepsilon'\right]$. Let H^* denote the dual of H in the sense of Young, then we have (see [28] for more details)

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left[\left(H'\right)^{-1}(s)\right], \qquad \forall s \in \mathbb{R}_+.$$

By using Jensen's inequality, we deduce

$$\begin{aligned}\int_{\Omega^-} b(x)\left(\psi_t^2 + g^2(\psi_t)\right) dx &\leq c\int_{\Omega^-} b(x)H^{-1}(\psi_t g(\psi_t))\, dx \\ &\leq c\int_{\Omega^-} H^{-1}(b(x)\psi_t g(\psi_t))\, dx \\ &\leq cH^{-1}\left(\int_{\Omega^-} b(x)\psi_t g(\psi_t)\, dx\right) \\ &\leq cH^{-1}\left(-cE'(t)\right). \end{aligned} \tag{2.112}$$

Thus, it follows from (2.107) , (2.110) and (2.112) that

$$\mathcal{F}'(t) \leq -cE(t) + cH^{-1}\left(-cE'(t)\right) - cE'(t) + c(h\circ\psi_x), \forall t \geq t_0.$$

By using Young's inequality and the fact that

$$H^{*}(s) \leq s\left(H'\right)(s), \; E^{'}(t) \leq 0, \; H^{''} \geq 0,$$

we obtain by the same method as in [28] (we omit the details)

$$H'\left(\varepsilon_0 E(t)\right)\left(\mathcal{F}'(t) + cE'(t) + c_0 E'(t)\right) \leq -cH_2\left(E(t)\right) + c(h \circ \psi_x) \tag{2.113}$$

where ε_0 is a small positive constant and c_0 is a large positive constant. Now, let us define the following functional:

$$\mathcal{L}(t) = \begin{cases} \mathcal{F}(t) + cE(t) & \text{if } H \text{ is linear on } \left[0, \varepsilon^{'}\right] \\ H^{'}\left(\varepsilon_0 E(t)\right)\left(\mathcal{F}(t) + cE(t)\right) + c_0 E(t) & \text{if } H^{'}(0) = 0 \text{ and } H^{''} > 0 \text{ on } (0, \varepsilon^{'}]. \end{cases}$$

We can easily show that

$$\mathcal{L} \sim E.$$

On the other hand, by making use of (2.111) and (2.113) , we easily deduce that the following inequality

$$\mathcal{L}^{'}(t) \leq -cH_2\left(E(t)\right) + c(h \circ \psi_x)$$

holds for all $t \geq t_0$. By using (2.60) and (H4) , we obtain

$$\begin{aligned} \left(\xi(t)\mathcal{L}(t)\right)^{'} &= \xi^{'}(t)\mathcal{L}(t) + \xi(t)\mathcal{L}^{'}(t) \\ &\leq -c\xi(t) H_2\left(E(t)\right) - cE^{'}(t). \end{aligned}$$

Next, let $\mathcal{K}(t) = \varepsilon\left(\xi(t)\mathcal{L}(t) + cE(t)\right)$, where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\xi(t)\mathcal{L}(t) + cE(t) \leq \frac{1}{\bar{\varepsilon}}E(t), \quad \forall t \geq 0.$$

We can also show that

$$\mathcal{K} \sim E$$

and, for $t \geq t_0$,

$$\mathcal{K}^{'}(t) \leq -c\varepsilon\xi(t) H_2\left(\mathcal{K}(t)\right).$$

A simple integration of the above inequality over (t_0, t) yields

$$\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon \int_0^t \xi(s)\, ds + H_1\left(\mathcal{K}(t_0)\right) - c\varepsilon \int_0^{t_0} \xi(s)\, ds\right), \forall t \geq t_0,$$

where $H_1(t) = \int_t^1 \left(\frac{1}{H_2(t)}\right) ds$. Since $\lim_{t\to 0^+} H_1(t) = \infty$ and

$$0 \leq \mathcal{K}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(0).$$

We may choose ε small enough such that

$$H_1(F(t_0)) - c\varepsilon \int_0^{t_0} \xi(s)\, ds \geq 0.$$

Therefore, $\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon \int_0^t \xi(s)\, ds\right)$, for $t \geq t_0$. Consequently, there exist two positive constants $c^{'}$, and $c^{''}$ for which

$$\mathcal{K}(t) \leq c^{''} H_1^{-1}\left(c^{'} \int_0^t \xi(s)\, ds\right), \ \forall t \geq 0,$$

since $\mathcal{K}$ is bounded, which gives (2.58)).

This completes the proof of the Theorem (2.13) □

Chapter 3

Global existence and exponential stability of a Timoshenko system in thermoelasticity of second sound with a delay term in the internal feedback

3.1 Introduction

We investigate in this chapter, the effect of time delay and the forcing term on the following system solution's behavior :

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + f(\psi) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_t(x,t) + \kappa q_x(x,t) + \gamma\psi_{tx}(x,t) = 0, \\ \tau_0 q_t(x,t) + \delta q(x,t) + \kappa\theta_x(x,t) = 0, \end{cases} \tag{3.1}$$

where $t \in (0,\infty)$ repreents the time variable, and the space varible is represented by $x \in (0,1)$, the transverse displacement of the solid elastic material and the rotation angle are respectively represented by the functions φ and ψ. Furthermore, θ is the the temperature difference function, $q = q(t,x) \in \mathbb{R}$ is the heat flux. Moreover, ρ_1, ρ_2, ρ_3, γ, τ_0, δ, κ, μ_1, μ_2 and K are positive constants and $\tau > 0$ represents the time delay. We consider the following initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x), \quad \varphi_t(x,t-\tau) = f_0(x,t-\tau), \end{cases} \tag{3.2}$$

where $x \in (0,1)$ and $t \in (0,\tau)$.
And we have as the boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \qquad \forall t \geq 0. \tag{3.3}$$

Through this chapter seek the proof the existence and the asymptotic behavior of the solution to problem (3.1)-(3.3). Before going on, let us first review some related results which seem to us interesting.

A lot of effort have beenmade regarding stability/instability of wave equations with delay. Existing works depict that delays can destabilize a system that is asymptotically stable in their absence (see [19] for more details).

Datko [18, Example 3.5], proved that the system in the following form : form

$$\begin{cases} w_{tt} - w_{xx} - aw_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = -kw_t(1,t-\tau), & t > 0, \end{cases}$$

become unstable for any arbitrary reduced values of τ and any values of a, k, where a, k and τ are positive constants.

Afterward, they addressed [19] the following one-dimentional problem :

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, & 0 < x < 1,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases} \tag{3.4}$$

which models the vibrations of a string clamped at one end and free at the other end, where $u(x,t)$ is the displacement of the string. Also, the string is controlled by a boundary control force (with a delay) at the free end. They showed that, if the positive constants a and k satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1,$$

then the delayed feedback system (3.4) is stable for all sufficiently small delays. On the other hand, if

$$k\frac{e^{2a}+1}{e^{2a}-1} > 1,$$

then there exists a dense open set D in $(0,+\infty)$ such that for each $\tau \in D$, system (3.4) admits exponentially instable solutions.

Nicaise and Pignotti [71] examined the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0 \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau). \end{cases} \tag{3.5}$$

Using an observability inequality obtained with a Carleman estimate, they proved that, under the assumption

$$\mu_2 < \mu_1, \tag{3.6}$$

the energy is exponentially stable. On the contrary, if (3.6) does not hold, they found a sequence of delays for which the corresponding solution of (3.5) is unstable. The same results were shown if both the damping and the delay act in the boundary of the domain.

Said-Houari and Laskri [82] considered the following Timoshenko system with a delay term in the internal feedback:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{3.7}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{3.8}$$

The work in [82] has been extended to the case of time-varying delay of the form $\psi_t(x, t-\tau(t))$ by Kirane, Said-Houari and Anwar [40]. First, by using the variable norm technique of Kato, and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$, the system has been shown to be well-posed. Second, under a hypothesis between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

The Timoshenko system goes back to Timoshenko [92] in 1921 who proposed a coupled hyperbolic system which is similar to (3.7) (with $\mu_1 = \mu_2 = 0$), describing the transverse vibration of a beam, but without the presence of any damping. For a physical derivation of Timoshenko's system, we refer the reader to [23].

In the absence of the delay in system (3.7), that is for $\mu_2 = 0$, the question of the stability of the Timoshenko-type systems has received a lot of attention in the last years, and quite a number of results concerning uniform and asymptotic decay of energy have been established.

An important issue of research is to look for a minimum dissipation by which solutions of the Timoshenko system decay uniformly to zero as time goes to infinity. In this regard, several types of dissipative mechanisms have been introduced, such as: frictional damping, viscoelastic damping and thermal dissipation. We recall here only some results related to the thermal dissipation in the Timoshenko systems. The interested reader is referred to [4, 49, 50, 65, 68, 89] for the Timoshenko systems with frictional damping and to [5, 28, 48, 69] for Timoshenko systems with viscoelastic damping.

To the best of our knowledge, the paper [64] is the first paper in which the authors dealt with the Timoshenko system with thermal dissipation. More precisely, they treated the problem

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, L) \times (0, +\infty), \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } (0, L) \times (0, +\infty), \end{cases} \tag{3.9}$$

where φ, ψ and θ are functions of (x, t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions on $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

Modeling heat conduction with the so-called Fourier law (as in (3.9)), which assumes the flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q + \kappa \nabla \theta = 0, \qquad (\kappa > 0),$$

leads to the phenomenon of infinite heat propagation speed. To overcome this physical paradox in the Fourier, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made. The common feature of these theories is that all lead to hyperbolic differential equation and the speed of propagation becomes finite. See [16] for more details. Among them Cattaneo's law,

$$\tau q_t + q + \kappa \nabla \theta = 0,$$

leading to the system with *second sound*, ([52], [78], [90]) and a suggestion by Green and Naghdi [24], [25], for thermoelastic systems introducing what is called *thermoelasticity of type III*, where the constitutive equations for the heat flux is characterized by

$$q + \kappa^* p_x + \tilde{\kappa} \nabla \theta = 0, \qquad (\tilde{\kappa} > \kappa^* > 0, \quad p_t = \theta).$$

Messaoudi *et al.* [53] studied the following problem

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu \varphi_t = 0, \\ \rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \beta \theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa \theta_x = 0, \end{cases} \tag{3.10}$$

where $(x, t) \in (0, L) \times (0, \infty)$, $\varphi = \varphi(t, x)$ is the displacement vector, $\psi = \psi(t, x)$ is the rotation angle of the filament, $\theta = \theta(t, x)$ is the temperature difference, $q = q(t, x)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = k$$

and

$$\sigma_{\varphi_x \varphi_x}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

Concerning the Timoshenko systems in thermoelasticity of type III, we have the papers of Messaoudi and Said-Houari [51, 55] in which the authors proved several stability results.

More precisely, in [51], they investigated the asymptotic behavior of the problem

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, \end{cases} \tag{3.11}$$

in $(0, \infty) \times (0.1)$ and proved an exponential decay result similar to the one in [64]. We recall that the heat conduction in (3.11) is given by Green and Naghdi's theory. The same problem (3.11) with an additional damping of history type of the form

$$\int_0^\infty g(s)\psi_{xx}(x, t-s)ds \tag{3.12}$$

acting in the second equation has been analyzed in [55]. The authors of [55] proved an exponential and polynomial stability results for the equal and nonequal wave-speed propagation respectively and under conditions on the relaxation function g weaker than those in [4] and [69].

In the present chapter our objective is to extend the result of D. Ouchenane [73] to a nonlinear framework by adding a forcing term $f(\psi)$.

3.2 Well-posedness of the problem

First we give some hypotheses on the forcing term $f(\psi(x,t))$, we assume $f : \mathbb{R} \to \mathbb{R}$ satisfying .

$$\left|f\left(\psi^2\right) - f\left(\psi^1\right)\right| \leq k_0\left(\left|\psi^1\right|^\theta - \left|\psi^2\right|^\theta\right)\left|\psi^1 - \psi^2\right| \tag{3.13}$$

for all $\psi^1, \psi^2 \in \mathbb{R}$ where $k_0 > 0, \theta > 0$. In addition we assume that

$$0 \leq \tilde{f}(\psi) \leq f(\psi)\psi \quad \text{for all } \psi \in \mathbb{R} \tag{3.14}$$

with

$$\tilde{f}(z) = \int_0^z f(s)\,ds.$$

Assumptions (3.13) and (3.14) include nonlinear term of the form

$$f(\psi) \approx |\psi|^{\rho}\psi \pm |\psi|^{\alpha}\psi, \quad 0 < \alpha < \rho$$

In order to prove the well-posedness result we proceed as in [70] (see also [82]). Let us introduce the following new dependent variable

$$z(x,\rho,t) = \varphi_t(x, t-\tau\rho), \qquad x \in (0,1), \ \rho \in (0,1), \ t > 0.$$

Then, we obtain the following equation

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad (x,\rho,t) \in (0,1) \times (0,1) \times (0,+\infty).$$

Therefore, problem (3.1) can be rewritten as

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + f(\psi) + \gamma\theta_x = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{cases} \tag{3.15}$$

where $x \in (0,1)$, $\rho \in (0,1)$, and $t > 0$. The above system subjected to the following initial conditions

$$\left.\begin{array}{ll} \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), & \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), & q(x,0) = q_0(x), \end{array}\right\} \quad x \in (0,1) \tag{3.16}$$

$$z(x,0,t) = \varphi_t(x,t), \qquad x \in (0,1), \quad t > 0$$

$$z(x,1,t) = f_0(x, t-\tau), \qquad (x,t) \in (0,1) \times (0,\tau).$$

In addition to the above initial conditions, we consider the following boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \qquad \forall t \geq 0. \tag{3.17}$$

The main question to be asked here is whether problem (3.15)-(3.17) is well posed?. Our main goal in this section is to give a positive answer to this question. In other words, we give sufficient conditions that guarantee the well-posedness of problem (3.15)-(3.17). To prove this, we adopt the steps used in the paper [82] in which a Thimoshenko problem with a frictional damping has been investigated.

In order to use the semigroup approach, we rewrite system (3.15)-(3.17) as a first order system. To this end, let $U = (\varphi, \varphi_t, \psi, \psi_t, \theta, q, z)^T$, and rewrite (3.15)-(3.17) as

$$\begin{cases} U' = \mathscr{A}U + \tilde{F}, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_{1,}\theta, q, f_0(., -.\tau))^T, \end{cases} \tag{3.18}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \theta \\ q \\ z \end{pmatrix} = \begin{pmatrix} u \\ K/\rho_1(\varphi_{xx}+\psi_x) - \mu_1/\rho_1 u - \mu_2/\rho_1 z(.,1) \\ v \\ b/\rho_2\psi_{xx} - K/\rho_2(\varphi_x+\psi) - \gamma/\rho_2\theta_x \\ -\kappa/\rho_3 q_x - \gamma/\rho_3 v_x \\ -\delta/\tau_0 q - \kappa/\tau_0\theta_x \\ -(1/\tau) z_\rho \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/\rho_2 f(\psi) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with domain

$$D(\mathscr{A}) = \left\{(\varphi, u, \psi, v, \theta, q, z)^T \in H : \ u \equiv z(.,0), \text{ in } (0,1)\right\}, \tag{3.19}$$

where

$$\begin{aligned} H \ : \ &= \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \\ &\times H^1(0,1) \times H_0^1(0,1) \times L^2\left((0,1); H^1(0,1)\right). \end{aligned}$$

The energy space $\mathscr{H}$ is defined as

$$\begin{aligned} \mathscr{H} \ : \ &= H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \\ &\times L^2(0,1) \times L^2\left((0,1); L^2(0,1)\right). \end{aligned}$$

For $U = (\varphi, u, \psi, v, \theta, q, z)^T$, $\overline{U} = \left(\overline{\varphi}, \overline{u}, \overline{\psi}, \overline{v}, \overline{\theta}, \overline{q}, \overline{z}\right)^T$ and for ζ a positive constant satisfying

$$\tau\mu_2 \leq \xi \leq \tau\left(2\mu_1 - \mu_2\right), \tag{3.20}$$

we define the following inner product in $\mathscr{H}$

$$\begin{aligned}\langle U, \overline{U}\rangle_{\mathscr{H}} &= \int_0^1 \left\{\rho_1 u\overline{u} + \rho_2 v\overline{v} + K\left(\varphi_x + \psi\right)\left(\overline{\varphi}_x + \overline{\psi}\right) + b\psi_x\overline{\psi}_x + \rho_3\theta\overline{\theta}\right\} dx \\ &\quad + \int_0^1 \tau_0 q\overline{q}dx + \xi\int_0^1\int_0^1 z\left(x,\rho\right)\overline{z}\left(x,\rho\right)d\rho dx.\end{aligned}$$

Our existence and uniqueness result reads as follows.

Theorem 3.1. *Assume that (3.13), (3.14) and $\mu_2 \leq \mu_1$, then for any $U_0 \in \mathscr{H}$, there exists a unique solution $U \in C\left([0,+\infty), \mathscr{H}\right)$ of problem (3.15)-(3.17). Moreover if $U_0 \in D\left(\mathscr{A}\right)$, then*

$$U \in C\left([0,+\infty), D\left(\mathscr{A}\right)\right) \cap C^1\left([0,+\infty), \mathscr{H}\right).$$

Proof. In order to prove Theorem 3.1, we use the semigroup approach. That is, we show that the operator $\mathscr{A}$ generates a C_0-semigroup in $\mathscr{H}$. In this step, we prove that the operator $\mathscr{A}$ is dissipative. Indeed, for $U = (\varphi, u, \psi, v, \theta, q, z)^T \in D\left(\mathscr{A}\right)$, we have

$$\begin{aligned}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} &= -\delta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \mu_2\int_0^1 z\left(x,1\right)udx \\ &\quad -\frac{\xi}{\tau}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)d\rho dx.\end{aligned} \tag{3.21}$$

Looking now to the last term in the right-hand side of (3.21), we have

$$\begin{aligned}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)d\rho dx &= \int_0^1\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z^2\left(x,\rho\right)d\rho dx \\ &= \frac{1}{2}\int_0^1 \left\{z^2\left(x,1\right) - z^2\left(x,0\right)\right\}dx.\end{aligned} \tag{3.22}$$

Consequently, (3.21) becomes

$$\begin{aligned}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} &= -\delta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \int_0^1 z\left(x,1\right)udx \\ &\quad -\frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx + \frac{\xi}{2\tau}\int_0^1 u^2\left(x\right)dx.\end{aligned} \tag{3.23}$$

By using Young's inequality we obtain, from (3.23),

$$\begin{aligned}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} &\leq -\delta \int_0^1 q^2 dx + \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\right)\int_0^1 v^2(x)\,dx \\ &\quad + \left(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)\int_0^1 z^2(x,1)\,dx.\end{aligned}$$

Keeping in mind condition (3.20), we observe that

$$-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \leq 0, \qquad \frac{\mu_2}{2} - \frac{\xi}{2\tau} \leq 0.$$

Consequently, the operator $\mathscr{A}$ is dissipative.

Now we prove that the operator $\lambda I - \mathscr{A}$ is surjective for $\lambda > 0$. For this purpose, we take an element $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathscr{H}$, we seek $U = (\varphi, u, \psi, v, \theta, q, z)^T \in D(\mathscr{A})$, solution to the problem

$$\lambda U - \mathscr{A}U = F \tag{3.24}$$

or equivalently

$$\begin{cases}\lambda\varphi - u = f_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) + \dfrac{\mu_1}{\rho_1}u + \dfrac{\mu_2}{\rho_1}z(.,1) = f_2, \\ \lambda\psi - v = f_3, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{1}{\rho_2}f(\psi) + \dfrac{\gamma}{\rho_2}\theta_x = f_4, \\ \lambda\theta + \dfrac{\kappa}{\rho_3}q_x + \dfrac{\gamma}{\rho_3}v_x = f_5 \\ \lambda q + \dfrac{\delta}{\tau_0}q + \dfrac{\kappa}{\tau_0}\theta_x = f_6 \\ \lambda z + \dfrac{1}{\tau}z_\rho = f_7.\end{cases} \tag{3.25}$$

Suppose that we have found φ and ψ with the appropriate regularity. Therefore, the first and the third equations in (3.25) yield

$$\begin{cases}u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_3.\end{cases} \tag{3.26}$$

It is clear that $u \in H_0^1(0,1)$, and $v \in H_0^1(0,1)$. Furthermore, we can find z as

$$z(x,0) = u(x), \qquad \text{for } x \in (0,1). \tag{3.27}$$

Following the same approach as in [70], we obtain, by using the last equation in (3.25),

$$z(x,\rho) = u(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^{\rho} f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma.$$

From (3.26), we obtain

$$z(x,\rho) = \lambda\varphi(x)e^{-\lambda\rho\tau} - f_1 e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^{\rho} f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma. \tag{3.28}$$

From (3.28), we have

$$z(x,1) = \lambda\varphi(x)e^{-\lambda\tau} + z_0(x),$$

where $x \in (0,1)$ and

$$z_0(x) = -f_1 e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^{\rho} f_7(x,\sigma)e^{\lambda\sigma\tau}d\sigma. \tag{3.29}$$

It is clear from the above formula that z_0 depends only on f_i, $i = 1,7$.

By using (3.25) and (3.26) the functions φ, ψ, θ and q satisfying the following system

$$\begin{cases} \left(\lambda^2 + \dfrac{\mu_1}{\rho_1}\lambda + \lambda e^{-\lambda\tau}\dfrac{\mu_2}{\rho_1}\right)\varphi - \dfrac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2 + \left(\lambda + \dfrac{\mu_1}{\rho_1}\right)f_1 - \dfrac{\mu_2}{\rho_1}z_0(x), \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) + \dfrac{1}{\rho_2}f(\psi) + \dfrac{\gamma}{\rho_2}\theta_x = f_4 + \lambda f_3, \\ \lambda\theta + \dfrac{\kappa}{\rho_3}q_x + \dfrac{\gamma\lambda}{\rho_3}\psi_x = f_5 + \dfrac{\gamma}{\rho_3}f_{3x}, \\ \lambda q + \dfrac{\delta}{\tau_0}q + \dfrac{\kappa}{\tau_0}\theta_x = f_6. \end{cases} \tag{3.30}$$

Solving system (3.30) is equivalent to finding $(\varphi,\psi,\theta,q) \in H^2(0,1)\cap H_0^1(0,1)\times H^2(0,1)\cap H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$ so that

$$\begin{cases} \int_0^1 \left(\left(\lambda^2\rho_1+\mu_1\lambda+\lambda e^{-\lambda\tau}\mu_2\right)\varphi w+K(\varphi_x+\psi)w_x\right)dx=\int_0^1(\rho_1 f_2+(\lambda\rho_1+\mu_1)f_1-\mu_2 z_0(x))\,w dx,\\ \int_0^1\left(\rho_2\lambda^2\psi\chi+b\psi_x\chi_x+K(\varphi_x+\psi)\chi+f(\psi)\chi+\gamma\theta_x\chi\right)dx=\int_0^1\rho_2(f_4+\lambda f_3)\chi dx,\\ \int_0^1(\rho_3\lambda\theta w_1+\kappa q_x w_1+\gamma\lambda\psi_x w_1)\,dx=\int_0^1(\rho_3 f_5+\gamma f_{3x})\,w_1 dx,\\ \int_0^1(\tau_0\lambda+\delta)\,q\chi_1+\kappa\theta_x\chi_1=\int_0^1\tau_0 f_6\chi_1 dx, \end{cases} \tag{3.31}$$

for all $(w,\chi,w_1,\chi_1)\in H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$.

Consequently, problem (3.31) is equivalent to the problem

$$\zeta\left((\varphi,\psi,\theta,q),(w,\chi,w_1,\chi_1)\right)=l\,(w,\chi,w_1,\chi_1), \tag{3.32}$$

where the bilinear from $\zeta:\left(H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)\right)^2\to\mathbb{R}$ and the linear from $l: H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)\to\mathbb{R}$ are defined by

$$\begin{aligned} \zeta\left((\varphi,\psi,\theta,q),(w,\chi,w_1,\chi_1)\right) &= \int_0^1\left(\left(\lambda^2\rho_1+\mu_1\lambda+\lambda e^{-\lambda\tau}\mu_2\right)\varphi w+K(\varphi_x+\psi)(w_x+\chi)\right)dx\\ &\quad+\int_0^1\left(\rho_2\lambda^2\psi\chi+b\psi_x\chi_x+f(\psi)\chi+\gamma\theta_x w_{1x}\right)dx\\ &\quad+\int_0^1(\rho_3\lambda\theta w_1+\kappa q_x\chi_{1x}+\gamma\lambda\psi_x\chi_x)\,dx\\ &\quad+\int_0^1(\tau_0\lambda+\delta)\,q\chi_1+\kappa\theta_x w_{1x}, \end{aligned}$$

and

$$\begin{aligned} l\,(w,\chi,w_1,\chi_1) &= \int_0^1(\rho_1 f_2+(\lambda\rho_1+\mu_1)f_1-\mu_2 z_0(x))\,w dx+\int_0^1\rho_2(f_4+\lambda f_3)\chi dx\\ &\quad+\int_0^1(\rho_3 f_5+\gamma f_{3x})\,w_1 dx+\int_0^1\tau_0 f_6\chi_1 dx, \end{aligned} \tag{3.33}$$

where $z_0(x)$ satisfies the equation in (3.29).

From (3.13) and (3.14) it is easy to verify that ζ is continuous and coercive, and l is continuous, so applying the Lax-Milgram theorem, we deduce that for all $(w,\chi,w_1,\chi_1)\in H_0^1(0,1)\times H_0^1(0,1)\times H^1(0,1)\times H_0^1(0,1)$, problem (3.32) admits a unique solution

$(\varphi, \psi, \theta, q) \in H_0^1(0,1) \times H_0^1(0,1) \times H^1(0,1) \times H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (3.31) that $(\varphi, \psi, \theta, q) \in H^2(0,1) \times H^2(0,1) \times H^1(0,1) \times H_0^1(0,1)$. Therefore, the operator $\lambda I - \mathscr{A}$ is surjective for any $\lambda > 0$.
Consequently, we can infer that the operator $\mathscr{A}$ is m-dissipative in $\mathscr{H}$.
Now, we prove that the operator $\tilde{F}$ defined in (3.18) is locally Lipschitz in $\mathscr{H}$.
Let $U = (\varphi, u, \psi, v, \theta, q, z)^T$ and $U_1 = (\varphi_1, u_1, \psi_1, v_1, \theta_1, q_1, z_1)^T$, then we have

$$\|\tilde{F}(U) - \tilde{F}(U_1)\|_{\mathscr{H}} \leq \|f(\psi) - f(\psi_1)\|_{L^2}.$$

By using (3.13), Hlder and Poincar inqualities, we get

$$\|f\left(\psi^2\right) - f\left(\psi^1\right)\|_{L^2} \leq k_0 \left(\|\psi^1\|^\theta - \|\psi^2\|^\theta\right) \|\psi^1 - \psi^2\| \leq C\|\psi_x^1 - \psi_x^2\|_{\mathscr{H}},$$

which gives us

$$\|\tilde{F}(U) - \tilde{F}(U_1)\|_{\mathscr{H}} \leq \|U - U_1\|_{\mathscr{H}}.$$

Then the operator $\tilde{F}$ is locally Lipschitz in $\mathscr{H}$.

Since $D(\mathscr{A})$ is dence in $\mathscr{H}$, thus we can conclude that the operator $\mathscr{A}$ is the infinitesimal generator of a C^0-semigroup in $\mathscr{H}$ by the Lumer-Phillips theorem (see, for exaple Pazy [77]). The proof of Theorem 3.1 is complete. □

3.3 Exponential stability for $\mu_1 > \mu_2$

In this section, we show that, under the assumption $\mu_1 > \mu_2$, the solution of problem (3.15)-(3.17) decays exponentially, independently of the wave speed assumption[1]. To achieve our goal we use the energy method to produce a suitable Lyapunov functional which leads to an exponential decay result.

In order to use the Poincaré inequality for θ, we introduce, as in [78],

$$\overline{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

[1] The wave speed assumption is significant only from the mathematical point of view since in practice the velocities of waves propagations may be different, see [47]. So, it is very interesting to obtain some stability results for the Timoshenko systems without the wave speed condition.

Then by the third equation in (3.1) we easily verify that

$$\int_0^1 \overline{\theta}(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\overline{\theta}$. On the other hand $(\varphi, \psi, \overline{\theta}, q, z)$ satisfies the same system (3.15) and the boundary conditions (3.17). For ξ satisfying

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2), \tag{3.34}$$

we define the functional energy of the solution of problem (3.15)-(3.17) as

$$\begin{aligned} E(t) &= E(t, z, \varphi, \psi, \theta, q) \\ &= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + \frac{1}{2}\int_0^1 \left\{K(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2\right\}dx \\ &\quad + \frac{1}{2}\int_0^1 \tau_0 q^2 dx + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx + \int_0^1 \tilde{f}(\psi(t))dx. \end{aligned} \tag{3.35}$$

We multiply the first equation in (3.15) by φ_t, the second equation by ψ_t, the third equation in (3.15) by θ, and the fourth equation in (3.15) by q, we integrate by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \left\{K(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\right\}dx + \frac{d}{dt}\tilde{f}(\psi(t)) \\ &= -\delta\int_0^1 q^2 dx - \mu_1\int_0^1 \varphi_t^2(x,t)\,dx - \mu_2\int_0^1 \varphi_t(x,t)\,z(x,1,t)\,dx + f(\psi)\psi_t. \end{aligned} \tag{3.36}$$

Now, multiplying the last equation in (3.15) by $(\xi/\tau)\,z$, integrate the result over $(0,1) \times (0,1)$ with respect to ρ and x respectively, we obtain

$$\begin{aligned} &\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx \\ &= -\frac{\xi}{\tau}\int_0^1\int_0^1 z z_\rho(x,\rho,t)\,d\rho dx = -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho dx \\ &= \frac{\xi}{2\tau}\int_0^1 \left(z^2(x,0,t) - z^2(x,1,t)\right)dx. \end{aligned} \tag{3.37}$$

From (3.35), (3.36) and (3.37), we get

$$\begin{aligned} \frac{dE(t)}{dt} &= -\delta\int_0^1 q^2 dx - \left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^1 \varphi_t^2(x,t)\,dx \\ &\quad -\frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx - \mu_2\int_0^1 \varphi_t(x,t)\,z(x,1,t)\,dx + f(\psi)\psi_t. \end{aligned} \tag{3.38}$$

Now, using Young's inequality, (3.38) can be rewritten as

$$\begin{aligned}\frac{dE(t)}{dt} &\leq -\delta\int_0^1 q^2dx - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 \varphi_t^2(x,t)\,dx \\ &\quad -\left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)\,dx + f(\psi)\psi_t.\end{aligned}$$

Then, by using (3.14),(3.34), we deduce that there exists $C > 0$ such that

$$\frac{dE(t)}{dt} \leq -\delta\int_0^1 q^2dx - C\left\{\int_0^1 \varphi_t^2(x,t)\,dx + \int_0^1 z^2(x,1,t)\,dx\right\}. \tag{3.39}$$

The last inequality implies that the energy E is a non-increasing function with respect to t.

Let us now state our main result:

Theorem 3.2. *Assume that* (3.13)*,* (3.14) *and* $\mu_2 < \mu_1$*. Then there exist two positive constants* C *and* γ *independent of t such that for any solution of problem (3.15)-(3.17), we have*

$$E(t) \leq Ce^{-\gamma t}, \qquad \forall t \geq 0. \tag{3.40}$$

To derive the exponential decay of the solution, it is enough to construct a functional $L(t)$, equivalent to the energy $E(t)$, and satisfying

$$\frac{dL(t)}{dt} \leq -\Lambda L(t), \qquad \forall t \geq 0,$$

for some constant $\Lambda > 0$.

In order to obtain such functional L, we need several Lemmas.

First, let us consider the functional I_1 given by

$$I_1(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx. \tag{3.41}$$

Then we have the following estimate.

Lemma 3.3. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of (3.15)-(3.17). Then we have for any* $\varepsilon_1 > 0$,

$$\begin{aligned} I_1(t) \quad &\leq \quad \left(-K + \varepsilon_1 \left(\frac{K}{2} + \frac{\mu_2 c}{2}\right)\right) \int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_1} \int_0^1 \psi_x^2 dx \\ &+ \frac{\mu_2}{2\varepsilon_1} \int_0^1 z^2(x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx, \end{aligned} \tag{3.42}$$

where $c = 1/\pi^2$ *is the Poincaré constant.*

Proof. By taking the derivative of (3.41) with respect to t, we conclude

$$\frac{dI_1(t)}{dt} = \int_0^1 \rho_1 \varphi_{tt} \varphi dx + \rho_1 \int_0^1 \varphi_t^2 dx + \mu_1 \int_0^1 \varphi \varphi_t dx.$$

Then, by using the first equation in (3.15), we find

$$\frac{dI_1(t)}{dt} = K \int_0^1 (\varphi_x + \psi)_x \varphi dx - \mu_2 \int_0^1 \varphi z(x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

Consequently, we arrive at

$$\frac{dI_1(t)}{dt} = -K \int_0^1 (\varphi_x + \psi) \varphi_x dx - \mu_2 \int_0^1 \varphi z(x, 1, t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

Applying Young's inequality and Poincaré's inequality, we find (3.41). This completes the proof of Lemma 3.3. □

Now, Let w be the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0. \tag{3.43}$$

then we get

$$w(x, t) = -\int_0^x \psi(y, t)\, dy + x \left(\int_0^1 \psi(y, t)\, dy \right).$$

We have the following inequalities.

Lemma 3.4. *The solution of (3.43) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx$$

Proof. We multiply equation (3.43) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

Next, we differentiate (3.43) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 3.4. □

Let w be the solution of (3.43). We introduce the following functional

$$I_2(t) := \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma\tau_0}{\kappa}\psi q\right) dx. \tag{3.44}$$

Then we have the following estimate.

Lemma 3.5. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of (3.15)-(3.17). Then we have for any* $\varepsilon_2 > 0$,

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq & \left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{c\mu_2\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2\kappa} + c_1\right)\int_0^1 \psi_x^2 dx + \frac{\mu_2}{2\varepsilon_2}\int_0^1 z^2(x,1,t) \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\int_0^1 q^2 dx \end{aligned} \tag{3.45}$$

where $c, c_1 > 0$

Proof. By taking the derivative of (3.44), we conclude

$$\begin{aligned} \frac{dI_2(t)}{dt} = & -b\int_0^1 \psi_x^2 dx + K\int_0^1 \varphi\psi_x dx - K\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx - K\int_0^1 \varphi_x w_x dx \\ & -K\int_0^1 \psi w_x dx - \int_0^1 f(\psi)\psi dx - \mu_1\int_0^1 \varphi_t w dx - \mu_2\int_0^1 z(x,1,t)\, w dx + \rho_1\int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx. \end{aligned}$$

Then, using (3.43) and the first inequality in Lemma 3.4, we get

$$
\begin{aligned}
\frac{dI_2(t)}{dt} \leq\ & -b\int_0^1 \psi_x^2 dx + K\int_0^1 \varphi\psi_x dx - K\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx + K\int_0^1 \varphi_x\psi dx \\
& + K\int_0^1 \psi^2 dx - \int_0^1 f(\psi)\psi dx - \mu_1\int_0^1 \varphi_t w dx - \mu_2\int_0^1 z(x,1,t)\, w dx + \rho_1\int_0^1 \varphi_t w_t dx \\
& - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx.
\end{aligned}
$$

We apply Young's inequality, Poincaré's inequality and using the inequalities in Lemma 3.4, we find (3.45), such that by using (3.13) we obtain

$$\begin{aligned}\int_0^1 |f(\psi)\psi| dx &\leq \int_0^1 |\psi|^\theta |\psi||\psi| dx \\ &\leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \\ &\leq c_1 \int_0^1 \psi_x^2 dx.\end{aligned}$$

This completes the proof of Lemma 3.5. □

Now, following [82], we define the functional

$$I_3(t) := \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx. \tag{3.46}$$

Then the following result holds.

Lemma 3.6. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (3.15)-(3.17), then we have*

$$\frac{dI_3(t)}{dt} \leq -I_3(t) - \frac{c_1}{2\tau} \int_0^1 z^2(x,1,t)\, dx + \frac{1}{2\tau} \int_0^1 \psi_t^2(x,t)\, dx, \tag{3.47}$$

where c_1 is a positive constant.

Proof. Differentiating (3.46) with respect to t and using the last equation in (3.15), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx\right) &= -\frac{1}{\tau} \int_0^1 \int_0^1 e^{-2\tau\rho} z z_\rho(x,\rho,t)\, d\rho dx \\ &= -\int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx \\ &\quad -\frac{1}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho dx.\end{aligned}$$

The above estimate implies that there exists a positive constant c_1 such that (3.47) holds. □

In order to obtain a negative term of $\int_0^1 \psi_t^2 dx$, we introduce, the following functional: (see [53])

$$I_4(t) := \rho_2 \rho_3 \int_0^1 \left(\int_0^x \theta(t,y) dy\right) \psi_t(t,x)\, dx. \tag{3.48}$$

Then we have the following estimate.

Lemma 3.7. *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.15)-(3.17). Then for any ε_4, $\varepsilon_4'>0$, we have*

$$\begin{aligned}\frac{d}{dt}I_4(t) \leq\ & \left(-\gamma\rho_2+\frac{\varepsilon_4\rho_2\kappa}{2}\right)\int_0^1\psi_t^2dx+\left(\frac{\varepsilon_4'\rho_3}{2}(b+\kappa c)+\frac{\kappa\rho_3 c}{\varepsilon_5}\right)\int_0^1\psi_x^2dx\\ &+\frac{\varepsilon_4'\kappa\rho_3 c}{2}\int_0^1\varphi_x^2dx+\left(\gamma\rho_3+\frac{\rho_3}{2\varepsilon_4'}(b+2\kappa)+\kappa\rho_3\varepsilon_5\right)\int_0^1\theta^2dx\\ &+\frac{\rho_2\kappa}{2\varepsilon_4}\int_0^1 q^2dx.\end{aligned}\tag{3.49}$$

Proof. Differentiating (3.48) and using the third equation in (3.15), we have

$$\begin{aligned}\frac{d}{dt}I_4(t) &= \int_0^1\left(\int_0^x\rho_3\theta_t dy\right)\rho_2\psi_t dx+\int_0^1\left(\int_0^x\rho_3\theta dy\right)\rho_2\psi_{tt}dx\\ &= -\int_0^1\left(\int_0^x(\kappa q_x+\gamma\psi_{tx})\,dy\right)\rho_2\psi_t dx\\ &\quad+\int_0^1\left(\int_0^x\rho_3\theta dy\right)(b\psi_{xx}-\kappa(\varphi_x+\psi)-f(\psi)-\gamma\theta_x)\,dx,\\ &= -\gamma\rho_2\int_0^1\psi_t^2dx-\rho_2\kappa\int_0^1 q\psi_t dx-b\rho_3\int_0^1\theta\psi_x dx\\ &\quad+\kappa\rho_3\int_0^1\theta\varphi dx-\kappa\rho_3\int_0^1\left(\int_0^x\theta dy\right)\psi dx-\int_0^1\left(\int_0^x\rho_3\theta dy\right)f(\psi)dx+\gamma\rho_3\int_0^1\theta^2dx,\end{aligned}$$

By using Young's inequality and Poincaré's inequality, we obtain (3.49). □

Now, in order to obtain a negative term of $\int_0^1\theta^2dx$ we introduce the following functional

$$I_5(t):=-\tau_0\rho_3\int_0^L q(t,x)\left(\int_0^x\theta(t,y)dy\right)dx.\tag{3.50}$$

Then we have the following estimate.

Lemma 3.8. *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.15)-(3.17). Then for any $\varepsilon_5,\varepsilon_5'>0$, we have*

$$\begin{aligned}\frac{dI_5(t)}{dt} \leq\ & \left(-\rho_3\kappa+\frac{\varepsilon_5\rho_3\delta c}{2}\right)\int_0^1\theta^2dx+\frac{\varepsilon_5'\tau_0\gamma}{2}\int_0^1\psi_t^2dx\\ &+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_5}+\frac{\tau_0\gamma}{2\varepsilon_5'}\right)\int_0^1 q^2dx.\end{aligned}\tag{3.51}$$

The above Lemma was proved in [53, Inequality (33)].

Proof of Theorem 3.2

To prove Theorem 3.2, we define for $N,\ N_2,\ N_4,\ N_5 > 0$, the Lyapunov functional L:

$$L(t) := NE(t) + I_1(t) + N_2 I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t). \tag{3.52}$$

Now, combining (4.20), (3.42), (3.45), (3.47), (3.49) and (3.51), we get

$$\begin{aligned}
\frac{d}{dt}L(t) \leq\ & \left\{\frac{K}{2\varepsilon_1} + N_2\left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{c\mu_2\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2\kappa} + c_1\right) + N_4\left(\frac{\varepsilon_4'\rho_3}{2}(b+\kappa c) + \frac{\kappa\rho_3 c}{\varepsilon_5}\right)\right\}\int_0^1 \psi_x^2 dx \\
& + \left\{-K + \varepsilon_1\left(\frac{K}{2} + \frac{\mu_2 c}{2}\right) + N_4\frac{\varepsilon_4'\kappa\rho_3 c}{2}\right\}\int_0^1 \varphi_x^2 dx - I_3(t) \\
& + \left\{-CN + \frac{\mu_2}{2\varepsilon_1} + N_2\frac{\mu_2}{2\varepsilon_2} - \frac{c_1}{2\tau}\right\}\int_0^1 z^2(x,1,t)\,dx \\
& + \left\{-CN + N_2\left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right) + \rho_1\right\}\int_0^1 \varphi_t^2 dx \\
& + \left\{N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{1}{2\tau} + N_4\left(-\gamma\rho_2 + \frac{\varepsilon_4\rho_2\kappa}{2}\right) + N_5\frac{\varepsilon_5'\tau_0\gamma}{2}\right\}\int_0^1 \psi_t^2 dx \\
& + \left\{-N\delta + N_2\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right) + N_4\frac{\rho_2\kappa}{2\varepsilon_4} + N_5\left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\right\}\int_0^1 q^2 dx \\
& + \left\{N_4\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2\kappa) + \kappa\rho_3\varepsilon_5\right) + N_5\left(-\rho_3\kappa + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\right\}\int_0^1 \theta^2 dx.
\end{aligned} \tag{3.53}$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_1, \varepsilon_2, \varepsilon_4$ and ε_5 small enough, such that

$$\varepsilon_2\left(\frac{c\mu_1}{2} + \frac{c\mu_2}{2} + \frac{\delta\gamma c}{2\kappa}\right) \leq \frac{b - c_1}{2}, \quad \varepsilon_1\left(\frac{K}{2} + \frac{\mu_2 c}{2}\right) \leq \frac{K}{2}, \quad \varepsilon_4 \leq \frac{\gamma}{\kappa}, \quad \varepsilon_5 \leq \frac{\kappa}{\delta c}.$$

After that, we can choose N_2 large enough such that

$$N_2 \geq \frac{2K}{b\varepsilon_1}.$$

Moreover, we pick N_4 large enough so that

$$N_4\frac{\gamma\rho_2}{4} \geq N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{1}{2\tau}.$$

Once N_2 and N_4 are fixed, we take ε_4' small enough such that

$$\varepsilon_4' \leq \min\left\{\frac{N_2 b}{4N_4\left(\rho_3(b+\kappa c) + \kappa\rho_3 c/\varepsilon_5\right)}, \frac{K}{2N_4\kappa\rho_3 c}\right\}.$$

Next, let N_5 be large enough such that

$$\frac{N_5\rho_3\kappa}{4} \geq N_4\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2\kappa) + \kappa\rho_3\varepsilon_5\right).$$

After that, we fix ε_5' small enough such that

$$\varepsilon_5' \leq \frac{N_4\gamma\rho_2}{4N_5\tau_0\gamma}.$$

Finally, once all the above constants are fixed, we choose N large enough such that

$$\begin{cases} \dfrac{CN}{2} \geq \max\left\{\dfrac{\mu_2}{2\varepsilon_1} + N_2\dfrac{\mu_2}{2\varepsilon_2}, N_2\left(\dfrac{\mu_1}{2\varepsilon_2} + \dfrac{\rho_1}{2\varepsilon_2}\right) + \rho_1\right\}, \\ \dfrac{N\delta}{2} \geq N_2\left(\dfrac{\gamma\tau_0}{2\kappa\varepsilon_2} + \dfrac{\delta\gamma}{2\kappa\varepsilon_2}\right) + N_4\dfrac{\rho_2\kappa}{2\varepsilon_4} + N_5\left(\tau_0\kappa + \dfrac{\rho_3\delta}{2\varepsilon_5} + \dfrac{\tau_0\gamma}{2\varepsilon_5'}\right). \end{cases}$$

Consequently, there exists a positive constant η_1, such that (3.53) becomes

$$\frac{d}{dt}L(t) \leq -\eta_1\int_0^1\left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x+\psi)^2 + \theta^2 + q^2\right)dx - \eta_1\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx, \tag{3.54}$$

which implies by (3.35), that there exists also $\eta_2 > 0$, such that

$$\frac{d}{dt}L(t) \leq -\eta_2 E(t), \quad \forall t \geq 0. \tag{3.55}$$

Moreover, we have the following:

Lemma 3.9. *For N large enough, there exist two positive constants β_1 and β_2 depending on N, N_1, N_2, N_4, N_5, ε_1, ε_2, ε_4, ε_4', ε_5 and ε_5' such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{3.56}$$

Proof. The proof of Lemma 3.9, can be shown with the same method as in [53, Inequality (29)], with small modifications. For convenience of the reader, we give the proof here. Indeed, let

$$H(t) = I_1(t) + N_2I_2(t) + I_3(t) + N_4I_4(t) + N_5I_5(t)$$

and show that

$$|H(t)| \leq CE(t),$$

for some constant $C > 0$. From (3.41), (3.44), (3.46), (3.48) and (3.50) we obtain

$$\begin{aligned} H(t) \leq & \left| \int_0^1 \rho_1 \varphi_t \varphi dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx \right| + N_2 \left| \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q \right) dx \right| \\ & + \left| \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t) \, d\rho dx \right| + N_4 \left| \rho_2 \rho_3 \int_0^1 \left(\int_0^x \theta(t,y) dy \right) \psi_t(t,x) \, dx \right| \\ & + N_5 \left| -\tau_0 \rho_3 \int_0^1 q(t,x) \left(\int_0^x \theta(t,y) dy \right) dx \right|. \end{aligned}$$

By using, the trivial relation

$$\int_0^1 \varphi^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 \, dx + 2 \int_0^1 \psi_x^2 dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| \leq & \ \alpha_1 \int_0^1 \varphi_t^2 dx + \alpha_2 \int_0^1 \psi_t^2 dx + \alpha_3 \int_0^1 (\varphi_x + \psi)^2 \, dx + \alpha_4 \int_0^1 \psi_x^2 dx + \alpha_5 \int_0^1 \theta^2 dx \\ & + \alpha_6 \int_0^1 q^2 dx + \int_0^1 \int_0^1 z^2(x,\rho,t) \, d\rho dx, \end{aligned} \tag{3.57}$$

where the positive constants $\alpha_1, ..., \alpha_6$ are determined as follows:

$$\begin{cases} \alpha_1 = \frac{1}{2}(\rho_1 + N_2\rho_1), \quad \alpha_2 = \frac{1}{2}(N_2\rho_2 + N_4\rho_2\rho_3), \quad \alpha_3 = \rho_1 c, \\ \alpha_4 = \frac{1}{2}\left(\frac{N_2\gamma\tau_0 c}{\kappa} + N_2\rho_1 c^2 + N_2\rho_2 c\right), \quad \alpha_5 = \frac{1}{2}(N_4\rho_2\rho_3 c + N_5\tau_0\rho_3 c), \\ \alpha_6 = \frac{1}{2}\left(N_2\frac{\gamma\tau_0}{\kappa} + N_5\tau_0\rho_3\right). \end{cases}$$

According to (3.57), we have

$$|H(t)| \leq \hat{C} E(t)$$

for

$$\hat{C} = \frac{\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}}{\min\{\rho_1, \rho_2, \rho_3, K, b, \kappa, 1, \gamma, \delta, \tau_0\}}.$$

Thus, we obtain

$$L - NE(t) \leq \hat{C} E(t).$$

So, we can choose N large enough so that $\beta_1 = N - \hat{C}$, $\beta_2 = N + \hat{C} > 0$. Then (3.56) holds true.

Combining (3.55) and (3.56), we conclude that there exists $\Lambda > 0$, such that

$$\frac{d}{dt}L(t) \leq -\Lambda L(t), \qquad \forall t \geq 0. \tag{3.58}$$

A simple integration of (3.58) leads to

$$L(t) \leq L(0)\, e^{-\Lambda t}, \qquad \forall t \geq 0. \tag{3.59}$$

Again, the use of (3.56) and (3.59) yields the desired result (3.40). This completes the proof of Theorem 3.2. □

Remark 3.10. It is an interesting open problem to look whether or not the heat conduction is strong enough to stabilize system (3.15)-(3.17) (at least polynomially) in the case when $\mu_2 \geq \mu_1$.

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Chapter 4

Global nonexistence of solution of a system wave equations with nonlinear damping and source terms

4.1 Introduction

The study of the interaction between the source term and the damping term in the wave equation

$$u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \text{ in } \Omega \times (0, T), \tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history.
It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters m, p and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $|u|^{p-2}\,u$ then a uniform estimate of the form

$$\|u_t\,(t)\|_2 + \|\nabla u\,(t)\|_2 \leq C, \tag{4.2}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1\,(\Omega) \times L^2\,(\Omega)$, where C is a positive constant independent of t. The estimate (4.2) shows that any local solution u of problem (4.1) can be continued in time as long as (4.2) is verified. This result has been proved by several authors. See for example [34, 38]. On the other hand in the absence of the damping term $|u_t|^{m-2}\,u_t$, the solution of (4.1) ceases to exist and there exists a finite value T^* such that

$$\lim_{t \to T^*} \|u(t)\|_p = +\infty, \tag{4.3}$$

the reader is refereed to Ball [8] and Kalantarov & Ladyzhenskaya [38] for more details.

When both terms are present in equation (4.1), the situation is more delicate. This case has been considered by Levine in [43, 44], where he investigated problem (4.1) in the linear damping case ($m = 2$) and showed that any local solution u of (4.1) cannot be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [43] and [44] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim_{t \to T^*} \theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [22] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (4.1) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function L which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{4.4}$$

In $[0, \infty)$, where $\nu > 0$. Inequality (4.4) leads to a blow up of the solutions in finite tim $t \geq L(0)^{-\nu} \xi^{-1} \nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [22] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [22] (see for example [42, 45, 62, 93]. In particular, Vitillaro in [93] combined the arguments in [22] and [42] to extend the result in [22] to situations where the damping is nonlinear and the solution has positive initial energy.

In [95], Young, studied the problem

$$u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) + a\,|u_t|^{m-2}\,u_t = b|u|^{p-2}u, \tag{4.5}$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [95][Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$([95][Remark 2]).

Messaoudi and Said-Houari [60] improved the result in [95] and showed that the blow up of solutions of problem (4.5) takes place for negative initial data only regardless of the size of Ω.

To the best of our knowledge, the system of wave equations is not well studied, and only few results are available in literature. Let us mention some of them. Milla Miranda and Medeiros [63] considered the following system

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{\rho+2}\,|u|^{\rho}\,u = f_1(x) \\ v_{tt} - \Delta v + v - |u|^{\rho+2}\,|v|^{\rho}\,v = f_2(x), \end{cases} \tag{4.6}$$

in $\Omega\times(0,T)$. By using the method of potential well, the authors determined the existence of weak solutions of system (4.6). Some special cases of system (4.6) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field. See [87] and [36]. Agre and Rammaha [3] studied the system

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}\,u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1}\,v_t = f_2(u,v), \end{cases} \tag{4.7}$$

in $\Omega\times(0,T)$ with initial and boundary conditions of Dirichlet type and the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ satisfying

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{4.8}$$

They proved, under some appropriate conditions on $f_1(u;v)$, $f_1(u;v)$ and the initial data, several results on local and global existence, but no rate of decay has been discussed. They also showed that any weak solution with negative initial energy blows up in finite time, using the same techniques as in [22]. Recently, the blow up result in [3] has been improved by Said-Houari [83] by considering certain class of initial data with positive initial energy. Subsequently, the paper [83] has been followed by [85], where the author proved that if the initial data are small enough, then the solution of (4.7) is global and decays with an exponential rate if $m = r = 1$ and with a polynomial rate like $t^{-2/(\max(m,r)-1)}$ if $\max(m, r) > 1$. Several authors and many results appeared in the literature see for example [[9],[75]]

In this chapter, we consider the following system of wave equations

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) + a_1|u_t|^{m-2}u_t &= f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) + a_2|v_t|^{r-2}v_t &= f_2(u,v), \end{cases} \tag{4.9}$$

where the functions $f_1(u, v)$ and $f_2(u, v)$ satisfying (4.8). In (4.9), $u = u(t, x)$, $v = v(t, x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and $a_1, a_2, b_1, b_2 > 0$ and $\beta_1, \beta_2, m, r \geq 2$, $\alpha > 2$. System (4.9) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), \ x \in \Omega \\ u(x) = v(x) = 0 \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega, \end{cases} \tag{4.10}$$

Our main interest in this chapter is to prove a global nonexistence result of solutions of system (4.9) - (4.10) for large initial data. We use the method in [83] with the necessary modification imposed by the nature of our problem. The core of this method relies on the use of an auxiliary function L in order to obtain a differential inequality of the form (4.4) which leads to the desired result.

4.2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm. The constants $C, c, c_1, c_2, \ldots,$ used throughout this paper are positive generic constants, which may be different in various occurrences. We define

$$F(u,v) = \frac{1}{2(\rho+2)}\left[b_1|u+v|^{2(\rho+2)} + 2b_2|uv|^{\rho+2}\right].$$

Then , it is clear that, from (4.8),we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v). \quad (4.11)$$

The following lemma was introduced and proved in [58]

Lemma 4.1. *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right). \quad (4.12)$$

And the energy functional

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) - \int_\Omega F(u,v)\,dx. \quad (4.13)$$

Let us know define a constant r_α as follows :

$$r_\alpha = \frac{N\alpha}{N-\alpha}, \quad if\ N > \alpha,\ r_\alpha > \alpha\ if\ N = \alpha,\ and\ r_\alpha = \infty\ if\ N < \alpha. \quad (4.14)$$

The inequality below is a key element in proving the global existence of solution. A similar version of this lemma was first introduced in [83]

Lemma 4.2. *Suppose that $\alpha > 2$, and $2 < 2(\rho+2) < r_\alpha$. Then there exists $\eta > 0$ such that the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}} \quad (4.15)$$

holds.

Proof. It is clear that by using the Minkowski inequality, we get

$$\|u+v\|_{2(\rho+2)}^2 \leq 2(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2),$$

the embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, gives

$$\|u\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^2 \leq C(\|\nabla u\|_\alpha^\alpha)^{\frac{2}{\alpha}} \leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}},$$

and similary , we have

$$\|v\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}$$

Thus, we deduce from the above estimates that

$$\|u+v\|_{2(\rho+2)}^{2} \leq C(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha})^{\frac{2}{\alpha}} \tag{4.16}$$

also, Hölder's and Young's inequalities give us

$$\|uv\|_{(\rho+2)} \leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \leq C(\|\nabla u\|_{2(\rho+2)}^{2}+\|\nabla v\|_{2(\rho+2)}^{2}) \leq C(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha})^{\frac{2}{\alpha}}. \tag{4.17}$$

Collecting the estimates (4.16) and (4.17), then (4.15) holds. This completes the proof of lemma (4.2) □

Lemma 4.3. *Let $\nu > 0$ be a real positive number and L be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{4.18}$$

defined in $[0,\infty)$.
If $L(0) > 0$, then the solution ceases to existe for $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$.

Proof. Direct integration of (4.18) gives:

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi\nu t,$$

Thus we obtain the following estimate:

$$L^{\nu}(t) \geq \left[L^{-\nu}(0) - \xi\nu t\right]^{-1}. \tag{4.19}$$

It is clear that the right-hand side of (4.19) is unbounded when

$$\xi\nu t = L^{-\nu}(0).$$

This completes the proof of lemma 4.3 □

Lemma 4.4. *Let (u, v) be the solution of system* (4.9) - (4.10) *then the energy functional is a non-increasing function, that is for all $t \geqslant 0$*

$$\frac{dE(t)}{dt} = -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|\nabla u_t\|_{\beta_1}^{\beta_1} - \|\nabla v_t\|_{\beta_2}^{\beta_2} - a_1\|u_t\|_m^m - a_2\|v_t\|_r^r \tag{4.20}$$

Proof. We multiply the first equation in (4.9) by u_t and second equation by v_t and integrate over Ω, using integration by parts, we obtain (4.20) □

4.3 Global nonexistence result

In this section, we prove that, under some restrictions on the initial data and under som restrictions on the parameter α,β_1,β_2,m, r then the lifespan of solution of problem (4.9)-(4.10) is finite

Theorem 4.5. *Suppose that* $\beta_1, \beta_2, m, r \geq 2, \alpha > 2, \rho > -1$ *such that* $\beta_1, \beta_2 < \alpha$, *and* $\max\{m, r\} < 2(\rho+2) < r_\alpha$, *where* r_α *is the Sobolev critical exponent of* $W_0^{1,\alpha}(\Omega)$. *defined in* (4.14).*Assume further that*

$$E(0) < E_1, \qquad (\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} > \zeta_1$$

Then, any weak solution of (4.9)-(4.10) *cannot exist for all time .Here the constants* E_1 *and* ζ_1 *are defined in* (4.5).

In ordre to prove our result and for the sake of simplicity , we take $b_1 = b_2 = 1$ and introduce the following :

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \zeta_1 = B^{\frac{-2(\rho+2)}{2(\rho+2)-\alpha}}, \qquad E_1 = \left(\frac{1}{\alpha} - \frac{1}{2(\rho+2)}\right)\zeta_1^\alpha, \tag{4.21}$$

where η is the optimal constant in (4.15).

The following lemma allows us to prove a blow up result for a large class of initial data. This lemma is similar to the one in [83] and has its origin in [93]

Lemma 4.6. *Let* (u, v) *be a solution of* (4.9)-(4.10). *Assume that* $\alpha > 2, \rho > -1$. *Assume further that* $E(0) < E_1$ *and*

$$(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} > \zeta_1. \tag{4.22}$$

Then there exists a constant $\zeta_2 > \zeta_1$ *such that*

$$(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{1}{\alpha}} > \zeta_2, \tag{4.23}$$

and

$$\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\zeta_2, \ \forall t \geq 0. \tag{4.24}$$

Proof. We first note that, by (4.13) and the definition of B, we have

$$\begin{aligned} E(t) &\geq \frac{1}{\alpha}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)-\frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)}+2|uv|^{\rho+2}\right] \\ &\geq \frac{1}{\alpha}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)-\frac{\eta}{2(\rho+2)}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)^{\frac{2(\rho+2)}{\alpha}} \\ &\geq \frac{1}{\alpha}\zeta^{\alpha}-\frac{\eta}{2(\rho+2)}\zeta^{2(\rho+2)}, \end{aligned} \tag{4.25}$$

where $\zeta=\left[\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right]^{\frac{1}{\alpha}}$. It is not hard to verify that g is increasing for $0<\zeta<\zeta_1$, decreasing for $\zeta>\zeta_1$, $g(\zeta)\to-\infty$ as $\zeta\to+\infty$, and

$$g(\zeta_1)=\frac{1}{\alpha}\zeta_1^{\alpha}-\frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_1^{2(\rho+2)}=E_1,$$

where ζ_1 is given in (4.21). Therefore, since $E(0)<E_1$, there exists $\zeta_2>\zeta_1$ such that $g(\zeta_2)=E(0)$.

If we set $\zeta_0=\left[\|\nabla u(0)\|_{\alpha}^{\alpha}+\|\nabla v(0)\|_{\alpha}^{\alpha}\right]^{\frac{1}{\alpha}}$, then by (4.25) we have $g(\zeta_0)\leq E(0)=g(\zeta_2)$, which implies that $\zeta_0\geq\zeta_2$.

Now, establish (4.23), we suppose by contradiction that

$$\left(\|\nabla u_0\|_{\alpha}^{\alpha}+\|\nabla v_0\|_{\alpha}^{\alpha}\right)^{\frac{1}{\alpha}}<\zeta_2,$$

for some $t_0>0$; by the continuity of $\|\nabla u(.)\|_{\alpha}^{\alpha}+\|\nabla v(.)\|_{\alpha}^{\alpha}$ we can choose t_0 such that

$$\left(\|\nabla u(t_0)\|_{\alpha}^{\alpha}+\|\nabla v(t_0)\|_{\alpha}^{\alpha}\right)^{\frac{1}{\alpha}}>\zeta_1.$$

Again, the use of (4.25) leads to

$$E(t_0)\geq g\left(\|\nabla u(t_0)\|_{\alpha}^{\alpha}+\|\nabla v(t_0)\|_{\alpha}^{\alpha}\right)>g(\zeta_2)=E(0).$$

This is impossible since $E(t)\leq E(0)$, for all $t\in[0,T)$. Hence, (4.23) is established.

To prove (4.24), we make use of (4.13) to get

$$\frac{1}{\alpha}\left(\|\nabla u_0\|_{\alpha}^{\alpha}+\|\nabla v_0\|_{\alpha}^{\alpha}\right)\leq E(0)+\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right].$$

Consequently, (4.23) yields

$$\begin{aligned}\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right] &\geq \frac{1}{\alpha}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)-E(0)\\ &\geq \frac{1}{\alpha}\zeta_2^{\alpha}-E(0)\\ &\geq \frac{1}{\alpha}\zeta_2^{\alpha}-g(\zeta_2)\\ &= \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_2^{2(\rho+2)}.\end{aligned} \tag{4.26}$$

Therefore, (4.26) and (4.21) yield the desired result. □

Proof. Proof of Theorem 4.5

We suppose that the solution exists for all time and set

$$H(t)=E_1-E(t). \tag{4.27}$$

By using (4.13) and (4.27) we get

$$H'(t)=\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\|\nabla u_t\|_{\beta_1}^{\beta_1}+\|\nabla v_t\|_{\beta_2}^{\beta_2}+a_1\|u_t\|_m^m+a_2\|v_t\|_r^r.$$

From (4.20) , It is clear that for all $t\geq 0$, $H'(t)>0$. Therefore , we have

$$\begin{aligned}0 &< H(0)\leq H(t)\\ &= E_1-\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)-\frac{1}{\alpha}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)\\ &\quad+\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right].\end{aligned} \tag{4.28}$$

From (4.13) and (4.23), we obtain, for all $t\geq 0$,

$$E_1-\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)-\frac{1}{\alpha}\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right)<E_1-\frac{1}{\alpha}\zeta_1^{\alpha}=-\frac{1}{2(\rho+2)}\zeta_1^{\alpha}<0.$$

Hence,

$$0<H(0)\leq H(t)\leq\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right],\ \forall t\geq 0.$$

Then by (4.12), we have

$$0<H(0)\leq H(t)\leq\frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right],\ \forall t\geq 0. \tag{4.29}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega (uu_t + vv_t))dx, \tag{4.30}$$

for ε small to be chosen later and

$$\begin{aligned} 0 \quad < \quad \sigma \leq \min\left\{\frac{1}{2}, \frac{\alpha - m}{2(\rho+2)(m-1)}, \frac{\alpha - r}{2(\rho+2)(r-1)},\right. \\ \left.\frac{(\alpha-2)}{2(\rho+2)}, \frac{\alpha-\beta_1}{2(\rho+2)(\beta_1-1)}, \frac{\alpha-\beta_2}{2(\rho+2)(\beta_2-1)}\right\} \end{aligned} \tag{4.31}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (4.4). Indeed, taking the derivative of (4.30), using (4.9)and adding subtracting $\varepsilon kH(t)$, we obtain

$$\begin{aligned} L'(t) \quad = \quad & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon kH(t) + \varepsilon\left(1 + \frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon(1-k)\int_\Omega F(u,v) - \varepsilon kE_1 \\ & - \varepsilon\int_\Omega \nabla u \nabla u_t dx - \varepsilon \int_\Omega \nabla v \nabla v_t dx \\ & + \varepsilon\left(\frac{k}{\alpha} - 1\right)(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) \\ & - \varepsilon\int_\Omega |\nabla u_t|^{\beta_1 - 2}\nabla u_t \nabla u dx - \varepsilon\int_\Omega |\nabla v_t|^{\beta_2-2}\nabla v_t \nabla v dx \\ & - \varepsilon a_1 \int_\Omega |u_t|^{m-2} u_t u dx - \varepsilon a_2 \int_\Omega |v_t|^{r-2} v_t v dx. \end{aligned} \tag{4.32}$$

We then exploit Young's inequality to get for $\mu_i, \lambda_i, \delta_i > 0$ $i = 1, 2$

$$\begin{aligned} \int_\Omega \nabla u \nabla u_t dx \leq \frac{1}{4\mu_1}\|\nabla u\|_2^2 + \mu_1 \|\nabla u_t\|_2^2 \\ \int_\Omega \nabla v \nabla v_t dx \leq \frac{1}{4\mu_2}\|\nabla v\|_2^2 + \mu_2 \|\nabla v_t\|_2^2 \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \int_\Omega |\nabla u_t|^{\beta_1-1}\nabla u dx \leq \frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1} + \frac{\beta_1 - 1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1} \\ \int_\Omega |\nabla v_t|^{\beta_2-1}\nabla v dx \leq \frac{\lambda_2^{\beta_2}}{\beta_2}\|\nabla v\|_{\beta_2}^{\beta_2} + \frac{\beta_2 - 1}{\beta_2}\lambda_2^{-\beta_2/(\beta_2-1)}\|\nabla v_t\|_{\beta_1}^{\beta_1} \end{aligned} \tag{4.34}$$

and also

$$\int_{\Omega} |u_t|^{m-2} u_t u dx \leq \frac{\delta_1^m}{m} \|u\|_m^m + \frac{m-1}{m}\delta_1^{-m/(m-1)} \|u_t\|_m^m$$

$$\int_{\Omega} |v_t|^{r-2} v_t v dx \leq \frac{\delta_2^r}{r} \|v\|_r^r + \frac{r-1}{r}\delta_2^{-r/(r-1)} \|v_t\|_r^r \tag{4.35}$$

A substitution of (4.33)-(4.35)) in (4.32) and using (4.12) yields

$$\begin{aligned} L^{'}(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon k H(t) + \varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)} - \frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\ & -\frac{\varepsilon}{4\mu_1}\|\nabla u\|_2^2 - \mu_1\varepsilon\|\nabla u_t\|_2^2 - \frac{\varepsilon}{4\mu_2}\|\nabla v\|_2^2 - \varepsilon\mu_2\|\nabla v_t\|_2^2 \\ & +\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) - \varepsilon\frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon\frac{\beta_1-1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1} \\ & -\varepsilon\frac{\lambda_2^{\beta_2}}{\beta_2}\|\nabla v\|_{\beta_2}^{\beta_2} - \varepsilon\frac{\beta_2-1}{\beta_2}\lambda_2^{-\beta_2/(\beta_2-1)}\|\nabla v_t\|_{\beta_1}^{\beta_1} - a_1\varepsilon\frac{\delta_1^m}{m}\|u\|_m^m \\ & -a_1\varepsilon\frac{m-1}{m}\delta_1^{-m/(m-1)}\|u_t\|_m^m - a_2\varepsilon\frac{\delta_2^r}{r}\|v\|_r^r - a_2\varepsilon\frac{r-1}{r}\delta_2^{-r/(r-1)}\|v_t\|_m^m. \end{aligned} \tag{4.36}$$

Let us choose $\delta_1, \delta_2, \mu_1, \mu_2, \lambda_1$, and λ_2 such that

$$\begin{cases} \delta_1^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \delta_2^{-r/(r-1)} = M_2 H^{-\sigma}(t) \\ \mu_1 = M_3 H^{-\sigma}(t) \\ \mu_2 = M_4 H^{-\sigma}(t) \\ \lambda_1^{-\beta_1/(\beta_1-1)} = M_5 H^{-\sigma}(t) \\ \lambda_2^{-\beta_2/(\beta_2-1)} = M_6 H^{-\sigma}(t) \end{cases} \tag{4.37}$$

for M_1, M_2, M_3, M_4, M_5 and M_6 large constants to be fixed later. Thus, by using (4.37),and for

$M = M_3 + M_4 + (\beta_1-1)M_5/\beta_1 + (\beta_2-1)M_6/\beta_2 + (m-1)M_1/m + (r-1)M_2/r$

then, inequality (4.36) takes the form

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t)+\varepsilon k H(t)+\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{k c_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right)-\varepsilon k E_1 \\ & +\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ & -\frac{\varepsilon}{4M_3} H^\sigma(t)\|\nabla u\|_2^2-\frac{\varepsilon}{4M_4} H^\sigma(t)\|\nabla v\|_2^2 \\ & -\frac{a_1\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t)\|u\|_m^m-\frac{a_2\varepsilon}{r} M_2^{-(r-1)} H^{\sigma(r-1)}(t)\|v\|_r^r \\ & -\varepsilon\frac{M_5^{-(\beta_1-1)}}{\beta_1} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1}-\varepsilon\frac{M_6^{-(\beta_2-1)}}{\beta_2} H^{\sigma(\beta_2-1)}(t)\|\nabla u\|_{\beta_2}^{\beta_2}, \end{aligned} \tag{4.38}$$

We then use the two embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^m(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (4.29) to get

$$\begin{aligned} H^{\sigma(m-1)}(t)\|u\|_m^m \leq\ & c_2\left(\|u\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)+m}+\|v\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)}\|u\|_{2(\rho+2)}^m\right) \\ \leq\ & c_2\left(\|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m}+\|\nabla v\|_\alpha^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_\alpha^m\right). \end{aligned} \tag{4.39}$$

Similarly, the embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (4.29) give

$$\begin{aligned} H^{\sigma(r-1)}(t)\|v\|_r^r \leq\ & c_3\left(\|v\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)+r}+\|u\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)}\|v\|_{2(\rho+2)}^r\right) \\ \leq\ & c_3\left(\|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r}+\|\nabla u\|_\alpha^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_\alpha^r\right). \end{aligned} \tag{4.40}$$

Furthermore, the two embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$, yields

$$\begin{aligned} H^\sigma(t)\|\nabla u\|_2^2 \leq\ & c_4\left(\|u\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2+\|v\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2\right) \\ \leq\ & c_4\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)+2}+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla u\|_\alpha^2\right) \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} H^\sigma(t)\|\nabla v\|_2^2 \leq\ & c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2\right) \\ =\ & c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)+2}\right). \end{aligned} \tag{4.42}$$

Since $max(\beta_1, \beta_2) < \alpha$ then we have

$$\begin{aligned} H^{\sigma(\beta_1-1)}(t)\, \|\nabla u\|_{\beta_1}^{\beta_1} &\leq c_6 \left(\|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} + \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} \right) \\ &= c_6 \left(\|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} + \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} \right). \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} H^{\sigma(\beta_2-1)}(t)\, \|\nabla v\|_{\beta_2}^{\beta_2} &\leq c_7 \left(\|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} + \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} \right) \\ &= c_7 \left(\|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} + \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)+\beta_2} \right). \end{aligned} \tag{4.44}$$

for some positive constants c_2, c_3, c_4, c_5, c_6 and c_7. By using (4.31) and the algebraic inequality

$$z^{\nu} \leq (z+1) \leq \left(1 + \tfrac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a \geq 0, \tag{4.45}$$

we have, for all $t \geq 0$,

$$\begin{cases} \|\nabla u\|_{\alpha}^{2\sigma(m-1)(\rho+2)+m} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(0)\right) \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(r-1)(\rho+2)+r} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)+\beta_2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \end{cases} \tag{4.46}$$

where $d = 1 + 1/H(0)$. Also keeping in mind the fact that $max(m, r) < \alpha$, using Yong's inequality, the inequality (4.45) togrther withe (4.31), we conclude

$$\left\{\begin{array}{l} \|\nabla v\|_{\alpha}^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_{\alpha}^{m} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_{\alpha}^{r} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla u\|_{\alpha}^{2} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla v\|_{\alpha}^{2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_{\alpha}^{\beta_1} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_{\alpha}^{\beta_2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right), \end{array}\right. \tag{4.47}$$

where C is a generic positive constant. Taking into account (4.39)- (4.47) , then, (4.38) takes the form

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t)+\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right) \\ & +\varepsilon\Big(\left[k/\alpha-1-kE_1\zeta_2^{-a}\right]-CM_1^{-(m-1)}-CM_2^{-(r-1)} \\ & -\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\Big)\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\Big(k-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ & -CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\Big)H(t) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{4.48}$$

for some constant k. Using $k=c_0/c_1$, we arrive at

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t)+\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right) \\ & +\varepsilon\Big(\overline{c}-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ & -CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\Big)\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\Big(c_0/c_1-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ & -CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\Big)H(t), \end{aligned} \tag{4.49}$$

where $\overline{c} = k/\alpha - 1 - kE_1\zeta_2^{-2} = c_0/(c_1\alpha) - 1 - (c_0/c_1)E_1\zeta_2^{-2} > 0$ since $\zeta_2 > \zeta_1$.

At this point, and for large values of M_1, M_2, M_3, M_4, M_5 and M_6, we can find positive constants Λ_1 and Λ_2 such that (4.49) becomes

$$\begin{aligned} L'(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon\left(1 + \frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + \varepsilon\Lambda_1 \left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + \varepsilon\Lambda_2 H(t). \end{aligned} \tag{4.50}$$

Once M_1, M_2, M_3, M_4, M_5 and M_6 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $((1-\sigma) - M\varepsilon) \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \int_\Omega [u_0.u_t + v_0.v_t]\, dx > 0.$$

From these and (4.50) becomes

$$L'(t) \geq \varepsilon\Gamma\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right). \tag{4.51}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\, dx + \int_\Omega v.v_t(x,t)\, dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \\ \leq\ & C\left(\|\nabla u\|_\alpha^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|\nabla v\|_\alpha^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \end{aligned} \tag{4.52}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We take $s = 2(1-\sigma)$, to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$. By using (4.31) and (4.45) we get

$$\|\nabla u\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla u\|_\alpha^\alpha + H(t)\right),$$

and

$$\|\nabla v\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla v\|_\alpha^\alpha + H(t)\right),\ \forall t \geq 0.$$

Therefore, (4.52) becomes

$$\begin{aligned}
&\left(\int_{\Omega} u.u_t\,(x,t)\,dx + \int_{\Omega} v.v_t\,(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\ & C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha} + \|u_t\|_2^2 + \|v_t\|_2^2 + H\,(t)\right),\ \forall t \geq 0.
\end{aligned} \tag{4.53}$$

Also, since

$$\begin{aligned}
L^{\frac{1}{1-\sigma}}\,(t) &= \left(H^{1-\sigma}\,(t) + \varepsilon \int_{\Omega} (u.u_t + v.v_t)\,(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\
&\leq C\left(H\,(t) + \left|\int_{\Omega} (u.u_t\,(x,t) + v.v_t\,(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\
&\leq C\left[H\,(t) + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha} + \|u_t\|_2^2 + \|v_t\|_2^2\right],\ \forall t \geq 0,
\end{aligned} \tag{4.54}$$

combining withe (4.54) and (4.51), we arrive at

$$L^{'}\,(t) \geq a_0 L^{\frac{1}{1-\sigma}}\,(t)\,,\ \forall t \geq 0. \tag{4.55}$$

Finally, a simple integration of (4.55) gives the desired result.This completes the proof of Theorem (4.5) □

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