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Title

Optimal control for stochastic differential equations governed by normal martingales

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Abstract

This thesis presents two research topics, the first one being divided into two parts. In the first part, we study an optimal control problem where the state equation is driven by a normal martingale. We prove a sufficient stochastic maximum and we also show the relationship between stochastic maximum principle and dynamic programming in which the control of the jump size is essential and the corresponding Hamilton–Jacobi–Bellman (HJB) equation in this case is a mixed second order partial differential-difference equation. As an application, we solve explicitly a mean-variance portfolio selection problem. In the second part, we study a non smooth version of the relationship between MP and DPP for systems driven by normal martingales in the situation where the control domain is convex.

The second topic is to characterize sub-game perfect equilibrium strategy of a partially observed optimal control problems for mean-field stochastic differential equations (SDEs) with correlated noises between systems and observations, which is time-inconsistent in the sense that it does not admit the Bellman optimality principle.

Keys words. Normal martingales, structure equation, stochastic maximum principle, dynamic programming principle, time inconsistency, mean-field control problem, partial information, mean-variance criterion, stochastic systems with jumps.

Résumé

Cette thèse présente deux sujets de recherche, le premier étant divisé en deux parties. Dans la première partie, nous étudions un problème de contrôle optimal où l'équation d'état est gouvernée par une martingale normale. Nous démontrons le principe du maximum (conditions suffisantes d'optimalité) et nous montrons aussi la relation entre le principe maximum stochastique et la programmation dynamique dans laquelle le contrôle de la taille du saut est essentiel et l 'équation de Hamilton - Jacobi - Bellman correspondante (HJB) dans ce cas est une equation différentielle partielle de deuxième ordre mixte. Comme exemple, nous résolvons explicitement un problème de sélection de portefeuille de variance moyenne. Dans la deuxième partie nous montrons aussi la relation entre la fonction de valeur et le processus adjoint qui est liée à la solution de viscosité.

Le deuxième sujet est de caractériser la stratégie d'équilibre parfait du sous-jeu d'un problème de contrôle optimal partiellement observé pour les équations différentielles stochastiques de champ moyen (EDS) avec des bruits corrélés entre les systèmes et les observations, ce qui est incohérent dans le temps en ce sens qu'il n'admet pas le principe d'optimalité de Bellman.

Mots Clés. Martingales normales, équation de structure, principe du maximum, principe de programmation dynamique, inconsistance, problème de contrôle de champ moyen, information partielle, critère de variance moyenne, systèmes stochastiques avec sauts.

Contents

Introduction

The main objective of this thesis is to study two research topics about stochastic control problems. For the first topic, we study an optimal control problem where the state equation is governed by a normal martingale of the type

$$\begin{cases} dY(t) & = b(t, Y(t), u(t), \pi(t))\, dt + \sigma(t, Y(t-), u(t), \pi(t))\, d\mathcal{M}^u(t) \\ Y(0) & = y, \end{cases}$$

where b and σ are given deterministic functions, $\mathcal{M}^u$ is a martingale that satisfies the equation

$$[\mathcal{M}^u](t) = t + \int_0^t u(s)\, d\mathcal{M}^u(s), \quad t \geq 0.$$

The above equation is called Emery's structure equation, $[\mathcal{M}^u](\cdot)$ denotes the quadratic variation of $\mathcal{M}^u(\cdot)$. A control process that solves this problem is called optimal. Recently there has been increasing interest in the study of this type of stochastic control problems where the system is driven by normal martingales. In an optimal reinsurance and investment problems, the reserve process of the insurance company is described by a stochastic differential equation with jumps, see for example [18, 47]. According to the parameter u, we have distinct normal martingales satisfying Emery's structure equation as follows: When $u \equiv 0$, $\mathcal{M}(\cdot)$ corresponds to the standard model of Brownian motion, the case when $u \equiv \alpha \in \mathbb{R}^* = \mathbb{R} - \{0\}$ corresponds to the compensated poisson process and for $u \equiv -\mathcal{M}(t-)$ corresponds to the Azéma martingale, etc. The freedom of choice of coefficients for the stochastic differential equation giving rise to the normal martingale means that for each u we have a model as rich as the standard model. The construction of the solutions to structure equations are studied by many authors, see for example Émery [29] in two-dimensional case ($d = 2$). In a multidimensional case, Buckdahn et al [18] proved the existence of the solutions of the structure equations in a Wiener–Poisson space. Accordingly, they establish the dynamic programming principle and obtain the corresponding new form of HJB equation, which in this case is a mixed second-order partial differential-difference equation. For more information about normal martingales and its applications, we refer to [5, 28, 48, 55]. Stochastic maximum principle for diffusions (without jumps) were made by Kushner [43] and

Bismut [15]. Further progresses on the subject were subsequently given by Bensoussan [10], Peng [50], see also Yong and Zhou [64] and the references therein. For diffusions with jumps, the stochastic maximum principle was given by Tang and Li [60], Kabanov [39] and Kohlmann [41]. Framstad et al [32] formulated the stochastic maximum principle and applied it to a quadratic portfolio optimization problem. Zhang et al [66] proved the sufficient maximum principle where the state process is governed by a continuous-time Markov regime-switching jump-diffusion model.

The relationship between the maximum principle and the dynamic programming is essentially the relationship between the solution of the adjoint equation, with the spatial gradient of the value function evaluated along the optimal trajectory, see e.g. [64] in the classical case. For diffusions with jumps, the relationship between the maximum principle and dynamic programming, was given by Framstad et al. [32], [21] and [58]. For singular stochastic control, refer to Balahli et al. [6], for stochastic recursive control, refer to Shi and Yu [57], and for stochastic differential game, refer to Shi [56]. Within the framework of viscosity solution, Zhou [67] showed that $\mathcal{D}_y^{1,-}V\left(t,\bar{Y}\left(t\right)\right) \subset \left\{-p\left(t\right)\right\} \subset \mathcal{D}_y^{1,+}V\left(t,\bar{Y}\left(t\right)\right)$, where $\mathcal{D}_y^{1,-}V\left(t,\bar{Y}\left(t\right)\right)$ and $\mathcal{D}_y^{1,+}V\left(t,\bar{Y}\left(t\right)\right)$ denote the first order sub- and super-jets of V at $\left(t,\bar{Y}\left(t\right)\right)$, respectively. Yong and Zhou [64] showed that $\left\{-p\left(t\right)\right\} \times \left[-P\left(t\right),\infty\right) \subset \mathcal{D}_y^{2,+}V\left(t,\bar{Y}\left(t\right)\right)$ and $\mathcal{D}_y^{2,-}V\left(t,\bar{Y}\left(t\right)\right) \subset \left\{-p\left(t\right)\right\} \times \left(-\infty,-P\left(t\right)\right]$, where $\mathcal{D}_y^{2,-}V\left(t,\bar{Y}\left(t\right)\right)$ and $\mathcal{D}_y^{2,+}V\left(t,\bar{Y}\left(t\right)\right)$ denote the second-order sub- and super-jets of V at $\left(t,\bar{Y}\left(t\right)\right)$, and p, P are the first- and second-order adjoint processes, respectively.

For the second topic, we characterize sub-game perfect equilibrium strategy of a partially observed optimal control problems for mean-field stochastic differential equations (SDEs) with correlated noises between systems and observations, which is time-inconsistent in the sense that it does not admit the Bellman optimality principle. Peng [50] derived a general maximum principle for a fully observed forward stochastic control system. It is well known that an optimal control can be represented by an adjoint process which is the solution of a BSDE. In [2], [45], the stochastic maximum principle is proved for mean-field stochastic control problem where both the state dynamics and the cost functional are of a mean-field type. The mean-field coupling makes the control problem time-inconsistent in the sense that the Bellman Principle is no longer valid, which motivates the use of the stochastic maximum approach to solve this type of optimal control problems instead of trying extensions of the dynamic programming principle.

In practice, the controllers usually cannot be able to observe the full information, but the partial one with noise. For the forward stochastic control case, there are rich articles to study partially observed optimization problems like Bensoussan [10], Baras, Elliott and Kohlmann [7], Haussmann [37], Zhou [64], Li and Tang [60], etc.

Time-inconsistent stochastic control is a game-theoretic generalization of standard stochastic control, based on the notion of Nash equilibrium, with well-known applications in finance. It is has a long history starting with [59] where a deterministic Ramsay problem is studied. Further work which extend [59] are [54], [53], [52], [33]. Early financial mathematics papers in time-inconsistent stochastic control include Ekeland and Lazrak [26] and Ekeland and Pirvu [27], who study a classic time-inconsistent finance problem in continuous time (optimal consumption and investment under hyperbolic discounting). The work [12] extends the idea to the stochastic framework where the controlled process is quit general Markov process. In addition, an extended HJB equation is derived, along with a verification argument that characterizes a Markov subgame perfect Nash equilibrium. Keeping the same game perspective Basak and Chabakauri [9] obtained a time-consistent strategy to the dynamic mean– variance portfolio selection problem in continuous-time setting. Böjrk et al [13] studied the mean-variance portfolio selection with state dependent risk aversion.

In [24] the authors undertake a deep study of a class of dynamic decision problems of mean-field type driven by Brownian motion with time-inconsistent cost functionals and derive a stochastic maximum principle to characterize subgame perfect equilibrium points.

Let us briefly describe the contents of this thesis:

In **Chapter 1**, we give some background on optimal control theory, we present strong and weak formulations of stochastic optimal control problems and the existence of stochastic optimal controls for both strong and weak formulation, then, we use the dynamic programming principle and the stochastic maximum principle in the classical case where the system is governed by Brownian motion for solving stochastic control problems, see, Yong and Zhou [64].

In **Chapter 2,** we prove a sufficient stochastic maximum principle for the optimal control of systems driven by normal martingales. We also show the relationship between stochastic maximum principle and dynamic programming in which the control of the jump size is essential and the corresponding Hamilton–Jacobi–Bellman (HJB) equation in this case is a mixed second order partial differential-difference equation. As an application, we solve explicitly a mean-variance portfolio selection problem. The results obtained in this chapter, generalizes the well known result concerning Brownian motion and Poisson random measure in [64], [32].

In **Chapter 3,** we present a nonsmooth version of the relationship between the stochastic maximum principle and the dynamic programming principle for stochastic control problems. The state of the systems driven by normal martingales and the control domain is convex. By using the concepts of sub and super-jets, all inclusions are derived from the value function and the adjoint process.

In **Chapter 4,** we characterize sub-game perfect equilibrium strategy of a partially observed optimal control problems for mean-field stochastic differential equations (SDEs) with correlated noises between systems and observations, which is time-inconsistent in the sense that it does not admit the Bellman optimality principle.

Relevant Papers

The content of this thesis was the subject of the following papers:

1. F. Chighoub, I.E. Lakhdari, and J.T. Shi, 'Relationship between Maximum Principle and Dynamic Programming for Systems Driven by Normal Martingales', Mathematics in Engineering, Science & Aerospace (MESA), 8 (2017), pp. 91-107

2. F. Chighoub, I.E. Lakhdari, Relationship between MP and DPP for systems driven by normal martingales: viscosity solution. (Preprint).

3. F. Chighoub, I.E. Lakhdari, A Characterization of Sub-game Perfect Equilibria for SDEs of Mean-Field Type Under Partial Information. (Preprint).

Chapter 1

Stochastic Control Problem

1.1 Introduction

Optimal control is a branch of the control theory strictly related with optimization, for this kind of problems the aim is to find a control strategy such that a certain optimality criterion is achieved. This criterion is usually expressed by a cost, that is a functional depending on the choice of the control input.

Two main approaches can be found in literature for dealing with optimal control problems: the Stochastic Maximum Principle (SMP) and the Dynamic Programming Principle (DPP). By the use of the Bellman's Dynamic Programming Principle the study of optimal control problems can be linked with the solution of a particular class of nonlinear second-order partial differential equations: the Hamilton-Jacobi-Bellman equations. It is well known that the HJB equation does not necessarily admit smooth solution in general, we can give a meaning to this EDP with a concept of weak solution called viscosity solution. On the other hand, the Stochastic Maximum Principle is to derive a set of necessary and sufficient conditions that must be satisfied by any optimal control, the basic idea is by perturbing an optimal control on a small time interval of length ε. Performing a Taylor expansion with respect to ε and then sending ε to zero one obtains a variational inequality. By duality the maximum principle is obtained. It states that any optimal control must solve the Hamiltonian system associated with the control problem. The Hamiltonian system involves a linear differential equation, with terminal conditions, called the adjoint equation. The relationship between the (SMP) and (DPP) is essentially the

relationship between the solution of the HJB equation (the value function), and the solution of the adjoint equation in the optimal state. More precisely, the solution of the adjoint process can be expressed in terms of the derivatives of the value function.
This chapter will be organized as follows. In section 2, we present strong and weak formulations of stochastic optimal control problems and the existence of stochastic optimal control for both strong and weak formulation. In section 3, we study the dynamic programming principle . In Section 4, we derive necessary as well as sufficient optimality conditions. Then, we prove that the adjoint process is equal to the derivative of the value function evaluated at the optimal trajectory.

1.2 Formulation of the problem

In this section we present two mathematical formulations strong and weak formulations of stochastic optimal control problems.

1.2.1 Strong formulation

We consider a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ satisfying the usual conditions, on witch we define an m-dimensional standard Brownian motion $B(\cdot)$, denote by U the separable metric space, and $T \in (0, \infty)$ being fixed. The state $y(t)$ of a controlled diffusion is described by the following stochastic differential equation

$$\begin{cases} dy(t) & = b(t, y(t), u(t))\, dt + \sigma(t, y(t), u(t))\, dB(t) \\ y(0) & = y, \end{cases} \tag{1.1}$$

where $b : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n\times m}$, are given. The function $u(\cdot)$ is called the control representing the action of the decision-makers (controllers). At any time instant the controller knowledgeable about some information (as specified by the information filed $\{\mathcal{F}_t\}_{t\geq 0}$) of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ befor the time t really comes) witche can be expressed in mathematical term as " $u(\cdot)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted", the control u is taken from the set

$$\mathcal{U}\left[0,T\right] \triangleq \left\{u:\left[0,T\right]\times\Omega \longrightarrow U \mid u\left(\cdot\right) \text{ is } \{\mathcal{F}_t\}_{t\geq 0} \text{ adapted}\right\}.$$

Consider the cost functional as follows

$$J\left(u\left(\cdot\right)\right)=\mathbb{E}\left(\int_0^T f\left(t,y\left(t\right),u\left(t\right)\right)dt+g\left(y\left(T\right)\right)\right). \tag{1.2}$$

Definition 1.2.1 *Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ be given satisfying the usual conditions and let $B\left(t\right)$ be a given m-dimensional standard $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. A control $u(\cdot)$ called an admissible control, and $(y(\cdot),u(\cdot))$ an admissible pair, if*

1. *$u(\cdot)\in\mathcal{U}\left[0,T\right].$*
2. *$y(\cdot)$ is the unique solution of equation (1.1).*
3. *$f\left(\cdot,y\left(\cdot\right),u\left(\cdot\right)\right)\in L^1_{\mathcal{F}}\left(0,T,\mathbb{R}\right)$ and $g\left(y\left(T\right)\right)\in L^1_{\mathcal{F}}\left(\Omega,\mathbb{R}\right).$*

We denote by $\mathcal{U}_{ad}\left[0,T\right]$ the set of all admissible controls. The stochastic control problem is to find an optimal control $\widehat{u}(\cdot)\in\mathcal{U}_{ad}\left[0,T\right]$ (if it ever exists), such that

$$J\left(\widehat{u}\left(\cdot\right)\right)=\inf_{u(\cdot)\in\mathcal{U}^s_{ad}[0,T]} J\left(u\left(\cdot\right)\right), \tag{1.3}$$

where $\widehat{u}\left(\cdot\right)$ is called an optimal control and the state control pair $(\widehat{y}(\cdot),\widehat{u}(\cdot))$ are called an optimal state process.

1.2.2 Weak formulation

In the strong formulation the filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ on witch we define the Brownian motion B are all fixed. However in the weak formulation, where we consider them as a parts of the control.

Definition 1.2.2 $\pi=\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P},B\left(\cdot\right),u(\cdot)\right)$ *is called a w-admissible control, and $y\left(\cdot\right),u\left(\cdot\right)$ a w-admissible pair if*

1. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions;*

2. $B(\cdot)$ *is an* m*-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$;

3. $u(\cdot)$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in* U*;*

4. $y(\cdot)$ *is the unique solution of equation (1.1);*

5. $f(\cdot, y(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T, \mathbb{R})$ *and* $g(y(T)) \in L^1_{\mathcal{F}}(\Omega, \mathbb{R})$.

The set of all admissible controls is denoted by $\mathcal{U}^w_{ad}[0,T]$. Our stochastic optimal control problem under weak formulation is to find an optimal control $\widehat{u}(\cdot) \in \mathcal{U}^w_{ad}[0,T]$ (if it ever exists), such that

$$J(\widehat{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}^w_{ad}[0,T]} J(u(\cdot)). \tag{1.4}$$

1.2.3 Existence of optimal control

In this subsection we are going to discuss the existence of optimal controls, we use the theory that a lower semi-continuous function on a compact metric space reaches its minimum.

Existence under strong formulation

We are given a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ with $\mathbb{R}$ -valued standard Brownian motion B. Consider the following linear controlled system

$$\begin{cases} dy(t) &= [Ay(t) + Fu(t)]\,dt + [Cy(t) + Du(t)]\,dB(t), \quad t \in [0,T] \\ y(0) &= y_0, \end{cases} \tag{1.5}$$

where A, F, C, D are matrices. The state $y(\cdot)$ takes value in $\mathbb{R}^n$, and the control $u(\cdot)$ is in

$$\mathcal{U}^L[0,T] = \left\{u(\cdot) \in L^2_{\mathcal{F}}\left(0, T, \mathbb{R}^k\right) \mid u(\cdot) \in U,\ a.e.t \in [0,T],\ \mathbb{P}-a.s.\right\},$$

with $U \subseteq \mathbb{R}^k$, The cost functional is

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, y(t), u(t))\,dt + g(y(T))\right], \tag{1.6}$$

with $f : \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Let us assume the following assumptions

(H1) The set $U \subset \mathbb{R}^k$ is convex and closed, and the functions f and g are convex and for some $\delta, K > 0$,

$$f(y,u) \geq \delta |u|^2 - K, \quad g(y) \geq -K, \quad \forall (y,u) \in \mathbb{R}^n \times U.$$

(H2) The set $U \subset \mathbb{R}^k$ is convex and compact, and the functions f and g are convex.

The optimal control problem is to minimize (1.6) subject to (1.5) over $\mathcal{U}^L[0,T]$.

Theorem 1.2.1 (Existence of optimal control) *Under either **(H1)** and **(H2)**, if the problem is finite, then it admits an optimal control.*

Proof. See Theorem 5.2 in [55]. ■

Existence under weak formulation

Now we will study the existence of optimal control under weak formulation. We introduce the standing assumptions

(H3) (U,d) is a compact metric space and $T > 0$,

(H4) The maps b, σ, f and g are all continuous, and there exists a constant $L > 0$ such that for $\phi(t,y,u) = b(t,y,u), \sigma(t,y,u), f(t,y,u), g(y)$,

$$\begin{cases} |\phi(t,y,u) - \phi(t,\widehat{y},u)| \leq L|y - \widehat{y}|, \\ \phi(t,0,u) \leq L, \ \forall t \in [0,T], \ y, \widehat{y} \in \mathbb{R}^n, \ u \in U. \end{cases}$$

(H5) For every $(t,y) \in [0,T] \times \mathbb{R}^n$, the set

$$\begin{aligned} (b, \sigma\sigma^{\intercal}, f)(t,y,U) \quad &\triangleq \Big\{ (b_i(t,yu)), \ (\sigma\sigma^{\intercal})^{ij}(t,y,u), \ f(t,y,u) \ \mid \\ &\qquad u \in U, \ i = 1, \ldots, n, \ j = 1, \ldots, m \} \end{aligned}$$

is convex in $\mathbb{R}^{n+nm+1}$.

(H6) $y(t) \in \mathbb{R}^n$.

Theorem 1.2.2 (Existence of optimal control) *Under the conditions **(H3)**-**(H5)**, if the problem is finite, then it admits an optimal control.*

Proof. See Theorem 5.2 in [55]. ■

1.3 Dynamic programming principle (DPP)

In this section, we use the dynamic programming method for solving stochastic control problems. We present the HJB equation and introduced the standard class of stochastic control problem, the associated dynamic programming principle, and the resulting HJB equation describing the local behavior of the value function of the control problem. Throughout this first introduction to HJB equation the value function is assumed to be as smooth as required.

1.3.1 The Bellman principle

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P}\right)$ be a filtered probability space satisfying the usual conditions. Let $B(t)$ be a Brownian motion valued in $\mathbb{R}^d$. We denote by A the set of all progressively measurable processes $\{u(t)\}_{t\geq 0}$ valued in $U \subset \mathbb{R}^k$. The elements of A are called control processes.
We consider the following stochastic controlled system

$$\begin{cases} dy(t) &= b(t, y(t), u(t))\, dt + \sigma(t, y(t), u(t))\, dB(t) \\ y(0) &= y, \end{cases} \tag{1.7}$$

where $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ be two given functions satisfying, for some constant M

$$|b(t, y(t), u(t)) - b(t, x(t), u(t))| + |\sigma(t, y(t), u(t)) - \sigma(t, x(t), u(t))| \leq M|y - x|, \tag{1.8}$$

$$|b(t, y(t), u(t))| + |\sigma(t, y(t), u(t))| \leq M(1 + |y(t)|). \tag{1.9}$$

Under (1.8) and (1.9) the above equation has a unique solution y.
We define the cost functional $J : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, by

$$J(t, y, u) = \mathbb{E}^{t,y}\left[\int_t^T f(s, y(s), u(s))\, ds + g(y(T))\right], \tag{1.10}$$

where $\mathbb{E}^{t,y}$ is the expectation operator conditional on $y(t) = y$, and $f : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$,

$g : \mathbb{R}^n \longrightarrow \mathbb{R}$, we assume that

$$|f(t,y,u)| + |g(y)| \leq M\left(1 + |y|^2\right), \tag{1.11}$$

for some constant M. The quadratic growth condition (1.11), ensure that J is well defined. The purpose of this Section is to study the minimization problem

$$V(t,y) = \inf_{u \in U} J(t,y,u), \quad \text{for } (t,y) \in [0,T] \times \mathbb{R}^n, \tag{1.12}$$

which is called the value function of the problem (1.7) and (1.10).

The dynamic programming is a fundamental principle in the theory of stochastic control, we give a version of the stochastic Bellman's principle of optimality. For mathematical treatments of this problem , we refer the reader to Lions [44], Krylov [42], Yong and Zhou [64], Fleming and Soner [30].

Theorem 1.3.1 *Let $(t,y) \in [0,T] \times \mathbb{R}^n$ be given. Then, for every $h \in [0, T-t]$, we have*

$$V(t,y) = \inf_{u \in U} \mathbb{E}^{t,y}\left(\int_t^{t+h} f(s, y(s), u(s))\, ds + V(t+h, y(t+h))\right). \tag{1.13}$$

Proof. Suppose that for $h > 0$, we given by $\widehat{u}(s) = \widehat{u}(s,y)$ the optimal feedback control for the problem (1.7) and (1.10) over the time interval $[t,T]$ starting at point $y(t+h)$. i.e.

$$J(t+h, y(t+h), \widehat{u}(t+h)) = V(t+h, y(t+h)), \ \mathbb{P}-a.s. \tag{1.14}$$

Now, we consider

$$\widetilde{u} = \begin{cases} u(s,y), & t \leq s \leq t+h \\ \widehat{u}(s,y), & t+h \leq s \leq T, \end{cases}$$

for some control u. By definition of $V(t,y)$, and using (1.10), we obtain

$$\begin{aligned} V(t,y) &\leq J(t,y,\widetilde{u}) \\ &= \mathbb{E}^{t;y}\left(\int_t^{t+h} f(s, y(s), u(s))\, ds + \int_{t+h}^T f(s, y(s), \widehat{u}(s))\, ds + g(y(T))\right). \end{aligned}$$

By the unicity of solution for the SDE (1.7), we have for s$\geq t+h$, $y^{t+h,y^{t,y}(t+h)}(s)=y^{t,y}(s)$, then

$$\begin{aligned} J(t,y,\widetilde{u}) &= \mathbb{E}\left(\int_t^{t+h} f(s,y(s),u(s))\,ds \right. \\ &\quad \left. + \int_{t+h}^{T} f\left(s,y^{t+h,y^{t,y}(t+h)}(s),\widehat{u}(s)\right) ds + g\left(y^{t+h,y^{t,y}(t+h)}(T)\right)\right) \\ &= \mathbb{E}\left(\int_t^{t+h} f(s,y(s),u(s))\,ds \right. \\ &\quad \left. + \mathbb{E}\int_{t+h}^{T} f(s,y(s),\widehat{u}(s))\,ds + g(y(T)) \mid y^{t,y}(t+h)\right) \\ &= \mathbb{E}\left(\int_t^{t+h} f(s,y(s),u(s))\,ds + V\left(t+h,y^{t,y}(t+h)\right)\right). \end{aligned}$$

So we get

$$V(t,y) \leq \mathbb{E}\left(\int_t^{t+h} f(s,y(s),u(s))\,ds + V\left(t+h,y^{t,y}(t+h)\right)\right), \tag{1.15}$$

and the equality holds if $\widetilde{u}=\widehat{u}$, which proves (1.13). ■

1.3.2 The Hamilton Jacobi Bellman equation

Now, we introduce the HJB equation by deriving it form the dynamic programming principle under smoothness assumptions on the value function. Let $G:[0,T]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times d}$ into $\mathbb{R}$, be defined by

$$G(t,y,r,p,A) = b(t,y,u)^{\intercal}p + \frac{1}{2}tr\left[\sigma\sigma^{\intercal}(t,y,u)\,A\right] + f(t,y,u), \tag{1.16}$$

we also need to introduce the linear second order operator $\mathcal{L}^u$ associated to the controlled processes $y(t)$, $t\geq 0$, we consider the constant control u

$$\mathcal{L}^u\varphi(t,y) = b(t,y,u)^{\intercal}D_y\varphi(t,y) + \frac{1}{2}tr\left[\sigma\sigma^{\intercal}(t,y,u)\,D_{yy}(\varphi(t,y))\right], \tag{1.17}$$

where D_y, D_{yy} denote the gradient and the Hessian operator with respect to the y variable. Assume the value function $V\in C([0,T],\mathbb{R}^n)$, and $f(\cdot,\cdot,u)$ be continuous in (t,y) for all fixed

$u \in A$, then we have by Itô's formula

$$V(t+h, y(t+h)) = V(t, y) + \int_t^{t+h} \left(\frac{\partial V}{\partial s} + \mathcal{L}^u V \right) \left(s, y^{t,y}(s) \right) ds$$
$$+ \int_t^{t+h} D_y V \left(s, y^{t,y}(s) \right)^\intercal \sigma \left(s, y^{t,y}(s), u \right) dB(s),$$

by taking the expectation, we get

$$\mathbb{E}\left(V(t+h, y(t+h))\right) = V(t, y) + \mathbb{E}\left(\int_t^{t+h} \left(\frac{\partial V}{\partial s} + \mathcal{L}^u V \right) \left(s, y^{t,y}(s) \right) ds \right),$$

then, we have by (1.15)

$$0 \leq \mathbb{E}\left(\frac{1}{h} \int_t^{t+h} \left(\left(\frac{\partial V}{\partial s} + \mathcal{L}^u V \right) \left(s, y^{t,y}(s) \right) + f\left(s, y^{t,y}(s), u \right) \right) ds \right).$$

We now send h to zero, we obtain

$$0 \leq \frac{\partial V}{\partial t}(t, y) + \mathcal{L}^u V(t, y) + f(t, y, u),$$

this provides

$$-\frac{\partial V}{\partial t}(t, y) - \inf_{u \in U} \left[\mathcal{L}^u V(t, y) + f(t, y, u) \right] \leq 0. \tag{1.18}$$

Now we shall assume that $\widehat{u} \in U$, and using the same procedure as above, we conclude that

$$-\frac{\partial V}{\partial t}(t, y) - \mathcal{L}^{\widehat{u}} V(t, y) - f(t, y, u) = 0, \tag{1.19}$$

by (1.18), then the value function solves the HJB equation

$$-\frac{\partial V}{\partial t}(t, y) - \inf_{u \in U} \left[\mathcal{L}^u V(t, y) + f(t, y, u) \right] = 0, \qquad \forall (t, y) \in [0, T] \times \mathbb{R}^n. \tag{1.20}$$

We give sufficient conditions which allow to conclude that the smooth solution of the HJB equation coincides with the value functionm this is the so-called verification result.

Theorem 1.3.2 *Let W be a $C^{1,2}([0,T], \mathbb{R}^n) \cap C([0,T], \mathbb{R}^n)$ function. Assume that f and g are* *quadratic growth, i.e. there is a constant M such that*

$$|f(t,y,u)|+|g(y)|\leq M\left(1+|y|^2\right),\ \textit{for all}\ (t,y,u)\in[0,T]\times\mathbb{R}^n\times U.$$

(1) Suppose that $W(T,\cdot)\leq g$, and

$$\frac{\partial W}{\partial t}(t,y)+G\left(t,y,W(t,y),D_yW(t,y),D_{yy}(W(t,y))\right)\geq 0, \tag{1.21}$$

on $[0,T]\times\mathbb{R}^n$, then $W\leq V$ on $[0,T]\times\mathbb{R}^n$.

(2) Assume further that $W(T,\cdot)=g$, and there exists a minimizer $\widehat{u}(t,y)$ of

$$\mathcal{L}^uV(t,y)+f(t,y,u),$$

such that

$$\begin{aligned}0&=\frac{\partial W}{\partial t}(t,y)+G\left(t,y,W(t,y),D_yW(t,y),D_{yy}(W(t,y))\right)\\&=\frac{\partial W}{\partial t}(t,y)+\mathcal{L}^{\widehat{u}(t,y)}W(t,y)+f(t,y,u),\end{aligned} \tag{1.22}$$

the stochastic differential equation

$$dy(t)=b\left(t,y(t),\widehat{u}(t,y)\right)dt+\sigma\left(t,y(t),\widehat{u}(t,y)\right)dB(t), \tag{1.23}$$

defines a unique solution $y(t)$ for each given initial data $y(t)=y$, and the process $\widehat{u}(t,y)$ is a well-defined control process in U. Then $W=V$, and $\widehat{u}$ is an optimal Markov control process.

Proof. The function $W\in C^{1,2}([0,T],\mathbb{R}^n)\cap C([0,T],\mathbb{R}^n)$, then for all $0\leq t\leq s\leq T$, by Itô's Lemma we get

$$\begin{aligned}W\left(t,y^{t,y}(r)\right)=&\int_t^s\left(\frac{\partial W}{\partial t}+\mathcal{L}^{u(r)}W\right)\left(r,y^{t,y}(r)\right)dr\\&+\int_t^s D_yW\left(r,y^{t,y}(r)\right)^{\intercal}\sigma\left(r,y^{t,y}(r),u(r)\right)dB(r),\end{aligned}$$

the process $\int_t^s D_yW\left(r,y^{t,y}(r)\right)^{\intercal}\sigma\left(r,y^{t,y}(r),u(r)\right)$, is a martingale, then by taking expectation, it follows that

$$\mathbb{E}\left[W\left(s,y^{t,y}\left(s\right)\right)\right]=W\left(t,y\right)+\mathbb{E}\left(\int_{t}^{s}\left(\frac{\partial W}{\partial t}+\mathcal{L}^{u(r)}W\right)\left(r,y^{t,y}\left(r\right)\right)dr\right).$$

By (1.21), we get

$$\frac{\partial W}{\partial t}\left(r,y^{t,y}\left(r\right)\right)+\mathcal{L}^{u(r)}W\left(r,y^{t,y}\left(r\right)\right)+f\left(r,y^{t,y}\left(r\right),u\left(r\right)\right)\geq 0,\ \forall u\in A,$$

then

$$\mathbb{E}\left[W\left(s,y^{t,y}\left(s\right)\right)\right]\geq W\left(t,y\right)-\mathbb{E}\left(\int_{t}^{s}f\left(r,y^{t,y}\left(r\right),u\left(r\right)\right)dr\right),\ \forall u\in A,$$

we now take the limit as $s\longrightarrow T$, then by the fact that $W\left(T\right)\leq g$ we obtain

$$\mathbb{E}\left[g\left(y^{t,y}\left(T\right)\right)\right]\geq W\left(t,y\right)-\mathbb{E}\left(\int_{t}^{s}f\left(r,y^{t,y}\left(r\right),u\left(r\right)\right)dr\right),\ \forall u\in A,$$

then $W\left(t,y\right)\leq V\left(t,y\right),\ \forall\left(t,y\right)\in\left[0,T\right]\times\mathbb{R}^{n}$. Statement (2) is proved by repeating the above argument and observing that the control $\widehat{u}$ achieves equality at the crucial step (1.21).

We now state without proof an existence result for the HJB equation (1.20), together with the terminal condition $W\left(T,y\right)=g\left(y\right)$. ■

Theorem 1.3.3 *assume that*

1. $\exists C>0\diagup\xi^{\intercal}\sigma\sigma^{\intercal}\left(t,y,u\right)\xi\geq C\left|\xi\right|^{2}$, *for all* $\left(t,y,u\right)\in\left[0,T\right]\times\mathbb{R}^{n}\times U$,

2. U *is compact,*

3. b,σ *and* f *are in* $C_{b}^{1,2}\left(\left[0,T\right],\mathbb{R}^{n}\right)$,

4. $g\in C_{b}^{3}\left(\mathbb{R}^{n}\right)$,

Then the HJB equation (1.20), with the terminal data $V\left(T,y\right)=g\left(y\right)$, *has a unique solution* $V\in C_{b}^{1,2}\left(\left[0,T\right],\mathbb{R}^{n}\right)$.

Proof. See Fleming and Rischel [31]. ■

We conclude this section by the celebrated Merton's optimal management problem.

Exemple 1.3.1 *We consider a market with two securities, a bond whose price solves*

$$\begin{cases} dS^0(t) = rS^0(t)\,dt \\ S^0(0) = s. \end{cases} \tag{1.24}$$

and a stock whose price process satisfies the stochastic differential equation

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t). \tag{1.25}$$

The market parameters μ *and* σ *are, respectively, the mean rate of return and the volatility, it is assumed that* $\mu > r > 0$*, and* $\sigma > 0$*. The process* $B(t)$ *is a standard Brownian motion defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *The wealth process satisfies* $Y(s) = u^0(s) + u(s)$*, with the amounnters* $u^0(s)$ *and* $u(s)$ *representing the current holdings in the bond and the stock accounts. The state wealth equation is given by*

$$dY(s) = rY(s)\,ds + (\mu - r)\,u(s)\,ds + \sigma u(s)\,dB(s). \tag{1.26}$$

The wealth process must satisfy the state constraint

$$Y(s) \geq 0, \;\; a.e.\ t \leq s \leq T. \tag{1.27}$$

The control $u(s)$*, is admissible if it is* $\mathcal{F}_s$ *-progressively measurable, it satisfies* $\mathbb{E}\int_t^T u^2(s)\,ds$ *and it is such that the state constraint (1.27) is satisfied. We denote the set of admissible policies by* $\widetilde{A}$*. The value function is defined by*

$$V(t,y) = \sup_{\widetilde{A}} E\left(\frac{1}{\gamma} Y^{\gamma}(T) / Y(t) = y\right). \tag{1.28}$$

Using stochastic analysis and under appropriate regularity and growth conditions on the value *function, we get that* V *solves the associated HJB equation, for* $y \geq 0,$ *and* $t \in [0,T]$,

$$
\begin{aligned}
&V(t) + \max_u \left(\frac{1}{2}\sigma^2 u^2 V_{yy} + (\mu - r) V_y\right) + ryV_y = 0, \\
&V(T,y) = \frac{1}{\gamma} y^\gamma, \qquad (1.29) \\
&V(t,0) = 0, \ t \in [0,T].
\end{aligned}
$$

The homogeneity of the utility function and the linearity of the state dynamics with respect to both the wealth and the control portfolio process, suggest that the value function must be of the form

$$V(t,y) = \frac{1}{\gamma} y^\gamma f(t), \ \textit{with } f(T) = 1. \qquad (1.30)$$

Using the above form in (1.29), and after some cancellations, one gets that f *must satisfy the first order equation*

$$\begin{cases} f'(t) + \lambda f(t) = 0, \\ f(T) = 1, \end{cases}$$

where

$$\lambda = r\gamma + \frac{(\mu - r)}{2(1-\gamma)\sigma^2}. \qquad (1.31)$$

Therefore,

$$V(t,y) = \frac{1}{\gamma} y^\gamma \exp \lambda (T - t). \qquad (1.32)$$

Once the value function is determined, the optimal policy may be obtained in the so-called feedback form as follows: first, we observe that the maximum of the quadratic term appearing in (1.29) is achieved at the point

$$\widehat{u}(t,y) = -\frac{(\mu - r) V_y(t,y)}{\sigma^2 V_{yy}(t,y)}, \qquad (1.33)$$

or, otherwise,

$$\widehat{u}(t,y) = \frac{(\mu - r)}{\sigma^2 (1-\gamma)} y, \qquad (1.34)$$

where we used (1.32). Next, we recall classical Verification results, which yield that the candidate solution, given in (1.32) is indeed the value function and that, moreover, the policy $\widehat{u}(t,y) =$

$\frac{(\mu-r)}{\sigma^2(1-\gamma)}Y^*(t)$, *is the optimal investment strategy. In the other words,*

$$V(t,y) = E\left(\frac{1}{\gamma}Y^{*\gamma}(T) \ \ / \ Y^*(t) = y\right),$$

where $Y^*(s)$ *solves*

$$dY^*(s) = \left(r + \frac{(\mu-r)^2}{(1-\gamma)\sigma^2}\right)Y^*(s)\,ds + \frac{(\mu-r)}{(1-\gamma)\sigma}Y^*(s)\,dB(s). \tag{1.35}$$

The solution of the optimal state wealth equation is, for $Y(t) = y$,

$$Y^*(s) = y\exp\left[\left(r + \frac{(\mu-r)^2}{(1-\gamma)\sigma^2} - \frac{(\mu-r)^2}{2(1-\gamma)^2\sigma^2}\right)(s-t) + \frac{(\mu-r)}{(1-\gamma)\sigma}B(s-t)\right].$$

The Merton optimal strategy dictates that it is optimal tokeep a fixed proportion, namely $\frac{(\mu-r)}{(1-\gamma)\sigma^2}$, *of the current total wealth invested in the stock account.*

1.3.3 Viscosity solutions

It is well known that the HJB equation (1.20) does not necessarily admit smooth solution in general. This makes the applicability of the classical verification theorems very restrictive and is a major deficiency in dynamic programming theory. In recent years, the notion of viscosity solutions was introduced by Crandall and Lions [35] for first-order equations, and by Lions [44] for second-order equations. For a general overview of the theory we refer to the User's Guide by Crandall, Ishii and Lions [34] and the book by Fleming and Soner [30]. In this theory all the derivatives involved are replaced by the so-called superdifferentials and subdifferentials, and the solutions in the viscosity sense can be merely continuous functions. The existence and uniqueness of viscosity solutions of the HJB equation can be guaranteed under very mild and reasonable assumptions, which are satisfied in the great majority of cases arising in optimal control problems. For example, the value function turns out to be the unique viscosity solution of the HJB equation (1.20).

Definition 1.3.1 *A function* $V \in C([0,T] \times \mathbb{R}^n)$ *is called a viscosity subsolution of (1.20), if* $V(T,y) \leq g(y)$, $\forall y \in \mathbb{R}^n$, *and for any* $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$, *whenever* $V - \varphi$ *attains a local* *maximum at* $(t, y) \in [0, T] \times \mathbb{R}^n$, *we have*

$$-\frac{\partial \varphi}{\partial t}(t, y) + \sup_{u \in U} G\left(t, y, u, -D_y \varphi(t, y), -D_{yy} \varphi(t, y)\right) \geq 0. \tag{1.36}$$

A function $V \in C\left([0, T] \times \mathbb{R}^n\right)$ *is called a viscosity supersolution of (1.20), if* $V(T, x) \leq g(y)$, $\forall y \in \mathbb{R}^n$, *and for any* $\varphi \in C^{1,2}\left([0, T] \times \mathbb{R}^n\right)$, *whenever* $V - \varphi$ *attains a local minimum at* $(t, y) \in [0, T] \times \mathbb{R}^n$, *we have*

$$-\frac{\partial \varphi}{\partial t}(t, y) + \sup_{u \in U} G\left(t, y, u, -D_y \varphi(t, y), -D_{yy} \varphi(t, y)\right) \geq 0. \tag{1.37}$$

Further, if $V \in C\left([0, T] \times \mathbb{R}^n\right)$ *is both a viscosity subsolution and viscosity supersolution of (1.20), then it is called a viscosity solution of (1.20).*

Theorem 1.3.4 *Let (1.8) and (1.9) hold, then the value function* V *is a viscosity solution of (1.20).*

Proof. For any $\varphi \in C^{1,2}\left([0, T] \times \mathbb{R}^n\right)$, let $V - \varphi$ attains a local maximum at $(t, x) \in [0, T] \times \mathbb{R}^n$. Fix a $u \in U$, let $y(t)$ be the state trajectory with the control $u(t) = u$. Then by the dynamic programming principle, and Itô's formula, we have for $\widehat{s} > s$ with $\widehat{s} - s > 0$ small enough

$$\begin{aligned}
0 &\leq \frac{1}{\widehat{s} - s} \mathbb{E}\left(V(s, x) - \varphi(s, x) - V(\widehat{s}, y(\widehat{s})) + \varphi(\widehat{s}, y(\widehat{s}))\right) \\
&\leq \frac{1}{\widehat{s} - s} \mathbb{E}\left(\int_s^{\widehat{s}} f(t, y(t), u)\, dt + \varphi(s, x) + \varphi(\widehat{s}, y(\widehat{s}))\right) \\
&\underset{\widehat{s} \longrightarrow s}{\longrightarrow} -\frac{\partial \varphi}{\partial t}(t, x) - G\left(t, x, u, -D_y \varphi(t, y), -D_{yy} \varphi(t, y)\right).
\end{aligned}$$

This leads to

$$-\frac{\partial \varphi}{\partial t}(t, x) - G\left(t, x, u, -D_y \varphi(t, x), -D_{yy} \varphi(t, x)\right) \leq 0, \ \forall u \in U.$$

Hence

$$-\frac{\partial \varphi}{\partial t}(t, x) + \sup_{u \in U} G\left(t, x, u, -D_y \varphi(t, x), -D_{yy} \varphi(t, x)\right) \leq 0, \ \forall u \in U. \tag{1.38}$$

On the other hand, if $V-\varphi$ attains a local minimum at $(t,x)\in[0,T]\times\mathbb{R}^n$, then for any $\varepsilon>0$, and $\widehat{s}>s$ with $\widehat{s}-s>0$ small enough, we can find a $u(t)=u^{\varepsilon}(s)\in U$, such that

$$\begin{aligned}0 &\geq \mathbb{E}\left(V(s,x)-\varphi(s,x)-V(\widehat{s},y(\widehat{s}))+\varphi(\widehat{s},y(\widehat{s}))\right)\\ &\geq -\varepsilon(\widehat{s}-s)+\mathbb{E}\left(\int_s^{\widehat{s}} f(t,y(t),u(t))\,dt+\varphi(\widehat{s},y(\widehat{s}))-\varphi(s,y)\right),\end{aligned}$$

dividing by $(\widehat{s}-s)$, and applying Itô's formula to the process $\varphi(t,y(t))$, we get

$$\begin{aligned}-\varepsilon &\leq \frac{1}{\widehat{s}-s}\mathbb{E}\left(\int_s^{\widehat{s}} -\frac{\partial\varphi}{\partial t}(t,y(t))+G(t,y(t),u,-D_y\varphi(t,y(t)),-D_{yy}\varphi(t,y(t)))\,dt\right)\\ &\leq \frac{1}{\widehat{s}-s}\mathbb{E}\left(\int_s^{\widehat{s}} -\frac{\partial\varphi}{\partial t}(t,y(t))+\sup_{u\in U} G(t,y(t),u,-D_y\varphi(t,y(t)),-D_{yy}\varphi(t,y(t)))\,dt\right)\\ &\underset{\widehat{s}\longrightarrow s}{\longrightarrow} -\frac{\partial\varphi}{\partial t}(t,y(t))+\sup_{u\in U} G(t,x,u,-D_y\varphi(t,x),-D_{yy}\varphi(t,x)).\end{aligned} \tag{1.39}$$

Combining (1.38), and (1.39), we conclude that V is a viscosity solution of the HJB equation (1.20). ■

The following Theorem is devoted to a proof of uniqueness of the viscosity solution to the HJB equation

Theorem 1.3.5 *Let $V,W\in C_b^{1,2}([0,T]\times\mathbb{R}^n)$. We suppose that V is a supersolution of (1.20), with $V(T,y)\leq W(T,y)$ for all $y\in\mathbb{R}^n$, then $V(t,y)\leq W(t,y),\forall(t,y)\in[0,T]\times\mathbb{R}^n$.*

Proof. Let, for $(\alpha,M,N)\in\mathbb{R}_+^*\times\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$, we define

$$G(y,M)=-tr\left[A(y)A(y)^{\top}M\right],$$

then, we obtain

$$\begin{aligned}G(x,N)-G(y,M) &= tr\left[A(y)A(y)^{\top}M\right]-tr\left[A(x)A(x)^{\top}N\right]\\ &= tr\left[A(y)A(y)^{\top}M-A(x)A(x)^{\top}N\right]\\ &\leq 3\alpha\left|A(y)-A(x)\right|^2,\end{aligned}$$

because the matrix

$$C := \begin{pmatrix} A(x)A(x)^\top & A(x)A(y)^\top \\ A(y)A(x)^\top & A(y)A(y)^\top \end{pmatrix},$$

is a non negative matrix, we have

$$\begin{aligned} tr\left[A(y)A(y)^\top M - A(x)A(x)^\top N\right] &= tr\left[C\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix}\right] \\ &\leq 3\alpha tr\left[C\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}\right] \\ &\leq 3\alpha tr\left[(A(y)-A(x))\left(A(y)^\top - A(x)\right)^\top\right] \\ &\leq 3\alpha\left|A(y)-A(x)\right|^2. \qquad (1.40) \end{aligned}$$

Now, we consider the function

$$F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow F(y,x) = V(y) - W(x) - \frac{1}{2\varepsilon}|y-x|^2,$$

with $\varepsilon > 0$. Suppose that there exists a point $(\overline{y}, \overline{x})$ such that F attaints a maximum at $(\overline{y}, \overline{x})$, then $y \longrightarrow F(y, \overline{x})$ attaints a maximum at $\overline{x}$, hence

$$y \longrightarrow V(y) - \frac{1}{2\varepsilon}|y-\overline{x}|^2,$$

attaints a maximum at $\overline{y}$. Moreover, $y \longrightarrow -F(\overline{y}, x)$ attaints a minimum at $\overline{y}$, then we have

$$y \longrightarrow W(y) - \frac{1}{2\varepsilon}|\overline{y}-x|^2,$$

attaints a minimum at $\overline{x}$. By the definition of viscosity subsolution at point $\overline{y}$, we obtain for V with $\varphi(y) = \frac{1}{2\varepsilon}|y-\overline{x}|^2$, we get

$$-\frac{\partial V}{\partial t}(\overline{y}) + \sup_{u\in U}\left[-b(\overline{y},u)\left(\frac{1}{\varepsilon}|\overline{y}-\overline{x}|\right) - \frac{1}{2}tr\left(\sigma\sigma^\top((\overline{y},u)\left(-\frac{1}{\varepsilon}\right)\right) - f((\overline{y},u)\right] \geq 0.$$

By the definition of viscosity subsolution at point $\overline{x}$, we obtain for W with $\varphi(y) = \frac{1}{2\varepsilon}|\overline{y} - x|^2$, we get

$$-\frac{\partial W}{\partial t}(\overline{x}) + \sup_{u \in U}\left[-b(\overline{x},u)\left(\frac{1}{\varepsilon}|\overline{y}-\overline{x}|\right) - \frac{1}{2}tr\left(\sigma\sigma^\top((\overline{x},u)\left(-\frac{1}{\varepsilon}\right)\right) - f((\overline{x},u)\right] \leq 0.$$

Hence

$$\begin{aligned}
& -\frac{\partial}{\partial t}(V(\overline{y}) - W(\overline{x})) \\
& \leq \sup_{u \in A}\left\{|b(\overline{y},u) - b(\overline{x},u)|\left(\frac{1}{\varepsilon}|\overline{y}-\overline{x}|\right) + |f((\overline{y},u) - f((\overline{x},u)|\right. \\
& \left. + \frac{1}{2}tr\left(\sigma\sigma^\top((\overline{y},u)\left(-\frac{1}{\varepsilon}\right)\right) - tr\left(\sigma\sigma^\top((\overline{x},u)\left(-\frac{1}{\varepsilon}\right)\right)\right\},
\end{aligned}$$

the functions b, f, $\sigma\sigma^\top$ are Lipshitz on y uniformly on u then by (1.40), we get

$$\begin{aligned}
& \frac{\partial}{\partial t}(V(\overline{y}) - W(\overline{x})) \\
& \leq \left(\frac{c}{\varepsilon}|\overline{y}-\overline{x}|^2\right) + c|\overline{y}-\overline{x}| + 3\grave{\alpha}|\overline{y}-\overline{x}|^2.
\end{aligned}$$

On the other hand, $F(y,y) \leq F(\overline{y},\overline{x}), \forall y \in \mathbb{R}^n$.

$$\begin{aligned}
& V(y) - W(y) \leq V(\overline{y}) - W(\overline{x}) - \frac{1}{2\varepsilon}|\overline{y}-\overline{x}|^2 \\
& \leq V(\overline{y}) - W(\overline{x}).
\end{aligned} \tag{1.41}$$

Because $F(\overline{y},\overline{x}) \geq F(\overline{y},\overline{y})$, we get

$$V(\overline{y}) - W(\overline{x}) - \frac{1}{2\varepsilon}|\overline{y}-\overline{x}|^2 \geq V(\overline{y}) - W(\overline{y}). \tag{1.42}$$

Then

$$W(\overline{y}) - W(\overline{x}) - \frac{1}{2\varepsilon}|\overline{y}-\overline{x}|^2 \geq 0. \tag{1.43}$$

Moreover, $F(\overline{y},\overline{x}) \geq F(\overline{x},\overline{x})$, then

$$V(\overline{y}) - V(\overline{x}) - \frac{1}{2\varepsilon}|\overline{y}-\overline{x}|^2 \geq 0, \tag{1.44}$$

this proves that

$$\frac{1}{\varepsilon}\left|\overline{y}-\overline{x}\right|^{2}\leq\left(V+W\right)\left(\overline{y}\right)-\left(V+W\right)\left(\overline{x}\right), \tag{1.45}$$

where V,W are bonded, then $\frac{1}{\varepsilon}\left|\overline{y}-\overline{x}\right|^{2}\leq c$, which means that

$$m\left(\eta\right)=\sup\left\{\left|\left(V+W\right)\left(y\right)-\left(V+W\right)\left(x\right)\right|,\left|y-x\right|\leq\eta\right\}:m\left(\eta\right)\underset{\eta\longrightarrow0}{\longrightarrow}0.$$

By (1.41), we get

$$\frac{1}{\varepsilon}\left|\overline{y}-\overline{x}\right|^{2}\leq m\left(\left|\overline{y}-\overline{x}\right|\right), \tag{1.46}$$

under (1.44), on has

$$\frac{1}{\varepsilon}\left|\overline{y}-\overline{x}\right|^{2}\leq m\left(c\sqrt{\varepsilon}\right). \tag{1.47}$$

Combining (1.45), (1.46) and (1.47), we obtain

$$\frac{\partial}{\partial t}\left(V\left(\overline{y}\right)-W\left(\overline{x}\right)\right)\leq c\sqrt{\varepsilon}+\grave{c}\varepsilon+m\left(c\sqrt{\varepsilon}\right).$$

Finally, by (1.41) on has

$$V\left(y\right)-W\left(x\right)\leq V\left(\overline{y}\right)-W\left(\overline{x}\right)\underset{\varepsilon\longrightarrow0}{\longrightarrow}0.\quad\text{for all } y\in\mathbb{R}^{n},$$

hence $V\left(y\right)\leq W\left(x\right)$. ■

Definition 1.3.2 *Let* $V\in C\left(\left[0,T\right]\times\mathbb{R}^{n}\right)$, *the right superdifferential (resp., subdifferential) of* V *at* $(t,y)\in[0,T]\times\mathbb{R}^{n}$, *denoted by* $D_{t,y}^{1,2,+}V\left(t,y\right)$ $\left(\textit{resp.}, D_{t,y}^{1,2,-}V\left(t,y\right)\right)$, *is a set defined by*

$$D_{t,y}^{1,2,+}V\left(t,y\right)=\left\{\left(p,q,Q\right)\in\mathbb{R}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times n}\ /\ \lim_{\substack{x\longrightarrow y,s\longrightarrow t,\\ s\in[0,T]}}\sup\frac{I\left(s,x\right)}{\left|s-t\right|+\left|x-y\right|^{2}}\leq0\right\},$$

$$D_{t,y}^{1,2,-}V\left(t,y\right)=\left\{\left(p,q,Q\right)\in\mathbb{R}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times n}\ /\ \lim_{\substack{x\longrightarrow y,s\longrightarrow t,\\ s\in[0,T]}}\inf\frac{I\left(s,x\right)}{\left|s-t\right|+\left|x-y\right|^{2}}\geq0\right\},$$

where

$$I\left(s,x\right)=V\left(s,x\right)-V\left(t,y\right)-q\left(s-t\right)-\left\langle p,x-y\right\rangle-\frac{1}{2}\left(x-y\right)^{\top}P\left(x-y\right).$$

Definition 1.3.3 *A function $V \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity solution of the HJB equation (1.20) if*

$$
\begin{aligned}
& -p + \sup_{u \in U} G(t, y, u, q, Q) \leq 0, \quad \forall (p, q, Q) \in D_{t,y}^{1,2,+} V(t, y), \forall (t, y) \in [0,T] \times \mathbb{R}^n, \\
& -p + \sup_{u \in U} G(t, y, u, q, Q) \geq 0, \quad \forall (p, q, Q) \in D_{t,y}^{1,2,-} V(t, y), \forall (t, y) \in [0,T] \times \mathbb{R}^n, \\
& V(T, y) = g(y), \quad \forall y \in \mathbb{R}^n.
\end{aligned}
$$

Lemma 1.3.1 *The value function V satisfies*

$$
|V(t, y) - V(s, x)| \leq C\left(|t - s|^{\frac{1}{2}} + |y - x|\right).
$$

Proof. See Yong and Zhou [64]. ∎

Corollary 1.3.1 *We have*

$$
\lim_{(p,q,Q) \in D_{t,y}^{1,2,+} V(t,y) \times U} \{[p - G(t, y, u, q, Q)] \geq 0, \forall (t, y) \in [0,T] \times \mathbb{R}^n\}. \tag{1.48}
$$

Proof. See Yong and Zhou [64]. ∎

Lemma 1.3.2 *Let $g \in C[0,T]$. Suppose that there is $\rho \in L^1[0,T]$ such that for sufficiently small $h > 0$,*

$$
\frac{g(t+h) - g(t)}{h} \leq \rho(t), \quad a.e. t \in [0,T]. \tag{1.49}
$$

Then

$$
g(t) - g(0) \leq \int_0^t \overline{\lim_{h \longrightarrow 0+}} \frac{g(r+h) - g(r)}{h} dr, \quad \forall t \in [0,T]. \tag{1.50}
$$

Proof. First fix $t \in [0,T]$. By (1.49) we can apply Fatou's Lemma to get

$$
\begin{aligned}
\int_0^t \overline{\lim_{h \longrightarrow 0+}} \frac{g(r+h) - g(r)}{h} dr &\geq \overline{\lim_{h \longrightarrow 0+}} \int_0^t \frac{g(r+h) - g(r)}{h} dr, \\
&= \overline{\lim_{h \longrightarrow 0+}} \frac{\int_h^{h+t} g(r) dr - \int_0^t g(r) dr}{h} \\
&= g(t) - g(0).
\end{aligned}
$$

This proves (1.50) $\forall t\in[0,T]$, finally, the $t=T$ case is obtained by continuity. ■

Theorem 1.3.6 *Let $W\in C([0,T]\times\mathbb{R}^n)$ be a viscosity solution of the HJB equation (1.20), then*

(1) $W(s,x)\leq J(s,x,u)$ for any $(s,x)\in[0,T]\times\mathbb{R}^n$ and any $u\in U$.

(2) Let $(\widehat{y},\widehat{u})$ be a given admissible pair for the problem (1.7)-(1.10). Suppose that there exists

$$\left(\widehat{p},\widehat{q},\widehat{Q}\right)\in L_F^2(s,T,\mathbb{R})\times L_F^2(s,T,\mathbb{R}^n)\times L_F^2\left(s,T,\mathbb{R}^{n\times d}\right),$$

such that for a.e.$t\in[s,T]$,

$$\left(\widehat{p}(t),\widehat{q}(t),\widehat{Q}(t)\right)\in D_{t,y}^{1,2,+}W(t,\widehat{y}(t)),\quad \mathbb{P}-a.s., \tag{1.51}$$

and

$$-\widehat{p}(t)+G\left(t,\widehat{y}(t),\widehat{u}(t),\widehat{q}(t),\widehat{Q}(t)\right)=0,\quad \mathbb{P}-a.s., \tag{1.52}$$

then $(\widehat{y}(t),\widehat{u}(t))$ is an optimal pair for the problem (1.7)-(1.10).

Proof. Part (1) is trivial since $W=V$ in view of the uniqueness of the viscosity solutions. We prove only part (2) of the Theorem, set $\varphi(t,\widehat{y}(t),\widehat{u}(t))=\widehat{\varphi}(t)$, $\varphi=b,\sigma,f$, ect., to simplify the notation. Fix $t\in[0,T]$ such that (1.51) and (1.52) holds. Choose a test function $\phi\in C([0,T]\times\mathbb{R}^n)\cap C^{1,2}([0,T]\times\mathbb{R}^n)$ as determined by $\left(\widehat{p}(t),\widehat{q}(t),\widehat{Q}(t)\right)\in D_{t,y}^{1,2,+}W(t,\widehat{y}(t))$ and Lemma (1.3.1). Applying Ito's formula to ϕ, we have for any $h>0$,

$$\begin{aligned}&W(t+h,\widehat{y}(t+h))-W(t,\widehat{y}(t))\leq\phi(t+h,\widehat{y}(t+h))-\phi(t,\widehat{y}(t))\\&=\mathbb{E}\left(\int_t^{t+h}\left(\phi_t(r,\widehat{y}(r))+\phi_y(r,\widehat{y}(r)).\widehat{b}(r)+\frac{1}{2}tr\left(\widehat{\sigma}(r)^{\top}\phi_{yy}(r,\widehat{y}(r)).\widehat{\sigma}(r)\right)\right)dr\right).\end{aligned}\tag{1.53}$$

It is well known by the martingale property of stochastic integrals that there are constant C, indepedent of t,such that

$$\mathbb{E}\left|\widehat{y}\left(r\right)-\widehat{y}\left(t\right)\right|^{2}\leq C\left|r-t\right| \quad ,\forall r\geq t, \tag{1.54}$$

$$\mathbb{E}\left[\sup_{s\leq r\leq T}\left|\widehat{y}\left(r\right)\right|^{\alpha}\right]\leq C\left(\alpha\right), \quad ,\forall\alpha\geq T. \tag{1.55}$$

Hence, in view of Lemma (1.3.1), we have

$$\sup_{s\leq r\leq T}\left|\phi_{t}\left(r,\widehat{y}\left(r\right)\right)\right|^{2}\leq C^{2}\sup_{s\leq r\leq T}\mathbb{E}\left[1+\frac{\left|\widehat{y}\left(r\right)-\widehat{y}\left(t\right)\right|^{2}}{r-t}\right]\leq C. \tag{1.56}$$

or

$$\sup_{s\leq r\leq T}\mathbb{E}\left|\phi_{t}\left(r,\widehat{y}\left(r\right)\right)\right|\leq\sqrt{C},$$

Moreover, by Lemma (1.3.2), assumption (1.8) and (1.9), one can show that

$$\sup_{s\leq r\leq T}\mathbb{E}\left|\phi_{y}\left(r,\widehat{y}\left(r\right)\right)\widehat{b}\left(r\right)+\frac{1}{2}tr\left(\widehat{\sigma}\left(r\right)^{\top}.\phi_{yy}\left(r,\widehat{y}\left(r\right)\right).\widehat{\sigma}\left(r\right)\right)\right|\leq C.$$

It then follows from (1.54) that for sufficiently small $h>0$,

$$\frac{\mathbb{E}\left[W\left(t+h,\widehat{y}\left(t+h\right)\right)-W\left(t,\widehat{y}\left(t\right)\right)\right]}{h}\leq C. \tag{1.57}$$

Now we calculate, for any fixed $N>0$,

$$\begin{aligned}\frac{1}{h}\int_{t}^{t+h}\mathbb{E}\left(\phi_{t}\left(r,\widehat{y}\left(r\right)\right)-\widehat{p}\left(t\right)\right)dr&=\frac{1}{h}\int_{t}^{t+h}\mathbb{E}\left(\left(\phi_{t}\left(r,\widehat{y}\left(r\right)\right)-\widehat{p}\left(t\right)\right)\mathbf{1}_{\left|\widehat{y}(r)-\widehat{y}(t)\right|>N\left|r-t\right|^{\frac{1}{2}}}\right)dr\\&\quad+\frac{1}{h}\int_{t}^{t+h}\mathbb{E}\left(\left(\phi_{t}\left(r,\widehat{y}\left(r\right)\right)-\widehat{p}\left(t\right)\right)\mathbf{1}_{\left|\widehat{y}(r)-\widehat{y}(t)\right|\leq N\left|r-t\right|^{\frac{1}{2}}}\right)dr,\\&=I_{1}\left(N,h\right)+I_{2}\left(N,h\right).\end{aligned}$$

By virtue of (1.55) and (1.57), we have

$$\begin{aligned}I_{1}\left(N,h\right)&\leq\frac{1}{h}\int_{t}^{t+h}\mathbb{E}\left[\left(\phi_{t}\left(r,\widehat{y}\left(r\right)\right)-\widehat{p}\left(t\right)\right)^{2}\right]^{\frac{1}{2}}\left[P\left(\left|\widehat{y}\left(r\right)-\widehat{y}\left(t\right)\right|>N\left|r-t\right|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}dr\\&\leq\frac{C}{N}\longrightarrow0\text{ uniformly in }h>0\text{ as }N\longrightarrow\infty.\end{aligned}$$

On the other hand, for fixed $N > 0$, we apply Lemma (1.3.2) to get

$$\lim_{h\to 0+}\sup_{t\le r\le t+h}\left(\left(\phi_t\left(r,\widehat{y}\left(r\right)\right)-\widehat{p}\left(t\right)\right)\mathbf{1}_{|\widehat{y}(r)-\widehat{y}(t)|\le N|r-t|^{\frac{1}{2}}}\right)\longrightarrow 0 \text{ as } h\to 0+,\ \mathbb{P}-a.s.$$

Thus we conclude by the dominated convergence theorem that

$$\lim_{h\to 0+} I_2\left(N,h\right)\longrightarrow 0,\ \text{ as } h\to 0+,\ \text{ for each fixed } N.$$

Therefore, we have proved that

$$\lim_{h\to 0+}\frac{1}{h}\int_t^{t+h}\mathbb{E}\left(\phi_t\left(r,\widehat{y}\left(r\right)\right)\right)dr\longrightarrow\mathbb{E}\left[\widehat{p}\left(t\right)\right]. \tag{1.58}$$

Similarly (in fact, more easily),we can show that

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\int_t^{t+h}\mathbb{E}\left[\phi_y\left(r,\widehat{y}\left(r\right)\right).\widehat{b}\left(r\right)\right]dr &= \mathbb{E}\left[\phi_y\left(t,\widehat{y}\left(t\right)\right).\widehat{b}\left(t\right)\right],\\ &= \mathbb{E}\left[\widehat{q}\left(t\right).\widehat{b}\left(t\right)\right],\end{aligned} \tag{1.59}$$

and

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\int_t^{t+h}\mathbb{E}\left[\frac{1}{2}tr\left(\hat{\sigma}\left(r\right)^t.\phi_{yy}\left(r,\hat{x}_r\right).\hat{\sigma}\left(r\right)\right)\right]dr &= \mathbb{E}\left[\frac{1}{2}tr\left(\widehat{\sigma}\left(t\right)^\top.\phi_{yy}\left(t,\widehat{y}\left(t\right)\right).\widehat{\sigma}\left(t\right)\right)\right]\\ &= \mathbb{E}\left[\frac{1}{2}tr\left(\widehat{\sigma}\left(t\right)^\top.\widehat{Q}\left(t\right).\widehat{\sigma}\left(t\right)\right)\right].\end{aligned}$$

Consequently (1.54) gives

$$\begin{aligned}\overline{\lim_{h\to 0+}}\frac{\mathbb{E}\left[W\left(t+h,\widehat{y}\left(t+h\right)\right)-W\left(t,\widehat{y}\left(t\right)\right)\right]}{h} &\le \mathbb{E}\left(\widehat{p}\left(t\right)+\widehat{q}\left(t\right).\widehat{b}\left(t\right)+\frac{1}{2}tr\left(\widehat{\sigma}\left(t\right)^\top.\widehat{Q}\left(t\right).\widehat{\sigma}\left(t\right)\right)\right)\\ &= -\mathbb{E}\left[\widehat{g}\left(t\right)\right],\end{aligned} \tag{1.60}$$

where the last equality is due to (1.53). Noting (1.58) and applying Lemma (1.3.1) to the $g\left(t\right)=\mathbb{E}\left[W\left(t,\widehat{y}\left(t\right)\right)\right]$ we arrive at

$$\mathbb{E}\left[W\left(T,\widehat{y}\left(T\right)\right)-W\left(s,x\right)\right]\le\int_s^T\mathbb{E}\left[\widehat{g}\left(t\right)\right]dt, \tag{1.61}$$

which leads to $W(s,x) \geq J(s,x,\widehat{u})$. It follows that $(\widehat{y},\widehat{u})$ is an optimal pair for (1.7) and (1.10).

■

Remark 1.3.1 *In view of Corollary (1.3.1), the condition (1.53) impllies that* $\left(\widehat{p}(t), \widehat{q}(t), \widehat{Q}(t), \widehat{u}(t)\right)$ *achieves the infimum of* $p - G(t, \widehat{y}(t), u, q, Q)$ *over* $D_{t,y}^{1,2,+}W(t,\widehat{y}(t)) \times U$. *Meanwhile, it also shows that (1.52) is equivalent to*

$$\widehat{p}(t) \leq G\left(t, \widehat{y}(t), \widehat{u}(t), \widehat{p}(t), \widehat{q}(t), \widehat{Q}(t)\right). \tag{1.62}$$

Remark 1.3.2 *The condition (1.53) implies that*

$$\max_{u \in U} G\left(t, \widehat{y}(t), u, \widehat{p}(t), \widehat{q}(t), \widehat{Q}(t)\right) = G\left(t, \widehat{y}(t), \widehat{u}(t), \widehat{p}(t), \widehat{q}(t), \widehat{Q}(t)\right). \tag{1.63}$$

This easily seen by recalling the fact that V *is the viscosity solution of (1.20),hence*

$$-\widehat{p}(t) + \sup_{u \in U} G\left(t, \widehat{y}(t), u, \widehat{p}(t), \widehat{q}(t), \widehat{Q}(t)\right) \leq 0,$$

which yields (1.63) under (1.53).

1.4 Stochastic maximum principle (SMP)

The stochastic maximum principle is an important result in stochastic optimal control. The basic idea is to derive a set of necessary and sufficient conditions that must be satisfied by any optimal control. The first version of the (SMP) was extensively established in the 1970s by Bismut [14], Kushner [43], and Haussmann [37], under the condition that there is no control on the diffusion coefficient. Haussman [36], developed a powerful form of Stochastic Maximum Principle for the feedback class of controls by Girsanov's transformation, and applied it to solve some problems in stochastic control.

There is interest in applying the stochastic maximum principle in finance. The first use of the stochastic maximum principle in finance is probably due to Cadenillas and Karatzas [19]. Some attention has been paid to applying the stochastic maximum principle to mean-variance portfolio selection problems (see, for example, Yong and Zhou [64] and Zhou and Yin [65]), where the problem was formulated as a stochastic linear-quadratic problem.

1.4.1 Problem formulation and assumptions

In all what follows, we are given a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P}\right)$ such that $\mathcal{F}_0$ contains the $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon T, and $\{\mathcal{F}_t\}_{t\leq T}$ satisfies the usual conditions. We assume that $\{\mathcal{F}_t\}_{t\leq T}$ is generated by a d-dimensional standard Brownian motion B. We denote by $\mathcal{U}$ the set of all admissible controls. Any element $y \in \mathbb{R}^n$ will be identified to a column vector with n components, and the norm $|y| = \left|x^1\right| + ... + |x^n|$. The scalar product of any two vectors y and x on $\mathbb{R}^n$ is denoted by yx or $\sum_{i=1}^n y^i x^i$. For a function h, we denote by h_y (resp. h_{yy}) the gradient or Jacobian (resp. the Hessian) of h with respect to the variable y.

Definition 1.4.1 *An admissible control is a measurable, adapted processes $u : [0, T] \times \Omega \to U$, such that $\mathbb{E}\left[\int_0^T u(s)\, ds\right] < \infty$.*

Consider the following stochastic controlled system

$$\begin{cases} dy(t) &= b(t, y(t), u(t))\, dt + \sigma(t, y(t), u(t))\, dB(t) \\ y(0) &= y, \end{cases} \tag{1.64}$$

where $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$, are given.

Suppose we are given a performance functional $J(u)$ of the form

$$J(u) = \mathbb{E}\left[\int_0^T f(t, y(t), u(t))\,dt + g(y(T))\right], \tag{1.65}$$

where $f : [0,T] \times \mathbb{R}^n \times U_1 \to \mathbb{R}, \quad g : \mathbb{R}^n \to \mathbb{R}$.

The stochastic control problem is to find an optimal control $\widehat{u} \in \mathcal{U}$ such that

$$J(\widehat{u}) = \inf_{u\in\mathcal{U}} J(u), \tag{1.66}$$

Let us make the following assumptions about the coefficients b, σ, f, and g.

(H1) The maps b, σ, and f are continuously differentiable with respect to (y, u), and g is continuously differentiable in y.

(H2) The derivatives $b_y, b_u, \sigma_y, \sigma_u, f_y, f_u$, and g_y are continuous in (y, u) and uniformly bounded.

(H3) b, σ, f are bounded by $K_1(1 + |y| + |u|)$, and g is bounded by $K_1(1 + |y|)$, for some $K_1 > 0$.

1.4.2 The stochastic maximum principle

Now, define the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \overline{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \longrightarrow \mathbb{R}$, by

$$H(t, y, u, p, q) = f(t, y, u) + pb(t, y, u) + \textstyle\sum_{j=1}^n q^j \sigma^j(t, y, u), \tag{1.67}$$

where q^j and σ^j for $j = 1, .., n$, denote the jth column of the matrix q and σ, respectively.

Let $\widehat{u}$ be an optimal control and $\widehat{y}$ denote the corresponding optimal trajectory. Then, we consider a pair (p, q) of square integrable adapted processes associated to $\widehat{u}$, with values in $\mathbb{R}^n \times \mathbb{R}^{n\times d}$ such that

$$\begin{cases} dp(t) = & -H_y(t, \widehat{y}(t), \widehat{u}(t), p(t), q(t))dt + q(t)\,dB(t), \\ p(T) = & g_y(\widehat{y}(T)). \end{cases} \tag{1.68}$$

1.4.3 Necessary conditions of optimality

The purpose of this subsection is to find optimality necessary conditions satisfied by an optimal control, assuming that the solution exists. The idea is to use convex perturbation for the optimal control, jointly with some estimations of the state trajectory and performance functional, and by sending the perturbations to zero, one obtains some inequality, then by completing with martingale representation theorem's the maximum principle is expressed in terms of an adjoint process.

We can state the stochastic maximum principle in a stronger form.

Theorem 1.4.1 (Necessary conditions of optimality) *Let $\widehat{u}$ be an optimal control minimizing the performance functional J over $\mathcal{U}$, and let $\widehat{y}$ be the corresponding optimal trajectory, then there exists an adapted processes $(p, q) \in \mathbb{L}^2\left(\left([0, T]; \mathbb{R}^n\right)\right) \times \mathbb{L}^2\left(\left([0, T]; \mathbb{R}^{n\times d}\right)\right)$ which is the unique solution of the BSDE (1.68), such that for all $v \in U$*

$$H_u\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right), p\left(t\right), q\left(t\right)\right)\left(v_t - \widehat{u}\left(t\right)\right) \leq 0, \quad \mathbb{P} - a.s.$$

In order to give the proof of theorem 1.4.1, it is convenient to present the following.

1.4.4 Variational equation

Let $v \in \mathcal{U}$ be such that $(\widehat{u} + v) \in \mathcal{U}$, the convexity condition of the control domain ensure that, for $\varepsilon \in (0, 1)$ the control $(\widehat{u} + \varepsilon v)$ is also in $\mathcal{U}$. We denote by y^ε the solution of the SDE (1.64) correspond to the control $(\widehat{u} + \varepsilon v)$, then by standard arguments from stochastic calculus, it is easy to check the following convergence result.

Lemma 1.4.1 *Under assumption $(H1)$ we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \left|y^\varepsilon\left(t\right) - \widehat{y}\left(t\right)\right|^2\right] = 0. \tag{1.69}$$

Proof. From assumption $(H1)$, we get by using the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|y^{\varepsilon}(t)-\widehat{y}(t)\right|^{2}\right]\leq K\int_{0}^{t}\mathbb{E}\left[\sup_{\tau\in[0,s]}\left|y^{\varepsilon}(r)-\widehat{y}(r)\right|^{2}\right]ds$$
$$+K\varepsilon^{2}\left(\int_{0}^{t}\mathbb{E}\left[\sup_{r\in[0,s]}\left|v(r)\right|^{2}\right]ds\right). \tag{1.70}$$

From definition 1.4.1, and Gronwall's lemma, the result follows immediately by letting ε go to zero. ∎

We define the process $z(t)=z^{\widehat{u},v}(t)$ by

$$\begin{cases} dz(t) & =\left\{b_{y}(t,\widehat{z}(t),\widehat{u}(t))\,z(t)+b_{u}(t,\widehat{y}(t),\widehat{u}(t))\,v(t)\right\}dt \\ & \quad+\sum_{j=1}^{d}\left\{\sigma_{y}^{j}(t,\widehat{y}(t),\widehat{u}(t))\,z(t)+\sigma_{u}^{j}(t,\widehat{y}(t),\widehat{u}(t))\,v(t)\right\}dB^{j}(t), \\ z(0) & =0. \end{cases} \tag{1.71}$$

From $(H2)$ and definition 1.4.1, one can find a unique solution z which solves the variational equation (1.71), and the following estimation holds.

Lemma 1.4.2 *Under assumption* $(H1)$, *it holds that*

$$\lim_{\varepsilon\to0}\mathbb{E}\left|\frac{y^{\varepsilon}(t)-\widehat{y}(t)}{\varepsilon}-z(t)\right|^{2}=0. \tag{1.72}$$

Proof. Let

$$\Gamma^{\varepsilon}(t)=\frac{y^{\varepsilon}(t)-\widehat{y}(t)}{\varepsilon}-z(t).$$

Denoting $y^{\mu,\varepsilon}(t)=\widehat{y}(t)+\mu\varepsilon(\Gamma^{\varepsilon}(t)+z(t))$, and $u^{\mu,\varepsilon}(t)=\widehat{u}(t)+\mu\varepsilon v(t)$, for notational convenience. Then we have immediately that $\Gamma^{\varepsilon}(0)=0$ and $\Gamma^{\varepsilon}(t)$ fulfills the following SDE

$$\begin{aligned} d\Gamma^{\varepsilon}(t)=&\left\{\frac{1}{\varepsilon}\left(b(t,y^{\mu,\varepsilon}(t),u^{\mu,\varepsilon}(t))-b(t,\widehat{y}(t),\widehat{u}(t))\right)\right.\\ &\left.-\left(b_{y}(t,\widehat{y}(t),\widehat{u}(t))\,z(t)+b_{u}(t,\widehat{y}(t),\widehat{u}(t))\,v(t)\right)\right\}dt\\ &+\left\{\frac{1}{\varepsilon}\left(\sigma(t,y^{\mu,\varepsilon}(t),u^{\mu,\varepsilon}(t))-\sigma(t,\widehat{y}(t),\widehat{u}(t))\right)\right.\\ &\left.-\left(\sigma_{y}(t,\widehat{y}(t),\widehat{u}(t))\,z(t)+\sigma_{u}(t,\widehat{y}(t),\widehat{u}(t))\,v(t)\right)\right\}dB(t) \end{aligned}$$

Since the derivatives of the coefficients are bounded, and from definition 1.4.1, it is easy to verify by Gronwall's inequality that

$$\mathbb{E}\left|\Gamma^{\varepsilon}(t)\right|^{2} \leq K\mathbb{E}\int_{0}^{t}\left|\int_{0}^{1} b_{y}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right)\Gamma^{\varepsilon}(s)\, d\mu\right|^{2} ds + K\mathbb{E}\left|\rho^{\varepsilon}(t)\right|^{2} + K\mathbb{E}\int_{0}^{t}\left|\int_{0}^{1} \sigma_{y}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right)\Gamma^{\varepsilon}(s)\, d\mu\right|^{2} ds,$$

where $\rho^{\varepsilon}(t)$ is given by

$$\begin{aligned}\rho^{\varepsilon}(t) = &-\int_{0}^{t} b_{y}\left(s, \widehat{y}(s), \widehat{u}(s)\right) z(s)\, ds \\ &-\int_{0}^{t} \sigma_{y}\left(s, \widehat{y}(s), \widehat{u}(s)\right) z(s)\, dB(s) \\ &-\int_{0}^{t} b_{v}\left(s, \widehat{y}(s), \widehat{u}(s)\right) v(s)\, ds \\ &-\int_{0}^{t} \sigma_{v}\left(s, \widehat{y}(s), \widehat{u}(s)\right) v(s)\, dB(s) \\ &+\int_{0}^{t}\int_{0}^{1} b_{y}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right) z(s)\, d\mu ds \\ &+\int_{0}^{t}\int_{0}^{1} b_{v}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right) v(s)\, d\mu ds \\ &+\int_{0}^{t}\int_{0}^{1} \sigma_{y}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right) z(s)\, d\mu dB(s) \\ &+\int_{0}^{t}\int_{0}^{1} \sigma_{v}\left(s, y^{\mu,\varepsilon}(s), u^{\mu,\varepsilon}(s)\right) v(s)\, d\mu dB(s).\end{aligned}$$

Since b_y, σ_y are bounded, then

$$\mathbb{E}\left|\Gamma^{\varepsilon}(t)\right|^{2} \leq M\mathbb{E}\int_{0}^{t}\left|\Gamma^{\varepsilon}(s)\right|^{2} ds + M\mathbb{E}\left|\rho^{\varepsilon}(t)\right|^{2},$$

where M is a generic constant depending on the constant K and T. We conclude from lemma 1.4.2 that $\lim_{\varepsilon\to 0}\rho^{\varepsilon}(t) = 0$. Hence (1.72) follows from Gronwall lemma and by letting ε go to 0. ■

1.4.5 Variational inequality

Let Φ be the fundamental solution of the linear matrix equation, for $0 \leq s < t \leq T$

$$\begin{cases} d\Phi_{s,t} &= b_y\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)\Phi_{s,t}dt + \sum_{j=1}^{d}\sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)\Phi_{s,t}dB^j\left(t\right), \\ \Phi_{s,s} &= I_d, \end{cases}$$

where I_d is the $n \times n$ identity matrix, this equation is linear with bounded coefficients, then it admits a unique strong solution.

From Itô's formula we can easily check that $d\left(\Phi_{s,t}\Psi_{s,t}\right) = 0$, and $\Phi_{s,s}\Psi_{s,s} = I_d$, where Ψ is the solution of the following equation

$$\begin{cases} d\Psi_{s,t} &= -\Psi_{s,t}\left\{b_y\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) - \sum_{j=1}^{d}\sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)\sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)\right\}dt \\ &\quad - \sum_{j=1}^{d}\Psi_{s,t}\sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)dB^j\left(t\right), \\ \Psi_{s,s} &= I_d, \end{cases}$$

so $\Psi = \Phi^{-1}$, if $s = 0$ we simply write $\Phi_{0,t} = \Phi_t$, and $\Psi_{0,t} = \Psi_t$. By integrating by part formula we can see that, the solution of (1.71) is given by $z\left(t\right) = \Phi_t\eta_t$, where η_t is the solution of the stochastic differential equation

$$\begin{cases} d\eta_t &= \Psi_t\left\{b_u\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)v\left(t\right) - \sum_{j=1}^{d}\sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)\sigma_u^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right)v\left(t\right)\right\}dt \\ &\quad + \sum_{j=1}^{d}\Psi_t\sigma_u^j\left(t, x_t^{\star}, u_t^{\star}\right)v\left(t\right)dB^j\left(t\right), \\ \eta_0 &= 0. \end{cases}$$

Let us introduce the following convex perturbation of the optimal control $\widehat{u}$ by

$$u^{\varepsilon} = \widehat{u} + \varepsilon v, \tag{1.73}$$

for any $v \in \mathcal{U}$, and $\varepsilon \in (0, 1)$. Since $\widehat{u}$ is an optimal control, then $\varepsilon^{-1}\left(J\left(u^{\varepsilon}\right) - J\left(\widehat{u}\right)\right) \geq 0$. Thus a necessary condition for optimality is that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(J\left(u^{\varepsilon}\right)-J\left(\widehat{u}\right)\right)\geq 0. \tag{1.74}$$

The rest is devoted to the computation of the above limit. We shall see that the expression (1.74) leads to a precise description of the optimal control $\widehat{u}$ in terms of the adjoint process. First, it is easy to prove the following lemma

Lemma 1.4.3 *Under assumptions* $(H1)$, *we have*

$$\begin{aligned} I &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\left(J\left(u^{\varepsilon}\right)-J\left(\widehat{u}\right)\right) \\ &= \mathbb{E}\left[\int_0^T \left\{f_y\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)z\left(s\right)+f_u\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)v\left(s\right)\right\}ds+g_y\left(\widehat{y}\left(T\right)\right)z\left(T\right)\right]. \end{aligned} \tag{1.75}$$

Proof. We use the same notations as in the proof of lemma 1.4.2. First, we have

$$\begin{aligned} &\varepsilon^{-1}\left(J\left(u^{\varepsilon}\right)-J\left(\widehat{u}\right)\right) \\ &= \mathbb{E}\left[\int_0^T\int_0^1\left\{f_y\left(s,y^{\mu,\varepsilon}\left(s\right),u^{\mu,\varepsilon}\left(s\right)\right)z\left(s\right)+f_u\left(s,y^{\mu,\varepsilon}\left(s\right),u^{\mu,\varepsilon}\left(s\right)\right)v\left(s\right)\right\}d\mu ds \right.\\ &\qquad \left.+\int_0^1 g_y\left(y^{\mu,\varepsilon}\left(T\right)\right)z\left(T\right)d\mu\right]+\beta^{\varepsilon}\left(t\right), \end{aligned}$$

where

$$\beta^{\varepsilon}\left(t\right)=\mathbb{E}\left[\int_0^T\int_0^1 f_y\left(s,y^{\mu,\varepsilon}\left(s\right),u^{\mu,\varepsilon}\left(s\right)\right)\Gamma^{\varepsilon}\left(s\right)d\mu ds+\int_0^1 g_y\left(y^{\mu,\varepsilon}\left(T\right)\right)\Gamma^{\varepsilon}\left(T\right)d\mu\right].$$

By using the lemma 1.4.2, and since the derivatives f_y, f_u, and g_y are bounded, we have $\lim_{\varepsilon\to 0}\beta^{\varepsilon}\left(t\right)=0$. Then, the result follows by letting ε go to 0 in the above equality. ■

Substituting by $z\left(t\right)=\Phi_t\eta_t$ in (1.75), this leads to

$$I=\mathbb{E}\left[\int_0^T\left\{f_y\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_s\eta_s+f_u\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)v\left(s\right)\right\}ds+g_y\left(\widehat{y}\left(T\right)\right)\Phi_T\eta_T\right].$$

Consider the right continuous version of the square integrable martingale

$$M(t) := \mathbb{E}\left[\int_0^T f_y(s, \widehat{y}(s), \widehat{u}(s))\,\Phi_s ds + g_y(\widehat{y}(T))\,\Phi_T \,|\mathcal{F}_t\right].$$

By the representation theorem, there exist $Q = \left(Q^1, .., Q^d\right)$ where $Q^j \in \mathbb{L}^2$, for $j = 1, ..., d$,

$$M(t) = \mathbb{E}\left[\int_0^T f_y(s, \widehat{y}(s), \widehat{u}(s))\,\Phi_s ds + g_y(\widehat{y}(T))\,\Phi_T\right] + \sum_{j=1}^{d}\int_0^t Q^j(s)\,dB^j(s).$$

We introduce some more notation, write $\widehat{y}(t) = M(t) - \int_0^t f_y(s, \widehat{y}(s), \widehat{u}(s))\,\Phi_s ds$. The adjoint variable is the processes defined by

$$\begin{cases} p(t) & = \widehat{y}(t)\,\Psi_t, \\ q^j(t) & = Q^j(t)\,\Psi_t - p(t)\,\sigma_y^j(t, \widehat{y}(t), \widehat{u}(t)), \text{ for } j = 1, ..., d. \end{cases} \tag{1.76}$$

Theorem 1.4.2 *Under assumptions* $(H1)$, *we have*

$$I = \mathbb{E}\left[\int_0^T \left\{ f_u(s, \widehat{y}(s), \widehat{u}(s)) + p(s)\,b_u(s, \widehat{y}(s), \widehat{u}(s)) + \sum_{j=1}^{d} q^j \sigma_u^j(s, \widehat{y}(s), \widehat{u}(s)) \right\}\right].$$

Proof. From the integration by part formula, and by using the definition of $p(t), q^j(t)$ for $j = 1, .., d$, we easily check that

$$\begin{aligned} E\left[y(T)\,\eta(T)\right] &= \mathbb{E}\left[\int_0^T \left\{ p(t)\,b_u(s, \widehat{y}(s), \widehat{u}(s)) + \sum_{j=1}^{d} q^j(s)\,\sigma_u^j(s, \widehat{y}(s), \widehat{u}(s)) \right\} v(t)\,dt \right. \\ &\quad \left. - \int_0^T f_y(s, \widehat{y}(s), \widehat{u}(s))\,\eta_t \Phi_t dt. \right. \end{aligned} \tag{1.77}$$

Also we have

$$I = \mathbb{E}\left[y(T)\,\eta(T) + \int_0^T f_y(s, \widehat{y}(s), \widehat{u}(s))\,\Phi_t \eta_t dt + \int_0^T f_u(s, \widehat{y}(s), \widehat{u}(s))\,v(t)\,dt\right], \tag{1.78}$$

substituting (1.77) in (1.78), This completes the proof. ■

1.4.6 Sufficient conditions of optimality

Theorem 1.4.3 *Let $\widehat{u}$ be an admissible control, we denote $\widehat{y}$ the associated controlled state process, and let (p, q) be a solution to the corresponding BSDE (1.68). Let us assume that $H(t, y, u, p(t), q(t))$, and (y) are concave functions. Moreover suppose that for all $t \in [0, T]$,*

$$H(t, \widehat{y}(t), \widehat{u}(t), p(t), q(t)) = \inf_{u \in U} H(t, \widehat{y}(t), u(t), p(t), q(t)). \tag{1.79}$$

Then $\widehat{u}$ is an optimal control.

Proof. We consider the difference

$$\begin{aligned} J(\widehat{u}) - J(u) = \mathbb{E}\left[\int_0^T (f(t, \widehat{y}(t), \widehat{u}(t)) - f(t, y(t), u(t)))\, dt\right] \\ + \mathbb{E}\left[g(\widehat{y}(T)) - g(y(T))\right]. \end{aligned}$$

Since g is concave, we get

$$\begin{aligned} \mathbb{E}\left[g(\widehat{y}(T)) - g(y(T))\right] &\geq \mathbb{E}\left[(\widehat{y}(T) - y(T))\, g_y(\widehat{y}(T))\right] \\ &= \mathbb{E}\left[(\widehat{y}(T) - y(T))\, p(T)\right] \\ &= \mathbb{E}\left[\int_0^T (\widehat{y}(t) - y(t))\, dp(t) + \int_0^T p(t)\, d(\widehat{y}(t) - y(t))\right] \\ &\quad + \mathbb{E}\left[\int_0^T \sum_{j=1}^n \left(\sigma^j(t, \widehat{y}(t), \widehat{u}(t)) - \sigma^j(t, y(t), u(t))\right) q^j(t)\, dt\right], \end{aligned}$$

with

$$\begin{aligned} \mathbb{E}\left[\int_0^T (\widehat{y}(t) - y(t))\, dp(t)\right] = \mathbb{E}\left[\int_0^T (\widehat{y}(t) - y(t))\, (-H_y(t, \widehat{y}(t), \widehat{u}(t), p(t), q(t)))\, dt\right] \\ + \mathbb{E}\left[\int_0^T (\widehat{y}(t) - y(t))\, q(t)\, dB(t)\right], \end{aligned}$$

and

$$\mathbb{E}\left[\int_0^T p(t)\, d(\widehat{y}(t) - y(t))\right] = \mathbb{E}\left[\int_0^T p(t)\left(b(t, \widehat{y}(t), \widehat{u}(t)) - b(t, y(t), u(t))\right) dt\right]$$
$$+ \mathbb{E}\left[\int_0^T p(t)\left(\sigma(t, \widehat{y}(t), \widehat{u}(t)) - \sigma(t, y(t), u(t))\right) dB(t)\right].$$

On the other hand, the process

$$\mathbb{E}\left[\int_0^T \left\{p(t)\left(\sigma(t, \widehat{y}(t), \widehat{u}(t)) - \sigma(t, y(t), u(t))\right) + (\widehat{y}(t) - y(t))\, q(t)\right\} dB(t)\right]$$

is a continuous local martingale for all $0 < t \leq T$, by the fact that $(p, q) \in \mathbb{L}^2(([0, T]; \mathbb{R}^n)) \times \mathbb{L}^2\left(([0, T]; \mathbb{R}^{n\times d})\right)$, we deduce that the stochastic integrals with respect to the local martingales have zero expectation. By the concavity of the Hamiltonian H, we get

$$\mathbb{E}\left[g(\widehat{y}(T)) - g(y(T))\right] \geq -\mathbb{E}\left[\int_0^T \left(H(t, \widehat{y}(t), \widehat{u}(t), p(t), q(t)) - H(t, y(t), u(t), p(t), q(t))\right) dt\right]$$
$$+ \mathbb{E}\left[\int_0^T p(t)\left(b(t, \widehat{y}(t), \widehat{u}(t)) - b(t, y(t), u(t))\right) dt\right]$$
$$+ \mathbb{E}\left[\int_0^T \left(\sigma(t, \widehat{y}(t), \widehat{u}(t)) - \sigma(t, y(t), u(t))\right) q(t)\, dt\right].$$

By the definition of the Hamiltonian H, we obtain

$$J(\widehat{u}) - J(u) \geq 0,$$

then $\widehat{u}$ is an optimal control. ■

1.5 Relation to dynamic programming principle

In this section, we come back to the control problem studied in the section 1.3. We recall a verification theorem, which is useful to compute optimal controls. Then we show that the adjoint process defined in Section 1.4, as the unique solution to the BSDE (1.68), can be expressed as the gradient of the value function, which solves the HJB variational inequality.
Let $y^{t,y}(s)$ be the solution of the controlled SDE (1.7) for $s \geq t$, with initial value $y(t) = y$. We put the problem in a Markovian framework. Since our objective is to maximize this functional, the value function of the control problem becomes

$$V(t,x) = \sup_{u \in \mathcal{U}} J(t,y,u).$$

The infinitesimal generator $\mathcal{L}^u$, associated with (1.7) , acting on functions φ, coincides on $C_b^2(\mathbb{R}^n;\mathbb{R})$ with partial differential operator $\mathcal{L}^u$ given by

$$\mathcal{L}^u\varphi(t,y) = \sum_{i=1}^{n} b^i(t,y,u)\frac{\partial \varphi}{\partial y^i}(t,y) + \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(t,y,u)\frac{\partial^2 \varphi}{\partial y^i \partial y^j}(t,y),$$

where $a^{ij} = \sum_{h=1}^{d}\left(\sigma^{ih}\sigma^{jh}\right)$ denotes the generic term of the symmetric matrix $\sigma\sigma^{\intercal}$.

Theorem 1.5.1 *Let V be a classical solution of* (1.20). *Assume that $V \in C^{1,3}([0,T] \times \mathbb{R}^n)$, and there exists $\widehat{u} \in \mathcal{U}$. Then the solution of the BSDE* (1.68) *is given by*

$$\begin{cases} p(t) & = V_y(t, \widehat{y}(t)), \\ q(t) & = V_{yy}(t,\widehat{y}(t))\,\sigma(t,\widehat{y}(t),\widehat{u}(t)). \end{cases}$$

Proof. Using Itô's formula to $\frac{\partial V}{\partial y^k}(\cdot, \widehat{y}(\cdot))$, we obtain

$$\begin{aligned} \frac{\partial V}{\partial y^k}(T,\widehat{y}(T)) - \frac{\partial V}{\partial y^k}(0,\widehat{y}(0)) = \int_0^T & \left\{ \frac{\partial^2 V}{\partial t \partial y^k}(t,\widehat{y}) + \sum_{i=1}^{n} b^i(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^2 V}{\partial y^k \partial y^i}(t,\widehat{y}(t)) \right. \\ & \left. + \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(t)\left(\frac{\partial^3 V}{\partial y^k \partial y^i \partial y^j}(t,\widehat{y}(t))\right)\right\} dt \\ & + \int_0^T \sum_{j=1}^{n} \frac{\partial^2 V}{\partial y^k \partial y^i}(t,\widehat{y})\,\sigma^i(t,\widehat{y}(t),\widehat{u}(t))\,dB(t). \end{aligned}$$

On the other hand, define

$$\mathcal{L}(t,y,u)=\frac{\partial V}{\partial t}(t,y)+\sum_{i=1}^{n}b^{i}(t,y,u)\frac{\partial V}{\partial y^{i}}(t,y)+\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(t,y,u)\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,y)+f(t,y,u).$$

If we differentiate $\mathcal{L}(t,y,u)$ with respect to y^k, and evaluate the result at $(y,u)=(\widehat{y}(t),\widehat{u}(t))$, we deduce

$$\begin{aligned}
&\frac{\partial^{2}V}{\partial t\partial y^{k}}(t,\widehat{y}(t))+\sum_{i=1}^{n}b^{i}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{2}V}{\partial y^{k}\partial y^{i}}(t,\widehat{y}(t))\\
&\qquad+\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{3}V}{\partial y^{k}\partial y^{i}\partial y^{j}}(t,\widehat{y}(t))\\
&=-\sum_{i=1}^{n}\frac{\partial b^{i}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial V}{\partial y^{i}}(t,\widehat{y}(t))-\frac{\partial f}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\\
&\qquad-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a^{ij}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,\widehat{y}(t)).
\end{aligned}$$

Then

$$\begin{aligned}
d\left(\frac{\partial V}{\partial y^{k}}(t,\widehat{y}(t))\right)=&-\left\{\sum_{i=1}^{n}\frac{\partial b^{i}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial V}{\partial y^{i}}(t,\widehat{y}(t))+\frac{\partial f}{\partial x^{k}}(t,\widehat{y}(t),\widehat{u}(t))\right.\\
&\left.+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a^{ij}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,\widehat{y}(t))\right\}dt\\
&+\sum_{i=1}^{n}\frac{\partial^{2}V}{\partial y^{k}\partial y^{i}}(t,\widehat{y}(t))\,\sigma^{i}(t,\widehat{y}(t),\widehat{u}(t))\,dB(t).
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a^{ij}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,\widehat{y}(t))\\
&=\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial y^{k}}\left(\sum_{h=1}^{d}\sigma^{ih}(t)\,\sigma^{jh}(t)\right)\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,\widehat{y}(t))\\
&=\sum_{j=1}^{n}\sum_{h=1}^{d}\left(\sum_{i=1}^{n}\sigma^{ih}(t,\widehat{y}(t),\widehat{u}(t))\frac{\partial^{2}V}{\partial y^{i}\partial y^{j}}(t,\widehat{y}(t))\right)\frac{\partial\sigma^{ih}}{\partial y^{k}}(t,\widehat{y}(t),\widehat{u}(t)).
\end{aligned}$$

Now, from (1.67) we have

$$\frac{\partial H}{\partial y^{k}}(t,y,u,p,q)=\sum_{i=1}^{n}\frac{\partial b^{i}}{\partial y^{k}}(t,y,u)\,p^{i}+\sum_{h=1}^{d}\sum_{i=1}^{n}\frac{\partial\sigma^{ih}}{\partial y^{k}}(t,y,u)\,q^{ih}+\frac{\partial f}{\partial y^{k}}(t,y,u).$$

The kth coordinate $p^k(t)$ of the adjoint process $p(t)$ satisfies

$$\begin{cases} dp^k(t) & = -\dfrac{\partial H}{\partial y^k}\left(t,\widehat{y}(t),\widehat{u}(t),p(t),q(t)\right)dt + q_t^k dB(t), \qquad \text{for } t\in[0,T], \\ p^k(T) & = \dfrac{\partial g}{\partial y^k}\left(\widehat{y}(T)\right), \end{cases}$$

with $q_t^k dB_t = \sum_{h=1}^d q_t^{kh} dB_t^h$. Hence, the uniqueness of the solution of the above equation given by

$$p^k(t) = \frac{\partial V}{\partial y^k}\left(t,\widehat{y}(t)\right),$$
$$q^{kh}(t) = \sum_{i=1}^n \frac{\partial^2 V}{\partial y^k \partial y^i}\left(t,\widehat{y}(t)\right)\sigma^{ih}\left(t,\widehat{y}(t),\widehat{u}(t)\right).$$

■

Chapter 2

Relationship Between Maximum Principle and Dynamic Programming for Systems Driven by Normal Martingales

In this chapter we study a class of stochastic control problems of the type

$$\begin{cases} dY(t) & = b(t, Y(t), u(t), \pi(t))\, dt + \sigma(t, Y(t-), u(t), \pi(t))\, d\mathcal{M}^u(t) \\ Y(0) & = y, \end{cases} \tag{2.1}$$

where b and σ are given deterministic functions, y is the initial state, $\mathcal{M}^u$ is a martingale that satisfies the equation

$$[\mathcal{M}^u](t) = t + \int_0^t u(s)\, d\mathcal{M}^u(s), \quad t \geq 0. \tag{2.2}$$

The above equation is called Emery's structure equation, $[\mathcal{M}^u](\cdot)$ denotes the quadratic variation of $\mathcal{M}^u(\cdot)$, and $u(\cdot)$ is some predictable process which controls exactly the jumps of $\mathcal{M}^u(\cdot)$. Let U_1 and U_2 be two non-empty compact sets in $\mathbb{R}$, and set $U = U_1 \times U_2$. The control variable is a suitable process pair (u, π) where $u : [0, T] \times \Omega \to U_1 \subset \mathbb{R}$, $\pi : [0, T] \times \Omega \to U_2 \subset \mathbb{R}$.

The cost functional to be minimized over the class of admissible controls has the form

$$J(u,\pi) = \mathbb{E}\left[\int_0^T f(t, Y(t), u(t), \pi(t))\, dt + g(Y(T))\right]. \tag{2.3}$$

This chapter is organized as follows. In section 2, we formulate the problem and give the notations which are needed throughout this work. In section 3, we prove a sufficient stochastic maximum principle. Section 4 is devoted for the study of the relationship between the stochastic maximum principle and the dynamic programming principle and we show that the solution of the adjoint equation coincides with the derivative of the value function. In the last section, we apply the sufficient stochastic maximum principle to the mean-variance portfolio selection problem.

2.1 Assumptions and problem formulation

In this section, we present the basic notation to be used throughout the paper.

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P}\right)$ be a filtered probability space, satisfying the usual conditions. Any element $y \in \mathbb{R}^n$ will be identified to a column vector with n components. Denote by $A^\top$ the transpose of any vector or matrix A. For a function h, we denote by h_y the gradient or Jacobian of h with respect to the variable y. Let T be a fixed strictly positive real number, U_1 and U_2 be two nonempty compact sets in $\mathbb{R}$, set $U = U_1 \times U_2$.

We shall denote $\mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ to be the space of all $\mathbb{R}$-valued, square integrable martingales $\mathcal{M}(\cdot)$ defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$ such that $\mathcal{M}(0) = 0$.

Definition 2.1.1 *Return to [28] that a martingale $\mathcal{M}(\cdot) \in \mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ is called normal if $\langle \mathcal{M}\rangle(t) = t$. Here $\langle \mathcal{M}\rangle(\cdot)$ is the conditional quadratic variation process of $\mathcal{M}(\cdot)$, or the compensator of the bracket process $[\mathcal{M}](\cdot)$. Since the processes $[\mathcal{M}](\cdot)$ and $\langle \mathcal{M}\rangle(\cdot)$ differ by a martingale, if $\mathcal{M}(\cdot)$ also has the "representation property" then it is readily seen that there exists an $(\mathcal{F}_t)$-predictable process $u(\cdot)$ such that*

$$d[\mathcal{M}^u](t) = dt + u(t)\, d\mathcal{M}^u(t), \quad \forall t \geq 0.$$

In the above $[\mathcal{M}^u](\cdot)$ denotes the quadratic variation of $\mathcal{M}^u(\cdot)$ and $u(\cdot)$ is some predictable process representing the jump size of the process $\mathcal{M}^u(\cdot)$. The continuous and the pure jump part of the martingale $\mathcal{M}^u(\cdot)$, denoted by $\mathcal{M}^{u,c}(\cdot)$ and $\mathcal{M}^{u,d}(\cdot)$, satisfy respectively

$$d\mathcal{M}^{u,c}(t) = \mathbf{1}_{\{u(t)=0\}} d\mathcal{M}^u(t) \text{ and } d\mathcal{M}^{u,d}(t) = \mathbf{1}_{\{u(t)\neq 0\}} d\mathcal{M}^u(t), \quad \forall t \geq 0.$$

The state $Y(t)$, for $t \in [0,T]$ of a controlled diffusion is described by the following stochastic differential equation

$$\begin{cases} dY(t) & = b(t, Y(t), u(t), \pi(t))\, dt + \sigma(t, Y(t-), u(t), \pi(t))\, d\mathcal{M}^u(t) \\ y(0) & = y, \end{cases} \tag{2.4}$$

where $b : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, are given. The process $\mathcal{M}^u(\cdot) \in \mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ is a solution to the following structure equation driven by the process $u(\cdot)$

$$[\mathcal{M}^u](t) = t + \int_0^t u(s)\, d\mathcal{M}^u(s) \quad t \geq 0.$$

Noting that the jump of the state $Y(\cdot)$ at any jumping time t is defined by

$$\Delta Y(t) := \begin{cases} \sigma(t, Y(t-), u(t))\, \Delta\mathcal{M}^u(t) & \text{if } \mathcal{M}^u \text{ has a jump at } t, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Delta\mathcal{M}^u(t) = \mathcal{M}^u(t) - \mathcal{M}^u(t-) = u(t).$$

Finally, we recall that

$$[\mathcal{M}^u](t) = \sum_{0<s\leq t} (\Delta\mathcal{M}^u(s))^2 + \langle \mathcal{M}^{u,c} \rangle (t).$$

Definition 2.1.2 *An admissible control is a pair of measurable, adapted processes* $(u(\cdot), \pi(\cdot)) \in \mathcal{U}$, *where,* $u : [0,T] \times \Omega \rightarrow U_1 \subset \mathbb{R}$, $\pi : [0,T] \times \Omega \rightarrow U_2 \subset \mathbb{R}$, *such that*

$$\mathbb{E}\left[\int_0^T \left\{|u(s)|^2 + |\pi(s)|^2\right\} ds\right] < \infty.$$

We denote by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ *the set of all admissible controls. Here* $\mathcal{U}_1$ *(resp.* $\mathcal{U}_2$*) represents the set of the admissible controls* $u(\cdot)$ *(resp.* $\pi(\cdot)$*).*

Consider a performance criterion wich has the form

$$J\left(u\left(\cdot\right),\pi\left(\cdot\right)\right)=\mathbb{E}\left[\int_0^T f\left(t,Y\left(t\right),u\left(t\right),\pi\left(t\right)\right)dt+g\left(Y\left(T\right)\right)\right]. \tag{2.5}$$

Here $f:[0,T]\times\mathbb{R}^n\times U\rightarrow\mathbb{R}$, $g:\mathbb{R}^n\rightarrow\mathbb{R}$ are measurable functions. The stochastic control problem is to find an optimal control $\left(\hat{u}\left(\cdot\right),\hat{\pi}\left(\cdot\right)\right)\in\mathcal{U}$ such that

$$J\left(\hat{u}\left(\cdot\right),\hat{\pi}\left(\cdot\right)\right)=\inf_{\left(u\left(\cdot\right),\pi\left(\cdot\right)\right)\in\mathcal{U}} J\left(u\left(\cdot\right),\pi\left(\cdot\right)\right). \tag{2.6}$$

Let us assume the following conditions

(H1) The maps b,σ,f and g are continuously differentiable with respect to $\left(y,u,\pi\right)$.

(H2) All the derivatives of b,σ,f and g are continuous and uniformly bounded.

(H3) b,σ,f are bounded by $K\left(1+|y|+|u|+|\pi|\right)$, and g is bounded by $K\left(1+|y|^2\right)$, for some $K>0$.

2.2 Sufficient stochastic maximum principle

Here we state and prove the sufficient stochastic maximum principle where we apply it in section 5, to solve the mean-variance portfolio selection problem.
We define the Hamiltonian function $H:[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}$ by

$$H\left(t,y,u,\pi,p,q\right)=f\left(t,y,u,\pi\right)+b^{\top}\left(t,y,u,\pi\right)p+tr\left(\sigma^{\top}\left(t,y,u,\pi\right)q\right), \tag{2.7}$$

where $tr(A)$ denotes the trace of the matrix A. The adjoint equation is given in terms of the derivative of the Hamiltonian as

$$\begin{cases} dp\left(t\right)= & -H_y\left(t,Y\left(t\right),u\left(t\right),\pi\left(t\right),p\left(t\right),q\left(t\right)\right)dt+q\left(t\right)d\mathcal{M}^u\left(t\right),\\ p\left(T\right)= & g_y\left(Y\left(T\right)\right). \end{cases} \tag{2.8}$$

Theorem 2.2.1 *Let* $\left(\hat{u}\left(\cdot\right),\hat{\pi}\left(\cdot\right)\right)$ *be an admissible control and* $\hat{Y}\left(\cdot\right)$ *the associated controlled state process. Let* $\left(p\left(\cdot\right),q\left(\cdot\right)\right)$ *be the unique solution of the adjoint equation* (2.8). *Suppose that* *the Hamiltonian H is convex in (y,u,π), and the terminal cost function g is convex in y. Under the conditions (H1)–(H3), an admissible control $(\hat{u}(\cdot),\hat{\pi}(\cdot))$ is optimal if the following condition holds*

$$H\left(t,\hat{Y}(t),\hat{u}(t),\hat{\pi}(t),p(t),q(t)\right)=\inf_{(u,\pi)\in U} H\left(t,\hat{Y}(t),u,\pi,p(t),q(t)\right). \tag{2.9}$$

Proof. Let $(u(\cdot),\pi(\cdot))$ be an arbitrary admissible pair, denoting for any $t\in[0,T]$ by

$$\delta\Lambda\left(t,\hat{Y}(t)\right)=\Lambda\left(t,\hat{Y}(t),\hat{u}(t),\hat{\pi}(t)\right)-\Lambda\left(t,Y(t),u(t),\pi(t)\right),\ \text{for } \Lambda=b,f,\sigma,$$

and

$$\delta H(t)=H\left(t,\hat{Y}(t),\hat{u}(t),\hat{\pi}(t),p(t),q(t)\right)-H\left(t,Y(t),u(t),\pi(t),p(t),q(t)\right).$$

Then

$$J(\hat{u},\hat{\pi})-J(u,\pi)=\mathbb{E}\left[\int_0^T \delta f(t)\,dt+\left(g\left(\hat{Y}(T)\right)-g\left(Y(T)\right)\right)\right]. \tag{2.10}$$

By convexity of the Hamiltonian and (2.9) it yields

$$\mathbb{E}\left[\int_0^T \delta H(t)\,dt\right]\leq \mathbb{E}\left[\int_0^T \left(\hat{Y}(t)-Y(t)\right)^\top H_y\left(t,\hat{Y}(t),\hat{u}(t),\hat{\pi}(t),p(t),q(t)\right)dt\right]. \tag{2.11}$$

Since g is convex, we get

$$\begin{aligned}\mathbb{E}\left[g\left(\hat{Y}(T)\right)-g\left(Y(T)\right)\right]&\leq \mathbb{E}\left[\left(\hat{Y}(T)-Y(T)\right)^\top g_y\left(\hat{Y}(T)\right)\right],\\ &=\mathbb{E}\left[\left(\hat{Y}(T)-Y(T)\right)^\top p(T)\right].\end{aligned} \tag{2.12}$$

Using the integration by parts formula, we obtain by taking the expectations

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{Y}(T)-Y(T)\right)^\top p(T)\right]\\ &=\mathbb{E}\left[\int_0^T \left(\hat{Y}(t)-Y(t)\right)^\top dp(t)+\int_0^T p^\top(t)\,d\left(\hat{Y}(t)-Y(t)\right)+\int_0^T d\left[\hat{Y}-Y,p\right](t)\right].\end{aligned} \tag{2.13}$$

where $[\hat{Y}-Y,p](\cdot)$ stands for the quadratic covariation of $\hat{Y}(\cdot)-Y(\cdot)$ and $p(\cdot)$, also called the bracket process. We refer to Protter [55] for more detail in this topic. Noting that

$$\begin{aligned}\mathbb{E}\left[\int_0^T d\left[\hat{Y}-Y,p\right](t)\right] &= \mathbb{E}\left[\int_0^T tr\left(\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top q(t)\right) d\left[\mathcal{M}^{\hat{u}}(t)\right]\right], \\ &= \mathbb{E}\left[\int_0^T tr\left(\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top q(t)\right)\left(dt+d\mathcal{M}^{\hat{u}}(t)\right)\right],\end{aligned}$$

Then we get

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{Y}(T)-Y(T)\right)^\top p(T)\right] \\ &=\mathbb{E}\left[\int_0^T \left\{\left(\hat{Y}(t)-Y(t)\right)^\top \left(-H_y\left(t,\hat{Y}(t),\hat{u}(t),\hat{\pi}(t),p(t),q(t)\right)\right)\right.\right. \\ &\qquad\left. +p^\top(t)\left(\delta b\left(t,\hat{Y}(t)\right)\right)+tr\left(\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top q(t)\right)\right\} dt \\ &\quad+\int_0^T \left\{p(t)\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top + tr\left(\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top q(t)\right)\right. \\ &\qquad\left.\left. +\left(\hat{Y}(t)-Y(t)\right)^\top q(t)\right\} d\mathcal{M}^{\hat{u}}(t)\right]. \end{aligned} \tag{2.14}$$

Noting that the process

$$\int_0^T \left\{p(t)\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top + tr\left(\left(\delta\sigma\left(t,\hat{Y}(t-)\right)\right)^\top q(t)\right) + \left(\hat{Y}(t)-Y(t)\right)^\top q(t)\right\} d\mathcal{M}^{\hat{u}}(t),$$

is a martingale with zero expectation, then substitute (2.14) into the inequality (2.12) , we obtain by (2.11)

$$\begin{aligned}&\mathbb{E}\left[g\left(\hat{Y}(T)\right)-g(Y(T))\right] \\ &\le \mathbb{E}\left[\int_0^T \left\{-\delta\hat{H}(t)+p^\top(t)\left(\delta b\left(t,\hat{Y}(t)\right)\right)+tr\left(\left(\delta\sigma\left(t,\hat{Y}(t)\right)\right)^\top q(t)\right)\right\} dt\right].\end{aligned}$$

On the other hand, by the definition of the Hamiltonian, on has

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \delta f(t)\,dt\right] \\ &=\mathbb{E}\left[\int_0^T \left\{\delta\hat{H}(t)-p^\top(t)\left(\delta b\left(t,\hat{Y}(t)\right)\right)-tr\left(\left(\delta\sigma\left(t,\hat{Y}(t)\right)\right)^\top q(t)\right)\right\} dt\right].\end{aligned}$$

Adding the above inequalities up, we obtain $J(\hat{u}(\cdot),\hat{\pi}(\cdot)) - J(u(\cdot),\pi(\cdot)) \le 0$, which means that $(\hat{u}(\cdot), \hat{\pi}(\cdot))$ is an optimal control for the problem (2.6). ■

2.3 Relation to dynamic programming

In this section, we recall a verification theorem, which is useful to compute optimal controls. Then we show that the adjoint process defined in Section 3, as the unique solution to the BSDE (2.8), can be expressed in terms of the derivatives of the value function, which solves the HJB equation.

Let $Y(s) = Y^{t,y}(s)$ be the solution of the controlled SDE (2.4) for $s \geq t$, with initial value $Y(t) = y$. To put the problem in a Markovian framework so that we can apply dynamic programming principle, we define the performance criterion

$$J(t, y, u, \pi) = \mathbb{E}\left[\int_t^T f(s, Y(s), u(s), \pi(s))\, ds + g(Y(T)) \,|Y(t) = y\right]. \tag{2.15}$$

Since our objective is to minimize this functional, we define the value function of the control problem as follows

$$V(t, y) = \inf_{(u,\pi)\in\mathcal{U}} J(t, y, u, \pi). \tag{2.16}$$

Following [18] we introduce the infinitesimal generator $\mathcal{L}^{(u,\pi)}$, associated with (2.4) acting on functions φ in $C_b^2(\mathbb{R}^n, \mathbb{R})$ by

$$\begin{aligned}
&\mathcal{L}^{(u,\pi)}\varphi(t, y) \\
&\quad = \sum_{i=1}^n b^i(t, y, u, \pi) \frac{\partial \varphi}{\partial y^i}(t, y) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^n \mathbf{1}_{\{u=0\}} \frac{\partial^2 \varphi}{\partial y^i \partial y^j}(t, y) \sum_{l=1}^n \sigma^i(t, y, u, \pi)\, \sigma^j(t, y, u, \pi) \\
&\quad + \sum_{j=1}^n \mathbf{1}_{\{u\neq 0\}} \left(\varphi(t, y + u\sigma(t, y, u, \pi)) - \varphi(t, y) - u\frac{\partial \varphi}{\partial y^j}\sigma^j(t, y, u, \pi)\right)(u)^{-2},
\end{aligned}$$

where $\sigma^i(t, y, u, \pi)$ denotes the $i-$th component of the vector σ. From the standard dynamic programming principle (see, for example, [18]), the following Hamilton–Jacobi–Bellman equation holds

$$\frac{\partial W}{\partial t}(t,y)+\inf_{(u,\pi)}\left\{\mathcal{L}^{(u,\pi)}W(t,y)+f(t,y,u,\pi)\right\}=0,\quad \forall (t,y)\in[0,T]\times\mathbb{R}^n, \tag{2.17}$$

with the terminal condition

$$W(T,y)=g(y),\quad \forall y\in\mathbb{R}^n, \tag{2.18}$$

We start with the definition of classical solutions of (2.17).

Definition 2.3.1 *Let us consider a function $W\in C^{1,2}([0,T]\times\mathbb{R}^n)$, we say that W is a classical solution of* (2.17) *if*

$$\frac{\partial W}{\partial t}(t,y)+\inf_{(u,\pi)}\left\{\mathcal{L}^{(u,\pi)}W(t,y)+f(t,y,u,\pi)\right\}=0,\quad \forall (t,y)\in[0,T]\times\mathbb{R}^n,$$

and the terminal condition (4.18) holds.

Theorem 2.3.1 (Verification theorem) *Let W be a classical solution of* (2.17) *with the terminal condition* (2.18), *and satisfying a quadratic growth condition, i.e. there exists a constant C such that $|W(t,y)|\leq C(1+|y|^2)$. Then, for all $(t,y)\in[0,T]\times\mathbb{R}^n$ and $(u,\pi)\in\mathcal{U}$*

$$W(t,y)\geq J^{(u,\pi)}(t,y). \tag{2.19}$$

Furthermore, if there exists $(\hat{u}(\cdot),\hat{\pi}(\cdot))\in\mathcal{U}$ such that

$$(\hat{u}(t),\hat{\pi}(t))\in\arg\min_{(u,\pi)}\left\{\mathcal{L}^{(u,\pi)}W(t,Y(t))+f(t,Y(t),u(t),\pi(t))\right\}, \tag{2.20}$$

Then it follows that $W(t,y)=J(t,y,\hat{u},\hat{\pi})$.

Proof. Since $W\in C^{1,2}([0,T]\times\mathbb{R}^n)$, then for $0\leq t\leq s\leq T$, from Itô's formula to $W(\cdot,Y(\cdot))$, see for example [55], we obtain

$$\begin{aligned} W(s,Y(s)) &= W(t,y) + \int_t^s \frac{\partial W}{\partial t}(r,Y(r))\,dr \\ &\quad + \sum_{i=1}^n \int_t^s \frac{\partial W}{\partial y^i}(r,Y(r))\,dY^i(r) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_t^s \frac{\partial^2 W}{\partial y^i \partial y^j}(r,Y(r))\left(\sigma^i(r)\,\sigma^j(r)\right)dr \\ &\quad + \sum_{t\le r\le s}\left\{W(r,Y(r)) - W(r,Y(r-)) - \sum_{i=1}^n \frac{\partial W}{\partial y^i}(r,Y(r-))\,\Delta Y^i(r)\right\}, \end{aligned} \tag{2.21}$$

where

$$\Delta Y^i(r) = Y^i(r) - Y^i(r-) = \sigma^i(r-)\,\Delta\mathcal{M}^u(r)\,,\ \text{for } i = 1\cdots,n. \tag{2.22}$$

On the other hand, we can rewrite the last sum of (2.21) as

$$\begin{aligned} &\sum_{t\le r\le s}\left\{W(r,Y(r)) - W(r,Y(r-)) - \sum_{i=1}^n \frac{\partial W}{\partial y^i}(r,Y(r-))\,\Delta Y^i(r)\right\} \\ &= \sum_{t\le r\le s} \mathbf{1}_{\{u(r)\neq 0\}}\Big(W(r,Y(r-)+\sigma(r-)\,u(r)) - W(r,Y(r-)) \\ &\qquad\qquad - \sum_{i=1}^n \frac{\partial W}{\partial y^i}(r,Y(r-))\,\sigma^i(r-)\,u(t)\Big)\frac{\left(\Delta Y^i(r)\right)^2}{\left(u(r)\right)^2}, \\ &= \sum_{i=1}^n \int_t^s \mathbf{1}_{\{u(r)\neq 0\}}\Big(W(r,Y(r-)+\sigma(r-)\,u(r)) - W(r,Y(r-)) \\ &\qquad\qquad - \frac{\partial W}{\partial y^i}(r,Y(r-))\,\sigma^i(r-)\,u(r)\Big)\frac{\mathbf{1}_{\{u(r)\neq 0\}}dr + u(r)\,dY^i(r)}{u(r)^2}, \end{aligned}$$

then we get

$$\begin{aligned} W(s,Y(s)) &= W(t,y) + \int_t^s \left\{\frac{\partial W}{\partial t}(r,Y(r)) + \mathcal{L}^{(u,\pi)}W(r,Y(r))\right\}dr \\ &\quad + \int_t^s \sum_{j=1}^n \Big\{\mathbf{1}_{\{u(r)=0\}}\frac{\partial W}{\partial y^j}(r,Y(r))\,\sigma^j(r) \\ &\quad + \mathbf{1}_{\{u(r)\neq 0\}}\left(W(r,Y(r-)+\sigma(r-)\,u(r)) - W(t,Y(r-))\right)u(r)^{-1}\Big\}\,d\mathcal{M}^u(r)\,. \end{aligned}$$

Furthermore, the process

$$\int_t^s \sum_{j=1}^n \left\{ \mathbf{1}_{\{u(r)=0\}} \frac{\partial W}{\partial y^j} (r, Y(r)) \sigma^j(r) \right.$$
$$\left. + \mathbf{1}_{\{u(r)\neq 0\}} \left(W(r, Y(r-) + \sigma(r-) u(r)) - W(r, Y(r-)) \right) u(r)^{-1} \right\} d\mathcal{M}^u(r),$$

is a martingale, so its expected value is zero. Taking the expectation, we get

$$\mathbb{E}[W(s, Y(s))] = W(t, y) + \mathbb{E}\left[\int_t^s \left\{ \frac{\partial W}{\partial t}(r, Y(r)) + \mathcal{L}^{(u,\pi)} W(r, Y(r)) \right\} dr \right].$$

Using (2.17), we get

$$\frac{\partial W}{\partial t}(t, Y(t)) + \mathcal{L}^{(u,\pi)} W(t, Y(t)) + f(t, y, u, \pi) \leq 0, \quad \forall (u, \pi) \in \mathcal{U},$$

then

$$\mathbb{E}[W(T, Y(T))] \leq W(t, y) - \mathbb{E}\left[\int_t^T f(r, Y(r), u(r), \pi(r)) \, dr \right].$$

Apply the above argument to $(\hat{u}, \hat{\pi}) \in \mathcal{U}$, and take the limit as $s \longrightarrow T$, then by $(2.3), (2.4)$ and (2.20) we get

$$W(t, y) = \mathbb{E}\left[\int_t^T f\left(r, \hat{Y}(r), \hat{u}(r), \hat{\pi}(r)\right) dr + g\left(\hat{Y}(T)\right)\right].$$

■

Now we present a theorem which establishes the relationship between the stochastic maximum principle and the dynamic programming principle. Throughout the rest of this section we denote the vector functions $\left(t, \hat{Y}(t), \hat{u}(t), \hat{\pi}(t)\right)$ and $\left(t, \hat{Y}(t-), \hat{u}(t), \hat{\pi}(t)\right)$ by (t) and $(t-)$, respectively.

Theorem 2.3.2 *Let W be a classical solution of* (2.17), *with the terminal condition* (2.18). *Assume that $W \in C^{1,3}([0, T] \times \mathbb{R}^n)$, and there exists $(\hat{u}, \hat{\pi}) \in \mathcal{U}$ such that the condition* (2.20) *is satisfied. Then the solution of the BSDE* (2.8) *is given by*

$$p^k(t) = \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t)\right), \tag{2.23}$$

$$\begin{aligned} q^k(t) &= \mathbf{1}_{\{\hat{u}(t)=0\}} \sum_{j=1}^{n} \frac{\partial^2 W}{\partial y^k \partial y^j}\left(t, \hat{Y}(t)\right)\sigma^j(t) \\ &\quad + \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-) + \hat{u}(t)\sigma(t-)\right) - \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-)\right)\right)\hat{u}(t)^{-1}. \end{aligned} \tag{2.24}$$

Proof. Using Itô's formula to $\frac{\partial W}{\partial y^k}\left(\cdot, \hat{Y}(\cdot)\right)$, see e.g. [18], we obtain

$$\begin{aligned} &\frac{\partial W}{\partial y^k}\left(T, \hat{Y}(T)\right) \\ &= \frac{\partial W}{\partial y^k}\left(0, \hat{Y}(0)\right) + \int_0^T \left\{\frac{\partial^2 W}{\partial t \partial y^k}\left(t, \hat{Y}(t)\right) + \sum_{i=1}^{n} b^i(t)\frac{\partial^2 W}{\partial y^k \partial y^i}\left(t, \hat{Y}(t)\right)\right. \\ &\quad + \frac{1}{2}\mathbf{1}_{\{\hat{u}(t)=0\}}\sum_{i,j=1}^{n}\left(\frac{\partial^3 W}{\partial y^k \partial y^i \partial y^j}\left(t, \hat{Y}(t)\right)\sigma^i(t)\sigma^j(t)\right) \\ &\quad + \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right) - \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-)\right)\right. \\ &\qquad\qquad \left.\left. - \sum_{j=1}^{n}\frac{\partial^2 W}{\partial y^k \partial y^j}\left(t, \hat{Y}(t-)\right)\sigma^j(t-)\hat{u}(t)\right)\hat{u}(t)^{-2}\right\} dt \\ &\quad + \int_0^T \left\{\mathbf{1}_{\{\hat{u}(t)=0\}}\sum_{j=1}^{n}\frac{\partial^2 W}{\partial y^k \partial y^j}\left(t, \hat{Y}(t)\right)\sigma^j(t)\right. \\ &\qquad\qquad + \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right)\right. \\ &\qquad\qquad\qquad \left.\left. - \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-)\right)\right)\hat{u}(t)^{-1}\right\} d\mathcal{M}^{\hat{u}}(t). \end{aligned} \tag{2.25}$$

On the other hand, define

$$\begin{aligned} &\mathcal{A}(t, y, u, \pi) \\ &= \frac{\partial W}{\partial t}(t, y) + \sum_{i=1}^{n} b^i(t)\frac{\partial W}{\partial y^i}(t, y) + f(t, y, u, \pi) \\ &\quad + \frac{1}{2}\mathbf{1}_{\{u=0\}}\sum_{i,j=1}^{n}\frac{\partial^2 W}{\partial y^i \partial y^j}(t, y)\sum_{l=1}^{n}\sigma^j(t, y, u, \pi)\sigma^i(t, y, u, \pi) \\ &\quad + \sum_{j=1}^{n}\mathbf{1}_{\{u\neq 0\}}\left(W(t, y + u\sigma(t, y, u, \pi)) - W(t, y) - u\frac{\partial W}{\partial y^j}\sigma^j(t, y, u, \pi)\right)u^{-2}. \end{aligned} \tag{2.26}$$

Differentiate $\mathcal{A}(t, y, u, \pi)$ with respect to y^k, and evaluate the result at $(y, u, \pi) = \left(\hat{Y}, \hat{u}, \hat{\pi}\right)$, we get

$$\begin{aligned}0 &= \frac{\partial f}{\partial y^k}(t) + \frac{\partial^2 W}{\partial t \partial y^k}\left(t, \hat{Y}(t)\right) + \sum_{i=1}^n b^i(t) \frac{\partial^2 W}{\partial y^k \partial y^i}\left(t, \hat{Y}(t)\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^n \mathbf{1}_{\{\hat{u}(t)=0\}} \frac{\partial^3 W}{\partial y^k \partial y^i \partial y^j}\left(t, \hat{Y}(t)\right)\sigma^j(t)\sigma^i(t) \\ &+ \sum_{j=1}^n \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right) - \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-)\right)\right. \\ &\qquad\qquad \left. - \hat{u}(t)\frac{\partial^2 W}{\partial y^k \partial y^j}\left(t, \hat{Y}(t-)\right)\sigma^j(t-)\right)\hat{u}(t)^{-2} \\ &+ \sum_{i=1}^n \frac{\partial b^i}{\partial y^k}(t)\frac{\partial W}{\partial y^i}\left(t, \hat{Y}(t)\right) + \frac{1}{2}\sum_{i,j=1}^n \mathbf{1}_{\{\hat{u}(t)=0\}}\frac{\partial}{\partial y^k}\sigma^j(t)\sigma^i(t)\frac{\partial^2 W}{\partial y^i \partial y^j}\left(t, \hat{Y}(t)\right) \\ &+ \sum_{j=1}^n \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^j}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right)\right. \\ &\qquad\qquad \left. - \frac{\partial W}{\partial y^j}\left(t, \hat{Y}(t-)\right)\right)\frac{\partial \sigma^j}{\partial y^k}(t-)\hat{u}(t)^{-1}. \end{aligned} \tag{2.27}$$

Finally, substituting (2.27) into (2.25) we get

$$\begin{aligned}& d\left(\frac{\partial W}{\partial y_k}\left(t, \hat{Y}(t)\right)\right) \\ &= -\left\{\sum_{i=1}^n \frac{\partial b^i}{\partial y^k}(t)\frac{\partial W}{\partial y^i}\left(t, \hat{Y}(t)\right) + \frac{1}{2}\sum_{i,j=1}^n 1_{\{\hat{u}(t)=0\}}\frac{\partial}{\partial y^k}\left(\sigma^j(t)\sigma^i(t)\right)\frac{\partial^2 W}{\partial y^i \partial y^j}\left(t, \hat{Y}(t)\right)\right. \\ &\quad + \sum_{j=1}^n \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^j}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right)\right. \\ &\qquad\qquad \left.\left. - \frac{\partial W}{\partial y^j}\left(t, \hat{Y}(t-)\right)\right)\frac{\partial \sigma^j}{\partial y^k}(t-)\hat{u}(t)^{-1} + \frac{\partial f}{\partial y^k}(t)\right\} dt \\ &+ \left\{\sum_{j=1}^n \mathbf{1}_{\{\hat{u}(t)=0\}}\frac{\partial^2 W}{\partial y^k \partial y^j}\left(t, \hat{Y}(t)\right)\sigma^j(t)\right. \\ &\quad \left. + \mathbf{1}_{\{\hat{u}(t)\neq 0\}}\left(\frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-) + \sigma(t-)\hat{u}(t)\right) - \frac{\partial W}{\partial y^k}\left(t, \hat{Y}(t-)\right)\right)\hat{u}(t)^{-1}\right\} d\mathcal{M}^{\hat{u}}(t). \end{aligned} \tag{2.28}$$

Note that

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial y^{k}}\left(\sigma^{i}\left(t\right)\sigma^{j}\left(t\right)\right)\frac{\partial^{2}W}{\partial y^{i}\partial y^{j}}\left(t,\hat{Y}\left(t\right)\right)\\
&=\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial\sigma^{j}}{\partial y^{k}}\left(t\right)\sigma^{i}\left(t\right)+\sigma^{j}\left(t\right)\frac{\partial\sigma^{i}}{\partial y^{k}}\left(t\right)\right)\frac{\partial^{2}W}{\partial y^{k}\partial y^{i}}\left(t,\hat{Y}\left(t\right)\right)\\
&=\sum_{i,j=1}^{n}\sigma^{i}\left(t\right)\left(\frac{\partial^{2}W}{\partial y^{i}\partial y^{j}}\left(t,\hat{Y}\left(t\right)\right)\right)\frac{\partial\sigma^{j}}{\partial y^{k}}\left(t\right). \qquad (2.29)
\end{aligned}$$

On the other hand, from (2.8), we can rewrite the $k-$th coordinate of the adjoint process as

$$\begin{cases} dp^{k}\left(t\right)=-\dfrac{\partial H}{\partial y^{k}}\left(t,\hat{Y}\left(t\right),\hat{u}\left(t\right),\hat{\pi}\left(t\right),p\left(t\right),q\left(t\right)\right)dt+q^{k}\left(t\right)d\mathcal{M}^{\hat{u}}\left(t\right)\\ p^{k}\left(T\right)=\dfrac{\partial g}{\partial y^{k}}\left(\hat{Y}\left(T\right)\right). \end{cases} \qquad (2.30)$$

From the definition of the Hamiltonian H by (2.7) we have

$$\frac{\partial H}{\partial y^{k}}\left(t,y,u,\pi,p,q\right)=\frac{\partial f}{\partial y^{k}}\left(t,y,u,\pi\right)+\sum_{i=1}^{n}\frac{\partial b^{i}}{\partial y^{k}}\left(t,y,u,\pi\right)p^{i}+\sum_{i=1}^{n}\frac{\partial\sigma^{i}}{\partial y^{k}}\left(t\right)q^{i}.$$

Therefore, the uniqueness of the solution of (2.30) and the relations (2.28) and (2.29) allow us to get (2.23) and (2.24). ■

Remark 2.3.1 *The two basic examples of structural equations are obtained by taking a constant process u in (2.2).*

1) Consider the case where $(u\equiv 0)$, we can write the equation (2.2) as $[\mathcal{M},\mathcal{M}](t)=t$, and shows that $[\mathcal{M},\mathcal{M}](\cdot)$ is continuous, hence also $\mathcal{M}(\cdot)$. Being a continuous martingale with quadratic variation t, $\mathcal{M}(\cdot)$ is a Brownian motion. In this case the classical result on the relationship between SMP and DPP is proved by Bensoussan [10].

2) The case where $u\equiv\alpha\in\mathbb{R}^{*}\triangleq\mathbb{R}\backslash\{0\}$, the equation (2.2) is now $[\mathcal{M},\mathcal{M}](t)=t+\alpha\left(\mathcal{M}(t)-\mathcal{M}(0)\right)$, in which $\mathcal{M}(t)=\alpha(N\left(t/\alpha^{2}\right)-t/\alpha^{2})$, where $N(\cdot)$ is a standard Poisson process. In this case the relationship between the SMP and DPP was reported in Framstad et al. [32] where the systems is driven by a Brownian motion and Poisson random measure.

2.4 Application to mean-variance portfolio selection problem

Remark 2.4.1 *In this section, we use the stochastic maximum principle to solve the mean-variance portfolio selection problem where the system is governed by normal martingales.*

We consider a market with a risky asset and a risk free bank account. The risk-free asset price $S^0(t)$ at time $t \in [0, T]$ evolves according to

$$dS^0(t) = \rho(t) S^0(t) dt, \; S^0(0) = 1. \tag{2.31}$$

Following [25], we assume that the risky asset price $S^1(t)$ at time t evolves according to the equation

$$dS^1(t) = \sigma(t-) S^1(t) d\mathcal{M}^u(t), \tag{2.32}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ are deterministic functions. In what follows, we denote by $\pi(t)$ the amount of money invested in the risky asset at time t. The process $\mathcal{M}(\cdot)$ is a one-dimensional martingale that satisfies the following structure equation

$$[\mathcal{M}^u](t) = t + \int_0^t u d\mathcal{M}^u(s) \qquad \forall t \in [0, T],$$

where $u \in \mathbb{R}$. The wealth process $Y(\cdot)$ corresponding to the portfolio $\pi(\cdot)$ is described by

$$\begin{cases} dY(t) = (\rho(t)(Y(t) - \pi(t))) dt + \sigma(t-) \pi(t) d\mathcal{M}^u(t), \\ Y(0) = y_0. \end{cases} \tag{2.33}$$

The objective is to find an admissible portfolio $\pi(\cdot)$ such that the expected terminal wealth satisfies $\mathbb{E}(y(T)) = d$, for some $d \in \mathbb{R}$ while the risk measured by the variance of the terminal wealth

$$Var[y(T)] := \mathbb{E}(y(T) - \mathbb{E}(y(T)))^2 = \mathbb{E}(y(T) - d)^2$$

is minimized. Finding such a portfolio $\pi(\cdot)$ is referred to as the mean-variance portfolio selection problem. In particular, we formulate the mean-variance portfolio selection problem as follows.

Definition 2.4.1 *The mean-variance portfolio selection is the following constrained stochastic* *optimization problem, parameterized by* $d \in \mathbb{R}$

$$\begin{cases} \textit{minimize } J_{MV}\left(y_0, \pi\left(\cdot\right)\right) := \mathbb{E}_{y_0}\left(\left(y\left(T\right) - d\right)^2\right) \\ \textit{subject to} \begin{cases} \mathbb{E}\left[y\left(T\right)\right] = d, \\ \pi\left(\cdot\right) \in \mathcal{U}, \\ \left(Y\left(\cdot\right), \pi\left(\cdot\right)\right) \ \textit{satisfy} \ (2.33), \end{cases} \end{cases} \tag{2.34}$$

where $\mathbb{E}_{y_0}$ *is the expectation with respect to the probability measure*

$$\mathbb{P}_{y_0} := \mathbb{P}(.|Y(0) = y_0).$$

Note that the mean-variance problem (2.34) is a dynamic optimization problem with a constraint $\mathbb{E}\left(y\left(T\right)\right) = d$. Here we apply the Lagrange multiplier technique to handle this constraint. Define

$$J_{MV}\left(y_0, \pi\left(\cdot\right), \zeta\right) := \mathbb{E}_{y_0}\left(\left(Y\left(T\right) - d\right)^2\right) + 2\zeta\mathbb{E}_{y_0}\left(\left(Y\left(T\right) - d\right)\right).$$

In this way the mean-variance problem (2.34) can be solved via the following stochastic optimal control problem (for every fixed ζ)

$$\begin{cases} \text{minimize } J_{MV}\left(y_0, \pi\left(\cdot\right), \zeta\right) = \mathbb{E}_{y_0}\left(\left(Y\left(T\right) - \left(d - \zeta\right)\right)^2\right) - \zeta^2 \\ \text{subject to } \pi\left(\cdot\right) \in \mathcal{U} \text{ and } \left(Y\left(t\right), \pi\left(\cdot\right)\right) \text{ satisfy } (2.32). \end{cases} \tag{2.35}$$

Clearly this problem has the same optimal strategy as the following optimization problem

$$\begin{cases} \text{minimize } J_{MV}\left(y_0, \pi\left(\cdot\right), \vartheta\right) = \mathbb{E}_{y_0}\left(\left(Y\left(T\right) - \vartheta\right)^2\right) \\ \text{subject to } \pi\left(\cdot\right) \in \mathcal{U} \text{ and } \left(Y\left(\cdot\right), \pi\left(\cdot\right)\right) \text{ satisfy } (2.33) \end{cases} \tag{2.36}$$

where we let $\vartheta = d - \zeta$. Thus the above optimal control problem turns out to be a quadratic loss minimization problem and we shall solve it using the stochastic maximum principle.

2.4.1 Quadratic loss minimization problem

We start by writing down the Hamiltonian for this system

$$\mathcal{H}(t,y,u,\pi,p,q)=\rho(t)(Y(t)-\pi(t))p+\pi(t)\sigma(t)q.$$

Therefore, the adjoint equation (2.8) becomes

$$\begin{cases} dp(t)=-\rho(t)p(t)dt+q(t)d\mathcal{M}^u(t),\ 0\leq t<T, \\ p(T)=2(Y(T)-\vartheta), \end{cases} \tag{2.37}$$

We seek the solution $(p(\cdot),q(\cdot))$ to (2.37). We try a process $p(\cdot)$ of the following form

$$p(t)=\phi(t)Y(t)+\psi(t),\ \forall t\in[0,T], \tag{2.38}$$

where $\phi(\cdot)$ and $\psi(\cdot)$ are deterministic functions with $\phi(T)=2$ and $\psi(T)=-2\vartheta$.
Applying Itô's formula on $p(\cdot)$ to get

$$\begin{aligned} dp(t)=&\left\{\phi'(t)Y(t)+\phi(t)\rho(t)(Y(t)-\pi(t))+\psi'(t)\right\}dt \\ &+\left\{\mathbf{1}_{\{u=0\}}\phi(t)\sigma(t)\pi(t)+\mathbf{1}_{\{u\neq 0\}}\phi(t)\sigma(t)\pi(t)\right\}d\mathcal{M}^u(t). \end{aligned} \tag{2.39}$$

Comparing the coefficients with (2.37), we obtain

$$-\rho(t)\phi(t)Y(t)-\rho(t)\psi(t)=\phi'(t)Y(t)+\phi(t)\rho(t)(Y(t)-\pi(t))+\psi'(t), \tag{2.40}$$

$$q(t)=\phi(t)\sigma(t)\pi(t). \tag{2.41}$$

Since $\mathcal{H}$ is a linear expression in $\pi(\cdot)$, that yields

$$-\rho(t)p(t)+\sigma(t)q(t)=0. \tag{2.42}$$

Substituting (2.38) and (2.41) into (2.42), we get

$$\hat{\pi}(t)=\frac{\rho(t)}{\sigma^2(t)}\left[\frac{\psi(t)}{\phi(t)}+\hat{Y}(t)\right],\ \forall t\in[0,T]. \tag{2.43}$$

Therefore, inserting (2.43) into (2.40), we obtain

$$\phi^{'}(t) + (2\rho(t) - \Gamma(t))\,\phi(t) = 0, \qquad \phi(T) = 2, \tag{2.44}$$

$$\psi^{'}(t) + (\rho(t) - \Gamma(t))\,\psi(t) = 0, \qquad \psi(T) = -2\vartheta, \tag{2.45}$$

where

$$\Gamma(t) = \frac{\rho^2(t)}{\sigma^2(t)}. \tag{2.46}$$

Then the solutions to these system of differential equations are

$$\phi(t) = 2\exp\left\{\int_t^T (\Gamma(s) - 2\rho(s))\,ds\right\},\ \forall t \in [0,T], \tag{2.47}$$

$$\psi(t) = -2\vartheta\exp\left\{\int_t^T (\Gamma(s) - \rho(s))\,ds\right\},\ \forall t \in [0,T]. \tag{2.48}$$

In order to solve the original mean-variance problem, we need to determine the value function $V(t,y)$ of the quadratic loss minimization problem, which is defined by

$$V(t,y) := \inf_{\pi\in\mathcal{U}} \mathbb{E}\left[(Y(T) - \vartheta)^2 \,|Y(t) = y_0\right].$$

From the relationship between $p(\cdot)$ and the value function $V\left(\cdot, \hat{Y}(\cdot)\right)$ (see Theorem 2.3.2) and the expression of $p(\cdot)$ in (2.38), we get

$$V\left(t, \hat{Y}(t)\right) = \frac{1}{2}\phi(t)\left(\hat{Y}(t)\right)^2 + \psi(t)\,\hat{Y}(t) + k(t), \quad V(T, y_0) = (y_0 - \vartheta)^2, \tag{2.49}$$

where $k(\cdot)$ is function must be deterministic. From the bondary conditions in (2.44) and (2.45), it is easy to see

$$k(T) = \vartheta^2. \tag{2.50}$$

Using Itô's formula to $V\left(t, \hat{Y}(t)\right)$, $\forall t \in [0,T]$, we get

$$\begin{aligned} dV\left(t,\hat{Y}(t)\right) = & \left\{\frac{1}{2}\phi'(t)\left(\hat{Y}(t-)\right)^2 + \psi'(t)\hat{Y}(t-) + \acute{k}(t)\right. \\ & + \left(\phi(t)\hat{Y}(t-) + \psi(t)\right)\left[\left(\rho(t)\left(\hat{Y}(t-) - \hat{\pi}(t)\right)\right)\right] \\ & + \frac{1}{2}\mathbf{1}_{\{u=0\}}\phi(t)\hat{\pi}(t)^2\sigma^2(t) \\ & \left. + \frac{1}{2}\mathbf{1}_{\{u\neq 0\}}\ \phi(t)\hat{\pi}(t)^2\sigma^2(t)\right\} dt \\ & + \left\{\left(\mathbf{1}_{\{u=0\}}\left(\phi(t)\hat{Y}(t) + \psi(t)\right)\hat{\pi}(t)\sigma(t)\right.\right. \\ & + \mathbf{1}_{\{u\neq 0\}}\left(\phi(t)\hat{Y}(t-)\sigma(t)\hat{\pi}(t)u + \frac{1}{2}\phi(t)\hat{\pi}(t)^2\sigma^2(t)u^2\right. \\ & \left.\left. + \ \psi(t)\sigma(t)\hat{\pi}(t)u\right)u^{-1}\right\} d\mathcal{M}^u(t). \end{aligned} \tag{2.51}$$

Noting that $\phi(\cdot)$ and $\psi(\cdot)$ are solutions to the differential equations (2.47) and (2.48), we can rewrite (2.50) as

$$\begin{aligned} dV\left(t,\hat{Y}(t)\right) = & \left\{\acute{k}(t) - \frac{1}{2}\frac{\psi^2(t)}{\phi(t)}\Gamma(t)\right\} dt \\ & + \left\{\left(\mathbf{1}_{\{u=0\}}\left(\phi(t)\hat{Y}(t) + \psi(t)\right)\hat{\pi}(t)\sigma(t)\right.\right. \\ & + \mathbf{1}_{\{u\neq 0\}}\left(\phi(t)\hat{Y}(t-)\sigma(t)\hat{\pi}(t)u + \frac{1}{2}\phi(t)\hat{\pi}(t)^2\sigma^2(t)u^2\right. \\ & \left.\left. + \ \psi(t)\sigma(t)\hat{\pi}(t)u\right)u^{-1}\right\} d\mathcal{M}^u(t). \end{aligned} \tag{2.52}$$

Since $\hat{\pi}(\cdot)$ is the optimal strategy the value function $V\left(\cdot, \hat{Y}(\cdot)\right)$ should be a martingale. To ensure the martingale property of $V\left(\cdot, \hat{Y}(\cdot)\right)$ the dt part must be equal to 0, that is

$$\acute{k}(t) - \frac{1}{2}\frac{\psi^2(t)}{\phi(t)}\Gamma(t) = 0,\ \forall t \in [0,T]. \tag{2.53}$$

Combining the terminal boundary condition (2.50) and the standard procedure to the Feyman–Kac representation of a system of differential equations, we have the following expression for $k(\cdot)$

$$k(t) = \vartheta^2\left(1 - \mathbb{E}\left(\int_t^T \Gamma(s)\frac{\psi^2(s)}{\phi(s)}ds\right)\right),\ \forall t \in [0,T]. \tag{2.54}$$

The above analysis yields the following theorem for the quadratic loss minimization problem

(2.36).

Theorem 2.4.1 *The optimal strategy for the quadratic loss minimization problem* (2.36) *is given by*

$$\hat{\pi}(t) = \frac{\rho(t)}{\sigma^2(t)}\left[\frac{\psi(t)}{\phi(t)} + \hat{Y}(t-)\right],\ \forall t \in [0,T],$$

and the corresponding optimal value function is given by

$$V\left(t, \hat{Y}(t)\right) = \frac{1}{2}\phi(t)\left(\hat{Y}(t)\right)^2 + \psi(t)\hat{Y}(t) + k(t),\ \forall t \in [0,T],$$

where $\phi(t)$, $\psi(t)$, and $k(t)$ are given by (2.47), (2.48), and (2.54), respectively.

2.4.2 The solution of the mean-variance problem

Denote by $V_{MV}(0, y_0)$ and $V_{MVL}(0, y_0)$ the optimal value functions for problem (2.34) and problem (2.35), respectively. Observing the relationship between the control problem (2.34) and the control problem (2.35) and the solution of the control problem (2.35) established in the previous subsection, we have the following result

$$\begin{aligned} V_{MVL}(0, y_0) &= V(0, y_0) - \zeta^2 \\ &= \frac{1}{2}\phi(0)\,y_0^2 + \psi(0)\,y_0 + k(0) - \zeta^2. \end{aligned}$$

Write

$$\begin{aligned} \tilde{\phi}(t) &:= \frac{1}{2}\phi(t), \\ \tilde{\psi}(t) &:= -\frac{\psi(t)}{2\vartheta} = -\frac{\psi(t)}{2(d-\zeta)}, \\ \tilde{k}(t) &:= \frac{k(t)}{\vartheta^2} = -\frac{\psi(t)}{(d-\zeta)^2}. \end{aligned}$$

Then, we can rewrite $V_{MVL}(0, y_0)$ as

$$V_{MVL}(0, y_0) = \tilde{\phi}(t)\,y_0^2 - 2(d-\zeta)\,\tilde{\psi}(t)\,y_0 + (d-\zeta)^2\,\tilde{k}(t) - \zeta^2.$$

Note that $J_{MV}(y_0, \pi(.))$ is strictly convex in $\pi(.)$ and the constraint function $\mathbb{E}[Y(t)] - d$ is affine in $\pi(.)$. Therefore, we can apply the well-known Lagrange duality theorem (see Luenberger [11, Theorem 1, p. 224]) to obtain that

$$V_{MV}(0, y_0) = \sup_{\zeta \in \mathbb{R}} V_{MVL}(0, y_0).$$

Observing that $V_{MVL}(0, y_0)$ is a quadratic function in ζ and the quadratic coefficient is equal to

$$\tilde{k}(0) - 1 = -\mathbb{E}\left[\int_t^T -\Gamma(s)\frac{\psi^2(s)}{\phi(s)}ds\right] < 0,$$

so $V_{MVL}(0, y_0)$ attains its maximum at the point

$$\zeta^* = d + \frac{d - \tilde{\psi}(0)\, y_0}{\tilde{k}(0) - 1}.$$

Substituting ζ^* into $V_{MVL}(0, y_0)$, we obtain the maximum value as follows

$$\sup_{\zeta \in \mathbb{R}} V_{MVL}(0, y_0) = \frac{\tilde{k}(0)}{1 - \tilde{k}(0)}\left[d - \frac{\tilde{\psi}(0)}{\tilde{k}(0)}y_0\right]^2 + \frac{\tilde{\phi}(0)\,\tilde{k}(0) - \tilde{\psi}^2(0)}{\tilde{k}(0)}y_0^2.$$

That is,

$$V_{MV}(0, y_0) = \frac{\tilde{k}(0)}{1 - \tilde{k}(0)}\left[d - \frac{\tilde{\psi}(0)}{\tilde{k}(0)}y_0\right]^2 + \frac{\tilde{\phi}(0)\,\tilde{k}(0) - \tilde{\psi}^2(0)}{\tilde{k}(0)}y_0^2.$$

The above analysis yields the following theorem.

Theorem 2.4.2 *The efficient portfolio of the mean-variance problem* (2.34) *corresponding to the expected terminal value d, as a function of time t, the wealth level y, is*

$$\hat{\pi}(t, y) = \left[y - (d - \zeta^*)\frac{\tilde{\psi}(t)}{\tilde{\phi}(t)}\right]\frac{\rho(t)}{\sigma^2(t)},$$

where

$$\zeta^* = d + \frac{d - \tilde{\psi}(0)\, y_0}{\tilde{k}(0) - 1}.$$

Furthermore, the efficient frontier (or optimal value function) for the mean-variance problem

(2.34) *is*

$$\begin{aligned} Var\hat{Y}\left(T\right) &= V_{MV}(0,y_0) \\ &= \frac{\tilde{k}\left(0\right)}{1-\tilde{k}\left(0\right)}\left[d-\frac{\tilde{\psi}\left(0\right)}{\tilde{k}\left(0\right)}y_0\right]^2+\frac{\tilde{\phi}\left(0\right)\tilde{k}\left(0\right)-\tilde{\psi}^2\left(0\right)}{\tilde{k}\left(0\right)}y_0^2, \end{aligned}$$

where $\tilde{\phi}\left(t\right)$, $\tilde{\psi}\left(t\right)$, *and* $\tilde{k}\left(t\right)$ *are given by*

$$\begin{aligned} \tilde{\phi}\left(t\right) &= \mathbb{E}\left(\exp\left(\int_t^T\left(\Gamma\left(s\right)-2\rho\left(s\right)\right)ds\right)\right), \\ \tilde{\psi}\left(t\right) &= \mathbb{E}\left(\exp\left(\int_t^T\left(\Gamma\left(s\right)-\rho\left(s\right)\right)ds\right)\right), \\ \tilde{k}\left(t\right) &= 1-\mathbb{E}\left(\int_t^T\Gamma\left(s\right)\frac{\psi^2\left(s\right)}{\phi\left(s\right)}ds\right). \end{aligned}$$

Chapter 3

Relationship Between MP and DPP for Systems Driven by Normal Martingales: viscosity solution

In this chapter, we present a nonsmooth version of the relationship between the stochastic maximum principle and the dynamic programming principle for stochastic control problems where the state of the systems driven by normal martingales and the control domain is convex. By using the concepts of sub and super-jets, all inclusions are derived from the value function and the adjoint process.

3.1 Problem statement and preliminaries

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P}\right)$ be a filtered probability space, satisfying the usual conditions. Any element $y \in \mathbb{R}^n$ will be identified to a column vector with n components. Denote by $A^\top$ the transpose of any vector or matrix A. For a function h, we denote by h_y the gradient or Jacobian of h with respect to the variable y. Let T be a fixed strictly positive real number, U_1 and U_2 be two nonempty compact sets in $\mathbb{R}$, set $U = U_1 \times U_2$. For a given $s \in [0, T]$, we denote by $\mathcal{U}[s, T]$ the set of $(\mathcal{F}_t)_{t\leq T}$ adapted processes. We shall denote $\mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ to be the space of all $\mathbb{R}$-valued, square integrable martingales $\mathcal{M}(\cdot)$ defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$ such that $\mathcal{M}(0) = 0$.

Definition 3.1.1 *Return to [28] that a martingale* $\mathcal{M}(\cdot) \in \mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ *is called normal if* $\langle \mathcal{M} \rangle (t) = t$. *Here* $\langle \mathcal{M} \rangle (\cdot)$ *is the conditional quadratic variation process of* $\mathcal{M}(\cdot)$*, or the compensator of the bracket process* $[\mathcal{M}](\cdot)$*. Since the processes* $[\mathcal{M}](\cdot)$ *and* $\langle \mathcal{M} \rangle (\cdot)$ *differ by a martingale, if* $\mathcal{M}(\cdot)$ *also has the "representation property" then it is readily seen that there exists an* $(\mathcal{F}_t)$*-predictable process* $u(\cdot)$ *such that*

$$d[\mathcal{M}^u](t) = dt + u(t)\, d\mathcal{M}^u(t), \quad \forall t \geq 0.$$

In the above $[\mathcal{M}^u](\cdot)$ denotes the quadratic variation of $\mathcal{M}^u(\cdot)$ and $u(\cdot)$ is some predictable process representing the jump size of the process $\mathcal{M}^u(\cdot)$. The continuous and the pure jump part of the martingale $\mathcal{M}^u(\cdot)$, denoted by $\mathcal{M}^{u,c}(\cdot)$ and $\mathcal{M}^{u,d}(\cdot)$, satisfy respectively

$$d\mathcal{M}^{u,c}(t) = \mathbf{1}_{\{u(t)=0\}} d\mathcal{M}^u(t) \text{ and } d\mathcal{M}^{u,d}(t) = \mathbf{1}_{\{u(t)\neq 0\}} d\mathcal{M}^u(t), \quad \forall t \geq 0.$$

For any initial time and state $(s, y) \in [0, T] \times \mathbb{R}^n$, suppose that the state $Y^{s,y,u,\pi}(\cdot) \in \mathbb{R}^n$ of a controlled diffusion is described by the following stochastic differential equation

$$\begin{cases} dY^{s,y,u,\pi}(t) & = b\left(t, Y^{s,y,u,\pi}, u(t), \pi(t)\right) dt + \sigma\left(t, Y^{s,y,u,\pi}(t-), u(t), \pi(t)\right) d\mathcal{M}^u(t) \\ Y^{s,x,u,\pi}(s) & = y, \end{cases} \tag{3.1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, are given functions.
The process $\mathcal{M}^u(\cdot) \in \mathbb{M}_0^2\left((\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{R}\right)$ is a solution to the following structure equation driven by the process $u(\cdot)$

$$[\mathcal{M}^u](t) = t + \int_0^t u(s)\, d\mathcal{M}^u(s) \quad t \geq 0.$$

Noting that the jump of the state $Y(\cdot)$ at any jumping time t is defined by

$$\Delta Y(t) := \begin{cases} \sigma\left(t, Y(t-), u(t)\right) \Delta\mathcal{M}^u(t) & \text{if } \mathcal{M}^u \text{ has a jump at } t, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Delta\mathcal{M}^u(t) = \mathcal{M}^u(t) - \mathcal{M}^u(t-) = u(t).$$

Finally, we recall that

$$[\mathcal{M}^u](t) = \sum_{0<s\le t} (\Delta \mathcal{M}^u(s))^2 + \langle \mathcal{M}^{u,c}\rangle (t).$$

Definition 3.1.2 *An admissible control is a pair of measurable, adapted processes* $(u(\cdot), \pi(\cdot)) \in \mathcal{U}$*, where,* $u : [0,T] \times \Omega \to U_1 \subset \mathbb{R}$*,* $\pi : [0,T] \times \Omega \to U_2 \subset \mathbb{R}$*, such that*

$$\mathbb{E}\left[\int_0^T \left\{|u(s)|^2 + |\pi(s)|^2\right\} ds\right] < \infty.$$

We denote by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ *the set of all admissible controls. Here* $\mathcal{U}_1$ *(resp.* $\mathcal{U}_2$*) represents the set of the admissible controls* $u(\cdot)$ *(resp.* $\pi(\cdot)$*).*

We consider the cost functional

$$J(s, y, u(\cdot), \pi(\cdot)) = \mathbb{E}\left[\int_s^T f\left(t, Y^{s,y,u,\pi}(t), u(t), \pi(t)\right) dt + g\left(Y^{s,y,u,\pi}(T)\right)\right], \tag{3.2}$$

where $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given functions.

For any $t \in [0,T]$ and $y, \widehat{y} \in \mathbb{R}^n$, we make the following assumptions

(H1) b, σ are uniformly continuous in (t, y, u, π). There exists a constant $C > 0$, such that

$$\begin{cases} |b(t,y,u,\pi) - b(t,\widehat{y},u,\pi)| + |\sigma(t,y,u,\pi) - \sigma(t,\widehat{y},u,\pi)| \le C(|y-\widehat{y}|), \\ |b(t,y,u,\pi)| + |\sigma(t,y,u,\pi)| \le C(1+|y|). \end{cases}$$

(H2) f, g are uniformly continuous in (t, y, u, π). There exists a constant $C > 0$, such that

$$\begin{cases} |f(t,y,u,\pi) - f(t,\widehat{y},u,\pi)| + |g(y) - g(\widehat{y})| \le C(|y-\widehat{y}|), \\ |f(t,y,u,\pi)| + |g(y)| \le C(1+|y|). \end{cases}$$

(H3) b, σ, f, g are continuously differentiable in y and the partial derivatives are uniformly bounded. There exists a constant $C > 0$, such that

$$|b_y(t,y,u,\pi) - b_y(t,\widehat{y},u,\pi)| + |\sigma_y(t,y,u,\pi) - \sigma_y(t,\widehat{y},u,\pi)| \le C(|y-\widehat{y}|),$$

$$|f_y(t,y,u,\pi)| + |g_y(y)| \le C(1+|y|).$$

Remark 3.1.1 *Under assumption **(H1)**–**(H3)** and for any* $(u(\cdot),\pi(\cdot))\in\mathcal{U}[s,T]$, *SDE* (3.1) *has a unique solution* $Y^{s,y,u,\pi}(\cdot)$, *see Buckdahn, Ma and Rainer [18].*

The objective of the optimality problem, is to minimize (3.2) subject to (3.1) over $\mathcal{U}[s,T]$. Any admissible control $\bar{u}(\cdot),\bar{\pi}(\cdot)$ that achieves the minimum is called an optimal control, and it implies an associated optimal state evolution $\bar{Y}^{s,y,\bar{u},\bar{\pi}}(\cdot)$ from (3.1).

We define the value function $V:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$ as

$$\begin{cases} V(s,y):= \inf\limits_{(u(.),\pi(.))\in\mathcal{U}[s,T]} J(s,y,u(\cdot),\pi(\cdot)), \quad \forall (s,y)\in[0,T]\times\mathbb{R}^n, \\ V(T,y)= g(y). \end{cases} \tag{3.3}$$

We introduce the following generalized Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} -v_t(t,y)-\inf\limits_{(u,\pi)\in U} G(t,y,u,\pi,-v(t,y),-v_y(t,y),-v_{yy}(t,y))=0, \\ v(T,y)=g(y), \quad (t,y)\in[0,T]\times\mathbb{R}^n, \end{cases} \tag{3.4}$$

where the generalized Hamiltonian function G associated with a function $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ is defined as

$$\begin{aligned} &G(t,y,u,\pi,\varphi(t,y),\varphi_y(t,y),\varphi_{yy}(t,y)) \\ &\quad=\sum_{i=1}^{n} b^i(t,y,u,\pi)\frac{\partial\varphi}{\partial y^i}(t,y)-f(t,y,u,\pi) \\ &\quad+\frac{1}{2}\sum_{i,j=1}^{n}\mathbf{1}_{\{u=0\}}\frac{\partial^2\varphi}{\partial y^i\partial y^j}(t,y)\sum_{l=1}^{n}\sigma^i(t,y,u,\pi)\sigma^j(t,y,u,\pi) \\ &\quad+\sum_{j=1}^{n}\mathbf{1}_{\{u\neq 0\}}\left(\varphi(t,y+u\sigma(t,y,u,\pi))-\varphi(t,y)-u\frac{\partial\varphi}{\partial y^j}\sigma^j(t,y,u,\pi)\right)(u)^{-2}, \end{aligned} \tag{3.5}$$

where $\sigma^i(t,y,u,\pi)$ denotes the $i-$th component of the vector σ.

Remark 3.1.2 *In this case the HJB equation takes a new form which we shall name as a mixed second-order partial differential-difference equation(PDDE in short), see Buckdahn, Ma and Rainer [18].*

Lemma 3.1.1 *Let **(H1)**–**(H2)** hold. Then for any* $t \in [0,T]$ *and* $y, \acute{y} \in \mathbb{R}^n$*, we have*

$$\begin{cases} |V(t,y) - V(t,\acute{y})| \leq C\,|y - \acute{y}|\,, \\ \\ |V(t,y)| \leq C\,(1 + |y|)\,. \end{cases} \tag{3.6}$$

Note that such a second-order PDDE has not been studied systematically in the literature. We begin by introducing the notion of viscosity solution, following the approach by Barles, Buckdahn and Pardoux in [8].

Now we introduce the definition of the viscosity solution for HJB equation (3.4)

Definition 3.1.3 ***(i)****A continuous function* $v : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is called a viscosity subsolution of the partial differential-difference equation* (3.4) *if*

$$v(T,y) \leq g(y)\,, \quad \forall y \in \mathbb{R}^n,$$

and for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ *such that* $v - \varphi$ *attains a local maximum at* $(t,y) \in [0,T] \times \mathbb{R}^n$*, it holds that*

$$-\varphi_t(t,y) - \inf_{(u,\pi)\in\bar{U}} G^{\delta}(t,y,u,\pi,-v(t,y),-\varphi_y(t,y),-\varphi_{yy}(t,y)) \leq 0. \tag{3.7}$$

(ii)*A continuous function* $v : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is called a viscosity supersolution of the PDDE* (3.4) *if*

$$v(T,y) \geq g(y)\,, \quad \forall y \in \mathbb{R}^n,$$

and for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ *such that* $v - \varphi$ *attains a local minimum at* $(t,y) \in [0,T] \times \mathbb{R}^n$*, it holds that*

$$-\varphi_t(t,y) - \inf_{(u,\pi)\in\bar{U}} G^{\delta}(t,y,u,\pi,-v(t,y),-\varphi_y(t,y),-\varphi_{yy}(t,y)) \geq 0, \tag{3.8}$$

for all sufficiently small $\delta > 0$, *where*

$$
\begin{aligned}
&G^{\delta}\left(t,y,u,\pi,-v\left(t,y\right),-\varphi_{y}\left(t,y\right),-\varphi_{yy}\left(t,y\right)\right)\\
&:=\sum_{i=1}^{n}b^{i}\left(t,y,u,\pi\right)\frac{\partial\varphi}{\partial y^{i}}\left(t,y\right)+\frac{1}{2}\sum_{i,j=1}^{n}\mathbf{1}_{\{u=0\}}\frac{\partial^{2}\varphi}{\partial y^{i}\partial y^{j}}\left(t,y\right)\sum_{l=1}^{n}\sigma^{i}\left(t,y,u,\pi\right)\sigma^{j}\left(t,y,u,\pi\right)\\
&\qquad+\sum_{j=1}^{n}\mathbf{1}_{\{0<|u|\leq\delta\}}\left(\varphi\left(t,y+u\sigma\left(t,y,u,\pi\right)\right)-\varphi\left(t,y\right)-u\frac{\partial\varphi}{\partial y^{j}}\sigma^{j}\left(t,y,u,\pi\right)\right)\left(u\right)^{-2}\\
&\qquad\quad+\sum_{j=1}^{n}\mathbf{1}_{\{|u|>\delta\}}\left(v\left(t,y+u\sigma\left(t,y,u,\pi\right)\right)-v\left(t,y\right)-u\frac{\partial\varphi}{\partial y^{j}}\sigma^{j}\left(t,y,u,\pi\right)\right)\left(u\right)^{-2}-f\left(t,y,u,\pi\right),
\end{aligned}
\tag{3.9}
$$

(iii) *A function* v *is called a viscosity solution of* (3.4) *if it is both a viscosity subsolution and a supersolution of* (3.4).

Remark 3.1.3 *We note that the last two second-order difference quotients in (3.9) are designed to take away the possible singularity at* $u = 0$ *when* V *is not smooth. Such an idea was also used by Barles, Buckdahn and Pardoux in [8].*

Remark 3.1.4 *In the general theory of viscosity solutions one can often replace the local maximum and/or minimum in the definition above by the global ones.*

Lemma 3.1.2 *In Definition 3.2 one can consider only those test functions* $\varphi \in C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$ *such that* $v-\varphi$ *achieves a global maximum (for a viscosity subsolution) and a global minimum (for a viscosity supersolution), respectively, at* (t,y).
Furthermore, the operator $G^{\delta}\left(t,y,u,\pi,-v\left(t,y\right),-\varphi_{y}\left(t,y\right),-\varphi_{yy}\left(t,y\right)\right)$ *can be replaced by* $G\left(t,y,u,\pi,-\varphi\left(t,y\right),-\varphi_{y}\left(t,y\right),-\varphi_{yy}\left(t,y\right)\right)$ *defined by (3.5).*

Theorem 3.1.1 (Uniqueness of the viscosity solution) *Assume **(H1)**–**(H2)**–**(H3)**, and if there exists a compact set* $U_1 \in \mathbb{R}$ *such that*
(i) $0 \notin U_1$,
(ii) $U = U_1$ *or* $U = \{0\} \cup U_1$.
Then the value function $V : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}$ *defined by (3.3) is the unique viscosity solution of (3.4) among all bounded, continuous functions.*

Proof. The proof is adapted from the proof of Theorem 6.2, see Barles, Buckdahn and Pardoux in [8]. ■

Define the usual Hamiltonian function $H : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n$ by

$$H\left(t, y, u, \pi, p, q\right) = f\left(t, y, u, \pi\right) + b^\top\left(t, y, u, \pi\right) p + tr\left(\sigma^\top\left(t, y, u, \pi\right) q\right). \tag{3.10}$$

Associated with an optimal process $\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(.\right), \bar{u}\left(.\right), \bar{\pi}\left(.\right)\right)$, we introduce the adjoint equation

$$\begin{cases} dp\left(t\right) = -H_y\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right), \bar{u}\left(t\right), \bar{\pi}\left(t\right), p\left(t\right), q\left(t\right)\right) dt - q\left(t\right) \mathcal{M}^{\bar{u}}\left(t\right), \\ p\left(T\right) = -g_y\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(T\right)\right), \qquad t \in [0,T]. \end{cases} \tag{3.11}$$

Note that under **(H1)**–**(H2)** and **(H3)**, the linear BSDE (3.11) admits a unique solution $(p\left(.\right), q\left(.\right))$ which is called the adjoint process pair.

Theorem 3.1.2 (Sufficient condition of optimality) *Let $(\bar{u}(\cdot), \bar{\pi}(\cdot))$ be an admissible control and $\bar{Y}^{s,y,\bar{u},\bar{\pi}}(\cdot)$ the associated controlled state process. Let $(p(\cdot), q(\cdot))$ be the unique solution of the adjoint equation* (3.11)*. Suppose that the Hamiltonian H is convex in (y, u, π), and the terminal cost function g is convex in y. Under the conditions (H1)–(H2) and (H3), an admissible control $(\bar{u}(\cdot), \bar{\pi}(\cdot))$ is optimal if the following condition holds*

$$H\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right), \bar{u}\left(t\right), \bar{\pi}\left(t\right), p\left(t\right), q\left(t\right)\right) = \inf_{(u,\pi)\in U} H\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right), \bar{u}\left(t\right), \bar{\pi}\left(t\right), p\left(t\right), q\left(t\right)\right). \tag{3.12}$$

Remark 3.1.5 *Notice that theorem 3.1.1 is proved by Chighoub, Lakhdari and Shi in [22].*

3.2 Main result

Let us recall the notions of the first-order super- and sub-jets in the spatial variable y, see [23], [64]. Given $v \in C\left([0,T] \times \mathbb{R}^n\right)$, and $(t, \hat{y}) \in [0,T] \times \mathbb{R}^n$, we define the first-order super-jet by

$$\mathcal{D}_y^{1,+} v\left(t, \widehat{y}\right) := \left\{p \in \mathbb{R}^n \,|\, v\left(t, y\right) \leq v\left(t, \widehat{y}\right) + p\left(y - \widehat{y}\right) + o\left(|y - \widehat{y}|\right), \text{ as } y \longrightarrow \widehat{y}\right\}. \tag{3.13}$$

Similarly, we consider the first-order sub-jet of v by

$$\mathcal{D}_y^{1,-}v(t,\widehat{y}) := \{p \in \mathbb{R}^n \,|v(t,y) \geq v(t,\widehat{y}) + p(y-\widehat{y}) + o(|y-\widehat{y}|), \text{ as } y \longrightarrow \widehat{y}\} \tag{3.14}$$

The following result shows that the adjoint process p and the value function V relate to each other within the framework of the superjet and the subjet in the state variable y along an optimal trajectory.

Theorem 3.2.1 *Let **(H1)–(H2)–(H3)** hold and $(t,y) \in [0,T] \times \mathbb{R}^n$ be fixed. Suppose that $(\bar{u}(t), \bar{\pi}(t))$ is an optimal control for problem (3.3), and $\bar{Y}^{s,y,\bar{u},\bar{\pi}}(\cdot)$ is the corresponding optimal state. Let $(p(\cdot), q(\cdot))$ be the adjoint process. Then*

$$\mathcal{D}_y^{1,-}V\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right) \subset \{-p(t)\} \subset \mathcal{D}_y^{1,+}V\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right), \quad \forall t \in [s,T], \ \ \mathbb{P}-a.s. \tag{3.15}$$

where $V(\cdot,\cdot)$ is the value function defined by (3.3).

Proof. For simplicity, we introduce the following notations

$$\begin{aligned}
&\bar{b}(t) = b\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t), \bar{u}(t), \bar{\pi}(t)\right), \quad \bar{\sigma}(t-) = \sigma\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t-), \bar{u}(t), \bar{\pi}(t)\right), \\
&\bar{f}(t) = f\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t), \bar{u}(t), \bar{\pi}(t)\right), \quad \text{for all } t \in [0,T], (u,\pi) \in U,
\end{aligned}$$

and similar notations used for all their derivatives.

Fix a $t \in [s,T]$. For any $y^1 \in \mathbb{R}^n$, denote by $Y^{t,y^1,\bar{u},\bar{\pi}}(t)$ the solution of the following SDE

$$\begin{aligned}
Y^{t,y^1,u,\pi}(r) = \; & y^1 + \int_t^r b\left(r, Y^{t,y^1,u,\pi}(r), u(r), \pi(r)\right) dr \\
& + \int_t^r \sigma\left(r, Y^{t,y^1,u,\pi}(r-), u(r), \pi(r)\right) d\mathcal{M}^u(r).
\end{aligned} \tag{3.16}$$

It is clear that (3.16) can be regarded as an SDE on $\left(\Omega, \mathcal{F}, (\mathcal{F}_r^s)_{r\geq s}, \mathbb{P}(.\,|\mathcal{F}_t^s)(\omega)\right)$ for $\mathbb{P}-a.s.\omega$, where $\mathbb{P}(.\,|\mathcal{F}_t^s)(\omega)$ is the regular conditional probability given $\mathcal{F}_t^s$ defined on $(\Omega, \mathcal{F})$, see pp. 12–16 of Ikeda and Watanabe [38].

For any $t \leq r \leq T$, set $X(r) := Y^{t,y^1,\bar{u},\bar{\pi}}(r) - \bar{Y}^{t,y,\bar{u},\bar{\pi}}(r)$, we have for any integer $k \geq 1$,

$$\mathbb{E}\left[\sup_{t\leq r\leq T} |X(r)|^{2k} |\mathcal{F}_t^s\right] \leq C \left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|^{2k}, \quad \mathbb{P}-a.s. \tag{3.17}$$

The above inequality can be proved by a routine successive approximation argument, see Theorem 6.3, Chapter1, Yong and Zhou [64].

Now we write the equation for the process$X(\cdot)$ as

$$\begin{cases} dX(r) = \left\{\bar{b}_y(r)X(r) + \varepsilon_1(r)\right\}dr + \left\{\bar{\sigma}_y(r-)X(r) + \varepsilon_2(r)\right\}d\mathcal{M}^{\bar{u}}(r), \\ X(t) = y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t), \end{cases} \tag{3.18}$$

where

$$\begin{cases} \varepsilon_1(r) := \int_0^1 \left[b_y\left(r, \bar{Y}^{s,y,\bar{u},\bar{\pi}}(r) + \theta X(r), \bar{u}(r), \bar{\pi}(r)\right) - \bar{b}_y(r)\right] X(r)\, d\theta, \\ \varepsilon_2(r) := \int_0^1 \left[\sigma_y\left(r, \bar{Y}^{s,y,\bar{u},\bar{\pi}}(r) + \theta X(r), \bar{u}(r), \bar{\pi}(r)\right) - \bar{\sigma}_y(r-)\right] X(r)\, d\theta. \end{cases} \tag{3.19}$$

For any $k \geq 1$, there exists a deterministic continuous and increasing function $\delta : [0,\infty) \to [0,\infty)$, independent of $y^1 \in \mathbb{R}^n$, with $\frac{\delta(r)}{r}$ as $r \to 0$, such that

$$\begin{cases} \mathbb{E}\left[\int_t^T |\varepsilon_1(r)|^{2k} dr\, |\mathcal{F}_t^s\right] \leq \delta \left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|^{2k}, \quad \mathbb{P}-a.s. \\ \mathbb{E}\left[\int_t^T |\varepsilon_2(r)|^{2k} dr\, |\mathcal{F}_t^s\right] \leq \delta \left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|^{2k}, \quad \mathbb{P}-a.s. \end{cases} \tag{3.20}$$

Applying Itô's formula to $\langle X(\cdot), p(\cdot)\rangle$, noting (3.11), we get

$$\begin{aligned} &\mathbb{E}\left\{\int_t^T \langle \bar{f}_y(r), X(r)\rangle dr + \langle g_y\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}(T)\right), X(T)\rangle\, |\mathcal{F}_t^s\right\} \\ &= \langle -p(t), X(t)\rangle + \mathbb{E}\left\{\int_t^T \left[\langle \varepsilon_1(r), p(r)\rangle + \langle \varepsilon_2(r), q(r)\rangle\right] dr\, |\mathcal{F}_t^s\right\}, \quad \mathbb{P}-a.s. \end{aligned} \tag{3.21}$$

Noting (3.20), since $\mathbb{E}\left[\sup_{t\leq r\leq T}|p(r)|^{2k}dr\,|\mathcal{F}_t^s\right]<\infty$, $\mathbb{E}\left[\sup_{t\leq r\leq T}|q(r)|^{2k}dr\,|\mathcal{F}_t^s\right]<\infty$, it follows that

$$\begin{aligned}&\mathbb{E}\left[\int_t^T\langle\varepsilon_1(r),p(r)\rangle dr\,|\mathcal{F}_t^s\right]\leq\mathbb{E}\left[\sup_{t\leq r\leq T}p(r)\int_t^T\varepsilon_1(r)\,dr\,|\mathcal{F}_t^s\right]\\&\leq\left(\mathbb{E}\left[\sup_{t\leq r\leq T}|p(r)|^2\,|\mathcal{F}_t^s\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_t^T\varepsilon_1(r)\,dr\,|\mathcal{F}_t^s\right]\right)^{\frac{1}{2}}\leq o\left|y^1-\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_t^T\langle\varepsilon_2(r),q(r)\rangle dr\,|\mathcal{F}_t^s\right]\leq\mathbb{E}\left[\sup_{t\leq r\leq T}q(r)\int_t^T\varepsilon_2(r)\,dr\,|\mathcal{F}_t^s\right]\\&\leq\left(\mathbb{E}\left[\sup_{t\leq r\leq T}|q(r)|^2\,|\mathcal{F}_t^s\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_t^T\varepsilon_2(r)\,dr\,|\mathcal{F}_t^s\right]\right)^{\frac{1}{2}}\leq o\left|y^1-\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathbb{E}\left\{\int_t^T\langle\bar{f}_y(r),X(r)\rangle dr+\langle g_y\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}(T)\right),X(T)\rangle\,|\mathcal{F}_t^s\right\}\\&=\langle -p(t),X(t)\rangle+o\left|y^1-\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right|,\quad\mathbb{P}-a.s.\end{aligned}\tag{3.22}$$

Let us call a $y^1\in\mathbb{R}^n$ rational if all its coordinates are rational numbers. Since the set of all rational $y^1\in\mathbb{R}^n$ is countable, we may find a subset $\Omega_0\subseteq\Omega$ with $P(\Omega_0)=1$ such that for any $\omega_0\in\Omega_0$,

$$\begin{cases}V\left(t,\bar{Y}^{s,y,\bar{u},\bar{\pi}}(t,\omega_0)\right)=\mathbb{E}\left[\int_s^T\bar{f}(r)\,dr+g\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}(T)\right)|\mathcal{F}_t^s\right](\omega_0),\\ \text{(3.17)-(3.20)-(3.22), are satisfied for any rational } y^1, \text{ and } \bar{u}(\cdot),\bar{\pi}(\cdot)|_{[t,T]}\in\mathcal{U}[t,T].\end{cases}$$

Let $\omega_0 \in \Omega_0$ be fixed, then for any rational $y^1 \in \mathbb{R}^n$, noting (3.20), we have

$$\begin{aligned}
&V\left(t, y^1\right) - V\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right) \\
&= \mathbb{E}\left\{\int_t^T \left[f\left(r, Y^{t,y^1,\bar{u},\bar{\pi}}\left(r\right), \bar{u}\left(r\right), \bar{\pi}\left(r\right)\right) - \bar{f}\left(r\right)\right] dr \right.\\
&\quad \left. + \, g\left(Y^{t,y^1,\bar{u},\bar{\pi}}\left(T\right)\right) - g\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(T\right)\right) | \mathcal{F}_t^s\right\}\left(\omega_0\right) \\
&= \mathbb{E}\left\{\int_t^T \left\langle \bar{f}_y\left(r\right), X\left(r\right)\right\rangle dr + \left\langle g_y\left(\bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(T\right)\right), X\left(T\right)\right\rangle | \mathcal{F}_t^s\right\}\left(\omega_0\right) \\
&\quad + o\left(\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right|\right). \qquad (3.23)
\end{aligned}$$

By (3.22), we have

$$\begin{aligned}
&V\left(t, y^1\right) - V\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right) \\
&\leq -\left\langle p\left(t, \omega_0\right), X\left(t, \omega_0\right)\right\rangle + o\left(\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right|\right) \qquad (3.24)
\end{aligned}$$

Note that the term $o\left(\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right|\right)$ in the above depends only on the size of $\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t, \omega_0\right)\right|$, and it is independent of y^1. Therefore, by the continuity of $V(t, \cdot)$, we see that (3.24) holds for all $y^1 \in \mathbb{R}^n$, which by definition (3.1) proves

$$\{-p\left(t\right)\} \in \mathcal{D}_y^{1,+} V\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right), \quad \forall t \in [s, T], \quad \mathbb{P}\text{-}a.s. \qquad (3.25)$$

Let us now show $\{-p\left(t\right)\} \subset \mathcal{D}_y^{1,-} V\left(t, \bar{y}^{s,x,\bar{u},\bar{\pi}}\left(t\right)\right)$. Fix an $\omega \in \Omega$ such that (3.24) holds for any $y^1 \in \mathbb{R}^n$. For any $\xi \in \mathcal{D}_y^{1,-} V\left(t, \bar{y}^{s,x,\bar{u},\bar{\pi}}\left(t\right)\right)$, by definition (3.2) we have

$$\begin{aligned}
0 &\leq \lim_{y^1 \longrightarrow \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)} \left\{\frac{V\left(t, y^1\right) - V\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right)}{\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right|} - \frac{\left\langle \xi, y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right\rangle}{\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right|}\right\} \\
&\leq \lim_{y^1 \longrightarrow \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)} \frac{\left\langle -p\left(t\right) - \xi, y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right\rangle}{\left|y^1 - \bar{Y}^{s,y,\bar{u},\bar{\pi}}\left(t\right)\right|}.
\end{aligned}$$

Then, it is necessary that

$$\xi = -p\left(t\right), \quad \forall t \in [s, T], \quad \mathbb{P}\text{-}a.s.$$

■

Remark 3.2.1 *It is interesting to note that if V is differentiable with respect to y, then (3.15) reduces to*

$$-p(t) = V_y\left(t, \bar{Y}^{s,y,\bar{u},\bar{\pi}}(t)\right), \quad \mathbb{P}-a.s.$$

Chapter 4

A Characterization of Sub-game Perfect Equilibria for SDEs of Mean-Field Type Under Partial Information

In this chapter, we characterize sub-game perfect equilibrium strategy of a partially observed optimal control problems for mean-field stochastic differential equations (SDEs) with correlated noises between systems and observations, which is time-inconsistent.

4.1 Notation and statement of the problem

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a given filtered probability space on which there are defined two independent standard one-dimensional Brownian motion $W(.)$ and $Y(.)$. Let $\{\mathcal{F}_t^w\}$ and $\left\{\mathcal{F}_t^Y\right\}$ be the natural filtration generated by $W(.)$ and $Y(.)$ respectively. Set $\mathbb{F} := \{\mathcal{F}_s, 0 \leq s \leq T\}$ and $\mathbb{F}^Y := \left\{\mathcal{F}_t^Y, 0 \leq s \leq T\right\}$, where, $\mathcal{F}_t = \mathcal{F}_t^w \otimes \mathcal{F}_t^Y$, $\mathcal{F} = \mathcal{F}_T$. For a function f, we denote by f_x (resp., f_{xx}) the gradient or Jacobian (resp., the Hessian) of f with respect to the variable X, and by $|.|$ the norm of an Euclidean space; by $\mathbf{I}_A$ the indicator function of a set A.

An admissible strategy v is an $\mathbb{F}^{Y}$-adapted process with values in a non-empty subset U of $\mathbb{R}$ and satisfies $\sup\limits_{0\leq s\leq T}\mathbb{E}\left|v\left(s\right)\right|^{m}<\infty$ for $m=1,2\ldots$. The set of all admissible strategies over $[0,T]$ is denoted by $\mathcal{U}_{ad}$.

For each admissible strategy $v\in\mathcal{U}_{ad}$, we consider the dynamics given by the following SDE of mean-field type with correlated noises between systems and observations defined on $\left(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\right)$,

$$\begin{cases} dX^{v}\left(s\right)=b\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)ds \\ \qquad\qquad +\sigma\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)dW\left(s\right) \\ \qquad\qquad +\widehat{\sigma}\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)d\widehat{W}\left(s\right),\ 0<s\leq T, \\ X^{v}\left(0\right)=x_{0}\left(\in\mathbb{R}\right). \end{cases} \tag{4.1}$$

Suppose $X^{v}\left(s\right)$ can not be directly observed, while we can observe a related process $Y(.)$, which is governed by

$$\begin{cases} dY\left(s\right)=h\left(s,X\left(s\right)\right)ds+d\widehat{W}\left(s\right), \\ Y\left(0\right)=0,\ s\geq0, \end{cases} \tag{4.2}$$

where $h:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$, and $\widehat{W}\left(.\right)$ is a stochastic process depending on $v\left(.\right)$.

Throughout what follows we shall assume the following.

(H_{1}) The functions $b,\sigma,\widehat{\sigma},f:[0,T]\times\mathbb{R}\times\mathbb{R}\times U\rightarrow\mathbb{R}$, $h:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$, $\phi:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ are twice continuously differentiable in $(x,\widehat{x})$. Moreover, $b,\sigma,\widehat{\sigma},f$ and all their derivatives up to second order with respect to $(x,\widehat{x})$ are continuous in $(x,\widehat{x},v)$ and bounded.

Now, putting (4.2) into (4.1), we get

$$\begin{cases} dX^{v}\left(s\right)=b\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)ds \\ \qquad\qquad -\widehat{\sigma}\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)h\left(s,X^{v}\left(s\right)\right)ds \\ \qquad\qquad +\sigma\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)dW\left(s\right), \\ \qquad\qquad +\widehat{\sigma}\left(s,X^{v}\left(s\right),\mathbb{E}\left[X^{v}\left(s\right)\right],v\left(s\right)\right)dY\left(s\right) \\ X^{v}\left(0\right)=x_{0}\left(\in\mathbb{R}\right). \end{cases} \tag{4.3}$$

For any $v(.) \in \mathcal{U}_{ad}$, and under assumptions (H_1), the SDE (4.3) admits a unique solution. Define $d\mathbb{P}^v = Z^v(t)d\mathbb{P}$, with

$$Z(s) = \exp\left\{ \int_0^s h(t, X^v(t))\, dY(t) - \frac{1}{2}\int_0^s h^2(t, X^v(t))\, dt \right\}.$$

Using Itô's formula, we know that $Z^v(.)$ is the solution of

$$\begin{cases} dZ^v(s) = Z^v(s) h(s, X^v(s))\, dY(s), \\ Z^v(0) = 1. \end{cases} \tag{4.4}$$

From the boundedness of h, we know Novikov condition $\mathbb{E}\left\{\exp\left(\frac{1}{2}\int_0^s h^2(t, X^v(t))\, dt\right)\right\} < \infty$ naturally succeeds. Thus, by Girsanov's theorem, $\mathbb{P}^v$ is a new probability measure and $\left(W(.), \widehat{W}(.)\right)$ is a two-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$.

We introduce the following cost functional

$$J(t,x,v) = \mathbb{E}^v\left[\int_t^T f\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) ds + \phi\left(X^{v,t,x}(T), \mathbb{E}\left[X^{v,t,x}(T)\right]\right)\right], \tag{4.5}$$

where $\mathbb{E}^v$ denotes mathematical expectation in the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$, According to Bayes' formula, (4.5) can be rewritten as

$$\begin{aligned} J(t,x,v) = \mathbb{E}\bigg[\int_t^T & Z^{v,t,x}(s) f\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) ds \\ & + Z^{v,t,x}(T)\phi\left(X^{v,t,x}(T), \mathbb{E}\left[X^{v,t,x}(T)\right]\right)\bigg]. \end{aligned} \tag{4.6}$$

Equation (4.6) associated with the state process $X^{v,t,x}$, parameterized by $(t,x) \in [0,T] \times \mathbb{R}$, whose the dynamics is given by the SDE

$$\begin{cases} dX^{v,t,x}(s) = b\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) ds \\ \qquad - \widehat{\sigma}\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) h\left(s, X^{v,t,x}(s)\right) ds \\ \qquad + \sigma\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) dW(s) \\ \qquad + \widehat{\sigma}\left(s, X^{v,t,x}(s), \mathbb{E}\left[X^{v,t,x}(s)\right], v(s)\right) dY(s), \\ X^{v,t,x}(t) = x\,(\in \mathbb{R}), \quad t < s \leq T. \end{cases} \tag{4.7}$$

The dependence of (4.6)–(4.7) on the term $\mathbb{E}\left[X^{v,t,x}(s)\right]$ makes the system (4.6)–(4.7) time-inconsistent in the sense that the Bellman Principle for optimality does not hold, i.e., the t-optimal strategy $u(t,x,.)$ which minimizes $J(t,x,.)$ may not be optimal after t: The restriction of $u(t,x,.)$ on $\left[t', T\right]$ does not minimize $J\left(t', x', v\right)$ for some $t > t'$ when the state process is steered to x' by u.

Define the admissible strategy u^{ε} as the "local" spike variation of a given admissible strategy $u \in \mathcal{U}_{ad}$ over the set $[t, t+\varepsilon]$,

$$u^{\varepsilon}(s) = \begin{cases} v(s), & s \in [t, t+\varepsilon], \\ u(s), & s \in [t,T] \setminus [t, t+\varepsilon]. \end{cases}$$

Definition 4.1.1 *The admissible strategy u is a sub-game perfect equilibrium for the system (4.6)–(4.7) if*

$$\lim_{\varepsilon \downarrow 0} \frac{J(t,x,u) - J(t,x,u^{\varepsilon})}{\varepsilon} \leq 0, \tag{4.8}$$

for all $v \in \mathcal{U}_{ad}$, $x \in \mathbb{R}$ and a.e. $t \in [0,T]$. The corresponding equilibrium dynamics solves the SDE

$$\begin{cases} dX^{u}(s) = b\left(s, X^{u}(s), \mathbb{E}\left[X^{u}(s)\right], u(s)\right) ds \\ \qquad - \widehat{\sigma}\left(s, X^{u}(s), \mathbb{E}\left[X^{u}(s)\right], u(s)\right) h\left(s, X^{u}(s)\right) ds \\ \qquad + \sigma\left(s, X^{u}(s), \mathbb{E}\left[X^{u}(s)\right], u(s)\right) dW(s) \\ \qquad + \widehat{\sigma}\left(s, X^{u}(s), \mathbb{E}\left[X^{u}(s)\right], u(s)\right) dY(s), \\ X^{u}(0) = x_0. \end{cases} \tag{4.9}$$

For brevity, sometimes we simply call u an equilibrium point when there is no ambiguity.

Our objective in this study is to characterize sub-game perfect equilibria for the system (4.10)–(4.11) for the more general case where player t has a random variable $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}^v, \mathbb{R})$ as a state in terms of stochastic maximum principle criterion.
For a given admissible strategy $v \in \mathcal{U}_{ad}$, if player t has $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}^v, \mathbb{R})$ as its state, the dynamics is given by the SDE

$$\begin{cases} dX^{v,t,\xi}(s) = b\left(s, X^{v,t,\xi}(s), \mathbb{E}\left[X^{v,t,\xi}(s)\right], v(s)\right) ds \\ \qquad\qquad - \widehat{\sigma}\left(s, X^{v,t,\xi}(s), \mathbb{E}\left[X^{v,t,\xi}(s)\right], v(s)\right) h\left(s, X^{v,t,\xi}(s)\right) ds \\ \qquad\qquad + \sigma\left(s, X^{v,t,\xi}(s), \mathbb{E}\left[X^{v,t,\xi}(s)\right], v(s)\right) dW(s) \\ \qquad\qquad + \widehat{\sigma}\left(s, X^{v,t,\xi}(s), \mathbb{E}\left[X^{v,t,\xi}(s)\right], v(s)\right) dY(s), \\ X^{v,t,\xi}(t) = \xi, \end{cases} \tag{4.10}$$

and the associated cost functional

$$\begin{aligned} J(t, \xi, v) = \mathbb{E}\bigg[\int_t^T & Z^{v,t,\xi}(s) f\left(s, X^{v,t,\xi}(s), \mathbb{E}\left[X^{v,t,\xi}(s)\right], v(s)\right) ds \\ & + Z^{v,t,\xi}(T)\phi\left(X^{v,t,\xi}(T), \mathbb{E}\left[X^{v,t,\xi}(T)\right]\right)\bigg]. \end{aligned} \tag{4.11}$$

In view of Karatzas and Sherve ([40], pp. 289-290), under (H_1), for any $v \in \mathcal{U}_{ad}$, the SDE (4.10) admits a unique strong solution. Moreover, there exists a constant $C > 0$ which depends only on the bounds of $b, \sigma, \widehat{\sigma}$ and their first derivatives $x, \widehat{x}$, such that, for any $t \in [0, T]$, $v \in \mathcal{U}_{ad}$ and $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}^v, \mathbb{R})$, we also have the following estimates, $\mathbb{P} - a.s.$

$$\mathbb{E}\left[\sup_{t \le s \le T} \left|X^{v,t,\xi}(s)\right|^2 \mid \mathcal{F}_t\right] \le C\left(1 + |\xi|^2 + \mathbb{E}\left[|\xi|^2\right]\right),$$
$$\mathbb{E}\left[\sup_{t \le s \le T} \left|X^{v,t,\xi}(s) - X^{v,t,\xi'}(s)\right|^2 \mid \mathcal{F}_t\right] \le C\left(\left|\xi - \xi'\right|^2 + \mathbb{E}\left[\left|\xi - \xi'\right|^2\right]\right).$$

Moreover, the performance functional (4.11) is well defined and finite.

Remark 4.1.1 *Definitions 4.1.1 can be accordingly generalized by replacing (t, x) by (t, ξ) and* *the inequality condition takes the form*

$$\lim_{\varepsilon\downarrow 0}\frac{J\left(t,\xi,u\left(\cdot\right)\right)-J\left(t,\xi,u^{\varepsilon}\left(\cdot\right)\right)}{\varepsilon}\leq 0, \tag{4.12}$$

for all $v\in\mathcal{U}_{ad},\xi\in L^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P}^{v},\mathbb{R}\right)$ *and a.e.* $t\in[0,T]$.

To simplify our notation, we will denote by $X^{t,\xi}=X^{v,t,\xi}$ the solution of the SDE (4.10), associated with the strategy u.

For $\gamma=b,\sigma,\widehat{\sigma},f$ respectively, we define

$$\begin{cases}\delta\gamma\left(s\right)=\gamma\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],v\right)-\gamma\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],u\left(s\right)\right),\\ \gamma_{x}\left(s\right)=\frac{\partial\gamma}{\partial x}\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],u\left(s\right)\right),\quad \gamma_{\widehat{x}}\left(s\right)=\frac{\partial\gamma}{\partial\widehat{x}}\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],u\left(s\right)\right),\\ \gamma_{xx}\left(s\right)=\frac{\partial^{2}\gamma}{\partial x^{2}}\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],u\left(s\right)\right),\quad \gamma_{x\widehat{x}}\left(s\right)=\frac{\partial^{2}\gamma}{\partial x\partial\widehat{x}}\left(s,X^{t,\xi}\left(s\right),\mathbb{E}\left[X^{t,\xi}\left(s\right)\right],u\left(s\right)\right).\end{cases}$$

For h and ϕ, we define

$$\begin{cases}\phi\left(T\right)=\phi\left(X^{t,\xi}\left(T\right),\mathbb{E}\left[X^{t,\xi}\left(T\right)\right]\right),\quad \phi_{x}\left(T\right)=\phi_{x}\left(X^{t,\xi}\left(T\right),\mathbb{E}\left[X^{t,\xi}\left(T\right)\right]\right),\\ \phi_{xx}\left(T\right)=\phi_{xx}\left(X^{t,\xi}\left(T\right),\mathbb{E}\left[X^{t,\xi}\left(T\right)\right]\right),\quad h\left(s\right)=h\left(s,X^{t,\xi}\left(s\right)\right),\\ h_{x}\left(s\right)=h_{x}\left(s,X^{t,\xi}\left(s\right)\right),\quad h_{xx}\left(s\right)=h_{xx}\left(s,X^{t,\xi}\left(s\right)\right).\end{cases}$$

The Hamiltonian function H is defined by

$$\begin{aligned}&H\left(s,X^{t,\xi}\left(s\right),u\left(s\right),p^{t,\xi}\left(s\right),q^{t,\xi}\left(s\right),\widehat{q}^{t,\xi}\left(s\right),\widehat{\Psi}^{t,\xi}\left(s\right)\right)\\ &=\left(b\left(s\right)\right)p^{t,\xi}\left(s\right)+\sigma\left(s\right)q^{t,\xi}\left(s\right)+\widehat{\sigma}\left(s\right)\widehat{q}^{t,\xi}\left(s\right)+h\left(s\right)\widehat{\Psi}^{t,\xi}\left(s\right)+f\left(s\right).\end{aligned}$$

4.2 Adjoint equations and the stochastic maximum principle

In this section, we introduce the variational equations and adjoint equations involved in the SMP which characterize the equilibrium points $u\in\mathcal{U}_{ad}$ of our problem.

Let $S_1^{\varepsilon}(.), S_2^{\varepsilon}(.)$ be the solutions of the first and second variational equations

$$\begin{cases} dS_1^{\varepsilon}(s) = \{(b_x(s) - \widehat{\sigma}_x(s)h(s) - \widehat{\sigma}(s)h_x(s))S_1^{\varepsilon}(s) \\ \qquad + (b_{\widehat{x}}(s) - \widehat{\sigma}_{\widehat{x}}(s)h(s))\mathbb{E}[S_1^{\varepsilon}(s)] + (\delta b(s) - \delta\widehat{\sigma}(s)h(s))\mathbf{I}_{[t,t+\varepsilon]}(s)\big\}\,ds \\ \qquad + \big\{\sigma_x(s)S_1^{\varepsilon}(s) + \sigma_{\widehat{x}}(s)\mathbb{E}[S_1^{\varepsilon}(s)] + \delta\sigma(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\big\}\,dW(s) \\ \qquad + \big\{\widehat{\sigma}_x(s)S_1^{\varepsilon}(s) + \widehat{\sigma}_{\widehat{x}}(s)\mathbb{E}[S_1^{\varepsilon}(s)] + \delta\widehat{\sigma}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\big\}\,dY(s), \\ S_1^{\varepsilon}(t) = 0. \end{cases} \tag{4.13}$$

$$\begin{cases} dS_2^{\varepsilon}(s) = \{(b_x(s) - \widehat{\sigma}_x(s)h(s) - \widehat{\sigma}(s)h_x(s))S_2^{\varepsilon}(s) + (b_{\widehat{x}}(s) - \widehat{\sigma}_{\widehat{x}}(s)h(s))\mathbb{E}[S_2^{\varepsilon}(s)] \\ \qquad + \frac{1}{2}(b_{xx}(s) - \widehat{\sigma}_{xx}(s)h(s) - 2\widehat{\sigma}_x(s)h_x(s) - \widehat{\sigma}(s)h_{xx}(s))(S_1^{\varepsilon}(s))^2 \\ \qquad + \delta(b_x(s) - \widehat{\sigma}_x(s)h(s) - \widehat{\sigma}(s)h_x(s))S_1^{\varepsilon}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\big\}\,ds \\ \qquad + \Big\{\sigma_x(s)S_2^{\varepsilon}(s) + \sigma_{\widehat{x}}(s)\mathbb{E}[S_2^{\varepsilon}(s)] + \frac{1}{2}\sigma_{xx}(s)(S_1^{\varepsilon}(s))^2 + \delta\sigma_x(s)S_1^{\varepsilon}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\big\}\,dW(s) \\ \qquad + \{\widehat{\sigma}_x(s)S_2^{\varepsilon}(s) + \widehat{\sigma}_{\widehat{x}}(s)\mathbb{E}[S_2^{\varepsilon}(s)] + \frac{1}{2}\widehat{\sigma}_{xx}(s)(S_1^{\varepsilon}(s))^2 + \delta\widehat{\sigma}_x(s)S_1^{\varepsilon}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\Big\}\,dY(s), \\ S_2^{\varepsilon}(t) = 0. \end{cases} \tag{4.14}$$

Lemma 4.2.1 *Let (H_1) hold, then we have the following estimates, where $X^{\varepsilon,t,\xi}(.)$ is the solution of the state equation (4.10) corresponding to the admissible strategy $u^{\varepsilon}(.)$,*

$$\mathbb{E}\left[\sup_{s\in[t,T]}|S_1^{\varepsilon}(s)|^{2k}\right] \le C\varepsilon^{k},$$
$$\mathbb{E}\left[\sup_{s\in[t,T]}|S_2^{\varepsilon}(s)|^{2k}\right] \le C\varepsilon^{2k},$$
$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^{\varepsilon,t,\xi}(s) - X^{t,\xi}(s)\right|^{2k}\right] \le C\varepsilon^{k},$$
$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^{\varepsilon,t,\xi}(s) - X^{t,\xi}(s) - S_1^{\varepsilon}(s)\right|^{2k}\right] \le C\varepsilon^{2k},$$

and for some function $\tilde{\rho}:(0,\infty)\rightarrow(0,\infty)$, *such that* $\lim_{\varepsilon\rightarrow 0}\tilde{\rho}_k(\varepsilon)=0$, *we get*

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^{\varepsilon,t,\xi}(s)-X^{t,\xi}(s)-S_1^{\varepsilon}(s)-S_2^{\varepsilon}(s)\right|^{2k}\right]\leq\varepsilon^{2k}\tilde{\rho}_k(\varepsilon).$$

Now, we introduce the following variational equations

$$\begin{cases} dZ_1^{t,\xi}(s)=Z_1^{t,\xi}(s)h(s)+Z^{t,\xi}(s)h_x(s)\,S_1^{\varepsilon}(s)dY(s), \\ Z_1^{t,\xi}(t)=\xi, \end{cases} \tag{4.15}$$

and

$$\begin{cases} dZ_2^{t,\xi}(s)=\Big\{Z_2^{t,\xi}(s)h(s)+Z^{t,\xi}(s)h_x(s)\,S_2^{\varepsilon}(s)+Z_1^{t,\xi}(s)h_x(s)\,S_1^{\varepsilon}(s) \\ \qquad\qquad +\dfrac{1}{2}Z^{t,\xi}(s)h_{xx}(s)\,(S_1^{\varepsilon}(s))^2\Big\}dY(s), \\ Z_2^{t,\xi}(t)=\xi, \end{cases} \tag{4.16}$$

where $Z_1^{t,\xi}(.)$, $Z_2^{t,\xi}(.)$ are the solutions of (4.15), (4.16) respectively.

Lemma 4.2.2 *Let* (H_1) *hold, then we have the following estimates, where,* $Z^{\varepsilon,t,\xi}(.)$ *is the solutions of the observation equation (4.4) corresponding to the admissible strategy* $u^{\varepsilon}(.)$,

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|Z_1^{t,\xi}(s)\right|^{2k}\right]\leq C\varepsilon^{k},$$
$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|Z^{\varepsilon,t,\xi}(s)-Z^{t,\xi}(s)-Z_1^{t,\xi}(s)\right|^{2k}\right]\leq C\varepsilon^{2k},$$

and for some function $\tilde{\rho}:(0,\infty)\rightarrow(0,\infty)$, *such that* $\lim_{\varepsilon\rightarrow 0}\tilde{\rho}_k(\varepsilon)=0$, *we get*

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|Z^{\varepsilon,t,\xi}(s)-Z^{t,\xi}(s)-Z_1^{t,\xi}(s)-Z_2^{t,\xi}(s)\right|^{2k}\right]\leq\varepsilon^{2k}\tilde{\rho}_k(\varepsilon).$$

We consider the following adjoint equation

$$\begin{cases} -d\Phi^{t,\xi}(s)=f(s)\,ds-\Psi^{t,\xi}(s)dW(s)-\widehat{\Psi}^{t,\xi}(s)d\widehat{W}(s), \\ \Phi^{t,x}(T)=\phi(T). \end{cases} \tag{4.17}$$

Under (H_1), (4.17) admit unique solution. The first-order adjoint equation is the following linear backward SDE of mean-field type parameterized by $(t,\xi)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,\mathbb{P}^v,\mathbb{R})$, satisfied by the processes $\left(p^{t,\xi}(s), q^{t,\xi}(s), \widehat{q}^{t,\xi}(s)\right), s \in [t,T],$

$$\begin{cases} -dp^{t,\xi}(s) = \Big\{ \left(b_x(s) - \widehat{\sigma}(s) h_x(s)\right) p^{t,\xi}(s) + \sigma_x(s) q^{t,\xi}(s) + \widehat{\sigma}_x(s) \widehat{q}^{t,\xi}(s) + h_x(s) \widehat{\Psi}^{t,\xi}(s) \\ \qquad\qquad + f_x(s) + \mathbb{E}\left[\left(b_{\widehat{x}}(s)\right) p^{t,\xi}(s)\right] + \mathbb{E}\left[\sigma_{\widehat{x}}(s) q^{t,\xi}(s) + \widehat{\sigma}_{\widehat{x}}(s) \widehat{q}^{t,\xi}(s) + f_{\widehat{x}}(s)\right] \Big\} ds \\ \qquad\qquad - q^{t,\xi}(s) dW(s) - \widehat{q}^{t,\xi}(s) d\widehat{W}(s), \\ p^{t,x}(T) = \phi_x(T) + \mathbb{E}\left[\phi_{\widehat{x}}(T)\right]. \end{cases} \tag{4.18}$$

Under Assumption (H_1), equation (4.18) admits a unique $\mathbb{F}$-adapted solution $\left(p^{t,\xi}(s), q^{t,\xi}(s), \widehat{q}^{t,\xi}(s)\right)$. Moreover, there exists a constant $C > 0$ such that, for all $t \in [0,T]$ and $\left(\xi, \xi^{'}\right) \times L^2\left(\Omega, \mathcal{F}_t, \mathbb{P}^v, \mathbb{R}\right)$, we have the following estimate, $\mathbb{P} - a.s.$,

$$\mathbb{E}\left[\sup_{s\in[t,T]} \left|p^{t,\xi}(s)\right|^2 + \int_t^T \left(\left|q^{t,\xi}(s)\right|^2 + \left|\widehat{q}^{t,\xi}(s)\right|^2\right) ds \mid \mathcal{F}_t\right] \leq C\left(1 + |\xi|^2 + \mathbb{E}\left[\xi^2\right]\right). \tag{4.19}$$

The second order adjoint equation is the following linear backward SDE parameterized by $(t,\xi) \in [0,T] \times L^2\left(\Omega, \mathcal{F}_t, \mathbb{P}^v, \mathbb{R}\right)$

$$\begin{cases} -dP^{t,\xi}(s) = \Big\{ 2\left(b_x(s) - \widehat{\sigma}(s) h_x(s)\right) P^{t,\xi}(s) + \left(\sigma_x(s)\right)^2 P^{t,\xi}(s), \\ \qquad\qquad + \widehat{\sigma}_x(s)^2 P^{t,\xi}(s) + 2\sigma_x(s) Q^{t,\xi}(s) + 2\widehat{\sigma}_x(s) \widehat{Q}^{t,\xi}(s) + H_{xx}^{t,\xi}(s) \Big\} ds \\ \qquad\qquad - Q^{t,\xi}(s) dW(s) - \widehat{Q}^{t,\xi}(s) d\widehat{W}(s) \\ P^{t,\xi}(T) = \phi_{xx}(T), \end{cases} \tag{4.20}$$

where

$$H_{xx}^{t,\xi}(s) = b_{xx}(s) p^{t,\xi}(s) + \sigma_{xx}(s) q^{t,\xi}(s) + \widehat{\sigma}_{xx}(s) \widehat{q}^{t,\xi}(s) + h_{xx}(s) \widehat{\Psi}^{t,\xi}(s) + f_{xx}(s).$$

Under (H_1), it is easy to check that (4.20) admit unique $\mathbb{F}$-adapted solution $\left(P^{t,\xi}(s), Q^{t,\xi}(s), \widehat{Q}^{t,\xi}(s)\right)$ satisfies the following estimate: There exists a constan $C > 0$ such that, for all $t \in [0,T]$ and

$\left(\xi,\xi'\right) \times L^2\left(\Omega,\mathcal{F}_t,\mathbb{P}^v,\mathbb{R}\right)$,

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|P^{t,\xi}(s)\right|^2+\int_t^T\left(\left|Q^{t,\xi}(s)\right|^2+\left|\widehat{Q}^{t,\xi}(s)\right|^2\right)ds\mid\mathcal{F}_t\right]\leq C\left(1+|\xi|^2+\mathbb{E}\left[\xi^2\right]\right).\quad \mathbb{P}-a.s., \tag{4.21}$$

Moreover, on the premise that the system (4.1) is with full information, these adjoint equations are different from those classical ones in in Buckdahn et al. [17] due to the appearance of $\widehat{W}(.)$.

Theorem 4.2.1 (Characterization of equilibrium strategies) *Let assumptions (H_1) hold. Then $u(.)$ is an equilibrium strategy for the system (4.10)-(4.11), if and only tf there are pairs of $\mathbb{F}$-adapted solution $(p,q,\widehat{q})$ and $\left(P,Q,\widehat{Q}\right)$ which satisfy (4.18)-(4.19) and (4.18)-(4.19), respectively, and for which*

$$\begin{aligned}&\mathbb{E}^u\Big[\Big\{H\left(t,\xi,v,p^{t,\xi},q^{t,\xi},\widehat{q}^{t,\xi},\widehat{\Psi}^{t,\xi}\right)-H\left(t,\xi,u,p^{t,\xi},q^{t,\xi},\widehat{q}^{t,\xi},\widehat{\Psi}^{t,\xi}\right)\\&+\frac{1}{2}P^{t,\xi}(t)\left[\sigma\left(t,\xi,\mathbb{E}[\xi],v\right)-\sigma\left(t,\xi,\mathbb{E}[\xi],u(t)\right)\right]^2\\&+\frac{1}{2}P^{t,\xi}(t)\left[\widehat{\sigma}\left(t,\xi,\mathbb{E}[\xi],v\right)-\widehat{\sigma}\left(t,\xi,\mathbb{E}[\xi],u(t)\right)\right]^2\Big\}\mid\mathcal{F}_t^Y\Big]\geq 0,\end{aligned} \tag{4.22}$$

for all $v\in\mathcal{U}_{ad}$, $\xi\in L^2\left(\Omega,\mathcal{F}_t,\mathbb{P}^v,\mathbb{R}\right)$ a.e. $t\in[0,T]$, $\mathbb{P}^v-a.s,$.

Proof. We put $\Gamma_1^{t,\xi}(.)=Z_1^{t,\xi}(.)\left(Z^{t,\xi}(.)\right)^{-1}$ and $\Gamma_2^{t,\xi}(.)=Z_2^{t,\xi}(.)\left(Z^{t,\xi}(.)\right)^{-1}$.

Then, by Itô's formula we obtain

$$\begin{cases}d\Gamma_1^{t,\xi}(s)=h_x(s)\,S_1^{\varepsilon}(s)d\widehat{W}(s),\\ \Gamma_1^{t,\xi}(t)=\xi,\end{cases}$$

and

$$\begin{cases}d\Gamma_2^{t,\xi}(s)=\left\{h_x(s)\,S_2^{\varepsilon}(s)+\Gamma_1^{t,\xi}(s)h_x(s)\,S_1^{\varepsilon}(s)+\frac{1}{2}h_{xx}(s)\left(S_1^{\varepsilon}(s)\right)^2\right\}d\widehat{W}(s),\\ \Gamma_2^{t,\xi}(t)=\xi.\end{cases}$$

By (4.11), Taylor's expansion, and lemma 4.2.2, we have

$$
\begin{aligned}
&0 \leq J\left(t, \xi, u^{\varepsilon}\right)-J(t, \xi, u) \\
&=\mathbb{E}\left[\int_t^T\left\{Z^{t, \xi}(s)\left[f_x(s)\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right)+f_{\widehat{x}}(s) \mathbb{E}\left[\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right)\right]\right.\right.\right. \\
&\left.\left.\left.+\ \frac{1}{2} f_{x x}(s)\left(S_1^{\varepsilon, t, \xi}(s)\right)^2+\delta f(s) \mathbf{I}_{[t, t+\varepsilon]}(s)\right]+\left(Z_1^{t, \xi}(s)+Z_2^{t, \xi}(s)\right) l(s)\right\} d s\right] \\
&+\mathbb{E}\left[Z^{t, \xi}(T)\left[\phi_x(T)\left(S_1^{\varepsilon, t, \xi}(T)+S_2^{\varepsilon, t, \xi}(T)\right)+\phi_{\widehat{x}}(T)\left(S_1^{\varepsilon, t, \xi}(T)+S_2^{\varepsilon, t, \xi}(T)\right)\right]\right. \\
&\left.+\ \frac{1}{2} \phi_{x x}(T)\left(S_1^{\varepsilon, t, \xi}(T)\right)^2\right]+\left(Z_1^{t, \xi}(T)+Z_2^{t, \xi}(T)\right) \phi(T)\Big]+o(\varepsilon) \\
&=\mathbb{E}^u\left[\int_t^T\left\{f_x(s)\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right)+f_{\widehat{x}}(s) \mathbb{E}\left[\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right)\right]\right.\right. \\
&\left.\left.+\ \frac{1}{2} f_{x x}(s)\left(S_1^{\varepsilon, t, \xi}(s)\right)^2+\delta f(s) \mathbf{I}_{[t, t+\varepsilon]}(s)+\left(\Gamma_1^{t, \xi}(s)+\Gamma_2^{t, \xi}(s)\right) f(s)\right\} d s\right] \\
&+\mathbb{E}^u\left[\phi_x(T)\left(S_1^{\varepsilon, t, \xi}(T)+S_2^{\varepsilon, t, \xi}(T)\right)+\phi_{\widehat{x}}(T) \mathbb{E}\left[\left(S_1^{\varepsilon, t, \xi}(T)+S_2^{\varepsilon, t, \xi}(T)\right)\right]\right. \\
&\left.+\ \frac{1}{2} \phi_{x x}(T)\left(S_1^{\varepsilon, t, \xi}(T)\right)^2\right]+\left(\Gamma_1^{t, \xi}(T)+\Gamma_2^{t, \xi}(T)\right) \phi(T)\Big]+o(\varepsilon) .
\end{aligned} \tag{4.23}
$$

Applying Itô's formula to $p^{t, \xi}(.)\left(S_1^{\varepsilon, t, \xi}(.)+S_2^{\varepsilon, t, \xi}(.)\right)$,

$$
\begin{aligned}
&p^{t, \xi}(T)\left(S_1^{\varepsilon, t, \xi}(T)+S_2^{\varepsilon, t, \xi}(T)\right) \\
&=\int_t^T p^{t, \xi}(s) d\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right)+\int_t^T\left(S_1^{\varepsilon, t, \xi}(s)+S_2^{\varepsilon, t, \xi}(s)\right) d p^{t, \xi}(s) \\
&+\int_t^T q^{t, \xi}(s)\left(\sigma_x(s) S_1^{\varepsilon, t, \xi}(s)+\sigma_{\widehat{x}}(s) \mathbb{E}\left[S_1^{\varepsilon, t, \xi}(s)\right]+\delta \sigma(s) \mathbf{I}_{[t, t+\varepsilon]}(s)\right) d s \\
&+\int_t^T q^{t, \xi}(s)\left(\sigma_x(s) S_2^{\varepsilon, t, \xi}(s)+\sigma_{\widehat{x}}(s) \mathbb{E}\left[S_2^{\varepsilon, t, \xi}(s)\right]\right) d s \\
&+\int_t^T q^{t, \xi}(s)\left(\frac{1}{2} \sigma_{x x}(s)\left(S_1^{\varepsilon, t, \xi}(s)\right)^2+\delta \sigma_x(s) S_1^{\varepsilon, t, \xi}(s) \mathbf{I}_{[t, t+\varepsilon]}(s)\right) d s \\
&+\int_t^T \widehat{q}^{t, \xi}(s)\left(\widehat{\sigma}_x(s) S_1^{\varepsilon, t, \xi}(s)+\widehat{\sigma}_{\widehat{x}}(s) \mathbb{E}\left[S_1^{\varepsilon, t, \xi}(s)\right]+\delta \widehat{\sigma}(s) \mathbf{I}_{[t, t+\varepsilon]}(s)\right) d s \\
&+\int_t^T \widehat{q}^{t, \xi}(s)\left(\widehat{\sigma}_x(s) S_2^{\varepsilon, t, \xi}(s)+\widehat{\sigma}_{\widehat{x}}(s) \mathbb{E}\left[S_2^{\varepsilon, t, \xi}(s)\right]\right) d s \\
&+\int_t^T \widehat{q}^{t, \xi}(s)\left(\frac{1}{2} \widehat{\sigma}_{x x}(s)\left(S_1^{\varepsilon, t, \xi}(s)\right)^2+\delta \widehat{\sigma}_x(s) S_1^{\varepsilon, t, \xi}(s) \mathbf{I}_{[t, t+\varepsilon]}(s)\right) d s+M(s),
\end{aligned} \tag{4.24}
$$

where $M(s)$ is a zero-mean martingale. By taking expectations we are left with

$$
\begin{aligned}
&\mathbb{E}^u\left[p^{t,\xi}(T)\left(S_1^{\varepsilon,t,\xi}(T)+S_2^{\varepsilon,t,\xi}(T)\right)\right] \\
&=\mathbb{E}^u\left[\int_t^T\left\{-\left(f_x(s)+\mathbb{E}\left[f_{\widehat{x}}(s)\right]\right)\left(S_1^{\varepsilon,t,\xi}(s)+S_2^{\varepsilon,t,\xi}(s)\right)+r_1(s)\right\}ds\right] \\
&\quad+\mathbb{E}^u\left[\int_t^T\left\{\left(p^{t,\xi}(s)\delta b(s)+q^{t,\xi}(s)\delta\sigma(s)+\widehat{q}^{t,\xi}(s)\delta\widehat{\sigma}(s)\right)\mathbf{I}_{[t,t+\varepsilon]}(s)\right\}ds\right] \\
&\quad+\mathbb{E}^u\left[\int_t^T\left\{\frac{1}{2}\left(b_{xx}(s)\,p^{t,\xi}(s)+\sigma_{xx}(s)\,q^{t,\xi}(s)+\widehat{\sigma}_{xx}(s)\,\widehat{q}^{t,\xi}(s)\right)\left(S_1^{\varepsilon,t,\xi}(s)\right)^2\right\}ds\right],
\end{aligned}
\tag{4.25}
$$

where

$$
\begin{aligned}
r_1(s)&=p^{t,\xi}(s)\left[-\widehat{\sigma}_x(s)\,h_x(s)-\frac{1}{2}\widehat{\sigma}(s)\,h_{xx}(s)\left(S_1^{\varepsilon,t,\xi}(s)\right)^2\right] \\
&\quad+p^{t,\xi}(s)\left[\delta b_x(s)\,S_1^{\varepsilon,t,\xi}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)-\delta\widehat{\sigma}(s)\,h_x(s)\,S_1^{\varepsilon,t,\xi}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\right] \\
&\quad+\left[q^{t,\xi}(s)\delta\sigma_x(s)+\widehat{q}^{t,\xi}(s)\delta\widehat{\sigma}_x(s)\right]S_1^{\varepsilon,t,\xi}(s)\mathbf{I}_{[t,t+\varepsilon]}(s).
\end{aligned}
$$

Then, applying Itô's formula to $\Phi^{t,\xi}(.)\left(\Gamma_1^{t,\xi}(.)+\Gamma_2^{t,\xi}(.)\right)$ and taking conditional expectations, we have

$$
\begin{aligned}
&\mathbb{E}^u\left[\left(\Gamma_1^{t,\xi}(T)+\Gamma_2^{t,\xi}(T)\right)\phi(T))\right] \\
&=\mathbb{E}^u\left[\int_t^T\left\{-\left(\Gamma_1^{t,\xi}(s)+\Gamma_2^{t,\xi}(s)\right)f(s)+\widehat{\Psi}^{t,\xi}(s)h_x(s)\left(S_1^{\varepsilon}(s)+S_2^{\varepsilon}(s)\right)ds\right.\right. \\
&\qquad\left.\left.+\ \widehat{\Psi}^{t,\xi}(s)\left(\frac{1}{2}h_{xx}(s)\left(S_1^{\varepsilon}(s)\right)^2\right)+r_2(s)\right\}ds\right],
\end{aligned}
\tag{4.26}
$$

where

$$r_2(s)=h_x(s)\,\widehat{\Psi}^{t,\xi}(s)\Gamma_1(s)\,S_1^{\varepsilon}(s).$$

According to lemma 4.2.1 and 4.2.2, we get

$$\mathbb{E}^u\left[\int_t^T\left|r_1(s)+r_2(s)\right|ds\right]=o(\varepsilon^2).$$

Substituting (4.25) and (4.26) into (4.23), we obtain

$$J(t,\xi,u^{\varepsilon}) - J(t,\xi,u) = \mathbb{E}^{u}\left[\int_t^T \left\{\frac{1}{2}H_{xx}^{t,\xi}(s)\left(S_1^{\varepsilon}(s)\right)^2 + \delta H^{t,\xi}(s)\,\mathbf{I}_{[t,t+\varepsilon]}(s)\right\}ds + \frac{1}{2}\phi_{xx}^{t,\xi}(T)\left(S_1^{\varepsilon}(T)\right)^2\right] + o(\varepsilon). \quad (4.27)$$

Here

$$\delta H^{t,\xi}(s) = p^{t,\xi}(s)\delta b(s) + q^{t,\xi}(s)\delta\sigma(s) + \widehat{q}^{t,\xi}(s)\delta\widehat{\sigma}(s) + \delta f(s).$$

On the other hand note that $\mathcal{K}^{t,\xi}(.) = S_1^{\varepsilon}(.)^2$ satisfies

$$\begin{cases} d\mathcal{K}^{t,\xi}(s) = \Big\{\Big(2b_x(s) - 2\widehat{\sigma}(s)h_x(s) + (\sigma_x(s))^2 + (\widehat{\sigma}_x(s))^2\Big)\mathcal{K}^{t,\xi}(s) + \Lambda_1^{t,\xi}(s) \\ \qquad + \Big((\delta\sigma(s))^2 + (\delta\widehat{\sigma}(s))^2\Big)\mathbf{I}_{[t,t+\varepsilon]}(s) + \Lambda_2^{t,\xi}(s)\Big\}ds \\ \qquad + \left\{2\sigma_x(s)\mathcal{K}^{t,\xi}(s) + 2S_1^{\varepsilon}(s)\left(\sigma_{\widehat{x}}(s)\mathbb{E}\left[S_1^{\varepsilon}(s)\right] + \delta\sigma(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\right)\right\}dW(s), \\ \qquad + \left\{2\widehat{\sigma}_x(s)\mathcal{K}^{t,\xi}(s) + 2S_1^{\varepsilon}(s)\left(\widehat{\sigma}_{\widehat{x}}(s)\mathbb{E}\left[S_1^{\varepsilon}(s)\right] + \delta\widehat{\sigma}(s)\mathbf{I}_{[t,t+\varepsilon]}(s)\right)\right\}d\widehat{W}(s) \\ \mathcal{K}^{t,\xi}(t) = \xi, \end{cases}$$

with

$$\Lambda_1^{t,\xi}(s) = \left(2b_{\widehat{x}}(s) + 2\sigma_x(s)\sigma_{\widehat{x}}(s) + 2\widehat{\sigma}_x(s)\widehat{\sigma}_{\widehat{x}}(s)\right)S_1^{\varepsilon}(s)\mathbb{E}\left[S_1^{\varepsilon}(s)\right] + \Big((\sigma_{\widehat{x}}(s))^2 + (\widehat{\sigma}_{\widehat{x}}(s))^2\Big)\left(\mathbb{E}\left[S_1^{\varepsilon}(s)\right]\right)^2,$$

and

$$\Lambda_2^{t,\xi}(s) = \left(2\delta b(s) + 2\sigma_x(s)\delta\sigma(s) + 2\widehat{\sigma}_x(s)\widehat{\sigma}(s)\right)S_1^{\varepsilon}(s)\mathbf{I}_{[t,t+\varepsilon]}(s) + \left(2\sigma_{\widehat{x}}(s)\delta\sigma(s) + 2\widehat{\sigma}_{\widehat{x}}(s)\delta\widehat{\sigma}(s)\right)\mathbb{E}\left[S_1^{\varepsilon}(s)\right]\mathbf{I}_{[t,t+\varepsilon]}(s).$$

Noting that

$$\mathbb{E}^{u}\left[\int_t^T \left|P^{t,\xi}(s)\left\{\Lambda_1^{t,\xi}(s) + \Lambda_2^{t,\xi}(s)\right\}\right| ds\right] = o(\varepsilon)$$

Then, by applying Itô's formula to $P^{t,\xi}(.)\,\mathcal{K}^{t,\xi}(.)$ and taking expectation, we get from the estimates of lemma 4.2.1, we deduce

$$\begin{aligned}&\mathbb{E}^u\left[\phi_{xx}(T)\left(S_1^{\varepsilon}(T)\right)^2\right]\\&=\mathbb{E}^u\left[\int_t^T\left\{-H_{xx}^{t,\xi}(s)\left(S_1^{\varepsilon}(s)\right)^2+P^{t,\xi}(s)\left((\delta\sigma(s))^2+(\delta\widehat{\sigma}(s))^2\right)\mathbf{I}_{[t,t+\varepsilon]}(s)\right\}ds\right]+o(\varepsilon).\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28) yields

$$\begin{aligned}&J(t,\xi,u^{\varepsilon})-J(t,\xi,u)\\&=\mathbb{E}^u\left[\int_t^T\left\{\delta H^{t,\xi}(s)+\frac{1}{2}P^{t,\xi}(s)\left((\delta\sigma(s))^2+(\delta\widehat{\sigma}(s))^2\right)\right\}\mathbf{I}_{[t,t+\varepsilon]}(s)\,ds\right]+o(\varepsilon),\end{aligned} \tag{4.29}$$

Dividing both sides of (4.29) by ε and then passing to the limit $\varepsilon \downarrow 0$, in view of Assumption (H_1), (4.19) and (4.21), we obtain

$$0\leq\mathbb{E}^u\left[\delta H^{t,\xi}(s)+\frac{1}{2}P^{t,\xi}(s)\left((\delta\sigma(s))^2+(\delta\widehat{\sigma}(s))^2\right)\right].$$

■

4.3 An application to linear-quadratic control problem

It is well known that LQ control is one of the most important classes of optimal control, due to its wide theoretical and practical viewpoints, and the related Riccati equations have played an important role within the framework of the investigation concerning this problem. To this end, we can note the following advantages. First, solutions of LQ problems exhibit elegant properties due to their simple and nice structures. For example, many nonlinear problems can be approximated by LQ problems. Second, several problems in finance and economic can be modeled by LQ control problems, for example the mean-variance portfolio selection problem [64], and the stochastic differential recursive utility with linear generator [61]. Backward and forward–backward LQ stochastic control problems can be seen in [11], [16], [20], [51].

In this section, we will characterize the equilibrium control in general LQ control problem, and identify it in special case including that, the mean-variance portfolio selection mixed with a recursive utility functional optimization problem.

We consider the following controlled SDE

$$\begin{cases} dX^{t,\xi,v}(s) = \left(\alpha(s) X^{t,\xi,v}(s) + \widetilde{\alpha}(s)\mathbb{E}\left[X^{t,\xi,v}(s)\right] + \beta(s) v(s) + \widetilde{\beta}(s)\right) ds \\ \qquad\qquad + \sigma(s) dW(s), \; t < s \leq T, \\ X^{t,\xi,v}(t) = \xi. \end{cases} \tag{4.30}$$

We assume that the cost functional to be minimized, takes the form

$$J(t,\xi,u(\cdot)) = \mathbb{E}\left[\int_t^T \left(A(s) X^{t,\xi,v}(s)^2 + \widetilde{A}(s)\mathbb{E}\left[X^{t,\xi,v}(s)\right]^2 + B(s) v(s)^2\right) ds \right. \\ \left. + DX^{t,\xi}(T)^2 + \widetilde{D}\mathbb{E}\left[X^{t,\xi,v}(T)\right]^2\right], \tag{4.31}$$

subject to the state equation (4.30) and the observation equation given by

$$\begin{cases} dY(s) = \left(h_1(s) X^{t,\xi,v}(s) + h_2(s)\right) ds + d\widehat{W}(s), \; 0 < s \leq T, \\ Y(0) = 0. \end{cases} \tag{4.32}$$

We denote by $\mathcal{U}$ the set of admissible controls u valued in $\mathbb{R}$. Throughout this section we assume all parameters in the equations (4.30), (4.32) and the cost functional are bounded and deterministic functions, such that $\sigma(\cdot)$ is different to zero.

In this case the Hamiltonian reduces to

$$\begin{aligned}
&H\left(s, X, \mathbb{E}[X], v, p, q, \widehat{q}, \widehat{\Psi}\right) \\
&= \left(\alpha(s) X + \widetilde{\alpha}(s) \mathbb{E}[X] + \beta(s) v + \widetilde{\beta}(s)\right) p + \sigma(s) q \\
&\quad + \left(h_1(s) X(s) + h_2(s)\right) \widehat{\Psi} + \frac{1}{2}\left(A(s) X^2 + \widetilde{A}(s) \mathbb{E}[X]^2 + B(s) v^2\right).
\end{aligned}$$

Let $u(\cdot)$ be an equilibrium control, we denote the corresponding trajectory by $X^{t,\xi,u}(\cdot) = X^{t,\xi}(\cdot)$. We introduce the adjoint equations involved in the stochastic maximum principle which characterize the open-loop Nash equilibrium controls of the problem (4.31). In this case the first order adjoint process $p^{t,\xi}(\cdot)$ is given by

$$\begin{cases}
dp^{t,\xi}(s) = -\left\{\alpha(s)p^{t,\xi}(s) + \widetilde{\alpha}(s) \mathbb{E}\left[p^{t,\xi}(s)\right] + h(s) \widehat{\Psi}(s) \right. \\
\qquad\qquad \left. + \dfrac{1}{2}\left(A(s) X^{t,\xi}(s) + \widetilde{A}(s) \mathbb{E}\left[X^{t,\xi}(s)\right]\right)\right\} ds \\
\qquad\qquad + q^{t,\xi}(s) dW(s) + \widehat{q}^{t,\xi}(s) d\widehat{W}(s), \\
p^{t,\xi}(T) = D X^{t,\xi}(T) + \widetilde{D} \mathbb{E}\left[X^{t,\xi}(T)\right],
\end{cases} \tag{4.33}$$

and the second order adjoint process is

$$\begin{cases}
dP^{t,\xi}(s) = -\left\{2\alpha(s) P^{t,\xi}(s) + A(s)\right\} ds + Q^{t,\xi}(s) dB(s) + \widehat{Q}^{t,\xi}(s) d\widehat{B}(s), \\
P^{t,\xi}(T) = D.
\end{cases} \tag{4.34}$$

$$\begin{cases}
dP^{t,\xi}(s) = -\left\{2\alpha(s) P^{t,\xi}(s) + A(s)\right\} ds, \\
P^{t,\xi}(T) = D.
\end{cases}$$

If $u(\cdot)$ is an equilibrium control, it follows from the theorem (4.2.1) that

$$u(t) = -B(t)^{-1} \beta(t) \mathbb{E}\left[p^{t,\xi}(t) \mid \mathcal{F}^Y(t)\right]. \tag{4.35}$$

As in the classical LQ control problem, we attempt to look for a linear open-loop equilibrium control, then we need first to give an explicit representation of the process $\mathbb{E}\left[p^{t,\xi}(\cdot) \mid \mathcal{F}^Y(t)\right]$, noting that

$$\begin{cases} d\mathbb{E}\left[X^{t,\xi}(s)\right] = \Big((\alpha(s) + \widetilde{\alpha}(s)) \mathbb{E}\left[X^{t,\xi}(s)\right] - \beta^2(s) B(s)^{-1} \mathbb{E}\left[p^{s,\xi}(s)\right] + \widetilde{\beta}(s) \Big) ds, \\ d\mathbb{E}\left[p^{t,\xi}(s)\right] = -\Big(\Big(A(s) + \widetilde{A}(s) \Big) \mathbb{E}\left[X^{t,\xi}(s)\right] + (\alpha(s) + \widetilde{\alpha}(s)) \mathbb{E}\left[p^{t,\xi}(s)\right] \Big) ds, \\ \mathbb{E}\left[X^{t,\xi}(t)\right] = \mathbb{E}[\xi] = \mu, \ \mathbb{E}\left[p^{t,\xi}(T)\right] = \Big(D + \widetilde{D} \Big) \mathbb{E}\left[X^{t,\xi}(T)\right]. \end{cases} \tag{4.36}$$

Due to the terminal condition in (4.36), we try a solution for the second equation in (4.36) of the form

$$\mathbb{E}\left[p^{t,\xi}(s)\right] = \phi(s) \mathbb{E}\left[X^{t,\xi}(s)\right] + \psi(s), \tag{4.37}$$

for the deterministic and differentiable functions $\phi(\cdot)$ and $\psi(\cdot)$ such that $\phi(T) = D + \widetilde{D}$ and $\psi(T) = 0$. Applying the chain rule for (4.37), we get

$$\begin{aligned} d\mathbb{E}\left[p^{t,\xi}(s)\right] &= \phi(s)\, d\mathbb{E}\left[X^{t,\xi}(s)\right] + \mathbb{E}\left[X^{t,\xi}(s)\right] d\phi(s) + \dot{\psi}(s)\, ds, \\ &= \dot{\phi}(s) \mathbb{E}\left[X^{t,\xi}(s)\right] + (\alpha(s) + \widetilde{\alpha}(s)) \mathbb{E}\left[X^{t,\xi}(s)\right] \phi(s) \\ &\quad + \Big(\widetilde{\beta}(s) - \beta^2(s) B(s)^{-1} \mathbb{E}\left[p^{t,\xi}(s)\right] \Big) \phi(s) + \dot{\psi}(s), \end{aligned}$$

form the representation (4.37) we obtain by simple computations

$$\begin{aligned} d\mathbb{E}\left[p^{t,\xi}(s)\right] = &\Big(\dot{\phi}(s) + (\alpha(s) + \widetilde{\alpha}(s)) \phi(s) \Big) \mathbb{E}\left[X^{t,\xi}(s)\right] - \beta^2(s) B(s)^{-1} \phi(s)^2 \mu \\ &+ \widetilde{\beta}(s) \phi(s) + \dot{\psi}(s) - \beta^2(s) B(s)^{-1} \phi(s) \psi(s). \end{aligned} \tag{4.38}$$

By comparing the coefficients with the second equation in (4.36) we find that $\phi(\cdot)$ and $\psi(\cdot)$ should solve the following system of ODEs

$$\begin{cases} \dot{\phi}(s) = -2(\alpha(s) + \widetilde{\alpha}(s)) \phi(s) - \Big(A(s) + \widetilde{A}(s) \Big), \\ \dot{\psi}(s) = -\Big((\alpha(s) + \widetilde{\alpha}(s)) - \beta^2(s) B(s)^{-1} \phi(s) \Big) \psi(s) \\ \qquad\qquad + \beta^2(s) B(s)^{-1} \phi(s)^2 \mu - \widetilde{\beta}(s) \phi(s), \\ \phi(T) = D + \widetilde{D}, \ \psi(T) = 0. \end{cases}$$

If we replace by the values of $\mathbb{E}\left[p^{t,\xi}(t)\right]$ into the first equation of (4.36) we obtain $\forall s \in [t,T]$

$$\begin{cases} d\mathbb{E}\left[X^{t,\xi}(s)\right] = \Big(\left(\alpha(s) + \widetilde{\alpha}(s)\right)\mathbb{E}\left[X^{t,\xi}(s)\right] - \beta^2(s)\,B(s)^{-1}\left(\phi(s)\,\mu + \psi(s)\right) + \widetilde{\beta}(s)\Big)\,ds, \\ \mathbb{E}\left[X^{t,\xi}(t)\right] = \mu, \end{cases} \tag{4.39}$$

which can explicity computed. It follows from Liptser and Shiryayev [46] (see also Xiong [63]) that the filtering process

$$\left(\bar{X}^{t,\xi}(\cdot), \bar{p}^{t,\xi}(\cdot)\right) = \left(\mathbb{E}\left[X^{t,\xi}(\cdot) \mid \mathcal{F}^Y(t)\right], \mathbb{E}\left[p^{t,\xi}(\cdot) \mid \mathcal{F}^Y(t)\right]\right),\ \forall t \in [0,T],$$

with respect to the observations $Y(\cdot)$ up to time t, is the solution of the following FBSDE system $\forall s \in [t,T]$

$$\begin{cases} d\bar{X}^{t,\xi}(s) = \Big(\alpha(s)\,\bar{X}^{t,\xi}(s) - B(s)^{-1}\,\beta(s)^2\,\bar{p}^{s,\xi}(s) + \Gamma_1(s)\Big)\,ds + \Theta(s)\,h_1(s)\,d\overline{W}(s), \\ d\bar{p}^{t,\xi}(s) = -\left(A(s)\,\bar{X}^{t,\xi}(s) + \alpha(s)\,\bar{p}^{t,\xi}(s) + \Gamma_2(s)\right)ds + \overline{Q}^{t,\xi}d\overline{W}(s), \\ \bar{X}^{t,\xi}(t) = \mu,\ \bar{p}^{t,\xi}(T) = D\bar{X}^{t,\xi}(T) + \widetilde{D}\mathbb{E}\left[X^{t,\xi}(T)\right], \end{cases} \tag{4.40}$$

where $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ are given by

$$\Gamma_1(s) = \overline{\alpha}(s)\,\mathbb{E}\left[X^{t,\xi}(s)\right] + \overline{\beta}(s),$$

and

$$\Gamma_2(s) = \overline{A}(s)\,\mathbb{E}\left[X^{t,\xi}(T)\right] + \overline{\alpha}(s)\,\mathbb{E}\left[p^{t,\xi}(s)\right],$$

the function $\Theta(\cdot)$ is given by

$$\begin{cases} \dot{\Theta}(s) = 2\alpha(s)\,\Theta(s) - \Theta(s)^2\,h_1(s)^2 + \sigma(s)^2 = 0, \\ \Theta(0) = \sigma(0), \end{cases} \tag{4.41}$$

and $\overline{W}(\cdot)$ is a standard Brownian motion with value in $\mathbb{R}$ given by

$$d\overline{W}(s) = dY^0(s) - h_1(s)\,X^{t,\xi,0}(s)\,ds, \tag{4.42}$$

where $X^{t,\xi,0}(\cdot)$ is the solution of the following SDE

$$\begin{cases} dX^{t,\xi,0}(s) = \Big(\alpha(s) X^{t,\xi,0}(s) + \widetilde{\alpha}(s) \mathbb{E}\left[X^{t,\xi,0}(s)\right] + \widetilde{\beta}(s)\Big) ds + \sigma(s) dW(s), \ \forall s \in [t,T], \\ X^{t,\xi,0}(t) = \xi, \end{cases} \tag{4.43}$$

and $Y^0(\cdot)$ is the solution of the SDE

$$\begin{cases} dY(s) = \big(h_1(s) X^{t,\xi,0}(s) + h_2(s)\big) ds + d\widehat{W}(s), \ \forall s \in [0,T], \\ Y(0) = 0. \end{cases} \tag{4.44}$$

Similarly, to characterize an explicite solution for (4.40) we let

$$\bar{p}^{t,\xi}(s) = \Phi(s) \bar{X}^{t,\xi}(s) + \Psi(s), \ \forall s \in [t,T], \tag{4.45}$$

for two deterministic and differentiable functions $\Phi(\cdot)$ and $\Psi(\cdot)$ such that $\Phi(T) = D$ and $\Psi(T) = \widetilde{D}\mathbb{E}\left[X^{t,\xi}(T)\right]$. It follows from Itô's formula that

$$\begin{aligned} d\bar{p}^{t,\xi}(s) &= \Phi(s) d\bar{X}^{t,\xi}(s) + \bar{X}^{t,\xi}(s) \dot{\Phi}(s) ds + \dot{\Psi}(s) ds, \\ &= \Phi(s) \Big(\alpha(s) \bar{X}^{t,\xi}(s) - B(s)^{-1} \beta(s)^2 \overline{p}^{s,\xi}(s) + \Gamma_1(s)\Big) ds \\ &\qquad + \bar{X}^{t,\xi}(s) \dot{\Phi}(s) ds + \dot{\Psi}(s) ds + \Phi(s) \Theta(s) h_1(s) d\overline{W}(s). \end{aligned}$$

A simple computation show that

$$\begin{aligned} d\bar{p}^{t,\xi}(s) = \Big(&\Big(\dot{\Phi}(s) + \Phi(s)\alpha(s)\Big) \bar{X}^{t,\xi}(s) - B(s)^{-1} \beta(s)^2 \Phi(s)^2 \mu \\ &- \Phi(s) B(s)^{-1} \beta(s)^2 \Psi(s) + \Phi(s)\Gamma(s) + \dot{\Psi}(s)\Big) ds \\ &+ \Phi(s) \Theta(s) h_1(s) d\overline{W}(s). \end{aligned}$$

By comparing with the BSDE in (4.40) we get

$$\begin{cases} \dot{\Phi}(s) = -2\alpha(s)\Phi(s) - A(s), \\ \dot{\Psi}(s) = -\left(\alpha(s) - \Phi(s)B(s)^{-1}\beta(s)^2\right)\Psi(s) \\ \qquad\qquad + B(s)^{-1}\beta(s)^2\Phi(s)^2\mu - \Phi(s)\Gamma_1(s) - \Gamma_2(s), \\ \Phi(T) = D, \ \Psi(T) = \widetilde{D}\mathbb{E}\left[X^{t,\xi}(T)\right]. \end{cases}$$

Substituting (4.45) in (4.35) we get $\forall t \in [0, T]$

$$\begin{aligned} u(t) &= -B(t)^{-1}\beta(t)\left(\Phi(t)\bar{X}^{t,\xi}(t) + \Psi(t)\right), \\ &= -B(t)^{-1}\beta(t)\left(\Phi(t)\mu + \Psi(t)\right). \end{aligned}$$

The corresponding equilibrium dynamics solves the following SDE

$$\begin{cases} dX^{0,x}(t) = \left(\alpha(t)X^{0,x}(t) + \widetilde{\alpha}(t)\mathbb{E}\left[X^{0,x}(t)\right] - B(t)^{-1}\beta(t)^2\left(\Phi(t)\mu + \Psi(t)\right) + \widetilde{\beta}(t)\right)dt, \\ \qquad\qquad + \sigma(t)dW(t), \ 0 \leq t \leq T, \\ X^{0,x}(0) = x. \end{cases} \tag{4.46}$$

4.4 Extension to Mean-Field Game Models

In this section, we extend the SMP approach to an N-player stochastic differential game of mean-field type where the ith player would like to find a strategy to optimize her own cost functional regardless of the other players' cost functionals.
Let $X = (X_1, ..., X_N)$ describe the states of the N players and $v = (v_1, ..., v_N) \in \Pi_{i=1}^{N} \mathcal{U}_i[0, T]$ be the ensemble of all the individual admissible strategies. Each v_i takes values in a non-empty subset U_i of $\mathbb{R}$, and the class of admissible strategies is given by

$$\mathcal{U}_i[0, T] = \{v_i : [0, T] \times \Omega \to U_i;\ v_i \text{ is } \mathbb{F}\text{-adapted and square integrable}\}. \tag{4.47}$$

To simplify the analysis, we consider a population of uniform agents so that $U_i = U$ and they have the same initial state $X_i(0) = x_0$ at time 0 for all $i \in \{1, ..., N\}$. In this case, the N sets $\mathcal{U}_i[0, T]$ are identical and equal to $\mathcal{U}_{ad}$. Let the dynamics be given by the following SDE

$$\begin{aligned} dX_i(s) &= b(s, X_i(s), \mathbb{E}[X_i(s)], v_i(s))\, ds + \sigma(s, X_i(s), \mathbb{E}[X_i(s)])\, dW_i(s) \\ &\quad + \widehat{\sigma}(s, X_i(s), \mathbb{E}[X_i(s)])\, d\widehat{W}_i(s), \end{aligned} \tag{4.48}$$

where the strategy v_i does not enter the diffusion coefficient σ and $\widehat{\sigma}$. Specifically as follows, we assume that the state process $X_i(.)$ is not completely observable, instead, it is partially observed through the related process $Y_i(.)$, which is governed by the following equation

$$\begin{cases} dY_i(s) = h(s, X_i(s))\, ds + d\widehat{W}_i(s), \\ Y(0) = 0,\ s \geq 0. \end{cases}$$

For notational simplicity, we do not explicitly indicate the dependence of the state on the control by writing $X_i^{v_i}(s)$. We take $\mathbb{F}$ to be the natural filtration of the N-dimensional standard Brownian motion $(W_1, ..., W_N)$ augmented by $\mathbb{P}$-null sets of $\mathcal{F}$.
Denote

$$(v_{-i}, \pi) := (v_1, ..., v_{i-1}, \pi, v_{i+1}, ..., v_N),\ i = 1, ..., N.$$

Then, the ith player selects $v_i \in \mathcal{U}_{ad}$ to evaluate her cost functional

$$\pi \to J^{i,N}(t, x_i, v_{-i,}\pi) := J^{i,N}(t, x_i, v_1, ..., v_{i-1}, \pi, v_{i+1}, ..., v_N),$$

where

$$\begin{aligned} J^{i,N}(t, x_i, v) = \mathbb{E}\Bigg[\int_t^T & Z_i^{t,x_i}(s) f\left(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)], X^{(-i)}(s), v_i(s)\right) ds \\ & + Z_i^{t,x_i}(T)\, \phi\left(X_i^{t,x_i}(T), \mathbb{E}[X_i^{t,x_i}(T)], X^{(-i)}(T)\right)\Bigg]. \end{aligned} \tag{4.49}$$

The associated dynamics, parameterized by (t, x_i), is

$$\begin{cases} dX_i^{t,x_i}(s) & = b(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)], v_i(s))ds \\ & \quad + \sigma\left(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)]\right) dW_i(s) \\ & \quad + \widehat{\sigma}\left(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)]\right) d\widehat{W}_i(s), \quad t < s \leq T, \\ X^{t,x_i}(t) & = x_i. \end{cases} \tag{4.50}$$

The ith player interacts with others through the mean-field coupling term

$$X^{(-i)} = \frac{1}{N-1}\sum_{k\neq i}^{N} X_k, \;\; i \in \{1, ..., N\},$$

which models the aggregate impact of all other players.

Note that the ith player assesses her cost functional over $[t, T]$ seen from her local state $X_i(t) = x_i$ and she knows only the initial states of all other players at time 0, $(X_k(0) = x_0, k \neq i)$. Thus the game may be cast as a decision problem where each player has incomplete state information about other players. The development of a solution framework in terms of a certain exact equilibrium notion is challenging. Our objective is to address this incomplete state information issue and design a set of individual strategies which has a meaningful interpretation. This will be achieved by using the so-called *consistent mean-field approximation.*

For a large N, even if each player has full state information of the system, the exact characterization of the equilibrium points, based on the SMP, will have high complexity since each player leads to a variational inequality for the underlying Hamiltonian similar to (4.22) which is further coupled with the state processes of all other players. Therefore, we should rely on the mean-field approximation of our system.

We note that $J^{i,N}$ depends on not only v_i, but also all other players' strategies v_{-i} through the mean-field coupling term $X^{(-i)}$. This suggests that we extend Definition 4.1.1 to the N-player case as follows.

Definition 4.4.1 *The admissible strategy $u = (u_1, ..., u_N)$ is a δ_N -sub-game perfect equilibrium point for N players in the system (4.48)–(4.49) if for every $i \in \{1, ..., N\}$,*

$$\lim_{\varepsilon \downarrow 0} \frac{J^{i,N}(t, x_i, u) - J^{i,N}(t, x_i, u_{-i}, u_i^{\varepsilon})}{\varepsilon} \leq O(\delta_N), \tag{4.51}$$

for each given $v_i \in \mathcal{U}_i[0,T]$, $x_i \in \mathbb{R}$ and a.e. $t \in [0,T]$, where u_i^{ε} is the spike variation of the strategy u_i of the $i$thplayer using v_i and $0 \leq \delta_N \to 0$ *as* $N \to \infty$.

The error term $O(\delta_N)$ is due to the mean-field approximation to be introduced below for designing u.

4.4.1 The local limiting decision problem

Let $X^{(-i)}$ be approximated by a deterministic function $\bar{X}(s)$ on $[0,T]$. Denote the cost functional

$$\begin{aligned} \bar{J}^i(t, x_i, v_i) = \mathbb{E}\bigg[\int_t^T & Z_i^{t,x_i}(s) f\left(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)], \bar{X}(s), v_i(s)\right) ds \\ & + Z_i^{t,x_i}(T)\, \phi\left(X_i^{t,x_i}(T), \mathbb{E}[X_i^{t,x_i}(T)], \bar{X}(T)\right)\bigg], \end{aligned} \tag{4.52}$$

which is intended as an approximation of $J^{i,N}$. Note that once $\bar{X}$ is assumed fixed, $\bar{J}^i$ is affected only by v_i. The introduction of $\bar{X}$ as a fixed function of time is based on the freezing idea in mean-field games. The reason is that $X^{(-i)} = \dfrac{1}{N-1}\sum_{k\neq i}^{N} X_k$ is generated by many negligibly small players, and therefore, a given player has little influence on it.

The strategy selection of the ith player is based on finding a sub-game perfect equilibrium for $\bar{J}^i$ to which the method based on the Stochastic Maximum Principle [Peng [50]] can be applied under the following conditions:

Assumption 2

(i) The functions $b(s,x,\widehat{x},v), \sigma(s,x,\widehat{x}), f(s,x,\widehat{x},w,v), \phi(x,\widehat{x},w)$ are bounded.

(ii) The functions b,σ are differentiable with respect to $(x,\widehat{x})$. The derivatives are Lipschitz continuous in $(x,\widehat{x})$ and bounded.

(iii) The functions f,ϕ are differentiable with respect to $(x,\widehat{x},w)$, and their derivatives are continuous in $(x,\widehat{x},w,v)$ and $(x,\widehat{x},w)$, respectively, and bounded.

To simplify our notation, we will denote by

$$\begin{aligned}
&b^{t,x_i}(s) = b\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],u_i\right),\ \sigma^{t,x_i}(s) = \sigma\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)]\right),\\
&\widehat{\sigma}^{t,x_i}(s) = \widehat{\sigma}\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)]\right),\ f^{t,x_i}(s) = f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\bar{X}(s),u_i\right),\\
&h^{t,x_i}(s) = h\left(s,\widehat{X}_i^{t,x_i}(s)\right).
\end{aligned}$$

Let $u_i \in \mathcal{U}_{ad}$ be a sub-game perfect equilibrium point for (4.50) and (4.52) and denote the associated backward SDE

$$\begin{cases}
-dp^{t,x_i}(s) = -\Big\{\left(b_x^{t,x_i}(s) - \widehat{\sigma}_x^{t,x_i}h^{t,x_i}(s)\right)p^{t,x_i}(s) + \sigma_x^{t,x_i}(s)\,q^{t,x_i}(s) + \widehat{\sigma}_x^{t,x_i}(s)\,\widehat{q}^{t,x_i}(s)\\
\qquad\qquad + h_x^{t,x_i}(s)\,\widehat{\Psi}^{t,x_i}(s) + f_x^{t,x_i}(s) + \mathbb{E}\left[\left(b_{\widehat{x}}^{t,x_i}(s)\right)p^{t,x_i}(s) + \sigma_{\widehat{x}}^{t,x_i}(s)\,q^{t,x_i}(s)\right]\\
\qquad\qquad + \mathbb{E}\left[\widehat{\sigma}_{\widehat{x}}^{t,x_i}(s)\,\widehat{q}^{t,x_i}(s) + f_{\widehat{x}}^{t,x_i}(s)\right]\Big\}ds + q^{t,x_i}(s)\,dW_i(s) + \widehat{q}^{t,x_i}(s)\,d\widehat{W}_i(s),\\
p^{t,x_i}(T) = \phi_x^{t,x_i}(T) + \mathbb{E}\left[\phi_{\widehat{x}}^{t,x_i}(T)\right],
\end{cases} \tag{4.53}$$

where

$$\begin{aligned}
H_x^{t,x_i}(s) = &\left(b_x^{t,x_i}(s) - \widehat{\sigma}_x^{t,x_i}h^{t,x_i}(s)\right)p^{t,x_i}(s) + \sigma_x^{t,x_i}(s)\,q^{t,x_i}(s) + \widehat{\sigma}_x^{t,x_i}(s)\,\widehat{q}^{t,x_i}(s)\\
&+ h_x^{t,x_i}(s)\,\widehat{\Psi}^{t,x_i}(s) - f_x(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\bar{X}(s),u_i),
\end{aligned}$$

and

$$H_{\widehat{x}}^{t,x_i}(s) = \left(b_{\widehat{x}}^{t,x_i}(s) - \widehat{\sigma}_{\widehat{x}}^{t,x_i} h^{t,x_i}(s)\right) p^{t,x_i}(s) + \sigma_{\widehat{x}}^{t,x_i}(s)\, q^{t,x_i}(s) + \widehat{\sigma}_{\widehat{x}}^{t,x_i}(s)\, \widehat{q}^{t,x_i}(s)$$
$$+ f_{\widehat{x}}(s, \widehat{X}_i^{t,x_i}(s), \mathbb{E}[\widehat{X}_i^{t,x_i}(s)], \bar{X}(s), u_i),$$

for which

$$H\left(t, x_i, v, p^{t,x_i}(t), q^{t,x_i}(t), \widehat{q}^{t,x_i}(t), \widehat{\Psi}^{t,x_i}(t)\right) - H\left(t, x_i, u_i(t), p^{t,x_i}(t), q^{t,x_i}(t), \widehat{q}^{t,x_i}(t), \widehat{\Psi}^{t,x_i}(t)\right) \leq 0$$
$$\forall v \in U,\ x_i \in \mathbb{R},\quad a.e. t \in [0,T],\ \mathbb{P}-a.s. \tag{4.54}$$

The closed-loop equilibrium state associated to u_i of the ith player is given by

$$d\widehat{X}_i(s) = b(s, \widehat{X}_i(s), \mathbb{E}[\widehat{X}_i(s)], u_i(s))ds + \sigma(s, \widehat{X}_i(s), \mathbb{E}[\widehat{X}_i(s)])dW_i(s)$$
$$+ \widehat{\sigma}(s, \widehat{X}_i(s), \mathbb{E}[\widehat{X}_i(s)])d\widehat{W}_i(s). \tag{4.55}$$

We call u_i a decentralized strategy in that it has sample path dependence only on its local Brownian motion W_i and $\widehat{W}$. The processes $\{u_k, 1 \leq k \leq N\}$ are independent. Further, we impose

Assumption 3 All the processes $\{u_k, 1 \leq k \leq N\}$ have the same law.

This restriction ensures that $\{\widehat{X}_i, 1 \leq i \leq N\}$ are i.i.d. random processes. Since each u_i is obtained as a process adapted to the filtration generated by W_i and $\widehat{W}$, it can be represented as a non-anticipative functional $\widehat{F}(\{W_i(s)\}_{s\leq t})$ of W_i and $\widehat{F}(\{\widehat{W}_i(s)\}_{s\leq t})$ of $\widehat{W}_i$. For a given $\bar{X}$, if non-uniqueness of u_i arises, we stipulate that the same functional $\widehat{F}$ is used by all players applying their respective Brownian motions so that all the individual control processes have the same law. This means some coordination is necessary for the strategy selection under non-uniqueness. By the law of large numbers, the consistency condition on $\bar{X}$ reads

$$\bar{X}(s) = \mathbb{E}[\widehat{X}_1(s)], \forall s \in [0,T]. \tag{4.56}$$

A question of central interest is how to characterize the performance of the set of strategies

$u = (u_1, ..., u_N)$ when they are implemented and assessed according to the original cost functionals $\{J^{i,N}, 1 \le i \le N\}$. An answer is provided in the following theorem for which the proof is displayed in the next section. This is the second main result of the chapter.

Theorem 4.4.1 *Under Assumptions 2 and 3, suppose there exists a solution to (4.53), (4.55) and (4.56). Then we have*

$$J^{i,N}(t, x_i, u) - J^{i,N}(t, x_i, u_{-i}, u_i^\varepsilon) = \bar{J}^i(t, x_i, u_i) - \bar{J}^i(t, x_i, u_i^\varepsilon) + O\left(\frac{\varepsilon}{\sqrt{N-1}}\right). \tag{4.57}$$

Moreover, $u = (u_1, ..., u_N) \in \Pi_{i=1}^N \mathcal{U}[0,T]$ is a δ_N -sub-game perfect equilibrium for the system (4.48), (4.49) where $\delta_N \le \dfrac{C}{\sqrt{N}}$ and C depends only on (b, σ, f, ϕ, T).
If there exists a unique solution $(\bar{X}, u)$ to (4.53), (4.55) and (4.56), each player can locally construct its strategy. When there are multiple solutions, the players need to coordinate to choose the same $\bar{X}$ and further ensure that $\{u_i, 1 \le i \le N\}$ have the same law.

4.5 Proof of Theorem 4.4.1

This section is devoted to the proof of Theorem 4.4.1. We first establish some performance estimates which will be used to conclude the proof of the theorem.

4.5.1 The performance estimate

We have

$$\begin{aligned} J^{i,N}(t, x_i, u) = \mathbb{E}_{t,x_i} \Bigg[\int_t^T & \widehat{Z}_i^{t,x_i}(s) f\left(s, \widehat{X}_i^{t,x_i}(s), \mathbb{E}[\widehat{X}_i^{t,x_i}(s)], \widehat{X}^{(-i)}(s), u_i(s)\right) ds \\ &+ \widehat{Z}_i^{t,x_i}(T) \phi\left(\widehat{X}_i^{t,x_i}(T), \mathbb{E}[\widehat{X}_i^{t,x_i}(T)], \widehat{X}^{(-i)}(T)\right)\Bigg]. \end{aligned}$$

Now we fix $i \in \{1, ..., N\}$ and change u_i to u_i^ε when all other players apply u_{-i}, where

$$u_i^\varepsilon = \begin{cases} v_i(s), & s \in [t, t+\varepsilon], \\ u_i(s), & s \in [t, T] \backslash [t, t+\varepsilon], \end{cases}$$

and $v_i \in \mathcal{U}_{ad}$. We have

$$J^{i,N}(t, x_i, u_{-i}, u_i^{\varepsilon}) = \mathbb{E}\left[\int_t^T Z_i^{t,x_i}(s) f\left(s, X_i^{t,x_i}(s), \mathbb{E}[X_i^{t,x_i}(s)], \widehat{X}^{(-i)}(s), u_i^{\varepsilon}(s)\right) ds \right.$$
$$\left. + Z_i^{t,x_i}(T)\phi\left(X_i^{t,x_i}(T), \mathbb{E}[X_i^{t,x_i}(T)], \widehat{X}^{(-i)}(T)\right)\right], \tag{4.58}$$

where X_i^{t,x_i} is the solution of (4.50) with admissible strategy u_i^{ε}. The following estimates will be frequently used in the sequel.

Lemma 4.5.1 *For the ith player, let X_i and $\widehat{X}_i$ be the state processes corresponding to u_i^{ε} and v_i, respectively. Then*

$$\mathbb{E}\left[\sup_{t\le s\le T}\left|X_i^{t,x_i}(s) - \widehat{X}_i^{t,x_i}(s)\right|^2\right] \le C\varepsilon^2,$$

where C does not depend on (t, x_i).

Proof. The proof can be performed in two steps as in that of Lemma 1 in [24]. So we do not repeat it here. ■

Lemma 4.5.2 *We have*

$$\mathbb{E}\left[\sup_{0\le s\le T}\left|\widehat{X}_i(s)\right|^2\right] \le C\mathbb{E}\left[\left|\widehat{X}_i(0)\right|^2 + 1\right].$$

Proof. We write

$$\widehat{X}_i(s) = \widehat{X}_i(0) + \int_0^s b\left(\tau, \widehat{X}_i(\tau), \mathbb{E}[\widehat{X}_i(\tau)], u_i(\tau)\right) ds$$
$$+ \int_0^s \sigma\left(\tau, \widehat{X}_i(\tau), \mathbb{E}[\widehat{X}_i(\tau)]\right) dW_i(\tau)$$
$$+ \int_0^s \widehat{\sigma}\left(\tau, \widehat{X}_i(\tau), \mathbb{E}[\widehat{X}_i(\tau)]\right) d\widehat{W}_i(\tau). \tag{4.59}$$

Then, by Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E}\left[\sup_{0\le s\le T}\left|\widehat{X}_i(s)\right|^2\right] \le C\left(\mathbb{E}\left|\widehat{X}_i(0)\right|^2+\mathbb{E}\left[\int_0^T\left|b\left(s,\widehat{X}_i(s),\mathbb{E}[\widehat{X}_i(s)],u_i(s)\right)\right|\right]^2\right)$$
$$+C\mathbb{E}\int_0^T\left|\sigma\left(s,\widehat{X}_i(s),\mathbb{E}[\widehat{X}_i(s)]\right)\right|^2 ds$$
$$+C\mathbb{E}\int_0^T\left|\widehat{\sigma}\left(s,\widehat{X}_i(s),\mathbb{E}[\widehat{X}_i(s)]\right)\right|^2 ds.$$

By the Lipschitz condition on b,σ and $\widehat{\sigma}$, we further obtain

$$\mathbb{E}\left[\sup_{0\le s\le T}\left|\widehat{X}_i(s)\right|^2\right] \le C\left(\mathbb{E}\left|\widehat{X}_i(0)\right|^2+1+\int_0^T\mathbb{E}\left[\sup_{0\le \eta\le s}\left|\widehat{X}_i(s)\right|^2\right]^2\right),$$

which combined with Gronwall's lemma yields the desired estimate. ■

Corollary 4.5.1 *We have, for $N\ge 2$,*

$$\sup_{0\le s\le T}\mathbb{E}\left[\left|\widehat{X}_i^{(-i)}(s)-\overline{X}(s)\right|^2\right]\le\frac{C}{N-1},$$

where C does not depend on N.

Proof. Thanks to Assumption 3, $\widehat{X}_1,\dots\ \widehat{X}_N$ are i.i.d. processes. The estimate follows from Lemma 4.5.2. ■

4.5.2 Proof of Theorem 4.4.1

We estimate the cost difference

$$
\begin{aligned}
&J^{i,N}(t,x_i,u) - J^{i,N}(t,x_i,u_{-i},u_i^{\varepsilon}) \\
&= \mathbb{E}\left[\int_t^T \left\{\widehat{Z}_i^{t,x_i}(s) f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. -\; Z_i^{t,x_i}(s) f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^{\varepsilon}(s)\right)\right\} ds\right] \\
&\quad + \mathbb{E}\left[\widehat{Z}_i^{t,x_i}(T)\phi\left(\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right) - Z_i^{t,x_i}(T)\phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right] \\
&= \mathbb{E}\left[\int_t^T \widehat{Z}_i^{t,x_i}(s)\left\{f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. -\; f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^{\varepsilon}(s)\right)\right\} ds\right] \\
&\quad + \mathbb{E}\left[\widehat{Z}_i^{t,x_i}(T)\left(\phi\left(\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right) - \phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right)\right] \\
&\quad + \mathbb{E}\left[\int_t^T \left(\widehat{Z}_i^{t,x_i}(s) - Z_i^{t,x_i}(s)\right) f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^{\varepsilon}(s)\right) ds\right. \\
&\qquad\qquad \left. +\; \left(\widehat{Z}_i^{t,x_i}(T) - Z_i^{t,x_i}(T)\right)\phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right].
\end{aligned}
$$

By Girsanov's theorem, we obtain

$$
\begin{aligned}
&J^{i,N}(t,x_i,u) - J^{i,N}(t,x_i,u_{-i},u_i^{\varepsilon}) \\
&= \mathbb{E}^u\left[\int_t^T \left\{f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. -\; f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^{\varepsilon}(s)\right)\right\} ds\right] \\
&\quad + \mathbb{E}^u\left[\left(\phi\left(\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right) - \phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right)\right] \\
&\quad + \mathbb{E}\left[\int_t^T \left(\widehat{Z}_i^{t,x_i}(s) - Z_i^{t,x_i}(s)\right) f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^{\varepsilon}(s)\right) ds\right. \\
&\qquad\qquad \left. +\; \left(\widehat{Z}_i^{t,x_i}(T) - Z_i^{t,x_i}(T)\right)\phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right],
\end{aligned}
$$

then, we can write

$$
\begin{aligned}
&J^{i,N}(t,x_i,u) - J^{i,N}(t,x_i,u_{-i},u_i^\varepsilon)\\
&= \mathbb{E}^u\left[\int_t^T \left\{ f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\bar{X}(s),u_i(s)\right)\right.\right.\\
&\qquad\qquad\left.\left. -\ f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\bar{X}(s),u_i^\varepsilon(s)\right)\right\} ds\right]\\
&\quad + \mathbb{E}^u\left[\left(\phi\left(s,\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\bar{X}(T)\right) - \phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\bar{X}(T)\right)\right)\right]\\
&\quad + \mathbb{E}\left[\int_t^T I_1 ds + I_2\right] + \mathbb{E}^u\left[\int_t^T I_3 ds + I_4\right],
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \left(\widehat{Z}_i^{t,x_i}(s) - Z_i^{t,x_i}(s)\right) f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^\varepsilon(s)\right),\\
I_2 &= \left(\widehat{Z}_i^{t,x_i}(T) - Z_i^{t,x_i}(T)\right)\phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right),\\
I_3 &= \left(f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i(s)\right) - f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\widehat{X}^{(-i)}(s),u_i^\varepsilon(s)\right)\right)\\
&\quad - \left(f\left(s,\widehat{X}_i^{t,x_i}(s),\mathbb{E}[\widehat{X}_i^{t,x_i}(s)],\bar{X}(s),u_i(s)\right) - f\left(s,X_i^{t,x_i}(s),\mathbb{E}[X_i^{t,x_i}(s)],\bar{X}(s),u_i^\varepsilon(s)\right)\right),\\
I_4 &= \left(\phi\left(\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right) - \phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\widehat{X}^{(-i)}(T)\right)\right)\\
&\quad - \left(\phi\left(\widehat{X}_i^{t,x_i}(T),\mathbb{E}[\widehat{X}_i^{t,x_i}(T)],\bar{X}(T)\right) - \phi\left(X_i^{t,x_i}(T),\mathbb{E}[X_i^{t,x_i}(T)],\bar{X}(T)\right)\right).
\end{aligned}
$$

The cost difference satisfies

$$
\begin{aligned}
J^{i,N}(t,x_i,u) - J^{i,N}(t,x_i,u_{-i},u_i^\varepsilon) &= \bar{J}^i(t,x_i,u_i) - \bar{J}^i(t,x_i,u_i^\varepsilon)\\
&\qquad + \mathbb{E}\left[\int_t^T I_1 ds + I_2\right] + \mathbb{E}^u\left[\int_t^T I_3 ds + I_4\right].
\end{aligned}
$$

We proceed to estimate $\mathbb{E}\left[\int_t^T I_1 ds + I_2\right]$ and $\mathbb{E}^u\left[\int_t^T I_3 ds + I_4\right]$.

Lemma 4.5.3 *We have*

$$
\left|\mathbb{E}^u\left[\int_t^T I_3 ds + I_4\right]\right| \le \frac{C\varepsilon}{\sqrt{N-1}}.
$$

Proof. The proof is similar with that of Lemma 3 in [24]. ∎

Remark 4.5.1 *The process* $\mathbb{E}\left[\int_t^T I_1 ds + I_2\right]$ *have zero expectation. The proof is similar with* *that of Lemma 2.2 in [62].*

Conclusion

In this thesis, we have investigated about two stochastic optimal control problems which, in various ways. In the first one, we have studied a sufficient stochastic maximum and we also show the relationship between stochastic maximum principle and dynamic programming in which the control of the jump size is essential and the corresponding Hamilton–Jacobi–Bellman (HJB) equation in this case is a mixed second order partial differential-difference equation. On the other hand we have studied the non smooth version of the relationship between MP and DPP for systems driven by normal martingales where the control domain is convex.
The second one, we have studied the characterize sub-game perfect equilibrium strategy of a partially observed optimal control problems for mean-field stochastic differential equations (SDEs) with correlated noises between systems and observations, which is time-inconsistent in the sense that it does not admit the Bellman optimality principle.

Following this study, several perspectives are considered. It would be interesting to use the optimal control problem where the state equation is driven by a normal martingale

- A second order maximum principle for systems driven by a normal martingale.
- The non smooth version of the relationship between MP and DPP for systems driven by normal martingales where the control domain is not convex.

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