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Abstract

Throughout this thesis, we focused our aim on the problem of optimal control under a risk-sensitive performance functional, where the systems studied are given by a backward stochastic differential equation, fully coupled forward-backward stochastic differential equation, and fully coupled forward-backward stochastic differential equation with jump. As a preliminary step, we use the risk neutral which is an extension of the initial control system where the set of admissible controls are convex in all the control problems, and an optimal solution exists. Then, we study the necessary as well as sufficient optimality conditions for risk sensitive performance, we illustrate our main results by giving applied examples of risk sensitive control problem. The first is under linear stochastic dynamics with exponential quadratic cost function. The second example deals with an optimal portfolio choice problem in financial market specially the model of control cash flow of a firm or project. The last one is an example of mean-variance for risk sensitive control problem applied in cash flow market.

Key words: Fully coupled forward backward stochastic differential equation, Optimal control, Risk-sensitive, Necessary Optimality Conditions, Sufficient Optimality Conditions, Mean variance, Cash flow.

Résumé

Dans cette thèse, on s'interesse au problèmes du contrôle optimal avec une fonction de performance de risque sensible, où les systèmes étudiés sont définis par: des équations différentielles stochastiques rétrogrades, des équations différentielles stochastiques progréssivement rétrogrades fortement couplée et des équations différentielles stochastiques progréssivement rétrogrades fortement couplée avec saut. Au début du travail, on utilise le facteur de risque neutre, qui est l'extension du système de contrôle initial dans lequel l'ensemble de valeur est convexe dans tous les les problèmes de contrôle et où une solution optimale existe. Après, on étudie les conditions nécessaires et suffisantes d'optimalité pour une performance de risque sensible. A la fin on illustrate nos principaux résultats par donner trois exemples d'application de problème de contrôle de risque sensible. Le premier concerne la dynamique stochastique linéaire avec une fonction de coût quadratique exponentielle. Le deuxième traite d'un problème de choix de portefeuille optimal sur le marché financier, notamment le modèle de contrôle cash-flow d'une entreprise ou d'un projet. Le dernier est un exemple de variance moyenne pour un problème de contrôle risque sensible appliqué au marché cash-flow.

Mots clés: Equation différentielle stochastique progréssivement rétrograde fortement couplée, risque sensible, contrôle optimal, conditions nécessaires d'optimalités, conditions suffisantes d'optimalités, variance moyenne, cash-flow.

Contents

Symbols and Abbreviations

Here we give the different symbols and abbreviations used in this thesis.

Symbols

$(\mathcal{F}_t)_{t\geq 0}$	:	Filtration.
$(\Omega, \mathcal{F})$	:	Measurable space.
$\mathbb{P}$	:	Probability measure with respect to risk-neutral.
$\mathbb{P}^\theta$	:	Probability measure with respect to risk-sensitive.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$	:	Filtred probability space.
W	:	Brownian motion.
W^θ	:	$\mathbb{P}^\theta -$ Brownian motion.
$\tilde{N}^\theta$	:	$\mathbb{P}^\theta -$ compensator Poisson measure.
$\tilde{N}$	:	The compensated Poisson measure.
$\mathcal{F}_t^W$	:	Filtration generated by Brownian motion.
$\mathcal{F}_t^N$	:	Filtration generated by Poisson measure.
$\mathcal{F}_t^{W,\tilde{N}}$	:	Filtration generated by two mutually independent processes Brownian motion and Poisson measure.
$J^\theta(.)$	:	Risk-sensitive functional cost.
$\widetilde{J}^\theta(.)$	:	Risk-neutral functional cost.
$H^\theta(.)$	:	Risk-sensitive Hamiltonian functional.
$\widetilde{H}^\theta(.)$	:	Risk-neutral Hamiltonian functional.
U	:	The set of values taken by control v.
$\mathcal{U}$	:	The set of all admissible controls.
u	:	Optimal control.

Abbreviations

a.e	:	Almost everywhere.
a.s	:	Almost surely.
BSDE	:	Backward stochastic differential equation.
cadlàg	:	Continu à droite limité à gauche (right continuous with left limits).
FBSDE	:	Forward-backward stochastic differential equation.
HARA	:	Hyperbolic absolute risk aversion.
SMP	:	Stochastic maximum principle.

Introduction

Problems of optimization take their essence in the permanent for man to find the optimal solution to his difficulties. Whether in the world of finance, physics, economy, biology, games theory industry, or health...., the interest is often focused on optimizing systems that evolve over time.

The history can be traced too early in 1827, botanist Robert Brown [6] published his observation about micro objects that pollen particles suspended on the surface of water will traverse continuously in an unpredictable way. This kind of motion is named the Brownian motion to indicate its randomness and continuity.

After that, Albert Einstein [13] developed a physics model to support his statement that atoms exist, that means he used the notion of Brownian motion to describe the physics investigation and proved that the position of particle can be follow by some normal distribution. Unfortunately, the mathematical description is not very correct in view of mathematicians.

Besides the works of Einstein, in 1923, and Wiener [38] did provide a correct mathematical definition of the stochastic process observed by Brown and described by Einstein, which is the Brownian motion that we used.

The first version of the stochastic maximum principle was extensively established in the 1970's by Bismut [5, 4], Kushner [22], Bensoussan [3] and Haussmann [18].

Stochastic control problems for the forward-backward system have been studied by many authors. The first contribution of control problem of the forward-backward system is made by Peng [29], he obtained the stochastic maximum principle with the control domain being convex. Xu [39] established the maximum principle for this kind of problem in the case where the control domain is not necessary convex, with uncontrolled diffusion coefficient and a restricted cost functional. The work of Peng [29] (convex control domain) is generalized by Wu [36], where the system is governed by a fully coupled forward backward stochastic differential equation. Shi and Wu [33], have established stochastic maximum principle to the fully coupled FBSDE where the control domain is not necessary convex, and without controlled diffusion coefficient under some monotonicity assumptions. Ji and Zhou [19] used the Ekeland variational principle to establish a maximum principle of controlled FBSDE systems, while the forward state is constrained in a convex set at the terminal time, and apply the result to state constrained stochastic linear-quadratic control models, and a recursive utility optimization problem is investigated. Yong [40] completely solved the problem of finding necessary conditions for optimal control of fully coupled FBSDEs, he considered an optimal control problem for general coupled FBSDEs with mixed initial-terminal conditions and derived the necessary conditions for the optimal controls when the control domain is not assumed to be convex, and the control variable appears in the diffusion coefficient.

In this thesis we are interested in the stochastic optimal control resolution by Pontryagin's Stochastic Maximum Principle (SMP in short) type and under Risk-Sensitive performance. We solve the problem by using the approach developed by Djehiche, Tembine and Tempone [11]. Their contribution can be summarized as follows. They have established a stochastic maximum principle for a class of risk-sensitive mean-field type control problems, where the distribution enters only through the mean of state process, this means that the drift, diffusion, and terminal cost functions depend on the state, the control and the means of state process. In the risk-sensitive control problem, in our second study we extended both of the results of Chala [8] and of Djehiche et al. [11], to a fully coupled case, is to establish a necessary, as well as sufficient optimality conditions, of Pontryagin's maximum principle type, for risk-sensitive performance functionals. We solve the problem by using the approach developed by Djehiche, Tembine and Tempone in [11]. In particular the best view of the last paper that can be found is: We can establish the necessary optimality condition without using the dynamic programming principle.

The existence of an optimal solution for forward-backward system has been solved in [2] to achieve the objective of our paper [20], and to establish necessary as well as a sufficient optimality conditions for this model, we give the stochastic maximum principle for risk-sensitive performance functional. Or at first we translated the risk-sensitive control problem by using an auxiliary process, then we obtained the SMP associated to this translated problem by using yong's theorem in (Yong [40]) Theorem 3.1). After that, and according to the transformation that we did to the intermediate adjoint processes, and using the logarithmic transformation established by El Karoui and Hamadene [14] we established the necessary optimality conditions for risk sensitive problem.

Finally, the last work in this subject by used the fully coupled forward-backward with jump system, with financial application, can be found in the paper of Khallout and Chala [21]. The previous work has been established with risk sensitive performance functional. Besides of that, we note here that paper of Shi and Wu [31] was in the case where the set of admissible controls is convex, and [32] in the general case with application to finance. Ma and Liu [24] who deal with the risk-sensitive control problem for mean-field stochastic delay differential equations (MF-SDDEs in short) with partial information, and under the assumptions that the control domain is not convex and the value function is non-smooth, they have established a SMP.

Our goal in this thesis is to treat pontryagin's maximum principle under risk sensitive functional for different systems: Backward stochastic differential equation, fully coupled forward-backward stochastic differential equation, fully coupled backward-forward stochastic differential equation with jump. This problem under consideration is not a simple extension from the mathematical point of view, but also provides interesting models in many applications, such as: mathematical finance, optimal control...ect. The proofs of our main results is based on spike variation method based on theorem -as a preliminary step- of stochastic maximum principle for the risk neutral control problem.

This thesis is organized as follows:

Chapter 01: *(The content of this chapter has been used as the project of the chapter book in Chala et al [7].)*

In this first chapter, we develop the general framework used in this thesis. We start discussing the standard risk-sensitive structures, and how constructions of this kind can give a rigorous treatment. We investigate in this chapter the financial market of risk-sensitive for the dynamic with and without jumps diffusion, by using Girsanov's theorems, and in virtue of Itô's formula.

Chapter 02: (*The content of this chapter has been used as the project of the published paper Khallout & Chala [20] in Asian Journal of Control.)*

In the third chapter, we study the necessary as well as sufficient optimality conditions where the system is given by a fully coupled forward-backward stochastic differential equation with a risk-sensitive performance functional. At the end of this chapter, we illustrate our main result by giving two examples of risk sensitive control problem under linear stochastic dynamics with exponential quadratic cost function, the second example deals with an optimal portfolio choice problem in financial market specially the model of control cash flow.

Chapter 03: *(The content of this chapter has been used as the project of the paper of Khallout & Chala [21].)*

In the last chapter, we extend the result of the second chapter where the system is given by a fully coupled forward-backward stochastic differential equation with jump, and we illustrate our new main result by giving an example of mean-variance for risk sensitive control problem applied in cash flow market.

The content of this thesis is the subject of the following works:

1. Chala, A. Khallout, R. Hefayed, D. The use of Girsanov's Theorem to Describe Risk-Sensitive Problem and Application To Optimal Control. Stochastique Differential Equations, Tony G. Deangelo, ISBN: 978-1-53613-809-2, 117-154, Nova 2018.

2. Khallout, R. Chala, A. (2019). A Risk-Sensitive Stochastic Maximum Principle for Fully Coupled Forward-Backward Stochastic Differential Equations with Applications. Published in Asian Journal of Control, 1-12, DOI: 10.1002/asjc.2020.

3. Khallout, R. Chala, A. Risk-sensitive Necessary and Sufficient Optimality Conditions

for Fully Coupled Forward-Backward Stochastic Differential Equations with Jump diffusion and Finantial Applications. arXiv:1903.02072.

Chapter 1

Expected exponential utility and Girsanov's theorem

This chapter has been considered as a part of book's chapter [7], we develop the general framework used in our papers [8, 9, 17, 20, 21]. The starting point for the discussion will be the standard risk-sensitive structures, and how constructions of this kind can be given a rigorous treatment. We investigate in this chapter the financial market of risk-sensitive for the dynamic with and without jumps diffusion, by using Girsanov's theorems, and in virtue of Itô's formula, Lévy-Itô's formula.

1.1 Problem formulation

Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtred probability space satisfying the usual conditions, in which a $one-$dimensional Brownian motion $W = (W(t) : 0 \leq t \leq T)$ is defined. We assume that $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ is defined by $\forall t \geq 0$, $\mathcal{F}_t^W = \sigma\left(W(s);\ \text{for any } s \in [0,t]\right) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of $\mathbb{P}-$null sets. Let $\mathcal{M}^2([0,T];\mathbb{R})$ denote the set of *one* dimensional jointly measurable random processes $\{\varphi_t, t \in [0,T]\}$ which satisfy:
$(i) : \mathbb{E}\left[\int_0^T |\varphi_t|^2 \, dt\right] < \infty$, $(ii) : \varphi_t$ is $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ measurable, for any $t \in [0,T]$.
We denote similarly by $\mathcal{S}^2([0,T];\mathbb{R})$ the set of continuous *one* dimensional random processes which satisfy:

$(i) : \mathbb{E}\left[\sup_{0\leq t\leq T} |\varphi_t|^2\right] < \infty$, $(ii) : \varphi_t$ is $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ measurable, for any $t \in [0,T]$.
Let T be a strictly positive real number, and U be a non empty subset of $\mathbb{R}$.

In the next, we will discuss a result, which called the Girsanov's Theorem, it plays the important role in the application especially in economics, and optimal control. In Girsanov's theorem application, we can visit the papers [8, 11, 14, 17, 20]. We can now show the versions of the Girsanov's Theorem. In the application of Itô calculus, Girsanov's theorem get used frequently since it transforms a class of process to Brownian motion with an equivalent probability measure transformation see [16].

Definition 1.1 *Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\in[0,T]}, \mathbb{P}\right)$ be a probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $\mathcal{F}_T$. We say that $\mathbb{Q}$ is equivalent to $\mathbb{P} \mid \mathcal{F}_T$ if $\mathbb{P} \mid \mathcal{F}_T \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P} \mid \mathcal{F}_T$, or equivalently, if $\mathbb{P}$ and $\mathbb{P}^\theta$ have the same zero sets in $\mathcal{F}_T$.*

Remark 1.1 *By the Radon-Nikodym's theorem, this is the case if and only if we have*

$$d\mathbb{Q}(w) = Z(T)\, d\mathbb{P}(w) \ \text{ on } \mathcal{F}_T, \text{ and}$$
$$d\mathbb{P}(w) = Z^{-1}(T)\, d\mathbb{Q}(w) \ \text{ on } \mathcal{F}_T.$$

Where $Z(T)$ is called the Radon–Nikodym derivative

Theorem 1.1 ***(Girsanov, 1960, [16]):*** *Assume that $W(t)$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with underlying filtration $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$. Let $f(t)$ be a square integrable stochastic process adapts to $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$, such that*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T f^2(t)\, dt\right\}\right] < \infty, \tag{1.1}$$

for all $t \in [0,T]$, then $W_{\mathbb{Q}}(t) = W(t) - \int_0^t f(s)\, ds$ is a Brownian motion with respect to an equivalent probability measure $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T) =: \exp\left\{\int_0^T f(t)\, dW_t - \frac{1}{2}\int_0^T f^2(t)\, dt\right\}.$$

Remark 1.2 *Using differential form, we can also say, if* $dW_{\mathbb{Q}}(t) = dW(t) - f(t)\,dt$. *Then* $W_{\mathbb{Q}}(t)$ *is a Brownian motion with respect to (w.r.t) the probability measure* $\mathbb{Q}$.

Remark 1.3 *The condition* $\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T f^2(t)\,dt\right\}\right] < \infty$ *is sufficient and not necessary, called the Novikov's condition.*

For more details the reader can see the Øksendal's book [26] pages 155-160.

In the next, we will discuss some special cases, to be able to investigate the model of risk-sensitive control in next chapter

1.2 Financial market of the risk-sensitive

1.2.1 Factor dynamic without jump diffusion

We model the dynamics of the investor with diffusion process as a following SDE

$$dx(t) = b(t, x(t))\,dt + \Lambda dW(t), \quad \text{and } x_0 = x. \tag{1.2}$$

We consider a financial market in which two asset (securities) can be investment choices, the first one is risk-free is called also bond (foreign currency deposit for example), whose price $S_0(t)$ at time t is given by

$$\frac{dS_0(t)}{S_0(t)} = r(t)\,dt \text{ or } (=r(t, x(t))\,dt).$$

The second risky asset is called stock, whose price $S_1(t)$ at time t is given by

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)\,dt + \sigma(t)\,dW(t) \text{ or } (=\mu(t, x(t))\,dt + \sigma(t, x(t))\,dW_t),$$

where $r(t, x(t))$ is bond function interest rate, $\sigma(t, x(t))$ is function stock price volatility rate, and $\mu(t, x(t))$ is called the expected rate of return.

Now let us consider an investor wants who want to invested in the risk-free (foreign currency deposit for example) and the stock, and whose decisions cannot affect the prices in the financial market.

Definition 1.2 *(Self-Financial) The market is called self-financial if there is no infusion or withdrawal of funds over* $[0,T]$.

We assume also that our market is to be self-financial, we denote by $V(t)$ the amount of the investor's wealth, and $u(t)$ is the proportion of the wealth invested in the stock at time t, then $\pi(t) = u(t)V(t)$ is the amount stock and $(1-u(t))V(t)$ is the amount in the bond, that means the investor has $V(t) - u(t)V(t) = V(t) - \pi(t)$ savings in a bank.
Then wealth dynamics of the investor who wants to invest in the financial market has the following form

$$\frac{dV(t)}{V(t)} = (V(t) - \pi(t))\frac{dS_0(t)}{S_0(t)} + \pi(t)\frac{dS_1(t)}{S_1(t)}.$$

Honestly, the wealth of the investor is described by

$$\begin{aligned}
\frac{dV(t)}{V(t)} &= (V(t) - \pi(t))\, r(t, x(t))\, dt + \pi(t)\left(\mu(t, x(t))\, dt + \sigma(t, x(t))\, dW(t)\right) \\
&= (V(t) - \pi(t))\, r(t, x(t))\, dt + \pi(t)\,\mu(t, x(t))\, dt \\
&+ \pi(t)\,\sigma(t, x(t))\, dW(t) \\
&= V(t)\, r(t, x(t))\, dt - \pi(t)\, r(t, x(t))\, dt + \pi(t)\,\mu(t, x(t))\, dt \\
&+ \pi(t)\,\sigma(t, x(t))\, dW(t) \\
&= \{V(t)\, r(t, x(t)) + (\mu(t, x(t)) - r(t, x(t)))\,\pi(t)\}\, dt \qquad (1.3) \\
&+ \pi(t)\,\sigma(t, x(t))\, dW(t).
\end{aligned}$$

Definition 1.3 *An admissible strategy is an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted and square integrable process* π *with values in* $\mathbb{R}$ *such that* (1.3) *has a strong solution* $(V(t))_{t\in[0,T]}$ *that satisfies* $\mathbb{E}\left[\int_0^T |V(t)|\, dt\right] < \infty$, *the set of all the admissible strategies will be denoted by* $\mathcal{U}_{ad}$.

The investor wants to maximize his (or her) expected utility (HARA type) over the set $\mathcal{U}_{ad}$

in some terminal time $T > 0$:

$$J^{\theta}(\pi(.)) = \frac{1}{\theta}\mathbb{E}\left[V^{\theta}(T)\right]. \tag{1.4}$$

By choosing an appropriate portfolio choice strategy $\pi(.)$, where the exponent $\theta > 0$ is called risk-sensitive parameter. If we put $\theta = 1$ the utility (1.4) is reduced to the usual risk-neutral case, the expectation under the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$.

Lemma 1.1 *We can rewrite the expectation of* $\mathbb{E}\left[V^{\theta}(T)\right]$ *in* (1.4) *in term of the exponential expected of integral criterion as*

$$J^{\theta}(\pi(.)) = \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h(t, x(t), \pi(t))\,dt\right)\right].$$

$\mathbb{E}^{\theta}$ *is the new expectation with respect to the probability measure* $\mathbb{P}^{\theta}$.

Proof. Applying the Itô's formula to logarithmic wealth value
$\ln V^{\theta}(t) = \theta \ln V(t) = \theta f(V(t))$, we have

$$\begin{aligned}
\theta d(f(V(t))) &= \theta d(\ln V(t)) \\
&= \theta\tfrac{\partial f}{\partial t}(t, V(t))\,dt + \theta\tfrac{\partial f}{\partial x}(t, V(t))\,dV(t) \\
&+ \theta\tfrac{1}{2}\tfrac{\partial^2 f}{\partial x^2}(t, V(t))\,\langle dV(t), dV(t)\rangle \\
&= \theta\tfrac{1}{V(t)}dV(t) + \theta\tfrac{1}{2}\left(-\tfrac{1}{V^2(t)}\right)\pi^2(t)\,\sigma^2(t, x(t))\,V^2(t)\,dt \\
&= \theta\left(\{V(t)\,r(t, x(t)) + (\mu(t, x(t)) - r(t, x(t)))\,\pi(t)\}\,dt\right. \\
&\left. + \pi(t)\,\sigma(t, x(t))\,dW(t)\right) - \tfrac{1}{2}\theta\pi^2(t)\,\sigma^2(t, x(t))\,dt.
\end{aligned}$$

Then, by taking the integral from zero to T with respect to time, the exponential expectation gets the form

$$\begin{aligned}
J^{\theta}(\pi(.)) &= \tfrac{1}{\theta}\mathbb{E}\left[V^{\theta}(T)\right] \\
&= \tfrac{1}{\theta}\mathbb{E}\left[\exp\left(\ln V^{\theta}(T)\right)\right] \\
&= \tfrac{1}{\theta}\mathbb{E}\left[\exp\left(\theta \ln V(T)\right)\right] \\
&= \tfrac{1}{\theta}\mathbb{E}\Big[\exp\Big(\theta f(V(0)) + \theta \textstyle\int_0^T \{V(t)\, r(t,x(t)) + (\mu(t,x(t)) \\
&\quad -r(t,x(t)))\,\pi(t)\}\,dt + \theta \textstyle\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t) \\
&\quad -\tfrac{1}{2}\theta \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt\Big)\Big] \\
&= \tfrac{1}{\theta}\mathbb{E}\Big[\exp\Big(\ln V^{\theta}(0) + \theta \textstyle\int_0^T \{V(t)\, r(t,x(t)) + (\mu(t,x(t)) \\
&\quad +\theta \textstyle\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t) \\
&\quad +\theta \textstyle\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t) - \tfrac{1}{2}\theta \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt\Big)\Big] \\
&= \tfrac{1}{\theta}\exp\left(\ln V^{\theta}(0)\right)\mathbb{E}\Big[\exp\Big(\theta \textstyle\int_0^T \{V(t)\, r(t,x(t)) + (\mu(t,x(t)) \\
&\quad -r(t,x(t)))\,\pi(t)\}\,dt + \theta \textstyle\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t) \\
&\quad -\tfrac{1}{2}\theta \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt\Big)\Big].
\end{aligned}$$

Then, we get

$$\begin{aligned}
J^{\theta}(\pi(.)) &= \tfrac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\Big[\exp\Big(\theta \textstyle\int_0^T \{V(t)\, r(t,x(t)) + (\mu(t,x(t)) - r(t,x(t)))\,\pi(t)\}\,dt \\
&\quad +\theta\pi(t)\,\sigma(t,x(t))\,dW(t) - \tfrac{1}{2}\theta \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt \\
&\quad -\tfrac{1}{2}\theta^2 \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \tfrac{1}{2}\theta^2 \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt\Big)\Big] \\
&= \tfrac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\Big[\exp\Big\{\Big(-\tfrac{1}{2}\theta^2 \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt \\
&\quad +\theta \textstyle\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t)\Big) - \tfrac{1}{2}\theta \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt \\
&\quad +\tfrac{1}{2}\theta^2 \textstyle\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta \textstyle\int_0^T \{V(t)\, r(t,x(t)) \\
&\quad +(\mu(t,x(t)) - r(t,x(t)))\,\pi(t)\}\,dt\}] \\
&= \tfrac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\left[I_1 \times I_2\right],
\end{aligned}$$

where

$$I_1 = \exp\left(-\frac{1}{2}\theta^2 \int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta \int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t)\right),$$

$$\begin{aligned}
I_2 &= \exp\left(-\frac{1}{2}\theta \int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \frac{1}{2}\theta^2 \int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt\right.\\
&\left.+\theta \int_0^T \{V(t)\,r(t,x(t)) + (\mu(t,x(t)) - r(t,x(t)))\,\pi(t)\}\,dt\right)\\
&= \exp\left(\theta \int_0^T -\frac{1}{2}(\theta-1)\,\pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta \int_0^T \{V(t)\,r(t,x(t))\right.\\
&+ (\mu(t,x(t)) - r(t,x(t)))\,\pi(t)\}\,dt)\\
&= \exp\left(\theta \int_0^T h(t,x(t),\pi(t))\,dt\right),
\end{aligned}$$

and

$$\begin{aligned}
&h(t,x(t),\pi(t))\\
&= -\frac{1}{2}(\theta-1)\,\pi^2(t)\,\sigma^2(t,x(t)) + V(t)\,r(t,x(t)) + (\mu(t,x(t)) - r(t,x(t)))\,\pi(t).
\end{aligned}$$

In virtue of Novikov's condition (1.1) from Girsanov's Theorem 1.1, we get

$$\mathbb{E}\left[\exp \alpha\pi^2(t)\right] \le C. \tag{1.5}$$

By applying Girsanov's transformation (see the theorem 1.1), the stochastic integral term can be deleted, and according to the condition (1.5), we get

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\theta^2 \int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta \int_0^T \pi(t)\,\sigma(t,(t))\,dW(t)\right),$$

for some positive constants α and C .

Hence

$$\begin{aligned} J^{\theta}\left(\pi\left(.\right)\right) &= \frac{1}{\theta}\mathbb{E}\left[V^{\theta}\left(T\right)\right] \\ &= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}\left[I_1 \times I_2\right] \\ &= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}\left[\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}} \times \exp\left(\theta\int_0^T h\left(t, x\left(t\right), \pi\left(t\right)\right)dt\right)\right] \\ &= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h\left(t, x\left(t\right), \pi\left(t\right)\right)dt\right)\right]. \end{aligned}$$

$\mathbb{E}^{\theta}$ is the new expectation with respect to probability measure $\mathbb{P}^{\theta}$, and we denote by

$$W^{\theta}\left(t\right) = W\left(t\right) - \theta\int_0^t \pi\left(s\right)\sigma\left(s, x\left(t\right)\right)ds,$$

a standard Brownian motion under the probability measure $\mathbb{P}^{\theta}$.

As a conclusion, for every $0 \leq s \leq t \leq T$, our dynamics (1.2) satisfies the SDE

$$\begin{aligned} dx\left(t\right) &= b\left(t, x\left(t\right)\right)dt + \Lambda dW\left(t\right) \\ &= b\left(t, x\left(t\right)\right)dt + \Lambda d\left(W^{\theta}\left(t\right) + \theta\int_0^t \pi\left(s\right)\sigma\left(s, x\left(s\right)\right)ds\right) \\ &= b\left(t, x\left(t\right)\right)dt + \Lambda dW^{\theta}\left(t\right) + \Lambda\theta\pi\left(t\right)\sigma\left(t, x\left(t\right)\right)dt \\ &= \left(b\left(t, x\left(t\right)\right) + \Lambda\theta\pi\left(t\right)\sigma\left(t, x_t\right)\right)dt + \Lambda dW\left(t\right). \end{aligned}$$

An auxiliary criterion function of the expected utility, whose the investor want to maximize, is given by

$$\widetilde{J}^{\theta}\left(\pi\left(.\right)\right) = \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h\left(t, x\left(t\right), \pi\left(t\right)\right)dt\right)\right].$$

The proof is completed. ■

1.2.2 Factor dynamic with jump diffusion process

In all what follows, we will work on the classical probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t^{W,\widetilde{N}}\right\}_{t\leq T}, \mathbb{P}\right)$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}-$null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon T, and $\left\{\mathcal{F}_t^{W,\widetilde{N}}\right\}_{t\leq T}$ satisfies the usual conditions. We assume that the filtration $\left\{\mathcal{F}_t^{W,\widetilde{N}}\right\}$ is generated by the following two mutually independent processes:

(i) $\{W(t)\}_{t\geq 0}$ is a one-dimensional standard Weiner motion.

(ii) Poisson random measure N on $[0,T]\times\Gamma$, where $\Gamma\subset\mathbb{R}-\{0\}$. We denote by $\left(\mathcal{F}_t^W\right)_{t\leq T}$ (resp $\left(\mathcal{F}_t^N\right)_{t\leq T}$) the $\mathbb{P}-$augmentation of the natural filtration of W (resp N). Obviously, we have

$$\mathcal{F}_t := \sigma\left[\int_0^s \int_A N(d\lambda, dr);\ s\leq t,\ A\in\mathcal{B}(\Gamma)\right]\vee\sigma\left[W(s);\ s\leq t\right]\vee\mathcal{N},$$

where $\mathcal{N}$ contains all $\mathbb{P}-$null sets in $\mathcal{F}$, and $\sigma_1\vee\sigma_2$ denotes the $\sigma-$field generated by $\sigma_1\cup\sigma_2$. We assume that the compensator of N has the form $\mu(dt,d\lambda)=m(d\lambda)\,dt$, for some positive and $\sigma-$finite Lévy measure m on Γ, endowed with its Borel $\sigma-$field $\mathcal{B}(\Gamma)$. We suppose that $\int_\Gamma 1\wedge|\lambda|^2\, m(d\lambda)<\infty$, and write $\widetilde{N}=N-mdt$ for the compensated jump martingale random measure of N.

Notation 1.1 *We need to define some additional notations. Given $s\leq t$, let us introduce the following spaces*

$\mathcal{S}^2([0,T],\mathbb{R})$ *the set of* $\mathbb{R}$- *valued adapted cadlàg processes P such that*

$$\|P\|_{\mathcal{S}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\sup_{r\in[0,T]}|P(r)|^2\right]^{\frac{1}{2}} < +\infty.$$

$\mathcal{M}^2([0,T],\mathbb{R})$ *is the set of progressively measurable* $\mathbb{R}-$*valued processes Q such that*

$$||Q||_{\mathcal{M}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\int_0^T |Q(r)|^2\, dr\right]^{\frac{1}{2}} < +\infty.$$

$\mathcal{L}^2([0,T],\mathbb{R})$ *is the set of* $\mathcal{B}([0,T]\times\Omega)\otimes\mathcal{B}(\Gamma)$ *measurable maps* $K:[0,T]\times\Omega\times\Gamma\to\mathbb{R}$ *such that*

$$\|K\|_{\mathcal{L}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\int_0^T\int_\Gamma |K(r,\lambda)|^2\, m(d\lambda)\, dr\right]^{\frac{1}{2}} < +\infty,$$

we denote by $\mathbb{E}$ *the expectation with respect to* $\mathbb{P}$.

First of all, we must defined the Poison random measure $\widetilde{N}(dt, d\lambda)$ as

$$\widetilde{N}(dt, d\lambda) = \begin{cases} N(dt, d\lambda) \text{ if } |\lambda| \geq R, \\ N(dt, d\lambda) - m(d\lambda) \text{ if } |\lambda| < R, \end{cases} \tag{1.6}$$

where $(|\lambda| \geq R) \subset \mathcal{B}(\mathbb{R})$, such that $m(|\lambda| \geq R) < \infty$.

We denote here that $W(t)$ is the Brownian motion given in measurable space $(\Omega, \mathcal{F})$.

Now, we come to the important Itô's formula for Itô–Lévy processes: If $x(t)$ is given by

$$\begin{cases} dx(t) = A(t, x(t^-))\,dt + B(t, x(t^-))\,dW(t) + \int_{\mathbb{R}} \gamma(t, x(t), \lambda)\,\widetilde{N}(dt, d\lambda), \\ x(0) = x, \end{cases} \tag{1.7}$$

and $f : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^2$ function, is the process $Y(t) = f(t, x(t))$.

Let $x^c(t)$ be the continuous part of $x(t)$, i.e., $x^c(t)$ is obtained by removing the jumps term from $x(t)$. Then an increment in $Y(t)$ stems from an increment in $x^c(t)$ plus the jumps (coming from $\widetilde{N}(dt, d\lambda)$). Hence in view of the classical Itô formula we would guess that.

Lemma 1.2 ***(Lévy-Itô formula I for Lévy processes):*** *Let us consider the dynamic system with jump diffusion which given by* (1.7). *Then, the Lévy-Itô's formula with respect to model with jump is*

$$\begin{aligned} d(f(t, x(t))) = {} & \frac{\partial f}{\partial t}(t, x(t))\,dt + \frac{\partial f}{\partial x}(t, x(t))\,dx^c(t) \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x(t))\langle dx(t), dx(t)\rangle \\ & + \int_{\mathbb{R}} \left(f(t, x(t^-) + \gamma(\lambda, x(t^-))) - f(t, x(t^-))\right)\widetilde{N}(dt, d\lambda). \end{aligned}$$

It can be proved that our guess is correct. Since

$$\begin{cases} dx^c(t) = \left[A(t, x(t)) - \int_{(|\lambda|<R)} \gamma(\lambda, x(t^-))\,m(d\lambda)\right] dt + B(t, x(t^-))\,dW(t), \\ x(0) = x. \end{cases}$$

This gives the following Lemma.

Lemma 1.3 ***(Lévy-Itô formula II for Lévy processes):*** *Suppose* $x(t) \in \mathbb{R}$ *is an Itô–Lévy process of the form* (1.7), *where the condition* (1.6) *is satisfied for some* $R \geq 0$, *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a* $\mathcal{C}^2$ *function, is the process* $Y(t) = f(t, x(t))$. *Then* $Y(t)$ *is again an Itô–Lévy process and*

$$
\begin{aligned}
df(t, x(t)) = \;& \frac{\partial f}{\partial t}(t, x(t))\, dt + \frac{\partial f}{\partial x}(t, x(t))\, dx(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x(t)) \langle dx(t), dx(t) \rangle \\
& + \int_{|\lambda|<R} \left(f(t, x(t^-) + \gamma(t, \lambda)) - f(t, x(t^-)) - \frac{\partial f(t, x(t^-))}{\partial x} \gamma(t, \lambda) \right) m(d\lambda) \\
& + \int_{\mathbb{R}} \left(f(t, x(t^-) + \gamma(t, \lambda)) - f(t, x(t^-)) \right) \widetilde{N}(dt, d\lambda).
\end{aligned} \tag{1.8}
$$

Example 1.1 ***(The Geometric Lévy Processes):*** *Consider the following differential equation (SDE with jump diffusion)*

$$
\frac{dx(t)}{x(t^-)} = adt + bdW_t + \int_{\mathbb{R}} c(t, \lambda) \widetilde{N}(dt, d\lambda), \tag{1.9}
$$

where a, b *are constants and* $c(t, \lambda) \geq -1$. *If we put* $Y(t) = \ln x(t)$, *then by Lévy-Itô's formula* (1.8) *from Lemma* 1.3, *we get*

$$
\begin{aligned}
df(t, x(t)) \;& = \frac{\partial f}{\partial t}(t, x(t))\, dt + \frac{\partial f}{\partial x}(t, x(t))\, dx(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x(t)) \langle dx(t), dx(t) \rangle \\
& + \int_{|\lambda|<R} \left(f(t, x(t^-) + \gamma(t, \lambda)) - f(t, x(t^-)) - \frac{\partial f(t, x(t^-))}{\partial x} \gamma(t, \lambda) \right) m(d\lambda) \\
& + \int_{\mathbb{R}} \left(f(t, x(t^-) + \gamma(t, \lambda)) - f(t, x(t^-)) \right) \widetilde{N}(dt, d\lambda) \\
& = \frac{1}{x(t)} x(t) (adt + bdW(t)) - \frac{1}{2}\frac{1}{x^2(t)} b^2 x^2(t)\, dt \\
& + \int_{|\lambda|<R} \left(\ln(x(t^-) + c(t, \lambda) x(t^-)) - \ln(x(t^-)) - \frac{1}{x(t^-)} c(t, \lambda) x(t^-) \right) m(d\lambda) \\
& + \int_{\mathbb{R}} \left\{ \ln(x(t^-) + c(t, \lambda) x(t^-)) - \ln(x(t^-)) \right\} \widetilde{N}(dt, d\lambda) \\
& = \left(a - \frac{1}{2} b^2 \right) dt + bdW(t) + \int_{|\lambda|<R} \left(\ln(1 + c(t, \lambda)) - c(t, \lambda) \right) m(d\lambda) \\
& + \int_{\mathbb{R}} \left(\ln(1 + c(t, \lambda)) \right) \widetilde{N}(dt, d\lambda).
\end{aligned}
$$

Then

$$\begin{aligned}Y(t) &= Y(0) + \int_0^t \left(a - \frac{1}{2}b^2\right) ds + b\int_0^t dW(s) \\ &+ \int_0^t \int_{|\lambda|<R} \left(\ln\left(1 + c(s,\lambda)\right) - c(s,\lambda)\right) m(d\lambda)\, ds \\ &+ \int_0^t \int_{\mathbb{R}} \left(\ln\left(1 + c(s,\lambda)\right) - c(s,\lambda)\right) \widetilde{N}(ds, d\lambda).\end{aligned}$$

This gives the explicit solution of equation (1.9) *by the following expression*

$$\begin{aligned}x(t) = \quad & x(0)\exp\Big\{\left(a - \tfrac{1}{2}b^2\right) t + b dW(t) \\ & + \int_0^t \int_{|\lambda|<R} \left(\ln\left(1 + c(s,\lambda)\right) - c(s,\lambda)\right) m(d\lambda)\, ds \\ & + \int_0^t \int_{\mathbb{R}} \left(\ln\left(1 + c(s,\lambda)\right)\right) \widetilde{N}(ds, d\lambda)\Big\}.\end{aligned}$$

In the next, we will discuss a result, which called the Girsanov's theorem for Lévy processes, it plays also the important role in the application especially in economics, and in optimal control, see the application part for this transformation in the papers [10, 31]. We can now show the versions of the Girsanov's Theorem.

Theorem 1.2 ***(Girsanov's Theorem I for Itô-Lévy Processes):*** *The dynamics with jump diffusion process can be described as a following SDE with jumps diffusion*

$$\begin{cases} dx(t) = b(t, x(t^-), w)\, dt + \Lambda(t, x(t^-), w)\, dW(t) + \int_{\mathbb{R}} \gamma(t, x(t^-), \lambda, w)\, \widetilde{N}(dt, d\lambda), \\ x(0) = x \end{cases}. \tag{1.10}$$

Assume there exist predictable processes $u(t) = u(t,w)$, *and* $\beta(t,\lambda) = \beta(t,\lambda,w)$ *such that*

$$\Lambda(t)\, u(t) + \int_{\mathbb{R}} \gamma(t,\lambda)\, \beta(t,\lambda)\, m(d\lambda) = \alpha(t) \;\; \textit{for a.a} \;\; (t,w) \in [0,T] \times \Omega, \tag{1.11}$$

and such that the process

$$Z(t) := \exp\left\{ -\int_0^t u(s)\, dW(s) - \frac{1}{2}\int_0^t u^2(s)\, ds \right. \qquad (1.12)$$
$$+ \int_0^t \int_{\mathbb{R}} \ln(1-\beta(s,\lambda))\, \widetilde{N}(ds, d\lambda)$$
$$\left. + \int_0^t \int_{\mathbb{R}} \{\ln(1-\beta(s,\lambda)) + \beta(s,\lambda)\}\, m(d\lambda)\, ds \right\},$$

is well-defined and satisfied

$$\mathbb{E}[Z(T)] = 1. \qquad (1.13)$$

Define the probability measure $\mathbb{Q}$ *on* $\mathcal{F}_T$ *by* $d\mathbb{Q}(w) = Z(T)\, d\mathbb{P}(w)$. *Then* $x(t)$ *is local martingale with respect to* $\mathbb{Q}$.

Proof. See [26] theorem 1.31 page 15. ■

Theorem 1.3 ***(Girsanov's Theorem II for Itô-Lévy Processes):*** *Assume there exist predictable processes* $u(t)$, *and* $\beta(t,\lambda) \leq 1$ *such that the process* (1.12) *exist for* $0 \leq t \leq T$, *and satisfies* (1.13). *Define the probability measure* $\mathbb{Q}$ *on* $\mathcal{F}_T$ *by*

$$dW_{\mathbb{Q}}(t) = u(t)\, dt + dW(t), \qquad (1.14)$$

and

$$\widetilde{N}_{\mathbb{Q}}(dt, d\lambda) = \beta(t,\lambda)\, m(d\lambda)\, dt + \widetilde{N}(dt, d\lambda). \qquad (1.15)$$

Then $W_{\mathbb{Q}}(.)$ *is a Brownian motion with respect to* $\mathcal{F}_t$ *and* $\mathbb{Q}$, *and* $\widetilde{N}_{\mathbb{Q}}(.,.)$ *is* $(\mathcal{F}_t, \mathbb{Q})$*-compensator Poisson measure of* $N(.,.)$, *in the sense that the process*

$$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(s,\lambda)\, \widetilde{N}_{\mathbb{Q}}(ds, d\lambda) \;\; , \;\; 0 \leq t \leq T,$$

is a local $(\mathcal{F}_t, \mathbb{Q})-$ *martingale, for all predictable processes* $\gamma(s,\lambda)$ *such that*

$$\int_0^T \int_{\mathbb{R}} (\gamma(s,\lambda))^2 (1-\beta(t,\lambda))\, m(d\lambda)\, ds < \infty \;\; a.s.$$

Proof. See [26] theorem 1.33 pages 17-19. ■

Theorem 1.4 ***(Girsanov's Theorem III for Itô-Lévy Processes):*** *Let $x(t)$ be as* (1.10) *in theorem* 1.2. *Let $u(t)$, and $\beta(t,\lambda)$ be $\mathcal{F}_t-$predictable processes satisfying* (1.11). *Let $\mathbb{Q}$, $W_{\mathbb{Q}}$ and $\widetilde{N}_{\mathbb{Q}}$ be as defined in Theorem* 1.3 *Then in terms of $W_{\mathbb{Q}}$ and $\widetilde{N}_{\mathbb{Q}}$ the precess $x(t)$ can be represented by*

$$dx(t) = f(t)\,dt + \Lambda(t)\,dW_{\mathbb{Q}}(t) + \int_{\mathbb{R}} \gamma(t,\lambda)\,\widetilde{N}_{\mathbb{Q}}(dt,d\lambda),$$

where

$$f(t) = b(t) - \Lambda(t)\,u(t) + \int_{\mathbb{R}} \gamma(t,\lambda)\,\beta(t,\lambda)\,m(d\lambda).$$

Proof. See [26] theorem 1.35 page 20. ■

The application of the Girsanov's transformation can be found in economics, in fact as the dynamic of the wealth value. For this end, we will investigate some applications.

The dynamic state of the investor with jump diffusion process can be described as the following SDE with jumps diffusion

$$\begin{cases} dx(t) = b(t,x(t^-))\,dt + \Lambda(t,x(t^-))\,dW(t) + \int_{\mathbb{R}} \gamma(t,\lambda)\,\widetilde{N}(dt,d\lambda), \\ x(0) = x. \end{cases} \quad (1.16)$$

We consider a financial market in which two asset (securities) can be investment choices, the first one is called a globally risk-free asset called also bond (foreign currency deposit for example), whose price $S_0(t)$ at time t is given by

$$\begin{cases} \dfrac{dS_0(t)}{S_0(t)} = r(t,x(t))\,dt, \\ S_0(0) = s_0. \end{cases}$$

The second risky asset is called stock, whose price $S_1(t)$ at time t is given by

$$\begin{cases} \dfrac{dS_1(t)}{S_1(t)} = \mu(t,x(t))\,dt + \sigma(t,x(t))\,dW(t) + \int_{\mathbb{R}} \delta(t,\lambda)\,\widetilde{N}(dt,d\lambda), \\ S_1(0) = s_1. \end{cases}$$

Where $r(t, x(t))$ is bond function interest rate, $\sigma(t, x(t))$ is function stock price volatility rate, and $\mu(t, x(t))$ is called the expected rate of return, and $\delta(., \lambda) \in \mathbb{R}$, satisfies $-1 \leq \delta(., \lambda) \leq +\infty$, in additional, the function $\delta(., \lambda)$ satisfies $\int_{\Gamma_0} |\delta(., \lambda)|^2 m(d\lambda) < +\infty$. Now let us consider an investor who wants to invest in the risk-free (foreign currency deposit for example) and the stock, and whose decisions cannot affect the prices in the financial market. We assume also that our market be self-financial, we denote by $V(t)$ be the amount of the investor's wealth, and $u(t)$ is the proportion of the wealth invested in the stock at time t, then $\pi(t) = u(t) V(t)$ is the the amount stock, and $(1 - u(t)) V(t)$ is the amount in the bond, that's mean the investor has $V(t) - u(t) V(t) = V(t) - \pi(t)$ savings in a bank. Then, the wealth dynamics of the investor who want invests in the financial market has the following form

$$\frac{dV(t)}{V(t^-)} = (V(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} + \pi(t) \frac{dS_1(t)}{S_1(t)}.$$

In fact, the wealth of the investor is described by

$$\begin{aligned}
&\frac{dV(t)}{V(t^-)} \qquad (1.17)\\
&= (V(t) - \pi(t)) r(t, x(t)) dt \\
&+ \pi(t) \left(\mu(t, x(t)) dt + \sigma(t, x(t)) dW(t) + \int_{\mathbb{R}} \delta(t, \lambda) \widetilde{N}(dt, d\lambda) \right) \\
&= (V(t) - \pi(t)) r(t, x(t)) dt + \pi(t) \mu(t, x(t)) dt + \pi(t) \sigma(t, x(t)) dW(t) \\
&+ \int_{\mathbb{R}} \pi(t) \delta(t, \lambda) \widetilde{N}(dt, d\lambda) \\
&= V(t) r(t, x(t)) dt - \pi(t) r(t, x(t)) dt + \pi(t) \mu(t, x(t)) dt + \pi(t) \sigma(t, x(t)) dW(t) \\
&+ \int_{\mathbb{R}} \pi(t) \delta(t, \lambda) \widetilde{N}(dt, d\lambda) \\
&= \{V(t) r(t, x(t)) + (\mu(t, x(t)) - r(t, x(t))) \pi(t)\} dt + \pi(t) \sigma(t, x(t)) dW(t) \\
&+ \int_{\mathbb{R}} \pi(t) \delta(t, \lambda) \widetilde{N}(dt, d\lambda).
\end{aligned}$$

Definition 1.4 *An admissible strategy is an $(\mathcal{F}_t)_{t \geq 0}$-adapted and square integrable process π with values in $\mathbb{R}$ such that (1.17) has a strong solution $(V(t))_{t \in [0,T]}$ that satisfies*

$\mathbb{E}\left[\int_0^T |V(t)|\, dt\right] < \infty$. *Then, the set of all the admissible strategy is denoted by* $\mathcal{U}_{ad}$.

The investor wants to maximize his (or her) expected utility (hyperbolic absolute risk aversion) HARA type, over the set $\mathcal{U}_{ad}$ in some terminal time $T > 0$:

$$J^{\theta}(\pi(.)) = \frac{1}{\theta}\mathbb{E}\left[V^{\theta}(T)\right]. \tag{1.18}$$

By choosing an appropriate portfolio choice strategy $\pi(.)$, where the exponent $\theta > 0$ is called risk-sensitive parameter. If we put $\theta = 1$ the utility (1.18) reduced to the usual risk-neutral case, the expectation under the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$.

Lemma 1.4 *We can rewrite the expectation* $\mathbb{E}\left[V^{\theta}(T)\right]$ *described in the equation* (1.18) *in term of the exponential expected of integral criterion as*

$$J^{\theta}(\pi(.)) = \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h(t, x(t), \pi(t), \lambda)\, dt\right)\right].$$

$\mathbb{E}^{\theta}$ *is the new expectation with respect to probability measure* $\mathbb{P}^{\theta}$*, and the function* h *is given by*

$$\begin{aligned} h(t, x(t), \pi(t), \lambda) = & -\frac{1}{2}(\theta - 1)\pi^2(t)\sigma^2(t, x(t)) + V(t) r(t, x(t)) \\ & + (\mu(t, x(t)) - r(t, x(t)))\pi(t) \\ & + \int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[(1 + \pi(t)\delta(t, \lambda))^{\theta} - 1\right] - \pi(t)\delta(t, \lambda)\right\} m(d\lambda). \end{aligned}$$

Proof. Applying the Lévy-Itô's formula in Lemma 1.3 to logarithmic wealth value

$\ln V^{\theta}(t) = \theta \ln V(t) = \theta f(t, V(t))$, we have

$$
\begin{aligned}
&\theta d\left(f\left(t, V\left(t\right)\right)\right) \qquad (1.19)\\
&= \theta d\left(\ln V\left(t\right)\right)\\
&= \theta \frac{\partial f}{\partial t}\left(t, V\left(t\right)\right) dt + \theta \frac{\partial f}{\partial x}\left(t, V\left(t\right)\right) dV\left(t\right)\\
&+ \theta \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(t, V\left(t\right)\right)\left\langle dV\left(t\right), dV\left(t\right)\right\rangle\\
&+ \theta \int_{|\lambda|<R}\left(f\left(t, V\left(t^-\right) + \Sigma\left(t,\lambda\right)\right) - f\left(t, V\left(t^-\right)\right) - \frac{\partial f\left(t, V\left(t^-\right)\right)}{\partial x}\Sigma\left(t,\lambda\right)\right) m\left(d\lambda\right)\\
&+ \theta \int_{\mathbb{R}}\left(f\left(t, V\left(t^-\right) + \Sigma\left(t,\lambda\right)\right) - f\left(t, V\left(t^-\right)\right)\right)\widetilde{N}\left(dt, d\lambda\right)\\
&= \theta \frac{1}{V\left(t\right)} dV\left(t\right) + \theta\frac{1}{2}\left(-\frac{1}{V^2\left(t\right)}\right)\pi^2\left(t\right)\sigma^2\left(t, x_t\right)V^2\left(t\right) dt\\
&+ \theta \int_{|\lambda|<R}\left(\ln\left(V\left(t\right) + V\left(t\right)\pi\left(t\right)\delta\left(t,\lambda\right)\right) - \ln\left(V\left(t\right)\right) - \pi\left(t\right)\delta\left(\lambda\right)\right) m\left(d\lambda\right)\\
&+ \theta \int_{\mathbb{R}}\left(\ln\left(V\left(t\right) + V\left(t\right)\pi\left(t\right)\delta\left(t,\lambda\right)\right) - \ln\left(V\left(t\right)\right)\right)\widetilde{N}\left(dt, d\lambda\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\theta \int_{|\lambda|<R}\left(\ln\left(V\left(t^-\right) + V\left(t^-\right)\pi\left(t\right)\delta\left(t,\lambda\right)\right) - \ln\left(V\left(t^-\right)\right) - \pi\left(t\right)\delta\left(\lambda\right)\right) m\left(d\lambda\right)\\
&= \theta \int_{|\lambda|<R}\left(\ln\left(1 + \pi\left(t\right)\delta\left(t,\lambda\right)\right) - \pi\left(t\right)\delta\left(\lambda\right)\right) m\left(d\lambda\right)\\
&= \int_{|\lambda|<R}\left(\ln\left(1 + \pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta} - \theta\pi\left(t\right)\delta\left(\lambda\right)\right) m\left(d\lambda\right)\\
&= \int_{|\lambda|<R}\left(\ln\left(1 + \pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta} + 1 - \left(1 + \pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}\right) m\left(d\lambda\right)\\
&+ \theta \int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[\left(1 + \pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta} - 1\right] - \pi\left(t\right)\delta\left(t,\lambda\right)\right\} m\left(d\lambda\right). \qquad (1.20)
\end{aligned}
$$

By substituting the equation (1.20) into (1.19) , we get

$$\begin{aligned}\exp\theta\ln V(T) &= \exp\Big\{V^{\theta}(0)+\theta\int_0^T\{V(t)\,r(t,x(t))+(\mu(t,x(t))-r(t,x(t)))\,\pi(t)\}\,dt\\
&-\frac{1}{2}\theta\int_0^T\pi^2(t)\,\sigma^2(t,x(t))\,dt+\theta\int_0^T\pi(t)\,\sigma(t,x(t))\,dW(t)\\
&+\int_{|\lambda|<R}\Big(\ln(1+\pi(t)\,\delta(t,\lambda))^{\theta}+1-(1+\pi(t)\,\delta(t,\lambda))^{\theta}\Big)\,m(d\lambda)\\
&+\theta\int_{|\lambda|<R}\Big\{\frac{1}{\theta}\Big[(1+\pi(t)\,\delta(t,\lambda))^{\theta}-1\Big]-\pi(t)\,\delta(t,\lambda)\Big\}\,m(d\lambda)\\
&+\theta\int_{\mathbb{R}}\ln(1+\pi(t)\,\delta(t,\lambda))\,\widetilde{N}(dt,d\lambda)\Big\}.\end{aligned}$$

Then we get after taking the expectation

$$\begin{aligned}\frac{1}{\theta}\mathbb{E}[V(T)] &= \frac{1}{\theta}\mathbb{E}[\exp\theta\ln V(T)]\\
&= \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\Big[\exp\Big\{\theta\int_0^T\{V(t)\,r(t,x(t))+(\mu(t,x_t)-r(t,x(t)))\,\pi(t)\}\,dt\\
&-\frac{1}{2}\theta\int_0^T\pi^2(t)\,\sigma^2(t,x(t))\,dt+\theta\int_0^T\pi(t)\,\sigma(t,x(t))\,dW(t)\\
&+\int_{|\lambda|<R}\Big(\ln(1+\pi(t)\,\delta(t,\lambda))^{\theta}+1-(1+\pi(t)\,\delta(t,\lambda))^{\theta}\Big)\,m(d\lambda)\\
&+\theta\int_{|\lambda|<R}\Big\{\frac{1}{\theta}\Big[(1+\pi(t)\,\delta(t,\lambda))^{\theta}-1\Big]-\pi(t)\,\delta(t,\lambda)\Big\}\,m(d\lambda)\\
&+\theta\int_{\mathbb{R}}\ln(1+\pi(t)\,\delta(t,\lambda))\,\widetilde{N}(dt,d\lambda)\Big\}\Big].\end{aligned}$$

Then

$$
\begin{aligned}
&\frac{1}{\theta}\mathbb{E}\left[V\left(T\right)\right] \\
&= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}\left[\exp\left(\theta\int_0^T \left(\left\{V\left(t\right)r\left(t,x\left(t\right)\right)+\left(\mu\left(t,x\left(t\right)\right)-r\left(t,x\left(t\right)\right)\right)\pi\left(t\right)\right\}dt\right)\right.\right. \\
&+\theta\int_0^T \pi\left(t\right)\sigma\left(t,x\left(t\right)\right)dW\left(t\right)-\frac{1}{2}\theta\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt \\
&-\frac{1}{2}\theta^2\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt+\frac{1}{2}\theta^2\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt \\
&+\int_{|\lambda|<R}\left(\ln\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}+1-\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}\right)m\left(d\lambda\right) \\
&+\theta\int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}-1\right]-\pi\left(t\right)\delta\left(t,\lambda\right)\right\}m\left(d\lambda\right) \\
&\left.\left.+\theta\int_{\mathbb{R}}\ln\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)\widetilde{N}\left(dt,d\lambda\right)\right)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{1}{\theta}\mathbb{E}\left[V\left(T\right)\right] \\
&= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}\left[\exp\left(-\frac{1}{2}\theta^2\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt+\theta\int_0^T \pi\left(t\right)\sigma\left(t,x\left(t\right)\right)dW\left(t\right)\right.\right. \\
&+\int_0^T\int_{|\lambda|<R}\left(\ln\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}+1-\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}\right)m\left(d\lambda\right)dt \\
&\left.+\theta\int_0^T\int_{\mathbb{R}}\ln\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)\widetilde{N}\left(dt,d\lambda\right)\right) \\
&\times\exp\left(-\frac{1}{2}\theta\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt+\frac{1}{2}\theta^2\int_0^T \pi^2\left(t\right)\sigma^2\left(t,x\left(t\right)\right)dt\right. \\
&+\theta\int_0^T\int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[\left(1+\pi\left(t\right)\delta\left(t,\lambda\right)\right)^{\theta}-1\right]-\pi\left(t\right)\delta\left(t,\lambda\right)\right\}m\left(d\lambda\right)dt \\
&\left.\left.+\theta\int_0^T\left\{V\left(t\right)r\left(t,x\left(t\right)\right)+\left(\mu\left(t,x\left(t\right)\right)-r\left(t,x\left(t\right)\right)\right)\pi\left(t\right)\right\}dt\right)\right] \\
&= \frac{1}{\theta}V^{\theta}\left(0\right)\mathbb{E}\left[I_1\times I_2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \exp\left(-\frac{1}{2}\theta^2 \int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t)\right.\\
&+\int_0^T\int_{|\lambda|<R}\left(\ln\left(1+\pi(t)\,\delta(t,\lambda)\right)^{\theta}+1-\left(1+\theta\pi(t)\,\delta(t,\lambda)\right)^{\theta}\right)m(d\lambda)\,dt\\
&\left.+\theta\int_0^T\int_{\mathbb{R}}\ln\left(1+\pi(t)\,\delta(t,\lambda)\right)\widetilde{N}(dt,d\lambda)\right),\\
I_2 &= \exp\left(\theta\int_0^T -\frac{1}{2}(\theta-1)\,\pi^2(t)\,\sigma^2(t,x(t))\,dt\right.\\
&+\theta\int_0^T\left\{V(t)\,r(t,x(t))+\left(\mu(t,x(t))-r(t,x(t))\right)\pi(t)\right\}dt\\
&+\theta\int_0^T\int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[\left(1+\pi(t)\,\delta(t,\lambda)\right)^{\theta}-1\right]-\pi(t)\,\delta(t,\lambda)\right\}m(d\lambda)\,dt\\
&= \exp\left(\theta\int_0^T h(t,x(t),\pi(t),\lambda)\,dt\right),
\end{aligned}
$$

and

$$
\begin{aligned}
h(t,x(t),\pi(t),\lambda) &= -\frac{1}{2}(\theta-1)\,\pi^2(t)\,\sigma^2(t,x(t))\,dt+V(t)\,r(t,x(t))\\
&+\left(\mu(t,x(t))-r(t,x(t))\right)\pi(t)\\
&+\int_{|\lambda|<R}\left\{\frac{1}{\theta}\left[\left(1+\pi(t)\,\delta(t,\lambda)\right)^{\theta}-1\right]-\pi(t)\,\delta(t,\lambda)\right\}m(d\lambda).
\end{aligned}
$$

We have the Novikov's condition for Lévy processes (see Theorem 1.36 page 20 in [26]).

$$
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(t)\,dt+\int_0^T\int_{\mathbb{R}}\beta^2(t,\lambda)\,\widetilde{N}(dt,d\lambda)\right)\right]\leq\infty. \tag{1.21}
$$

By applying Girsanov's transformation, the stochastic integral term can be delated, and according to the assumption (1.21) , we get

$$\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}} = \exp\Big(-\frac{1}{2}\theta^2\int_0^T \pi^2(t)\,\sigma^2(t,x(t))\,dt + \theta\int_0^T \pi(t)\,\sigma(t,x(t))\,dW(t)$$
$$+\int_0^T\int_{|\lambda|<R}\left(\ln(1+\pi(t)\,\delta(t,\lambda))^{\theta} + 1 - (1+\pi(t)\,\delta(t,\lambda))^{\theta}\right)m(d\lambda)\,dt$$
$$+\theta\int_0^T\int_{\mathbb{R}}\ln(1+\pi(t)\,\delta(t,\lambda))\,\widetilde{N}(dt,d\lambda)\Big),$$

for some constants α, C are positive. Hence

$$\begin{aligned}
J^{\theta}(\pi(.)) &= \frac{1}{\theta}\mathbb{E}\left(V^{\theta}(T)\right)\\
&= \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\left[I_1\times I_2\right]\\
&= \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}\left[\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}}\exp\left(\theta\int_0^T h(t,x(t),\pi(t),\lambda)\,dt\right)\right]\\
&= \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h(t,x(t),\pi(t),\lambda)\,dt\right)\right].
\end{aligned}$$

$\mathbb{E}^{\theta}$ is the new expectation with respect to probability measure $\mathbb{P}^{\theta}$. ■

As a conclusion.

Lemma 1.5 *Our dynamics* (1.16) *satisfies the following SDE with jump*

$$dx(t) = f(t,x(t),\lambda,\pi(t))\,dt + \Lambda dW^{\theta}(t) + \int_{\mathbb{R}}\gamma(t,\lambda)\,\widetilde{N}^{\theta}(ds,d\lambda),$$

where the function f is given by

$$f(t,x(t),\lambda,\pi(t)) = b(t,x(t)) - \Lambda\theta\pi(t)\,\sigma(t,x(t)) - \int_{|\lambda|<R}(1+\pi(t)\,\delta(t,\lambda))^{\theta}\,m(d\lambda).$$

Proof. Applying the Girsanov's transformation given in theorem 1.3. We denote by

$$W^{\theta}(t) = W(t) + \theta\int_0^t \pi(s)\,\sigma(s,x(s))\,ds,$$

is a standard brownian motion under the probability measure $\mathbb{P}^{\theta}$, and the $\mathbb{P}^{\theta}-$compensated Poison random measure is given by

$$\int_0^t \int_{|\lambda|<R} \widetilde{N}^{\theta}(ds, d\lambda) = \int_0^t \int_{|\lambda|<R} \widetilde{N}(ds, d\lambda) + \int_0^t \int_{|\lambda|<R} (1 + \pi(t)\delta(s, \lambda))^{\theta} m(d\lambda) ds.$$

For every $0 \leq s \leq t$,

$$\begin{aligned}
dx(t) &= b(t, x(t)) dt + \Lambda dB(t) + \int_{\mathbb{R}} \gamma(t, \lambda) \widetilde{N}(dt, d\lambda) \\
&= b(t, x(t)) dt + \Lambda d\left(W^{\theta}(t) - \theta \int_0^t \pi(s) \sigma(s, x_s) ds\right) \\
&+ \int_{\mathbb{R}} \gamma(t, \lambda) \left(\int_{|\lambda|<R} \widetilde{N}^{\theta}(ds, d\lambda) - \int_{|\lambda|<R} (1 + \pi(t)\delta(t, \lambda))^{\theta} m(d\lambda) dt\right) \\
&= b(t, x(t)) dt + \Lambda dW^{\theta}(t) - \Lambda\theta\pi(t)\sigma(t, x(t)) dt - \int_{|\lambda|<R} (1 + \pi(t)\delta(t, \lambda))^{\theta} m(d\lambda) dt \\
&+ \int_{\mathbb{R}} \gamma(t, \lambda) \widetilde{N}^{\theta}(ds, d\lambda). \\
&= \left(b(t, x(t)) - \Lambda\theta\pi(t)\sigma(t, x(t)) - \int_{|\lambda|<R} (1 + \pi(t)\gamma(t, \lambda))^{\theta} m(d\lambda)\right) dt + \Lambda dW^{\theta}(t) \\
&+ \int_{\mathbb{R}} \gamma(t, \lambda) \widetilde{N}^{\theta}(ds, d\lambda).
\end{aligned}$$

If we denote by

$$f(t, x(t), \lambda, \pi(t)) = b(t, x(t)) - \Lambda\theta\pi(t)\sigma(t, x(t)) - \int_{|\lambda|<R} (1 + \pi(t)\delta(t, \lambda))^{\theta} m(d\lambda).$$

Then we get

$$\begin{cases} dx(t) = f(t, x(t), \lambda, \pi(t)) dt + \Lambda dW^{\theta}(t) + \int_{\mathbb{R}} \gamma(t, \lambda) \widehat{N}(ds, d\lambda), \\ x(0) = x_0. \end{cases}$$

The proof is completed. ■

In the next, we will give an auxiliary criterion function of the expected utility whose wants the investor maximized as

$$\widetilde{J}^{\theta}(\pi(.)) = \frac{1}{\theta}V^{\theta}(0)\,\mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h(t, x(t), \pi(t), \lambda)\,dt\right)\right].$$

We sum up, we have seen that the risk-sensitive asset problem is equivalent to the stochastic control problem of minimizing the cost function

$$\widetilde{J}^{\theta}(\pi(.)) = \mathbb{E}^{\theta}\left[\exp\left(\theta\int_0^T h(t, x(t), \pi(t), \lambda)\,dt\right)\right].$$

Here the value $\frac{1}{\theta}V^{\theta}(0)$ plays no role important any more for the optimization problem, so we can put $\frac{1}{\theta}V^{\theta}(0) = 1$.

1.2.3 Mean-Variance of loss functional

We require the following condition

$$A_T^{\theta} := \exp\theta\left\{\int_0^T h(t, x(t), \pi(t))\,dt\right\}, \tag{1.22}$$

and we can put also

$$\Psi(T) := \int_0^T h(t, x(t), \pi(t))\,dt. \tag{1.23}$$

The risk-sensitive of loss functional is given by

$$\begin{aligned} &\Phi(\theta) \qquad (1.24)\\ &:= \frac{1}{\theta}\log\left(\mathbb{E}\left[\exp\theta\left\{\int_0^T h(t, x(t), \pi(t))\,dt\right\}\right]\right)\\ &= \frac{1}{\theta}\log\left(\mathbb{E}\left[\exp\theta\Psi(T)\right]\right). \end{aligned}$$

Lemma 1.6 *Let $\Phi(\theta)$ be the loss functional has written as (1.24), where $\Psi(T)$ is given by (1.23). Then, if the risk-sensitive index θ is small, the loss functional $\Phi(\theta)$ can be expanded* *as*

$$\mathbb{E}\left[\Psi\left(T\right)\right]+\frac{\theta}{2}\mathbb{V}ar\left(\Psi\left(T\right)\right)+O\left(\theta^{2}\right). \tag{1.25}$$

Proof. The limited development of the function $f\left(x\right)=\exp\left(\theta x\right)$ with rang two in the neighborhood of zero is given by

$$f\left(x\right)=\exp\left(\theta x\right)=\sum_{k=0}^{2}\frac{\left(\theta x\right)^{k}}{k!}=1+\theta x+\frac{1}{2}\left(\theta x\right)^{2}+O\left(\theta^{2}\right).$$

Then, by replacing x by $\Psi\left(T\right)$, we get

$$\exp\left(\theta\Psi\left(T\right)\right)=1+\theta\Psi\left(T\right)+\frac{1}{2}\left(\theta\Psi\left(T\right)\right)^{2}+O\left(\theta^{2}\right).$$

By taking expectation, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\theta\Psi\left(T\right)\right)\right]&=\mathbb{E}\left[1+\theta\Psi\left(T\right)+\frac{1}{2}\left(\theta\Psi\left(T\right)\right)^{2}+O\left(\theta^{2}\right)\right]\\&=1+\theta\mathbb{E}\left[\Psi\left(T\right)\right]+\frac{\theta^{2}}{2}\mathbb{E}\left[\Psi^{2}\left(T\right)\right]+O\left(\theta^{2}\right).\end{aligned}$$

Then

$$\log\mathbb{E}\left[\exp\left(\theta\Psi\left(T\right)\right)\right]=\log\left(1+\theta\mathbb{E}\left[\Psi\left(T\right)\right]+\frac{\theta^{2}}{2}\mathbb{E}\left[\Psi^{2}\left(T\right)\right]+O\left(\theta^{2}\right)\right).$$

If we take $X=\theta\mathbb{E}\left[\Psi\left(T\right)\right]+\frac{\theta^{2}}{2}\mathbb{E}\left[\Psi^{2}\left(T\right)\right]+o\left(\theta^{2}\right)$, and by using the limited development of the function $g\left(X\right)=\ln(1+X)$, with rang two in neighborhood of zero

$$g\left(X\right)=\ln(1+X)=\sum_{k=1}^{2}\frac{\left(-1\right)^{k-1}}{k}X^{k}.$$

Then

$$
\begin{aligned}
&\log \mathbb{E}\left[\exp\left(\theta\Psi\left(T\right)\right)\right] \\
&= \theta\mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta^2}{2}\mathbb{E}\left[\Psi^2\left(T\right)\right] + O\left(\theta^2\right) \\
&+ \left(-1\right)\frac{1}{2}\left[\theta\mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta^2}{2}\mathbb{E}\left[\Psi^2\left(T\right)\right] + o\left(\theta^2\right)\right]^2 + O(\theta^2) \\
&= \theta\mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta^2}{2}\mathbb{E}\left[\Psi^2\left(T\right)\right] - \frac{\theta^2}{2}\left(\mathbb{E}\left[\Psi\left(T\right)\right]\right)^2 \\
&- \frac{\theta^4}{4}\left(\mathbb{E}\left[\Psi^2\left(T\right)\right]\right)^2 + \ldots + O\left(\theta^2\right) \\
&= \theta\mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta^2}{2}\left[\mathbb{E}\left[\Psi^2\left(T\right)\right] - \left(\mathbb{E}\left[\Psi\left(T\right)\right]\right)^2\right] + O\left(\theta^2\right) \\
&= \theta\mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta^2}{2}\mathbb{V}ar\left(\Psi\left(T\right)\right) + O\left(\theta^2\right).
\end{aligned}
$$

This implies that

$$
\Phi\left(\theta\right) = \frac{1}{\theta}\log \mathbb{E}\left[\exp\left(\theta\Psi\left(T\right)\right)\right] = \mathbb{E}\left[\Psi\left(T\right)\right] + \frac{\theta}{2}\mathbb{V}ar\left(\Psi\left(T\right)\right) + O\left(\theta^2\right).
$$

The proof is completed. ■

Chapter 2

Pontryagin's risk-sensitive stochastic maximum principle for fully coupled FBSDE with applications

In this chapter, we are interested in the problem of optimal control where the system is given by a fully coupled forward-backward stochastic differential equation with a risk-sensitive performance functional. As a preliminary step, we use the risk neutral which is an extension of the initial control problem where the admissible controls are convex, and an optimal solution exists.Then, we study the necessary as well as sufficient optimality conditions for risk sensitive performance. At the end of this chapter, we illustrate our main result by giving two examples of risk sensitive control problem under linear stochastic dynamics with exponential quadratic cost function.The second example deals with an optimal portfolio choice problem in financial market specially the model of control cash flow of a firm or project where, for instance, we can set the model of pricing and managing an insurance contract.

2.1 Formulation of the problem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtred probability space satisfying the usual conditions, in which a $d-$dimensional Brownian motion $W = (W_t : 0 \leq t \leq T)$ is defined. We assume that $(\mathcal{F}_t)$

is defined by $\forall t \geq 0$, $\mathcal{F}_t = \sigma(W(r); 0 \leq r \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ denote the totality of $\mathbb{P}-$null sets. For any $n \in \mathbb{N}$, let $\mathcal{M}^2([0,T];\mathbb{R}^n)$ denotes the set of n dimensional jointly measurable random processes $\{\varphi_t, t \in [0,T]\}$ which satisfy:

$(i) : \mathbb{E}\left[\int_0^T |\varphi_t|^2 \, dt\right] < \infty,$

$(ii) : \varphi_t$ is $\left(\mathcal{F}_t^{(W)}\right)$ measurable, for $a.e.$ $t \in [0,T]$.

We denote similarly by $\mathcal{S}^2([0,T];\mathbb{R}^n)$ the set of continuous n dimensional random processes which satisfy:

$(i) : \mathbb{E}\left[\sup_{0 \leq t \leq T} |\varphi_t|^2\right] < \infty,$

$(ii) : \varphi_t$ is $\left(\mathcal{F}_t^{(W)}\right)$ measurable, for any $t \in [0,T]$.

Let T be a strictly positive real number and U is a convex nonempty subset of $\mathbb{R}^k$.

Definition 2.1 *An admissible control v is a process with values in U such that* $\mathbb{E}\left[\int_0^T |v_t|^2 \, dt\right] < \infty$. *We denote by $\mathcal{U}$ the set of all admissible controls.*

For any $v \in \mathcal{U}$, we consider the following fully coupled forward-backward system

$$\begin{cases} dx_t^v = b(t, x_t^v, y_t^v, z_t^v, v_t)\, dt + \sigma(t, x_t^v, y_t^v, z_t^v, v_t)\, dW_t, \\ dy_t^v = -g(t, x_t^v, y_t^v, z_t^v, v_t)\, dt + z_t^v dW_t, \\ x_0^v = x(0), \qquad y_T^v = a, \end{cases} \tag{2.1}$$

where $b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times U \to \mathcal{M}_{n\times d}(\mathbb{R})$, $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}^m$.

We define the criterion to be minimized, with initial and terminal risk-sensitive cost functional, as follows

$$J^\theta(v) = \mathbb{E}\left[\exp\theta\left\{\Phi(x_T^v) + \Psi(y_0^v) + \int_0^T f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt\right\}\right], \tag{2.2}$$

where θ is the risk-sensitive index, and $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Psi : \mathbb{R}^m \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}$.

The control problem is to minimize the functional J^θ over $\mathcal{U}$, if $u \in \mathcal{U}$ is an optimal control, that is

$$J^{\theta}(u) = \inf_{v \in \mathcal{U}} J^{\theta}(v). \tag{2.3}$$

Next, we give some notations $\Gamma = (x_t^v, y_t^v, z_t^v)^*$,

and $M(t,\Gamma) =: \begin{pmatrix} -G^T g \\ G^T b \\ G^T \sigma \end{pmatrix}(t,\Gamma)$,

We use the Euclidean norm $|.|$ in $\mathbb{R}$. All the equalities and inequalities, mentioned in this paper, are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0,T] \times \Omega$. We assume that

Assumption 2.1 *For each* $\Gamma \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d}$, $M(t,\Gamma)$ *is an* $\mathcal{F}_t-$*measurable process defined on* $[0,T]$ *with* $M(t,\Gamma) \in \mathcal{M}^2\left([0,T];\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d}\right)$.

Assumption 2.2 $M(t,.)$ *satisfies Lipschitz conditions*: *There exists a constant* $k > 0$, *such that* $|M(t,\Gamma) - M(t,\Gamma')| \leq k|\Gamma - \Gamma'|$ $\forall$ Γ, $\Gamma' \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d}$, $\forall t \in [0,T]$.

The following monotonic conditions introduced in Peng and Wu [29], are the main assumptions in this paper.

The coefficients g, b and σ are $G-$monotone i.e., there exists a nondegenerate $m \times n-$matrix G such that, for every fixed (w,t) the mapping $M(t,\Gamma)$ is monotonous in (x,y,z) in the sense of the assumption

Assumption 2.3 $\langle M(t,\Gamma) - M(t,\Gamma'), \Gamma - \Gamma\rangle \leq \beta_1 |G(x-x')|^2 - \beta_2 |G(y-y')|^2$, $\forall$ $\Gamma = (x,y,z)^*$; $\Gamma' = (x',y',z')^* \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d}$, $\forall$ $t \in [0,T]$, *where* β_1 *and* β_2 *are given nonnegative constants with* $\beta_1 + \beta_2 > 0$.

Proposition 2.1 *For any given admissible control* $v(.)$, *we assume that* $2.1 - 2.3$ *hold. Then the fully coupled FBSDE* (2.1) *has the unique solution* $(x_t^v, y_t^v, z_t^v) \in \left(\mathcal{M}^2(0,T;\mathbb{R}^n \times \mathbb{R}^m)\right)^2 \times \mathcal{S}^2\left(0,T;\mathbb{R}^{n\times d}\right)$.

Proof. The proof can be seen in Peng and Wu [29]. ■

A control that solves the problem $\{(2.1),(2.2),(2.3)\}$ is called optimal. Our goal is to establish necessary, as well as sufficient conditions of optimality, satisfied by a given optimal control, in the form of stochastic maximum principle with a risk-sensitive performance functional type.

We also assume that

Assumption 2.4 *i)* b, σ, g, f, Φ *and* Ψ *are continuously differentiable with respect to* (x^v, y^v, z^v, v).

ii) The derivatives of b, σ, g and f are bounded by $C(1 + |x^v| + |y^v| + |z^v| + |v|)$.

iii) The derivatives of Φ, Ψ are bounded by $C(1 + |x^v|)$ and $C(1 + |y^v|)$ respectively.

Under the above assumptions, for every $v \in \mathcal{U}$ equation (2.1) has a unique strong solution and the function cost J^θ is well defined from $\mathcal{U}$ into $\mathbb{R}$.

2.2 Risk-sensitive stochastic maximum principle of fully coupled forward-backward control problem type

First of all, we may introduce an auxiliary state process ξ_t^v which is the solution of the following stochastic differential equation (SDE in short):

$$d\xi_t^v = f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt, \qquad \xi_0^v = 0.$$

From the above auxiliary process, the fully coupled forward-backward type control problem is equivalent to

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\Phi(x_T^v) + \Psi(y_0^v) + \xi_T^v\right\}\right] = \inf_{v\in\mathcal{U}} \mathbb{E}\left[\Gamma(x_T^v, y_0^v, \xi_T^v)\right], \\ \text{subject to} \\ d\xi_t^v = f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt, \\ dx_t^v = b(t, x_t^v, y_t^v, z_t^v, v_t)\, dt + \sigma(t, x_t^v, y_t^v, z_t^v, v_t)\, dW_t, \\ dy_t^v = -g(t, x_t^v, y_t^v, z_t^v, v_t)\, dt + z_t^v dW_t, \\ \xi_0^v = 0, \qquad x_0^v = x^v(0), \qquad y_T^v = a. \end{cases} \tag{2.4}$$

We denote by $A_T^\theta := \exp\theta\left\{\Phi(x_T^v) + \Psi(y_0^v) + \int_0^T f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt\right\}$, and we can put also

$\Theta_T := \Phi(x_T^v) + \Psi(y_0^v) + \int_0^T f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt$, the risk-sensitive loss functional is given by

$$\begin{aligned}\Theta_\theta &:= \frac{1}{\theta} \log \mathbb{E}\left[\exp \theta \left\{ \Phi(x_T^v) + \Psi(y_0^v) + \int_0^T f(t, x_t^v, y_t^v, z_t^v, v_t)\, dt \right\}\right] \\ &= \frac{1}{\theta} \log \mathbb{E}\left[\exp\{\theta \Theta_T\}\right].\end{aligned}$$

When the risk-sensitive index θ is small, the functional Θ_θ can be expanded as $\mathbb{E}[\Theta_T] + \frac{\theta}{2} Var(\Theta_T) + O(\theta^2)$, where, $Var(\Theta_T)$ denotes the variance of Θ_T. If $\theta < 0$, the variance of Θ_T, as a measure of risk, improves the performance Θ_θ, in which case the optimizer is called *risk seeker.* But, when $\theta > 0$, the variance of Θ_T worsens the performance Θ_θ, in which case the optimizer is called *risk averse.* The risk-neutral loss functional $\mathbb{E}(\Theta_T)$ can be seen as a limit of risk-sensitive functional Θ_θ when $\theta \to 0$, for more details the reader can see the chapter's book [7].

Notation 2.1 *We will use the following notation throughout the paper. For* $\phi \in \left\{b, \sigma, f, g, H^\theta, \widetilde{H}^\theta\right\}$, *we define*

$$\begin{cases} \phi(t) = \phi(t, x_t^v, y_t^v, z_t^v, v_t), \\ \partial\phi(t) = \phi(t, x_t^v, y_t^v, z_t^v, v_t) - \phi(t, x_t^u, y_t^u, z_t^u, u_t), \\ \phi_\zeta(t) = \frac{\partial \phi}{\partial \zeta}(t, x_t^v, y_t^v, z_t^v, v_t), \qquad \zeta = x,\ y,\ z. \end{cases}$$

Where v_t in an admissible control from $\mathcal{U}$.

We assume that the assumptions 2.1, 2.2, 2.3 and 2.4 hold, we may apply the stochastic maximum principle for risk-neutral of fully coupled forward-backward type control from Yong [40] to augmented state dynamics (ξ, x, y, z) to derive the adjoint equation. There exist unique $\mathcal{F}_t-$adapted pairs of processes $((p_1, q_1), (p_2, q_2), (p_3, q_3))$, which solve the following system matrix of backward SDEs

$$\begin{cases} d\overrightarrow{p}(t) = \begin{pmatrix} dp_1(t) \\ dp_2(t) \\ dp_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ \text{-}f_y(t) & \text{-}b_y(t) & \text{-}g_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} dt \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \text{-}\sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dt - \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dW_t, \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} = -\theta A_T \begin{pmatrix} 1 \\ \Phi_x(x_T^u) \end{pmatrix}, \qquad \text{and } p_3(0) = -\theta \Psi_y(y_0^u) A_T, \end{cases} \tag{2.5}$$

with $\mathbb{E}\left[\sum_{i=1}^{3} \sup_{0\le t\le T} |p_i(t)|^2 + \sum_{i=1}^{2} \int_0^T |q_i(t)|^2 dt\right] < \infty$, and

$q_3(t) = Tr\left[\begin{pmatrix} f_z(t) & b_z(t) \\ \sigma_z(t) & g_z(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix}\right]$.

To this end, we may define (2.5) in the compact form as:

$$\begin{cases} d\overrightarrow{p}(t) = \begin{pmatrix} dp_1(t) \\ dp_2(t) \\ dp_3(t) \end{pmatrix} = F(t)\,dt - \Sigma(t)\,dW_t \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} = -\theta A_T \begin{pmatrix} 1 \\ \Phi_x(x_T^u) \end{pmatrix}, \qquad \text{and } p_3(0) = -\theta \Psi_y(y_0^u) A_T, \end{cases}$$

where

$$\Sigma(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix},$$

$$F(t) = \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ -f_y(t) & -b_y(t) & -g_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & -\sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}.$$

We suppose here that $\widetilde{H}^\theta$ is the Hamiltonian associated with the optimal state dynamics (ξ^u, x^u, y^u, z^u), and the pair of adjoint processes $(\overrightarrow{p}(t), \overrightarrow{q}(t))$ given by

$$\widetilde{H}^\theta (t, \xi_t^u, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)) := \begin{pmatrix} f(t) \\ b(t) \\ g(t) \end{pmatrix} (\overrightarrow{p}(t))^* + \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \end{pmatrix} (\overrightarrow{q}(t))^* .$$

Theorem 2.1 *Assume that* 2.1, 2.2, 2.3 *and* 2.4 *hold. If* (ξ^u, x^u, y^u, z^u) *is an optimal solution of the risk-neutral control problem* (3.4), *then there exist pairs of* $\mathcal{F}_t-$*adapted processes* $((p_1, q_1), (p_2, q_2), (p_3, q_3))$ *that satisfy* (2.5), *such that*

$$\widetilde{H}_v^\theta (t) (v_t - u_t) \geq 0, \tag{2.6}$$

for all $u \in \mathcal{U}$, *almost every* $t \in [0, T]$, *and* $\mathbb{P}-$*almost surely, where* $\widetilde{H}_v^\theta (t)$ *is defined in notation* 2.1.

2.2.1 How to find the new adjoint equation ?

As we said, Theorem 2.1 is a good SMP for the risk-neutral of forward backward control problem. We follow the approach used in [8, 11], and suggest a transformation of the adjoint processes (p_1, q_1), (p_2, q_2) and (p_3, q_3) in such a way to omit the first component (p_1, q_1) in (2.5), and to obtain the SMP in terms of only the last two adjoint processes, that we denote by

$\left(\left(\widetilde{p}_2, \widetilde{q}_2\right), \left(\widetilde{p}_3, \widetilde{q}_3\right)\right)$. Noting that $dp_1(t) = q_1(t)\, dW_t$ and $p_1(T) = -\theta A_T^\theta$, the explicit solution of this backward SDE is

$$p_1(t) = -\theta \mathbb{E}\left[A_T^\theta \mid \mathcal{F}_t\right] = -\theta V_t^\theta, \tag{2.7}$$

where

$$V_t^\theta := \mathbb{E}\left[A_T^\theta \mid \mathcal{F}_t\right], \qquad 0 \le t \le T. \tag{2.8}$$

As a good look of (2.7), it would be natural to choose a transformation of $(\overrightarrow{p}, \overrightarrow{q})$ into an adjoint process $(\widetilde{p}, \widetilde{q})$, where $\widetilde{p}_1(t) = \dfrac{1}{\theta V_t^\theta} p_1(t) = -1$.

We consider the following transform

$$\widetilde{p}(t) = \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} := \frac{1}{\theta V_t^\theta} \overrightarrow{p}(t), \qquad 0 \le t \le T. \tag{2.9}$$

By using (2.5) and (2.9), we have

$$\widetilde{p}(T) := \begin{pmatrix} \widetilde{p}_1(T) \\ \widetilde{p}_2(T) \end{pmatrix} = -\begin{pmatrix} 1 \\ \Phi_x\left(x_T^u\right) \end{pmatrix}, \text{ and } \widetilde{p}_3(0) = -\Psi_y\left(y_0^u\right).$$

The following properties of the generic martingale V^θ are essential in order to investigate the properties of these new processes $\left(\widetilde{p}(t), \widetilde{q}(t)\right)$.

The process Λ^θ is the first component of the $\mathcal{F}_t-$adapted pair of processes $\left(\Lambda^\theta, l\right)$, which is the unique solution to the following quadratic backward SDE

$$\begin{cases} d\Lambda_t^\theta = -\left\{f(t) + \frac{\theta}{2}\left|l(t)\right|^2\right\} dt + l(t)\, dW_t, \\ \Lambda_T^\theta = \Psi\left(y_0^u\right), \end{cases} \tag{2.10}$$

where

$$\mathbb{E}\left[\int_0^T \left|l(t)\right|^2 dt\right] < \infty.$$

In the next, we will state and prove the necessary conditions of optimality for the system driven by fully coupled FBSDE with a risk sensitive performance functional type. To this end, let us summarize and prove some lemmas that we will use thereafter.

Lemma 2.1 *Suppose that* 2.4 *holds. Then*

$$\mathbb{E}\left[\sup_{0\leq t\leq T.}\left|\Lambda_t^{\theta}\right|^2\right]\leq C_T, \tag{2.11}$$

In particular, V^{θ} *solves the following linear backward SDE*

$$dV_t^{\theta}=\theta l\left(t\right)V_t^{\theta}dW_t,\qquad V_T^{\theta}=A_T^{\theta}. \tag{2.12}$$

Hence, the process defined on $\left(\Omega,\mathcal{F},\left(\mathcal{F}_t^{(W)}\right)_{t\geq0},\mathbb{P}\right)$ *by*

$$L_t^{\theta}:=\frac{V_t^{\theta}}{V_0^{\theta}}=\exp\left(\int_0^t\theta l\left(s\right)dW_s-\frac{\theta^2}{2}\int_0^t\left|l\left(s\right)\right|^2ds\right),\quad 0\leq t\leq T, \tag{2.13}$$

is a uniformly bounded $\mathcal{F}_t-$*martingale.*

Proof. First we prove (2.11). We assume that 2.4 holds, f, Φ and Ψ are bounded by a constant $C>0$, we have

$$0<e^{-(2+T)C\theta}\leq A_T^{\theta}\leq e^{(2+T)C\theta}. \tag{2.14}$$

Therefore, V^{θ} is a uniformly bounded $\mathcal{F}_t-$martingale satisfying

$$0<e^{-(2+T)C\theta}\leq V_t^{\theta}\leq e^{(2+T)C\theta},\qquad 0\leq t\leq T. \tag{2.15}$$

The sufficient conditions of the Logarithmic transform established in ([14], Proposition 3.1) can be applied to the martingale V^{θ} as follows

$$V_t^{\theta}=\exp\left(\theta\Lambda_t^{\theta}+\theta\int_0^t f\left(s\right)ds\right),\qquad 0\leq t\leq T,$$

and $V^{\theta}\left(0\right)=\exp\left(\theta\Lambda_0\right)=\mathbb{E}\left[A_T^{\theta}\right]$. It is easy to see from (2.15), and the boundedness of f

that

$$\mathbb{E}\left[\sup_{0\leq t\leq T.}\left|\Lambda_t^\theta\right|^2\right]\leq C_T,$$

where C_T is a positive constant that only depends on T, and the boundedness of f, and Ψ.

Second, we may find the explicit form of (2.12), by using the second Itô's formula to $V_t^\theta=\exp\left(\theta\Lambda_t^\theta+\theta\int_0^t f(s)\,ds\right)$.

$$\begin{aligned}
d\left(V_t^\theta\right) &= d\left[\exp\left(\theta\int_0^t f(s)\,ds\right).\exp\left(\theta\Lambda_t^\theta\right)\right] \\
&= \theta f(t)\exp\left(\theta\Lambda_t^\theta+\theta\int_0^t f(s)\,ds\right) \\
&+ \theta\left(d\Lambda_t^\theta\right).\exp\left(\theta\Lambda_t^\theta+\theta\int_0^t f(s)\,ds\right)+\frac{1}{2}\theta^2 l^2(t).\exp\left(\theta\Lambda_t^\theta+\theta\int_0^t f(s)\,ds\right)dt \\
&= \theta l(t)\,V_t^\theta dW_t.
\end{aligned}$$

Now, we can prove (2.13), by starting from the integral form of (2.12), such that $V_t^\theta=\theta\int_0^t l(s)\,V^\theta(s)\,dW_s$. On the other hand, we have

$$V_t^\theta=\exp\left(\theta\int_0^t f(s)\,ds\right).\exp\left(\theta\Lambda_t^\theta\right).$$

By replacing Λ_t^θ in (2.10), we have

$$\frac{V_t^\theta}{V_0^\theta}=\exp\left(\int_0^t \theta l(s)\,dW_s-\frac{\theta^2}{2}\int_0^t |l(s)|^2\,ds\right):=L_t^\theta.$$

In view of (2.11), the last expression (2.13) is a uniformly bounded $\mathcal{F}_t$-martingale. ■

Lemma 2.2 *The second and the third risk-sensitive adjoint equations for* $\left(\widetilde{p}_2(t),\widetilde{q}_2(t)\right)$,

$(\widetilde{p}_3(t), \widetilde{q}_3(t))$ and $\left(V^\theta(t), l(t)\right)$ become

$$\begin{cases} d\widetilde{p}_2(t) = H_x^\theta(t)\,dt + (\widetilde{q}_2(t) + \theta l(t)\widetilde{p}_2(t))\,dW_t^\theta, \\ d\widetilde{p}_3(t) = -H_y^\theta(t)\,dt - H_z^\theta(t)\,dW_t^\theta, \\ dV_t^\theta = \theta l(t) V_t^\theta dW_t, \\ V_T^\theta = A_T^\theta, \\ \widetilde{p}_2(T) = -\Phi_x(x_T) \qquad , \widetilde{p}_3(0) = -\Psi_y(y_0). \end{cases} \tag{2.16}$$

The solution $\left(\widetilde{p}(t), \widetilde{q}(t), V^\theta(t), l(t)\right)$ of the system (2.16) is unique, such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\widetilde{p}(t)|^2 + \sup_{0\le t\le T}\left|V_t^\theta\right|^2 + \int_0^T \left(|\widetilde{q}(t)|^2 + |l(t)|^2\right)dt\right] < \infty, \tag{2.17}$$

where

$$H^\theta\left(t, x_t, y_t, z_t, \begin{pmatrix} \widetilde{p}_2(t) \\ \widetilde{q}_2(t) \end{pmatrix}, \begin{pmatrix} \widetilde{p}_3(t) \\ 0 \end{pmatrix}, V_t^\theta, l(t)\right) \tag{2.18}$$
$$= b(t)\widetilde{p}_2(t) + \sigma(t)\widetilde{q}_2(t) + (g(t) - z_t\theta l(t))\widetilde{p}_3(t) - f(t).$$

Proof. We want to identify the processes $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$d\widetilde{p}(t) = -\widetilde{\alpha}(t)\,dt + \widetilde{\beta}(t)\,dW_t.$$

By applying Itô's formula to the process $\overrightarrow{p}(t) = \theta V_t^\theta \widetilde{p}(t)$, and using the expression of V^θ in (2.12), we obtain

$$
\begin{aligned}
d\widetilde{p}(t) &= \frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ -f_y(t) & -b_y(t) & -g_y(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} dt \\
&+ \frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & -\sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dt - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{\beta}(t)\, dt \\
&- \frac{1}{\theta V_t^\theta}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dW_t + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{p}(t)\, dW_t.
\end{aligned}
$$

By identifying the coefficients, we get the diffusion term

$$
\widetilde{\beta}(t) = -\frac{1}{\theta V_t^\theta}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{p}(t),
$$

and the drift term of the process $\widetilde{p}(t)$

$$
\begin{aligned}
\widetilde{\alpha}(t) &= \frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ -f_x(t) & -b_x(t) & -g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} \\
&+ \frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{\beta}(t).
\end{aligned}
$$

Using the relation $\widetilde{p}(t) = \dfrac{1}{\theta V_t^\theta}\overrightarrow{p}(t)$, the diffusion coefficient $\widetilde{\beta}(t)$ will be

$$\widetilde{\beta}(t) = -\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{p}(t),$$

and the drift term of the process $\widetilde{p}(t)$

$$\begin{aligned}\widetilde{\alpha}(t) = &\begin{pmatrix} 0 & 0 & 0 \\ \text{-}f_x(t) & \text{-}b_x(t) & \text{-}g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{-}\sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{\beta}(t).\end{aligned}$$

We finally obtain

$$\begin{aligned}d\widetilde{p}(t) = &- \begin{pmatrix} 0 & 0 & 0 \\ \text{-}f_x(t) & \text{-}b_x(t) & \text{-}g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt \\ &- \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{-}\sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{\beta}(t)\, dt \\ &+ \widetilde{\beta}(t)\, dW_t.\end{aligned}$$

It is easily verified that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\left[-\theta l_1(t)\, dt + dW_t\right], \qquad \widetilde{p}_1(T) = -1.$$

In view of (2.13), we may use Girsanov's theorem 1.3 to claim that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\, dW_t^\theta, \qquad \mathbb{P}^\theta - a.s. \quad \widetilde{p}_1(T) = -1,$$

where

$$dW_t^\theta = -\theta l(t)\, dt + dW_t, \tag{2.19}$$

is a $\mathbb{P}^\theta-$Brownian motion, where, $\left.\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} := L_t^\theta = \exp\left(\int_0^t \theta l(s)\, dW_s - \frac{\theta^2}{2}\int_0^t |l(s)|^2\, ds\right)$, $0 \leq t \leq T$. But according to (2.14) and (2.13), the probability measures $\mathbb{P}^\theta$ and $\mathbb{P}$ are in fact equivalent. Hence, noting that $\widetilde{p}_1(t) := \dfrac{1}{\theta V_t^\theta} p_1(t)$ is square-integrable, we get that $\widetilde{p}_1(t) = \mathbb{E}^{\mathbb{P}^\theta}[\widetilde{p}_1(T) \mid \mathcal{F}_t] = -1$. Thus, its quadratic variation $\int_0^T |\widetilde{q}_1(t)|^2\, dt = 0$. This implies that, for almost every $0 \leq t \leq T$, $\widetilde{q}_1(t) = 0$, $\mathbb{P}^\theta$ and $\mathbb{P}-$a.s.

$$\begin{aligned} d\widetilde{p}(t) = &-\begin{pmatrix} 0 & 0 & 0 \\ \text{-}f_x(t) & \text{-}b_x(t) & \text{-}g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{-}\sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt + \widetilde{\beta}(t)\, dW_t^\theta. \end{aligned}$$

Now we use the relation $\widetilde{\beta}(t) = -\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{H}_z(t) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{p}(t)$, in the equation above, to obtain

$$d\widetilde{p}(t) = -\begin{pmatrix} 0 & 0 & 0 \\ \text{-}f_x(t) & \text{-}b_x(t) & \text{-}g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt \tag{2.20}$$
$$-\begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{-}\sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt$$
$$+\left\{-\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ f_z(t)\widetilde{p}_1(t) + b_z(t)\widetilde{p}_2(t) + g_z(t)\widetilde{p}_3(t) + \sigma_z(t)\widetilde{q}_2(t) \end{pmatrix}\right.$$
$$\left. +\theta\begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{p}(t)\right\} dW_t^{\theta}.$$

Therefore, the second and third components of $\widetilde{p}_2$ and $\widetilde{p}_3$ given by (2.20), are given by

$$d\widetilde{p}_2(t) = \{b_x(t)\widetilde{p}_2(t) + g_x(t)\widetilde{p}_3(t) + \sigma_x(t)\widetilde{q}_2(t) - f_x(t)\} dt \tag{2.21}$$
$$+ \{\widetilde{q}_2(t) + \theta l_2(t)\widetilde{p}_2(t)\} dW_t^{\theta},$$

$$d\widetilde{p}_3(t) \tag{2.22}$$
$$= -\{b_y(t)\widetilde{p}_2(t) + g_y(t)\widetilde{p}_3(t) + \widetilde{q}_2(t)\sigma_y(t) - f_y(t)\} dt$$
$$- \{-f_z(t) + b_z(t)\widetilde{p}_2(t) + g_z(t)\widetilde{p}_3(t) + \sigma_z(t)\widetilde{q}_2(t) - \theta l_3(t)\widetilde{p}_3(t)\} dW_t^{\theta},$$

and the second and third risk-sensitive adjoint equations for $(\widetilde{p}_2, \widetilde{q}_2)$, $(\widetilde{p}_3, \widetilde{q}_3)$ and $\left(V^{\theta}, l\right)$

become

$$\begin{cases} d\widetilde{p}_2(t) = H_x^\theta(t)\,dt + \left(\widetilde{q}_2(t) + \theta l_2(t)\,\widetilde{p}_2(t)\right) dW_t^\theta, \\ d\widetilde{p}_3(t) = -H_y^\theta(t)\,dt - H_z^\theta(t)\,dW_t^\theta, \\ dV_t^\theta = \theta l(t)\,V_t^\theta dW_t, \\ V_T^\theta = A_T^\theta, \\ \widetilde{p}_2(T) = -\Phi_x(x_T) \qquad , \widetilde{p}_3(0) = -\Psi_y(y_0). \end{cases} \tag{2.23}$$

The solution $\left(\widetilde{p}, \widetilde{q}, V^\theta, l\right)$ of the system (2.23) is unique, such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\widetilde{p}(t)|^2 + \sup_{0\le t\le T}\left|V_t^\theta\right|^2 + \int_0^T \left(|\widetilde{q}(t)|^2 + |l(t)|^2\right) dt\right] < \infty,$$

where

$$H^\theta(t) := H^\theta\left(t, x_t, y_t, z_t, \begin{pmatrix} \widetilde{p}_2(t) \\ \widetilde{q}_2(t) \end{pmatrix}, \begin{pmatrix} \widetilde{p}_3(t) \\ 0 \end{pmatrix}, V_t^\theta, l(t)\right)$$
$$= b(t)\,\widetilde{p}_2(t) + \sigma(t)\,\widetilde{q}_2(t) + \left(g(t) - z_t\theta l(t)\right)\widetilde{p}_3(t) - f(t).$$

This completed the proof. ■

Theorem 2.2 *(Risk-Sensitive Stochastic Maximum Principle): We assume that* 2.4 *holds, if* (x^u, y^u, z^u, u) *is an optimal solution of the risk-sensitive control problem* $\{(2.1), (2.2), (2.3)\}$, *then there exist pairs of* $\mathcal{F}_t$*-adapted processes* $\left(V^\theta(t), l(t)\right)$, *and* $\begin{pmatrix} \widetilde{p}_2(t) \\ \widetilde{q}_2(t) \end{pmatrix}, \begin{pmatrix} \widetilde{p}_3(t) \\ 0 \end{pmatrix}$ *that satisfy* (2.16), (2.17) *such that*

$$H_v^\theta(t)(v - u) \ge 0, \tag{2.24}$$

for all $u \in \mathcal{U}$, *almost every* $0 \le t \le T$ *and* $\mathbb{P}$*-almost surely, where the Hamiltonian* $\widetilde{H}^\theta$

associated with (2.4), *is given by*

$$\widetilde{H}^{\theta}\left(t,\xi_t^u,x_t^u,y_t^u,z_t^u,\overrightarrow{p}_t,\overrightarrow{q}_t\right) \tag{2.25}$$
$$=\left\{\theta V_t^{\theta}\right\}H^{\theta}\left(t,x_t^u,y_t^u,z_t^u,\begin{pmatrix}\widetilde{p}_2(t)\\ \widetilde{q}_2(t)\end{pmatrix},\begin{pmatrix}\widetilde{p}_3(t)\\ 0\end{pmatrix},V_t^{\theta},\begin{pmatrix}l_2(t)\\ l_3(t)\end{pmatrix}\right),$$

and H^{θ} *is the risk-sensitive Hamiltonian given by* (2.18).

Proof. To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\widetilde{p}_2,\widetilde{q}_2)$, $(\widetilde{p}_3,\widetilde{q}_3)$ and $\left(V^{\theta},l\right)$, which solve (2.16). Hence, since $V^{\theta}>0$, the variational inequality (2.6) translates into $H_v^{\theta}(t)\geq 0$, for all $u\in\mathcal{U}$, almost every $0\leq t\leq T$ and $\mathbb{P}$-almost surely. ■

2.3 Risk sensitive sufficient optimality conditions

This section is concerned with a study of the necessary condition of optimality (2.24) when it becomes sufficient.

Theorem 2.3 *(Risk neutral sufficient optimality conditions) Assume that* $\Phi(.)$ *and* $\Psi(.)$ *are convex and for all* $(x,y,z,v)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{n\times d}\times U$ *the function* $\widetilde{H}^{\theta}(.,x,y,z,v,.,.)$ *is concave and for any* $v\in U$ *such that* $\mathbb{E}\left[\int_0^T |v|^2dt\right]<\infty$ *then, u is an optimal control of the problem* $\{(2.1),(2.2),(2.3)\}$, *if it satisfies* (2.6).

Proof. Let u be an admissible control (candidate to be optimal) for any $v\in U$, we have

$$J^{\theta}(v)-J^{\theta}(u)$$
$$=\mathbb{E}\left[\exp\left\{\theta\Psi\left(y_0^v\right)+\theta\Phi\left(x_T^v\right)+\theta\xi_T^v\right\}\right]-\mathbb{E}\left[\exp\left\{\theta\Psi\left(y_0^u\right)+\theta\Phi\left(x_T^u\right)+\theta\xi_T^u\right\}\right].$$

Since Ψ and Φ are convex and by applying Taylor's expansion, we get

$$\begin{aligned}
&J^{\theta}(v) - J^{\theta}(u) \\
&\geq \mathbb{E}\left[\theta A_T\left(\xi_T^v - \xi_T^u\right)\right] + \mathbb{E}\left[\theta \Phi_x\left(x_T^u\right) A_T\left(x_T^v - x_T^u\right)\right] + \mathbb{E}\left[\theta \Psi_y\left(y_0^u\right) A_T\left(y_0^v - y_0^u\right)\right].
\end{aligned}$$

According to (2.5), we remark that $p_1(T) = \theta A_T$, $p_2(T) = \theta \Phi_x\left(x_T^u\right) A_T$ and $p_3(0) = \theta \Psi_y\left(y_0^u\right) A_T$, then

$$\begin{aligned}
&J^{\theta}(v) - J^{\theta}(u) \\
&\geq \mathbb{E}\left[p_1(T)\left(\xi_T^v - \xi_T^u\right)\right] + \mathbb{E}\left[p_2(T)\left(x_T^v - x_T^u\right)\right] + \mathbb{E}\left[p_3(0)\left(y_0^v - y_0^u\right)\right].
\end{aligned} \tag{2.26}$$

We apply Itô's formula to $p_1(t)\left(\xi_t^v - \xi_t^u\right)$

$$\begin{aligned}
d\left(p_1(t)\left(\xi^v(t) - \xi^u(t)\right)\right) = \; & \left(\xi^v(t) - \xi^u(t)\right) dp_1(t) + p_1(t)\, d\left(\xi^v(t) - \xi^u(t)\right) \\
& + \left\langle\left(\xi^v - \xi^u\right), p_1\right\rangle_t dt
\end{aligned},$$

then

$$\begin{aligned}
\int_0^T \left(p_1(t)\left(\xi^v(t) - \xi^u(t)\right)\right) dt \; = & \int_0^T \left(f\left(t, x^v(t), y^v(t), z^v(t), v_t\right) -\right. \\
& \left. f\left(t, x^u(t), y^u(t), z^u(t), u_t\right)\right) q_1(t)\, dW_t \\
& + \int_0^T \left(f\left(t, x^v(t), y^v(t), z^v(t), v_t\right) -\right. \\
& \left. f\left(t, x^u(t), y^u(t), z^u(t), u_t\right)\right) p_1(t)\, dt
\end{aligned}$$

We apply expectation, we get

$$\mathbb{E}\left[p_1(T)\left(\xi_T^v - \xi_T^u\right)\right] = \mathbb{E}\left[\int_0^T \left(f\left(t, x_t^v, y_t^v, z_t^v, v_t\right) - f\left(t, x_t^u, y_t^u, z_t^u, u_t\right) p_1\right) dt\right], \tag{2.27}$$

We apply also Itô's formula to $p_2(t)\left(x_t^v - x_t^u\right)$

$$\begin{aligned}
d\left(p_2(t)\left(x^v(t) - x^u(t)\right)\right) = \; & \left(x^v(t) - x^u(t)\right) dp_2(t) + p_2(t)\, d\left(x^v(t) - x^u(t)\right) \\
& + \left\langle x^v - x^u, p_2\right\rangle_t dt
\end{aligned}$$

then

$$\begin{aligned}\int_0^T d\left(p_2(t)\left(x^v(t)-x^u(t)\right)\right)= & \int_0^T \left(b\left(t,x^v(t),y^v(t),z^v(t),v_t\right)-\right.\\ & \left.b\left(t,x^u(t),y^u(t),z^u(t),u_t\right)\right)p_2(t)\,dt \\ & +\int_0^T \left(\sigma\left(t,x^v(t),y^v(t),z^v(t),v_t\right)-\right.\\ & \left.\sigma\left(t,x^u(t),y^u(t),z^u(t),u_t\right)\right)p_2(t)\,dW_t \\ & +\int_0^T -\left(f_x(t)\,p_1+b_x(t)\,p_2+\sigma_x(t)\,q_2\right.\\ & \left.+g_x(t)\,p_3\right)\left(x_t^v-x_t^u\right)dt \\ & +\int_0^T q_2(t)\left(x_t^v-x_t^u\right)dW_t\end{aligned}$$

We apply expectation, we get

$$\begin{aligned}& \mathbb{E}\left[p_2(T)\left(x_T^v-x_T^u\right)\right] \\ = & \,\mathbb{E}\left[\int_0^T \left\{f_x\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_1(t)+b_x\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_2(t)\right.\right. \\ & +\sigma_x\left(t,x_t^u,y_t^u,z_t^u,u_t\right)q_2(t)+g_x\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_3(t) \\ & \left.\left.+\sigma_x\left(t,x_t^u,y_t^u,z_t^u,u_t\right)q_2(t)\right\}\left(x_t^v-x_t^u\right)dt\right] \\ & +\mathbb{E}\left[\int_0^T \left(b\left(t,x_t^v,y_t^v,z_t^v,v_t\right)-b\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_2(t)\right)dt\right] \\ & +\mathbb{E}\left[\int_0^T \left(\sigma\left(t,x_t^v,y_t^v,z_t^v,v_t\right)-\sigma\left(t,x_t^u,y_t^u,z_t^u,u_t\right)q_2(t)\right)dt\right],\end{aligned} \tag{2.28}$$

And We apply also Itô's formula to $p_3(t)\left(y_t^v-y_t^u\right)$,

$$\begin{aligned}d\left(p_3(t)\left(y^v(t)-y^u(t)\right)\right)= & \left(y^v(t)-y^u(t)\right)dp_3(t)+p_3(t)\,d\left(y^v(t)-y^u(t)\right) \\ & +\left\langle y^v-y^u,p_3\right\rangle_t dt\end{aligned}$$

then
$$\begin{aligned}\int_0^T d\left(p_3(t)\left(y^v(t)-y^u(t)\right)\right) = & \int_0^T \left(g\left(t,x^v(t),y^v(t),z^v(t),v_t\right)-\right.\\ & \left. g\left(t,x^u(t),y^u(t),z^u(t),u_t\right)\right)p_3(t)\,dt\\ & +\int_0^T \left(z^v(t)-z^u(t)\right)p_3(t)\,dW_t\\ & +\int_0^T -\left(f_y(t)\,p_1+b_y(t)\,p_2+\sigma_y(t)\,q_2\right.\\ & \left.+g_y(t)\,p_3\right)\left(y_t^v-y_t^u\right)dt\\ & +\int_0^T -\left(f_z(t)\,p_1+b_z(t)\,p_2+\sigma_z(t)\,q_2\right.\\ & \left.+g_z(t)\,p_3\right)\left(y_t^v-y_t^u\right)dW_t\end{aligned}$$

We apply expectation, we get

$$\begin{aligned}& \mathbb{E}\left[p_3(0)\left(y_0^v-y_0^u\right)\right] \\ & =\mathbb{E}\left[\int_0^T \left(g\left(t,x_t^v,y_t^v,z_t^v,v_t\right)-g\left(t,x_t^u,y_t^u,z_t^u,u_t\right)\right)p_3(t)\,dt\right]\\ & -\mathbb{E}\left[\int_0^T \left(f_y\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_1(t)+g_y\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_3(t)\right)\left(y_t^v-y_t^u\right)dt\right]\\ & -\mathbb{E}\left[\int_0^T \left(f_z\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_1(t)+g_z\left(t,x_t^u,y_t^u,z_t^u,u_t\right)p_3(t)\right)\left(z_t^v-z_t^u\right)dt\right].\end{aligned} \tag{2.29}$$

By replacing (2.27) ,(2.28) and (2.29) into (2.26), and using the fact that the Hamiltonian can written by the relationship (2.25), we have

$$\begin{aligned}& J^\theta(v)-J^\theta(u)\\ \geq & \mathbb{E}\left[\int_0^T \left(H^\theta\left(t,x_t^v,y_t^v,z_t^v,v_t,\widetilde{p}_t,\widetilde{q}_t\right)-H^\theta\left(t,x_t^u,y_t^u,z_t^u,u_t,\widetilde{p}_t,\widetilde{q}_t\right)\right)dt\right]\\ & +\mathbb{E}\left[\int_0^T H_x^\theta\left(t,x_t^u,y_t^u,z_t^u,v_t,\widetilde{p}_t,\widetilde{q}_t\right)\left(x_t^v-x_t^u\right)dt\right]\\ & +\mathbb{E}\left[\int_0^T H_y^\theta\left(t,x_t^u,y_t^u,z_t^u,u_t,\widetilde{p}_t,\widetilde{q}_t\right)\left(y_t^v-y_t^u\right)dt\right]\\ & +\left[\int_0^T H_z^\theta\left(t,x_t^u,y_t^u,z_t^u,u_t,\widetilde{p}_t,\widetilde{q}_t\right)\left(z_t^v-z_t^u\right)dt\right].\end{aligned}$$

Since the Hamiltonian $\widetilde{H}^{\theta}$ is concave with respect to (x, y, z, v), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(H^{\theta}\left(t, x_t^v, y_t^v, z_t^v, v_t, \widetilde{p}_t, \widetilde{q}_t\right) - H^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^T H_x^{\theta}\left(t, x_t^u, y_t^u, z_t^u, v_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(x_t^v - x_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_y^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(y_t^v - y_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_z^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(z_t^v - z_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_v^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(v_t - u_t\right) dt\right],
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T H_v^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(v_t - u_t\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \left(H^{\theta}\left(t, x_t^v, y_t^v, z_t^v, v_t, \widetilde{p}_t, \widetilde{q}_t\right) - H^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_x^{\theta}\left(t, x_t^u, y_t^u, z_t^u, v_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(x_t^u - x_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_y^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(y_t^u - y_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T H_z^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(z_t^u - z_t^u\right) dt\right].
\end{aligned}$$

Then

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T H_v^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \widetilde{p}_t, \widetilde{q}_t\right)\left(v_t - u_t\right) dt\right].$$

In virtue of the necessary condition of optimality (2.6), the last inequality implies that $J^{\theta}(v) - J^{\theta}(u) \geq 0$. Then, the theorem is improved. ■

Theorem 2.4 *(Risk sensitive sufficient optimality conditions) Assume that* $\Phi(.)$ *and* $\Psi(.)$ *are convex and for all* $(x, y, z, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times d} \times U$ *the function* $H^{\theta}(., x, y, z, v, ., .)$ *is concave and for any* $v \in U$ *such that* $\mathbb{E}\left[\int_0^T |v|^2 dt\right] < \infty$ *then,* u *is an optimal control of the problem* $\{(2.1), (2.2), (2.3)\}$*, if it satisfies* (2.24).

Remark 2.1 *This is not our aim to provide elaborate existence results for optimal controls.*

It should just be noted that usual existence results require some convexity on the dynamics since their proof usually relies on weak compactness properties. The result given below, whose early version was obtained by Filippov in [15], is standard.

2.4 Applications

2.4.1 Example 01: Application to the linear quadratic risk-sensitive control problem

We provide a concrete example of risk-sensitive forward backward stochastic linear quadratic problem, and we give the explicit optimal control and validate our major theoretical results in Theorem 2.3 (sufficient optimality conditions). Consider the following linear quadratic of risk-sensitive control problem

$$\begin{cases} \inf_{v \in \mathcal{U}} \mathbb{E}\left[\exp \theta \left\{ \frac{1}{2} y_0^v + \frac{1}{2} x_T^v + \int_0^T \frac{1}{2} v_t^2 dt \right\} \right], \\ \text{subject to} \\ dx_t^v = \left(A_t x_t^v + B_t v_t\right) dt + \left(C_t x_t^v + D_t v_t\right) dW_t, \\ dy_t^v = -\left(R_t x_t + a_t y_t + c_t z_t\right) dt + z_t dW_t, \\ x_0 = 0, \qquad y_T^v = a, \end{cases} \tag{2.30}$$

where A_t, B_t, C_t, and D_t are $n \times n$ bounded progressively measurable matrix-valued processes. We assume that the term $2A_t p_2(t)(x_t^v - x_t^u)$ is positive.
Recall that $A_T := \exp \theta \left\{ \frac{1}{2} y_0^v + \frac{1}{2} x_T^v + \int_0^T \frac{1}{2} v_t^2 dt \right\}$. Instantly, we give the Hamiltonian $\widetilde{H}^\theta$ defined by

$$\widetilde{H}^\theta(t, x_t, y_t, z_t, v_t, p(t), q(t)) = \frac{1}{2} v_t^2 p_1(t) + \left(A_t x_t^v + B_t v_t\right) p_2(t) + \left(C_t x_t^v + D_t v_t\right) q_2(t)$$
$$+ \left(R_t x_t + a_t y_t + c_t z_t\right) p_3(t).$$

Our adjoint equation to be defined in the current system as

$$\begin{cases} dp_1(t) = -q_1(t)\,dW_t, \\ dp_2(t) = \left(A_t p_2(t) + C_t q_2(t) + R_t p_3(t)\right) dt - q_2(t)\,dW_t, \\ dp_3(t) = -\left(a_t p_3(t)\right) dt - \left(c_t p_3(t)\right) dW_t, \\ p_1(T) = \theta A_T, \qquad p_2(T) = \theta x_T A_T, \qquad p_3(0) = \theta y_0 A_T. \end{cases} \tag{2.31}$$

We have $\widetilde{H}^{\theta}_x(t, x_t, y_t, z_t, v_t, p(t), q(t)) = A_t p_2(t) + C_t q_2(t) + R_t p_3(t)$, $\widetilde{H}^{\theta}_y(t, x_t, y_t, z_t, v_t, p(t), q(t)) = a_t p_3(t)$, $\widetilde{H}^{\theta}_z(t, x_t, y_t, z_t, v_t, p(t), q(t)) = c_t p_3(t)$, and $\widetilde{H}^{\theta}_v(t, x_t, y_t, z_t, v_t, p(t), q(t)) = v_t p_1(t) + B_t p_2(t) + D_t q_2(t)$. Maximizing the Hamiltonian yields

$$u_t = -\frac{1}{p_1(t)}\left(B_t p_2(t) + D_t q_2(t)\right). \tag{2.32}$$

Theorem 2.5 *(Risk-sensitive sufficient optimality condition for linear quadratic control problem): The function* (2.32)*, for all* $t \in [0, T]$*, is the unique optimal control for the linear quadratic control problem* (2.30)*, where* (x_t, y_t, z_t) *is the solution of the following FBSDE*

$$\begin{cases} dx^v_t = \left(A_t x^v_t + B_t v_t\right) dt + \left(C_t x^v_t + D_t v_t\right) dW_t, \\ dy^v_t = -\left(R_t x_t + a_t y_t + c_t z_t\right) dt + z_t dW_t, \\ x_0 = 0, \qquad y^v_T = a. \end{cases}$$

Proof. From the definition of the cost functional J^{θ}, we have

$$\begin{aligned} J^{\theta}(v_t) - J^{\theta}(u_t) &= \mathbb{E}\left[\exp\theta\left(\frac{1}{2}y^v_0 + \frac{1}{2}x^v_T + \int_0^T \frac{1}{2}v^2_t dt\right)\right] \\ &\quad - \mathbb{E}\left[\exp\theta\left(\frac{1}{2}y^u_0 + \frac{1}{2}x^u_T + \int_0^T \frac{1}{2}u^2_t dt\right)\right]. \end{aligned}$$

We put $\xi_T = \int_0^T \frac{1}{2}v^2_t dt$, and applying Taylor's expansion, we get

$$\begin{aligned} J^{\theta}(v_t) - J^{\theta}(u_t) &= \mathbb{E}\left[p_1(T)\left(\xi^v_T - \xi^u_T\right)\right] + \mathbb{E}\left[p_2(T)\left(x^v_T - x^u_T\right)\right] \\ &\quad + \mathbb{E}\left[p_3(0)\left(y^v_0 - y^u_0\right)\right]. \end{aligned} \tag{2.33}$$

We apply Itô's formula to $p_1(t)(\xi_t^v - \xi_t^u)$, $p_2(t)(x_t^v - x_t^u)$ and $p_3(t)(y_t^v - y_t^u)$, and used the explicit forms of the adjoint equations (2.31), to get

$$\mathbb{E}\left[p_1(T)(\xi_T^v - \xi_T^u)\right] = \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\left(v_t^2 - u_t^2\right)p_1(t)\right)dt\right], \tag{2.34}$$

$$\begin{aligned}
&\mathbb{E}\left[p_2(T)(x_T^v - x_T^u)\right] \\
&= \mathbb{E}\left[\int_0^T \left(A_t p_2(t) + C_t q_2(t) + R_t p_3(t)\right)(x_t^v - x_t^u)\,dt\right] \\
&+ \mathbb{E}\left[\int_0^T \left((A_t x_t^v + B_t v_t) - (A_t x_t^u + B_t u_t)\right)p_2(t)\,dt\right] \\
&- \mathbb{E}\left[\int_0^T \left((C_t x_t^v + D_t v_t) - (C_t x_t^u + D_t u_t)\right)q_2(t)\,dt\right],
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
&\mathbb{E}\left[p_3(0)(y_0^v - y_0^u)\right] \\
=&\mathbb{E}\left[\int_0^T \left((R_t x_t^v + a_t y_t^v + c_t z_t^v) - (R_t x_t^u + a_t y_t^u + c_t z_t^u)\right)p_3(t)\,dt\right] \\
&- \mathbb{E}\left[\int_0^T a_t p_3(t)(y_t^v - y_t^u)\,dt\right] - \mathbb{E}\left[\int_0^T c_t p_3(t)(z_t^v - z_t^u)\,dt\right].
\end{aligned} \tag{2.36}$$

We replace (2.34), (2.35), and (2.36) into (2.33), we obtain

$$\begin{aligned}
&J^\theta(v_t) - J^\theta(u_t) \\
&= \frac{1}{2}\mathbb{E}\left[\int_0^T \left((v_t - u_t)(v_t - u_t)p_1(t) + 2u_t(v_t - u_t)p_1(t)\right)dt\right] \\
&+ \mathbb{E}\left[\int_0^T B_t p_2(t)(v_t - u_t)\,dt\right] + \mathbb{E}\left[\int_0^T D_t q_2(t)(v_t - u_t)\,dt\right] \\
&+ \mathbb{E}\left[\int_0^T 2A_t p_2(t)(x_t^v - x_t^u)\,dt\right].
\end{aligned}$$

And then, because of $2A_t p_2(t)(x_t^v - x_t^u)$ being nonnegative, we have

$$J^\theta(v_t) - J^\theta(u_t) \geq \mathbb{E}\left[\int_0^T \left(u_t p_1(t) + B_t p_2(t) + D_t q_2(t)\right)(v_t - u_t)\,dt\right],$$

By replacing u_t with its value in (2.32), we obtain

$$\begin{aligned} J^{\theta}(v_t) - J^{\theta}(u_t) &\geq \mathbb{E}\left[\int_0^T \left(-\left(B_t p_2(t) + D_t q_2(t)\right)\right.\right. \\ &\quad \left. + B_t p_2(t) + D_t q_2(t)\right)(v_t - u_t)\, dt] \\ &= 0. \end{aligned}$$

Then, we get $J^{\theta}(v_t) - J^{\theta}(u_t) \geq 0$. The proof is completed. ■

2.4.2 Example 02: Application to risk sensitive stochastic optimal portfolio problem

Now we return to the problem of optimal portfolio stated in the motivating example, and deal with the linear quadratic risk sensitive stochastic optimal control problem shown in section 01, and apply the risk sensitive necessary optimality condition (Theorem 2.2).

Our state dynamics is

$$\begin{cases} dx_t = (\rho v_t + r x_t)\, dt + \sigma dW_t, \\ x_0 = m_0, \end{cases} \tag{2.37}$$

and

$$\begin{cases} dy_t = (-c x_t + \lambda y_t)\, dt + z_t dW_t, \\ y_T = 0. \end{cases} \tag{2.38}$$

The cost functional is the following expected exponential form

$$J(v(.)) = \mathbb{E}\left[\exp\left(\theta \int_0^T \left(\frac{(\theta - 1)\sigma^2}{2} v_t^2 + \left(\frac{1}{2}\sigma^2 + m_0 + (r - c)\, x_t\right) v_t + r\right) dt\right)\right]. \tag{2.39}$$

The investor wants to maximize (2.39) subject to (2.37) and (2.38) by taking $v(.)$ over $\mathcal{U}$.

The Hamiltonian function (3.18) gets the form

$$
\begin{aligned}
&H^{\theta}(t) := H^{\theta}\left(t, x_t, y_t, z_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), l(t), v_t\right) \\
&= b(t)\,\widetilde{p}_2(t) + \sigma(t)\,\widetilde{q}_2(t) + \{g(t) - \theta l(t)\, z_t\}\,\widetilde{p}_3(t) - f(t). \\
&= (\rho v_t + r x_t)\,\widetilde{p}_2(t) + \sigma \widetilde{q}_2(t) \\
&+ \{(-c x_t + \lambda y_t) - \theta l(t)\, z_t\}\,\widetilde{p}_3(t) \\
&- \left(\frac{(\theta-1)\,\sigma^2}{2} v_t^2 + \left(\frac{1}{2}\sigma^2 + m_0 + (r-c)\, x_t\right) v_t + r\right).
\end{aligned}
$$

Let $(x^u(t), u(t))$ be an optimal pair, the adjoint equations $(2.21), (2.22)$ are given by

$$
\begin{cases}
d\widetilde{p}_2(t) = H_x^{\theta}(t)\, dt + \{\theta l_2(t)\,\widetilde{p}_2(t) + \widetilde{q}_2(t)\}\, dW_t^{\theta}, \\
\widetilde{p}_2(T) = 0,
\end{cases}
\tag{2.40}
$$

and

$$
\begin{cases}
d\widetilde{p}_3(t) = -H_y^{\theta}(t)\, dt - H_z^{\theta}(t)\, dW_t^{\theta}, \\
\widetilde{p}_3(0) = 0.
\end{cases}
\tag{2.41}
$$

Maximizing the Hamiltonian yields

$$
u_t = \frac{\rho}{(\theta-1)\,\sigma^2}\widetilde{p}_2(t) + \frac{(r-c)}{(\theta-1)\,\sigma^2} x_t^u - \frac{1}{(\theta-1)\,\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right). \tag{2.42}
$$

By substituting (2.42) into the SDE (2.37) and (2.40) gives

$$
\begin{cases}
dx_t^u = \left[\left(r - \frac{\rho\,(r-c)}{(\theta-1)\,\sigma^2}\right) x_t^u + \frac{\rho^2}{(\theta-1)\,\sigma^2}\widetilde{p}_2(t)\right] dt + \sigma dW_t \\
-\frac{\rho}{(\theta-1)\,\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right) dt, \\
x^u(0) = m_0,
\end{cases}
\tag{2.43}
$$

and

$$\begin{cases} d\widetilde{p}_2^u(t) = \left\{\left(r - \frac{\rho(r-c)}{(\theta-1)\sigma^2}\right)\widetilde{p}_2^u(t) - c\widetilde{p}_3^u(t) + \frac{(r-c)^2}{(\theta-1)\sigma^2}x_t^u + \frac{\rho(r-c)}{(\theta-1)\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right)\right\}dt \\ + \left(\theta l_2(t)\widetilde{p}_2^u(t) + \widetilde{q}_2^u(t)\right)dW_t^\theta, \\ \widetilde{p}_2^u(T) = 0. \end{cases} \tag{2.44}$$

Therefore, an optimal solution $\left(x_t^u, \widetilde{p}_2^u(t), u_t\right)$ can be obtained by solving the system of fully coupled FBSDE (2.43) and (2.44), unfortunately, in such a system it is difficult to find the explicit solution. To solve the fully coupled FBSDE $\{(2.43), (2.44)\}$, we use the similar technique as in [41], we conjecture the solution to (2.43) and (2.44) is related by

$$\widetilde{p}_2^u(t) = A(t)\,x_t^u + B(t), \tag{2.45}$$

for some deterministic differentiable functions $A(t)$ and $B(t)$. Applying Itô's formula to (2.45), we get

$$\begin{cases} d\widetilde{p}_2^u(t) = \left\{\left[\dot{A}(t) + A(t)\left(r - \frac{\rho(r-c)}{(1-\theta)\sigma^2}\right) + A^2(t)\frac{\rho}{(1-\theta)\sigma^2}\right]x_t^u \right. \\ \left. + A(t)\frac{\rho}{(1-\theta)\sigma^2}B(t) + \dot{B}(t) - \frac{\rho}{(1-\theta)\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right)\right\}dt + A(t)\,\sigma dW_t, \\ \widetilde{p}_2^u(T) = 0. \end{cases} \tag{2.46}$$

On the other hand, by substituting (2.45) into (2.44), and using (2.19), we obtain

$$\begin{cases} d\widetilde{p}_2^u(t) = \left\{\left[A(t)\left(r - \frac{\rho(r-c)}{(\theta-1)\sigma^2}\right) - \frac{(r-c)^2}{(\theta-1)\sigma^2}\right]x_t^u \right. \\ \left. + B(t)\left(r - \frac{\rho(r-c)}{(\theta-1)\sigma^2}\right) - c\widetilde{p}_3^u(t) + \frac{\rho(r-c)}{(\theta-1)\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right) - \widetilde{q}_3^u(t)\,\theta l_1(t)\right\}dt \\ + \widetilde{q}_3^u(t)\,dW_t, \\ \widetilde{p}_2^u(T) = 0. \end{cases} \tag{2.47}$$

By equating the coefficients of (2.47) and (2.46), we have $\widetilde{q}_3^u(t) = \sigma A(t)$, where $A(t)$ is the solution of the following Riccati type equation

$$\begin{cases} \overset{\bullet}{A}(t) - A^2(t)\frac{\rho}{(1-\theta)\sigma^2} + 2A(t)\left(r - \frac{\rho(r-c)}{(\theta-1)\sigma^2}\right) - \frac{(r-c)^2}{(\theta-1)\sigma^2} = 0, \\ A(T) = 0, \end{cases} \tag{2.48}$$

and $B(t)$ is the solution of the following equation

$$\begin{cases} \overset{\bullet}{B}(t) + B(t)\left(A(t)\frac{\rho}{(\theta-1)\sigma^2} - r + \frac{\rho(r-c)}{(\theta-1)\sigma^2}\right) \\ +A(t)\sigma\theta l_1(t) \\ +c\widetilde{p}_3^u(t) - \frac{\rho(r-c-1)}{(1-\theta)\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right) = 0, \\ B(T) = 0. \end{cases} \tag{2.49}$$

Finally, we can have the optimal control in the following state feedback form by using (2.42) and (2.45), then

$$u_t = \frac{1}{(\theta-1)\sigma^2}(\rho A(t) - (r-c))x_t^u + \frac{\rho}{(\theta-1)\sigma^2}B(t) + \frac{1}{(\theta-1)\sigma^2}\left(\frac{1}{2}\sigma^2 + m_0\right), \tag{2.50}$$

where $A(t)$, $B(t)$ are determined by (2.48) and (2.49) respectively.

Theorem 2.6 *We assume that the pair* $(A(t), B(t))$ *has the unique solution given by* (2.48) *and* (2.49). *Then the optimal control of the problem* (2.37) – (2.39) *has the state feedback form* (2.50).

Remark 2.2 *It is important to remark that the solution of the function* $B(t)$ *in the form* (2.49) *is dependent on the solution of* $\widetilde{p}_3(t)$. *If we put* $\widetilde{p}_3(t) = \psi(t)y_t + \varphi(t)$, *for smooth deterministic functions* ψ, *and* φ, *by using the similar technique as optimal solution in the last paragraph, to the optimal solution* $(y^u(t), \widetilde{p}_3^u(t), u(t))$, *then the solutions of functions* ψ, *and* φ *yield respectively the equations*

$$\begin{cases} \overset{\bullet}{\psi}(t) + 2\lambda\psi(t) = 0, \\ \overset{\bullet}{\varphi}(t) + 2\lambda\varphi(t) - \psi(t)(z_t + x_t) = 0, \\ \psi(0) = 0, \ \text{and } \varphi(0) = 0. \end{cases} \tag{2.51}$$

2.4.3 Solution of the deterministic functions $A(t)$ and $B(t)$ via Riccati equation

In the best of our knowledge, it is hard to find the explicit solution to Riccati equation in general. But in our case, we can find the explicit solution of (2.48) for its constant coefficients. For simplicity, we rewrite the Riccati equation (2.48) as follows

$$\begin{cases} \overset{\bullet}{A}(t) - A^2(t)\,\alpha + 2A(t)\,\beta + \gamma = 0, \\ A(T) = 0, \end{cases}$$

where we denote $\alpha = -\frac{\rho}{(1-\theta)\sigma^2}$, $\beta = r - \frac{\rho(r-c)}{(1-\theta)\sigma^2}$, and $\gamma = -\frac{(r-c)^2}{(\theta-1)\sigma^2}$. For convenience, we suppose that $\Delta' = \beta^2 + \alpha\gamma > 0$, we obtain

$$dt = \frac{dA(t)}{\alpha A^2(t) - 2\beta A(t) - \gamma}.$$

By derivation for both terms in above equation, we get

$$\begin{aligned} T - t &= \int_t^T \frac{dA(s)}{A^2(s)\,\alpha - 2A(s)\,\beta - \gamma} \\ &= \frac{1}{\alpha}\int_t^T \frac{dA(s)}{A^2(s) - 2\frac{\beta}{\alpha}A(s) - \frac{\gamma}{\alpha}} \\ &= \frac{1}{\alpha}\int_t^T \frac{dA(s)}{\left(A(s) - \frac{\beta-\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right)\left(A(s) - \frac{\beta+\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right)} \\ &= \frac{1}{2\sqrt{\beta^2+\alpha\gamma}}\int_t^T \left(\frac{1}{A(s) - \frac{\beta+\sqrt{\beta^2+\alpha\gamma}}{\alpha}} - \frac{1}{A(s) - \frac{\beta-\sqrt{\beta^2+\alpha\gamma}}{\alpha}}\right) dA(s). \end{aligned}$$

Using the simple technique of integration calculus, we have

$$\begin{aligned} 2\sqrt{\beta^2+\alpha\gamma}\,(T-t) = &\left[\ln\left|\frac{\beta+\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right| - \ln\left|\frac{\beta-\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right|\right] \\ &- \left[\ln\left|A(t) - \frac{\beta+\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right| - \ln\left|A(t) - \frac{\beta-\sqrt{\beta^2+\alpha\gamma}}{\alpha}\right|\right]. \end{aligned}$$

This implies

$$\frac{\alpha A(t) - \left(\beta + \sqrt{\beta^2 + \alpha\gamma}\right)}{\alpha A(t) - \left(\beta - \sqrt{\beta^2 + \alpha\gamma}\right)} = \pm \frac{\beta - \sqrt{\beta^2 + \alpha\gamma}}{\beta + \sqrt{\beta^2 + \alpha\gamma}} \exp\left(2\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right).$$

This concludes

$$A_1(t) = \frac{\left(\beta + \sqrt{\beta^2 + \alpha\gamma}\right)\left(1 + \exp\left(-2\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)\right)}{\alpha\left(1 + \frac{\beta + \sqrt{\beta^2 + \alpha\gamma}}{\beta - \sqrt{\beta^2 + \alpha\gamma}} \exp\left(-2\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)\right)}.$$

In fact the Riccati equation (2.48) has another solution

$$A_2(t) = \frac{\left(\beta + \sqrt{\beta^2 + \alpha\gamma}\right)\left(1 - \exp\left(-\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)\right)}{\alpha\left(1 - \frac{\beta + \sqrt{\beta^2 + \alpha\gamma}}{\beta - \sqrt{\beta^2 + \alpha\gamma}} \exp\left(-\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)\right)}.$$

We must reject this solution because of the portfolio choice problem, if we denote

$$\delta = \frac{\left(\beta + \sqrt{\beta^2 + \alpha\gamma}\right)}{\alpha},\ \eta = \frac{\beta + \sqrt{\beta^2 + \alpha\gamma}}{\beta - \sqrt{\beta^2 + \alpha\gamma}},\ \text{and } \kappa = \frac{\left(\beta - \sqrt{\beta^2 + \alpha\gamma}\right)}{\alpha}. \tag{2.52}$$

Then we have

$$A(t) = \frac{\delta + \eta\kappa \exp\left(-\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)}{1 - \eta \exp\left(-\sqrt{\beta^2 + \alpha\gamma}\,(T-t)\right)}. \tag{2.53}$$

We put

$$\begin{aligned}
\mu(t) &= A(t)\frac{\rho^2}{(1-\theta)\sigma^2} - c\left(1 + \frac{\rho}{(1-\theta)\sigma^2}\right), \qquad (2.54)\\
\xi(t) &= A(t)\left(\frac{\rho}{(1-\theta)\sigma^2}\left(\frac{1}{2}\sigma^2 + m - r\right) + \sigma\theta l_1(t)\right)\\
&\quad - \left[\widetilde{p}_3(t) + \frac{c}{(1-\theta)\sigma^2}\left(\frac{1}{2}\sigma^2 + m - r\right)\right],
\end{aligned}$$

we rewrite (2.49) as follows

$$\begin{cases} \overset{\bullet}{B}(t) + B(t)\,\mu(t) + \xi(t) = 0, \\ B(T) = 0. \end{cases} \tag{2.55}$$

The explicit solution of equation (2.55) is

$$B(t) = \exp\left(\int_t^T \mu(s)\,ds\right) \int_t^T \xi(s) \exp\left(-\int_s^T \mu(r)\,dr\right) ds, \tag{2.56}$$

where $\mu(t)$ and $\xi(t)$ are determined by (2.54).

Corollaire 2.1 *The explicit solution of Riccati equation* (2.48) *is given by* (2.53), *and the equation* (2.49) *has an explicit solution given by* (2.56), *where the constants coefficients* $\delta, \eta,$ *and* κ *are given by* (2.52), $\mu(t)$ *and* $\xi(t)$ *are determined by* (2.54), *and the system* (2.51).

Corollaire 2.2 *We assume that the pair* $(A(t), B(t))$ *has the unique solution given by* (2.53), (2.56) *and* (2.51). *Then the optimal control of the problem* (2.37) – (2.39) *has the state feedback form* (2.50), *where the constants coefficients* δ, η, κ, $\mu(t)$ *and* $\xi(t)$ *are given by* (2.52), *and* (2.54) *respectively.*

Chapter 3

Pontryagin's risk-sensitive stochastic maximum principle for fully coupled FBSDE with jump diffusion and finantial application

Throughout this chapter, we focus our aim on the problem of optimal control under a risk-sensitive performance functional, where the system is given by a fully coupled forward-backward stochastic differential equation with jump. The risk neutral control system has been used as preliminary step, where the set of admissible controls is convex, and the optimal solution exists. The necessary as well as sufficient optimality conditions for risk-sensitive performance are proved. At the end, we illustrate our main result by giving an example of mean-variance for risk sensitive control problem applied in cash flow market.

3.1 Problem formulation and assumptions

In all what follows, we will be worked on the classical probability space $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\leq T}, \mathbb{P}\right)$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}-$null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon T, and $(\mathcal{F}_t)_{t\leq T}$ satisfies the usual conditions. We assume that the filtration $(\mathcal{F})_{t\leq T}$ is generated by the following two mutually independent processes

(i) $(W(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion.

(ii) Poisson random measure N on $[0,T]\times\Gamma$, where $\Gamma\subset\mathbb{R}-\{0\}$. We denote by $\left(\mathcal{F}_t^W\right)_{t\leq T}$ (resp. $\left(\mathcal{F}_t^N\right)_{t\leq T}$) the $\mathbb{P}-$augmentation of the natural filtration of W (resp. N). Obviously, we have

$$\mathcal{F}_t := \sigma\left[\int_0^s\int_A N(d\lambda, dr);\ s\leq t,\ A\in\mathcal{B}(\Gamma)\right]\vee\sigma\left[W(s);\ s\leq t\right]\vee\mathcal{N},$$

where $\mathcal{N}$ contains all $\mathbb{P}-$null sets in $\mathcal{F}$, and $\sigma_1\vee\sigma_2$ denotes the $\sigma-$field generated by $\sigma_1\cup\sigma_2$. We assume that the compensator of N has the form $\mu(dt,d\lambda)=m(d\lambda)\,dt$, for some positive and $\sigma-$finite Lévy measure m on Γ, endowed with its Borel $\sigma-$field $\mathcal{B}(\Gamma)$. We suppose that $\int_\Gamma 1\wedge|\lambda|^2\, m(d\lambda)<\infty$, and write $\tilde{N}=N-mdt$ for the compensated jump martingale random measure of N.

Notation 3.1 *We need to define some additional notations. Given $s\leq t$, let us introduce the following spaces*

$\mathcal{S}^2([0,T],\mathbb{R})$ *the set of* $\mathbb{R}$- *valued adapted cadlàg processes P such that*

$$\|P\|_{\mathcal{S}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\sup_{r\in[0,T]}|P(r)|^2\right]^{\frac{1}{2}} < +\infty.$$

$\mathcal{M}^2([0,T],\mathbb{R})$ *is the set of progressively measurable* $\mathbb{R}-$*valued processes Q such that*

$$||Q||_{\mathcal{M}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\int_0^T|Q(r)|^2\,dr\right]^{\frac{1}{2}} < +\infty.$$

$\mathcal{L}^2([0,T],\mathbb{R})$ *is the set of* $\mathcal{B}([0,T]\times\Omega)\otimes\mathcal{B}(\Gamma)$ *measurable maps* $R:[0,T]\times\Omega\times\Gamma\to\mathbb{R}$ *such that*

$$\|R\|_{\mathcal{L}^2([0,T],\mathbb{R})} := \mathbb{E}\left[\int_0^T\int_\Gamma|R(r)|^2\,m(d\lambda)\,dr\right]^{\frac{1}{2}} < +\infty,$$

we denote by $\mathbb{E}$ *the expectation with respect to* $\mathbb{P}$

Let T be a strictly positive real number and U is a convex nonempty subset of $\mathbb{R}$.

Definition 3.1 *Let U be a nonempty closed subset in $\mathbb{R}$. An admissible control is a $U-$valued measurable $\mathcal{F}_t-$adapted process v, such that $\|v\|_{S^2} < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.*

For all $v \in \mathcal{U}$, we consider the following fully coupled forward-backward with jump system

$$\begin{cases} dx(t) = & b(t, x(t), y(t), z(t), r(t,.), v(t)) dt \\ & +\sigma(t, x(t), y(t), z(t), r(t,.), v(t)) dW(t) \\ & +\int_\Gamma \gamma(t-, x(t-), y(t-), z(t-), r(t-,\lambda), v(t-), \lambda) \tilde{N}(dt, d\lambda) \\ dy(t) = & -g(t, x(t), y(t), z(t), r(t,.), v(t)) dt + z(t) dW(t) \\ & +\int_\Gamma r(t,\lambda) \tilde{N}(dt, d\lambda) \\ x(0) = & d, \quad y(T) = a, \qquad\qquad t \in [0,T] \end{cases} \tag{3.1}$$

where $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}$, $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}$. and $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \longrightarrow \mathbb{R}$ are given maps. If $(x(.), y(.), z(.), r(.,.))$ is the unique solution of (3.1) associated with $v(.) \in \mathcal{U}$.
The functional cost of the risk-sensitive type is given by

$$\begin{aligned} & J^\theta(v) \\ & = \mathbb{E}\left[\exp\theta\left\{\int_0^T f(t, x(t), y(t), z(t), r(t,.), v(t)) dt + \Phi(x^v(T)) + \Psi(y^v(0))\right\}\right], \end{aligned} \tag{3.2}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$, $\Psi : \mathbb{R} \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}$ are given maps, and $\theta > 0$ is called the risk-sensitive parameter.
Our risk-sensitive stochastic optimal control problem is stated as follows: For given $(t, x(t), y(t), z(t), r(t,.)) \in [0,T] \times \mathbb{R}^4$, minimize (3.2) subject to (3.1) over $\mathcal{U}$.

$$\inf_{v \in \mathcal{U}} J^\theta(v) = J^\theta(u). \tag{3.3}$$

A control that solves the problem $\{(3.1), (3.2), (3.3)\}$ is called optimal. Our goal is to establish a necessary optimality conditions as well as a sufficient optimality conditions, satisfied by a given optimal control, in the form of stochastic maximum principle (SMP in short).

We give some notations $\Upsilon = \left(x^v(t), y^v(t), z^v(t), r^v(t,.)\right)^\top$, where $(.)^\top$ denotes the transport of the matrix,

and $M(t,\Upsilon) = \begin{pmatrix} b \\ \sigma \\ -g \end{pmatrix} (t,\Upsilon)$.

We introduce the following assumptions.

$\mathbf{H}_1$:

For each $\Upsilon \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $M(t,\Upsilon)$ is an $\mathcal{F}_t-$measurable process defined on $[0,T]$ with $M(t,\Upsilon) \in \mathcal{M}^2([0,T];\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Gamma)$.

$\mathbf{H}_2$:

$M(t,.)$ satisfies Lipschitz conditions: There exists a constant $k>0$, such that

$$|M(t,\Upsilon) - M(t,\Upsilon')| \leq k\,|\Upsilon - \Upsilon'|\, \forall\Upsilon,\Upsilon' \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Gamma, \forall t \in [0,T].$$

The following monotonic conditions introduced in [37], are the main assumptions in this paper.

$\mathbf{H}_3$:

$\langle M(t,\Upsilon) - M(t,\Upsilon'), \Upsilon - \Upsilon\rangle \leq \beta\,|\Upsilon - \Upsilon'|^2$, for every $\Upsilon = (x,y,z,r)^*$ and $\Upsilon' = (x',y',z',r')^* \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Gamma$, $\forall\ t \in [0,T]$, where β is a positive constant.

$\mathcal{U}$ is a convex subset of $\mathbb{R}$.

Proposition 3.1 *For any given admissible control* $v(.)$ *and under the assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$, *the fully coupled FBSDE with jump diffusion* (3.1) *admits an unique solution* $\left(x^v(t), y^v(t), z^v(t), r^v(t,.)\right) \in \left(\mathcal{M}^2([0,T];\mathbb{R}\times\mathbb{R}\times\Gamma)\right)^2 \times \mathcal{S}^2([0,T];\mathbb{R}\times\Gamma)$.

Proof. The proof can be seen in [37]. ■

Next, we assume that

$\mathbf{H}_4$:

$i)b$, σ, g, f, Φ and Ψ are continuously differentiable with respect to $(x^v, y^v, z^v, r^v(.))$.

ii) All the derivatives of b, σ, g and f are bounded by

$C\left(1+|x^v|+|y^v|+|z^v|+|v|+|r^v|\right)$.

iii) The derivatives of Φ, Ψ are bounded by $C\left(1+|x^v|\right)$ and $C\left(1+|y^v|\right)$ respectively.

Under the above assumptions, for every $v\in\mathcal{U}$ equation (3.1) has a unique strong solution and the function cost J^θ is well defined from $\mathcal{U}$ into $\mathbb{R}$.

3.2 Risk-neutral necessary optimality conditions

First of all, we may introduce an auxiliary state process $\xi^v(t)$ which is solution of the following stochastic differential equation (SDE in short):

$$d\xi^v(t)=f\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dt,\ \ \xi^v(0)=0.$$

From the above auxiliary process, the fully coupled forward-backward type control problem is equivalent to

$$\begin{cases}\inf\limits_{v\in\mathcal{U}} & \mathbb{E}\left[\exp\theta\left\{\Phi\left(x^v(T)\right)+\Psi\left(y^v(0)\right)+\xi(T)\right\}\right],\\ \text{subject to} & \\ d\xi^v(t)= & f\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dt,\\ dx^v(t)= & b\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dt\\ & +\sigma\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dW(t)\\ & +\displaystyle\int_\Gamma\gamma\left(t,x(t-),y(t-),z(t-),r(t-,\lambda),v(t-),\lambda\right)\tilde{N}(dt,d\lambda),\\ dy^v(t)= & -g\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dt+z^v(t)\,dW(t)\\ & +\displaystyle\int_\Gamma r^v(t,\lambda)\,\tilde{N}(dt,d\lambda),\\ \xi^v(0)= & 0,\ \ x^v(0)=d,\ y^v(T)=a.\end{cases}\tag{3.4}$$

We denote by

$$A_T^\theta:=\exp\theta\left\{\Phi\left(x^v(T)\right)+\Psi\left(y^v(0)\right)+\int_0^T f\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v(t)\right)dt\right\},$$

and we can put also

$$\Theta_T := \Phi\left(x^v(T)\right) + \Psi\left(y^v(0)\right) + \int_0^T f\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v(t)\right) dt,$$

the risk-sensitive loss functional is given by

$$\begin{aligned}\Theta_\theta &:= \frac{1}{\theta}\log \mathbb{E}\left[\exp\left\{\Phi\left(x^v(T)\right) + \Psi\left(y^v(0)\right)\right.\right.\\ &\left.\left.+\int_0^T f\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v(t)\right) dt\right\}\right]\\ &= \frac{1}{\theta}\log \mathbb{E}\left[\exp\left\{\theta\Theta_T\right\}\right].\end{aligned}$$

When the risk-sensitive index θ is small, the functional Θ_θ can be expanded as $\mathbb{E}\left(\Theta_T\right) + \frac{\theta}{2}Var\left(\Theta_T\right) + O\left(\theta^2\right)$, where, $Var\left(\Theta_T\right)$ denotes the variance of Θ_T. If $\theta < 0$, the variance of Θ_T, as a measure of risk, improves the performance Θ_θ, in which case the optimizer is called *risk seeker.* But, when $\theta > 0$, the variance of Θ_T worsens the performance Θ_θ, in which case the optimizer is called *risk averse.* The risk-neutral loss functional $\mathbb{E}\left(\Theta_T\right)$ can be seen as a limit of risk-sensitive functional Θ_θ when $\theta \to 0$, for more details the reader can see the papers [11], [7].

Notation 3.2 *We will use the following notation throughout this paper.*
For $\phi \in \left\{b, \sigma, f, g, H^\theta, \widetilde{H}^\theta\right\}$, *we define*

$$\begin{cases}\phi(t) = \phi\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v(t)\right),\\ \partial\phi(t) = \phi\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v(t)\right)\\ -\phi\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), u(t)\right),\\ \phi_\zeta(t) = \frac{\partial\phi}{\partial\zeta}\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v(t)\right), \ \zeta = x, \ y, \ z, \ r(.).\end{cases}$$

and $\gamma(t-, \lambda)$ *it means that the function* γ *is càdlag.*

Where v_t in an admissible control from $\mathcal{U}$.

We assume that $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ and $(\mathbf{H_4})$ hold, we might apply the SMP for risk-neutral of fully coupled forward-backward type control from Yong [40], to augmented state dynamics

(ξ, x, y, z, r) and derive the adjoint equation. There exist unique $\mathcal{F}_t-$adapted of processes $(p_1, q_1, \pi_1), (p_2, q_2, \pi_2), (p_3, q_3, \pi_3)$, which solve the following system matrix of backward SDEs

$$\begin{cases} d\overrightarrow{p}(t) & = \begin{pmatrix} dp_1(t) \\ dp_2(t) \\ dp_3(t) \end{pmatrix} \\ & = -\begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} dt \\ & \quad - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dt \\ & \quad + \int_\Gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} m(d\lambda)\, dt \\ & \quad + \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dW(t) + \int_\Gamma \begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} \tilde{N}(dt, d\lambda) \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} & = \theta A_T \begin{pmatrix} 1 \\ \Phi_x(x_T^u) \end{pmatrix} \\ p_3(0) & = \theta \Psi_y(y^u(0)) A_T, \end{cases} \tag{3.5}$$

with $\mathbb{E}\left[\sum_{i=1}^{3} \sup_{0\leq t\leq T} |p_i(t)|^2 + \sum_{i=1}^{2} \int_0^T |q_i(t)|^2 dt\right] < \infty$, and

$$q_3(t) = -Tr\left[\begin{pmatrix} f_z(t) & b_z(t) \\ \sigma_z(t) & g_z(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix}\right] + \int_\Gamma \gamma_z(t-,\lambda)\, \pi_2(t,\lambda)\, m(d\lambda),$$

$$\pi_3(t,\lambda) = -Tr\left[\begin{pmatrix} f_r(t) & b_r(t) \\ \sigma_r(t) & g_r(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix}\right] + \int_\Gamma \gamma_r(t-,\lambda)\, \pi_2(t,\lambda)\, m(d\lambda).$$

To this end we may define (3.5) in the compact form as

$$\begin{cases} d\overrightarrow{p}(t) = \begin{pmatrix} dp_1(t) \\ dp_2(t) \\ dp_3(t) \end{pmatrix} = -F(t)\,dt + \Sigma(t)\,dW(t) + \int_\Gamma R(t,\lambda)\,\tilde{N}(dt,d\lambda) \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} = \theta A_T \begin{pmatrix} 1 \\ \Phi_x(x_T^u) \end{pmatrix}, \text{ and } p_3(0) = \theta \Psi_y(y^u(0))\,A_T, \end{cases}$$

where

$$\begin{aligned} F(t) = & \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} \\ & - \int_\Gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} m(d\lambda), \end{aligned}$$

$$\Sigma(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix},$$

and

$$R(t,.) = \begin{pmatrix} \pi_1(t,.) \\ \pi_2(t,.) \\ \pi_3(t,.) \end{pmatrix}.$$

We suppose here that $\widetilde{H}^\theta$ be the Hamiltonian associated with the optimal state dynamics

$(\xi^u, x^u, y^u, z^u, r^u(.))$, and the triplet of adjoint processes $(\overrightarrow{p}(t), \overrightarrow{q}(t), \overrightarrow{\pi}(t,.))$ is given by

$$\begin{aligned}&\widetilde{H}^\theta\left(t, \xi^u(t), x^u(t), y^u(t), z^u(t), r(t,.), u(t), \overrightarrow{p}(t), \overrightarrow{q}(t), \overrightarrow{\pi}(t,.)\right)\\ &= \begin{pmatrix} f(t) \\ b(t) \\ g(t) \end{pmatrix} \left(\overrightarrow{p}(t)\right)^\top + \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \end{pmatrix} \left(\overrightarrow{q}(t)\right)^\top \\ &-\int_\Gamma \begin{pmatrix} 0 \\ \gamma(t-,\lambda) \\ 0 \end{pmatrix} \left(\overrightarrow{\pi}(t,\lambda)\right)^\top m(d\lambda).\end{aligned}$$

Theorem 3.1 *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2}), (\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *hold.*
If $(\xi^u(.), x^u(.), y^u(.), z^u(.), r(.,.))$ *is an optimal solution of the risk-neutral control problem* (3.4), *then there exist* $\mathcal{F}_t-$*adapted processes* $((p_1, q_1, \pi_1), (p_2, q_2, \pi_2), (p_3, q_3, \pi_3))$ *that satisfy* (3.5), *such that*

$$\widetilde{H}_v^\theta(t)(v_t - u_t) \geq 0, \tag{3.6}$$

for all $u \in \mathcal{U}$, *almost every* t *and* $\mathbb{P}-$*almost surely, where* $\widetilde{H}_v^\theta(t)$ *is defined in notation* (3.2).

Proof. For more details the reader can see paper [40] with the result of paper [35]. ■

3.2.1 Steps to find the transformed adjoint equation

As we said, Theorem 3.1 is a good SMP for the risk-neutral of forward backward control problem. We follow the same approach used in [8, 11], and suggest a transformation of the adjoint processes $(p_1, q_1, \pi_1(.))$, $(p_2, q_2, \pi_2(.))$, $(p_3, q_{3,}\pi_3(.))$ in such a way to omit the first component $(p_1, q_1, \pi_1(.))$ in (3.5), and to obtain the SMP (3.6) in terms of only the last two adjoint processes, that we denote them by $((\widetilde{p}_2, \widetilde{q}_2, \widetilde{\pi}_2(.)), (\widetilde{p}_3, \widetilde{q}_3,, \widetilde{\pi}_3(.)))$. Noting that $dp_1(t) = q_1(t) dW_t + \int_\Gamma \pi_1(t,\lambda) \tilde{N}(dt, d\lambda)$ and $p_1(T) = \theta A_T^\theta$, the explicit solution of this backward SDE is

$$p_1(t) = \theta \mathbb{E}\left[A_T^\theta \mid \mathcal{F}_t\right] = \theta V^\theta(t), \tag{3.7}$$

where

$$V^{\theta}(t) := \mathbb{E}\left[A_T^{\theta} \mid \mathcal{F}_t\right], \; 0 \leq t \leq T. \tag{3.8}$$

As a good look of (3.7), it would be natural to choose a transformation of $(\widetilde{p}, \widetilde{q}, \widetilde{\pi}(.))$ instead of $(\overrightarrow{p}, \overrightarrow{q}, \overrightarrow{\pi}(.))$, where $\widetilde{p}_1(t) = \dfrac{1}{\theta V^{\theta}(t)} p_1(t) = 1.$

We consider the following transform

$$\widetilde{p}(t) = \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} := \frac{1}{\theta V^{\theta}(t)} \overrightarrow{p}(t), \; 0 \leq t \leq T. \tag{3.9}$$

By using (3.5) and (3.9), we have

$$\widetilde{p}(T) := \begin{pmatrix} \widetilde{p}_1(T) \\ \widetilde{p}_2(T) \end{pmatrix} = \begin{pmatrix} 1 \\ \Phi_x(x^u(T)) \end{pmatrix}, \text{ and } \widetilde{p}_3(0) = \Psi_y(y^u(0)).$$

The following properties of the generic martingale V^{θ} are essential in order to investigate the properties of these new processes $(\widetilde{p}(t), \widetilde{q}(t), \widetilde{\pi}(t,.))$.

As is proved in [21], the process Λ^{θ} is the first component of the $\mathcal{F}_t$-adapted pair of processes $\left(\Lambda^{\theta}, l, L(.)\right)$ which is the unique solution to the following quadratic backward SDE with jump diffusion

$$\begin{cases} d\Lambda^{\theta}(t) = & -\left\{ f(t) + \frac{\theta}{2}|l(t)|^2 + \frac{\theta}{2}\int_{\Gamma}|L(t,\lambda)|^2 m(d\lambda) \right. \\ & \left. + \int_{\Gamma}\left(\dfrac{\exp(\theta r(t,\lambda)) - 1}{\theta} - r(t,\lambda)\right) m(d\lambda) \right\} dt + l(t)\, dW(t) \\ & - \int_{\Gamma}\left\{\dfrac{\exp(\theta r(t,\lambda)) - 1}{\theta}\right\} \tilde{N}(dt, d\lambda) + \int_{\Gamma} L(t,\lambda)\, \tilde{N}(dt, d\lambda), \\ \Lambda^{\theta}(T) = & \Phi_x(x^u(T)) + \Psi(y^u(0)), \end{cases} \tag{3.10}$$

where

$$\mathbb{E}\left[\int_0^T |l(t)|^2 dt + \int_0^T \int_{\Gamma} |L(t,\lambda)|^2 m(d\lambda) dt\right] < \infty.$$

Lemma 3.1 *Suppose that* $(\mathbf{H}_4)$ *holds. Then*

$$\mathbb{E}\left[\sup_{0\leq t\leq T.}\left|\Lambda^{\theta}(t)\right|^{2}\right]\leq C_{T}. \tag{3.11}$$

In particular, V^{θ} *solves the following linear backward SDE*

$$dV^{\theta}(t)=\theta l(t)V^{\theta}(t)dW(t)+\theta V^{\theta}(t)\int_{\Gamma}L(t,\lambda)\tilde{N}(dt,d\lambda),\ V^{\theta}(T)=A_{T}^{\theta}. \tag{3.12}$$

Hence, the process defined on $\left(\Omega,\mathcal{F},\left(\mathcal{F}_{t}^{(W,N)}\right)_{t\geq0},\mathbb{P}\right)$ *by*

$$\begin{aligned} L_{t}^{\theta}:=\tfrac{V^{\theta}(t)}{V^{\theta}(0)}= &\exp\Bigg(\int_{0}^{t}\theta l(s)dW(s)-\tfrac{\theta^{2}}{2}\int_{0}^{t}|l(s)|^{2}ds+\int_{0}^{t}\int_{\Gamma}L(s,\lambda)\tilde{N}(ds,d\lambda)\\ &-\int_{\Gamma}\left\{\tfrac{\exp(\theta r(t,\lambda))-1}{\theta}\right\}\tilde{N}(dt,d\lambda)-\tfrac{\theta^{2}}{2}\int_{0}^{t}\int_{\Gamma}|L(s,\lambda)|^{2}m(d\lambda)ds\\ &-\int_{\Gamma}\left(\tfrac{\exp(\theta r(t,\lambda))-1}{\theta}-r(t,\lambda)\right)m(d\lambda)\Bigg),\qquad 0\leq t\leq T, \end{aligned} \tag{3.13}$$

is a uniformly bounded $\mathcal{F}-$*martingale.*

Proof. First we prove (3.11). We assume that $(\mathbf{H_4})$ holds, f, Φ and Ψ are bounded by a constant $C>0$, we have

$$0<e^{-(2+T)C\theta}\leq A_{T}^{\theta}\leq e^{(2+T)C\theta}. \tag{3.14}$$

Therefore, V^{θ} is a uniformly bounded $\mathcal{F}_{t}-$martingale satisfying

$$0<e^{-(2+T)C\theta}\leq V^{\theta}(t)\leq e^{(2+T)C\theta},\ 0\leq t\leq T. \tag{3.15}$$

The complete proof see the Lemma 3.1 page 405 [8]. ■

In the next, we will state and prove the necessary optimality conditions for the system driven by fully coupled FBSDE with jumps diffusion with a risk sensitive performance functional type. To this end, let us summarize and prove some lemmas that we will use thereafter.

Lemma 3.2 *The second and the third risk-sensitive adjoint equations of the solution* $(\widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{\pi}_2(t,\lambda))$, $(\widetilde{p}_3(t), \widetilde{q}_3(t), \widetilde{\pi}_3(t,.))$ *and* $\left(V^\theta(t), l(t), L(t,.)\right)$ *become*

$$\begin{cases} d\widetilde{p}_2(t) = -H_x^\theta(t)\,dt + (\widetilde{q}_2(t) - \theta l(t)\widetilde{p}_2(t))\,dW_t^\theta + \int_\Gamma (\widetilde{\pi}_2(t,\lambda) - \theta L(t,\lambda)\widetilde{p}_2(t))\,\tilde{N}^\theta(dt,d\lambda), \\ d\widetilde{p}_3(t) = -H_y^\theta(t)\,dt - \left(H_z^\theta(t) - \theta l(t)\widetilde{p}_3(t)\right)dW_t^\theta - \int_\Gamma (\nabla H_r(t) - \theta L(t,\lambda)\widetilde{p}_3(t))\,\tilde{N}^\theta(dt,d\lambda), \\ dV^\theta(t) = \theta V^\theta(t) l(t)\,dW_t + \theta V^\theta(t)\int_\Gamma L(t,\lambda)\,\tilde{N}(dt,d\lambda), \\ V^\theta(T) = A^\theta(T), \\ \widetilde{p}_2(T) = \Phi_x(x_T),\ \widetilde{p}_3(0) = \Psi_y(y(0)). \end{cases} \tag{3.16}$$

The solution $\left(\widetilde{p}(t), \widetilde{q}(t), \widetilde{\pi}(t,.), V^\theta(t), l(t), L(t,.)\right)$ *of the system* (3.16) *is unique, such that*

$$\begin{aligned} \mathbb{E}\Bigg[\sup_{0\le t\le T}|\widetilde{p}(t)|^2 + \sup_{0\le t\le T}\left|V^\theta(t)\right|^2 + \int_0^T \Big(|\widetilde{q}(t)|^2 + |l(t)|^2 \\ + \int_\Gamma \left(|\widetilde{\pi}(t,\lambda)|^2 + |L(t,\lambda)|^2\right) m(d\lambda)\Big)\,dt\Bigg] < \infty, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} &H^\theta\left(t, x(t), y(t), z(t), r(t,.), \widetilde{p}(t), \widetilde{q}(t), \widetilde{\pi}(t,\lambda), V^\theta(t), l(t), L(t,.)\right) \\ &= f(t) + b(t)\widetilde{p}_2 + \sigma(t)\widetilde{q}_2 + (g(t) - \theta z(t) l(t))\widetilde{p}_3 \\ &+ \int_\Gamma \{\gamma(t^-,\lambda)\widetilde{\pi}_2(t,\lambda) - (g(t) - \theta r(t,\lambda) L(t,\lambda))\widetilde{p}_3\}\,\lambda m(d\lambda). \end{aligned} \tag{3.18}$$

Proof. We want to identify the processes $\widetilde{\alpha}, \widetilde{\beta}$ and $\widetilde{\gamma}$ such that

$$d\widetilde{p}(t) = -\widetilde{\alpha}(t)\,dt + \widetilde{\beta}(t)\,dW(t) + \int_\Gamma \widetilde{\gamma}\left(t^-,\lambda\right)\widetilde{N}(d\lambda, dt)$$

By applying Itô's formula to the process $\overrightarrow{p}(t) = \theta V^\theta(t)\widetilde{p}(t)$, and using the expression of

V^{θ} in (3.12) , we obtain

$$
\begin{aligned}
d\widetilde{p}(t) = & -\left[\frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}\right. \\
& +\frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\left(\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} - \theta\begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{\beta}(t)\right) \\
& -\frac{1}{\theta V^{\theta}(t)}\int_{\Gamma}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix}\left(\begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix}\right.\right. \\
& \left.\left.\left. -\theta\int_{\Gamma}\begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix}\widetilde{\gamma}(t)\right)m(d\lambda)\right]dt \\
& +\left[\frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} - \theta\begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{p}(t)\right]dW(t) \\
& +\left[\frac{1}{\theta V^{\theta}(t)}\int_{\Gamma}\begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} - \theta\int_{\Gamma}\begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix}\widetilde{p}(t)\right]\widetilde{N}(d\lambda,dt)
\end{aligned}
$$

By identifying the coefficients, and using the relation $\widetilde{p}(t) = \dfrac{1}{\theta V^{\theta}(t)}\overrightarrow{p}(t)$, the diffusion coefficient $\widetilde{q}(t)$ will be

$$
\widetilde{\beta}(t) = \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} - \theta\begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{p}(t),
$$

the drift term of the process $\widetilde{p}(t)$

$$\widetilde{\alpha}(t) = \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} +\theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{\beta}(t) - \int_\Gamma \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} - \theta \begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix} \widetilde{\gamma}(t^-,\lambda) \right) m(d\lambda).$$

the jump diffusion gets the form

$$\widetilde{\gamma}(t^-,.) = \begin{pmatrix} \widetilde{\pi}_1(t,.) \\ \widetilde{\pi}_2(t,.) \\ \widetilde{\pi}_3(t,.) \end{pmatrix} - \theta \begin{pmatrix} L_1(t,.) \\ L_2(t,.) \\ L_3(t,.) \end{pmatrix} \widetilde{p}(t).$$

Finally, we obtain

$$
\begin{aligned}
d\widetilde{p}(t) = & -\left[\begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right. \\
& + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\left(\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{\beta}(t)\right) \\
& - \int_\Gamma \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix}\left(\begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix}\right.\right. \\
& \left.\left.\left. -\theta \begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix}\widetilde{\gamma}(t,\lambda)\right)m(d\lambda)\right] dt \\
& + \left[\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{p}(t)\right] dW(t) \\
& + \int_\Gamma \left[\begin{pmatrix} \widetilde{\pi}_1(t,\lambda) \\ \widetilde{\pi}_2(t,\lambda) \\ \widetilde{\pi}_3(t,\lambda) \end{pmatrix} - \theta \begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix}\widetilde{p}(t)\right] \widetilde{N}(d\lambda, dt).
\end{aligned}
$$

It is easily verified that

$$
\begin{cases}
d\widetilde{p}_1(t) = & \widetilde{q}_1(t)\left[-\theta l_1(t)\,dt + dW(t)\right] + \int_\Gamma \widetilde{\pi}_1(t,\lambda)\left[-\theta L_1(t,\lambda)\,m(d\lambda)dt + \widetilde{N}(d\lambda, dt)\right] \\
\widetilde{p}_1(T) = & 1.
\end{cases}
$$

In view of (3.13), we may use Girsanov's Theorem to claim that

$$
\begin{cases}
d\widetilde{p}_1(t) = & \widetilde{q}_1(t)\,dW^\theta(t) + \int_\Gamma \widetilde{\pi}_1(t,\lambda)\,\widetilde{N}^\theta(d\lambda, dt) \\
\widetilde{p}_1(T) = & 1,
\end{cases}, \quad \mathbb{P}^\theta - as,
$$

where,

$$\begin{aligned} dW^{\theta}(t) &= -\theta l(t)\, dt + dW(t) \\ \widetilde{N}^{\theta}(d\lambda, dt) &= -\theta L(t,\lambda)\, m(d\lambda) + \widetilde{N}(d\lambda, dt), \end{aligned} \tag{3.19}$$

$W^{\theta}(t)$ is a $\mathbb{P}^{\theta}-$Brownian motion and $\widetilde{N}^{\theta}(\lambda, t)$ is a $\mathbb{P}^{\theta}-$compensator Poisson measure, where,

$$\begin{aligned} \left.\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \quad L_t^{\theta} &= \exp\Bigg(\Bigg(\int_0^t \theta l(s)\, dW(s) - \tfrac{\theta^2}{2}\int_0^t |l(s)|^2\, ds + \int_0^t\int_{\Gamma} L(s,\lambda)\, \tilde{N}(ds, d\lambda) \\ &- \int_{\Gamma}\left\{\frac{\exp(\theta r(t,\lambda)) - 1}{\theta}\right\}\tilde{N}(dt, d\lambda) - \tfrac{\theta^2}{2}\int_0^t\int_{\Gamma}|L(s,\lambda)|^2 m(d\lambda)ds \\ &- \int_{\Gamma}\left(\frac{\exp(\theta r(t,\lambda)) - 1}{\theta} - r(t,\lambda)\right) m(d\lambda)\Bigg) \qquad 0 \leq t \leq T. \end{aligned}$$

But according to (3.13) and (3.14), the probability measures $\mathbb{P}^{\theta}$ and $\mathbb{P}$ are in fact equivalent. Hence, noting that $\widetilde{p}_1(t) := \dfrac{1}{\theta V^{\theta}(t)} p_1(t)$ is square-integrable, we get that

$$\widetilde{p}_1(t) = \mathbb{E}^{\mathbb{P}^{\theta}}\left[\widetilde{p}_1(T) \mid \mathcal{F}_t\right] = 1.$$

Thus, its quadratic variation $\displaystyle\int_0^T |\widetilde{q}_1(t)|^2\, dt = 0$. This implies that, for almost every $0 \leq t \leq T$, $\widetilde{q}_1(t) = 0$, $\mathbb{P}^{\theta}$ and $\mathbb{P}-$a.s. Now we use the relations

$$\widetilde{q}(t) = \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ -\widetilde{H}_z(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \widetilde{p}(t),$$

and

$$\widetilde{\pi}(t,.) = \begin{pmatrix} \widetilde{\pi}_1(t,.) \\ \widetilde{\pi}_2(t,.) \\ -\nabla_r \widetilde{H}(t) \end{pmatrix} - \theta \begin{pmatrix} L_1(t,.) \\ L_2(t,.) \\ L_3(t,.) \end{pmatrix} \widetilde{p}(t),$$

in the equation above, to obtain

$$
\begin{aligned}
d\widetilde{p}(t) = & -\left\{\begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right. \\
& + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} \\
& \left. + \int_\Gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-,\lambda) & 0 \\ 0 & \gamma_y(t-,\lambda) & 0 \end{pmatrix}\begin{pmatrix} \pi_1(t,\lambda) \\ \pi_2(t,\lambda) \\ \pi_3(t,\lambda) \end{pmatrix} m(d\lambda)\right\} dt \\
& + \left\{\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ -f_z(t)\widetilde{p}_1 - b_z(t)\widetilde{p}_2 - g_z(t)\widetilde{p}_3 - \sigma_z(t)\widetilde{q}_2 \\ + \int_\Gamma \gamma_z(t-,\lambda)\widetilde{\pi}_2(t,\lambda)m(d\lambda) \end{pmatrix} + \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix}\widetilde{p}(t)\right\} dW^\theta(t) \\
& + \int_\Gamma \left\{\begin{pmatrix} \widetilde{\pi}_1(t,\lambda) \\ \widetilde{\pi}_2(t,\lambda) \\ -f_r(t)\widetilde{p}_1 - b_r(t)\widetilde{p}_2 - g_r(t)\widetilde{p}_3 - \sigma_r(t)\widetilde{q}_2 \\ + \int_\Gamma \gamma_r(t-,\lambda)\widetilde{\pi}_2(t,\lambda)m(d\lambda) \end{pmatrix} - \theta \begin{pmatrix} L_1(t,\lambda) \\ L_2(t,\lambda) \\ L_3(t,\lambda) \end{pmatrix}\widetilde{p}(t)\right\} \widetilde{N}^\theta(d\lambda, dt).
\end{aligned}
\tag{3.20}
$$

Therefore, the second and third components of $\widetilde{p}_2$ and $\widetilde{p}_3$ in (3.20), are given by

$$\begin{cases} d\widetilde{p}_2(t) = & -\{f_x(t) + b_x(t)\widetilde{p}_2(t) + g_x(t)\widetilde{p}_3(t) + \sigma_x(t)\widetilde{q}_2(t) \\ & -\int_\Gamma \gamma_x(t-,\lambda)\widetilde{\pi}_2(t,\lambda)m(d\lambda)\Big\} dt \\ & +\{\widetilde{q}_2(t) - \theta l_2(t)\widetilde{p}_2(t)\} dW^\theta(t) + \int_\Gamma \{\widetilde{\pi}_2(t,\lambda) - \theta L_2(t,\lambda)\widetilde{p}_2(t)\}\widetilde{N}^\theta(d\lambda,dt), \\ \widetilde{p}_2(T) = & \Phi_x(x_T), \end{cases} \tag{3.21}$$

and

$$\begin{cases} d\widetilde{p}_3(t) = & -\{f_y(t) + b_y(t)\widetilde{p}_2(t) + g_y(t)\widetilde{p}_3(t) + \sigma_y(t)\widetilde{q}_2(t) \\ & +\theta l_3(t)\widetilde{q}_3(t) - \int_\Gamma \gamma_y(t-,\lambda)\widetilde{\pi}_2(t,\lambda)m(d\lambda)\Big\} dt \\ & -\{\{f_z(t) + b_z(t)\widetilde{p}_2(t) + g_z(t)\widetilde{p}_3(t) + \sigma_z(t)\widetilde{q}_2(t)\} \\ & +\int_\Gamma \gamma_z(t-,\lambda)\widetilde{\pi}_2(t,\lambda)m(d\lambda) + \theta l_3(t)\widetilde{p}_3(t)\Big\} dW^\theta(t) \\ & -\int_\Gamma\{f_r(t) + b_r(t)\widetilde{p}_2(t) + g_r(t)\widetilde{p}_3(t) + \sigma_r(t)\widetilde{q}_2(t) \\ & -\int_\Gamma(\gamma_r(t-,\lambda)\widetilde{\pi}_2(t,\lambda) + \theta L_3(t,\lambda)\widetilde{p}_3(t))m(d\lambda)\Big\}\widetilde{N}^\theta(d\lambda,dt), \\ \widetilde{p}_3(0) = & \Psi_y(y(0)), \end{cases} \tag{3.22}$$

or in equivalent expression the adjoint equations for $(\widetilde{p}_2, \widetilde{q}_2)$, $(\widetilde{p}_3, \widetilde{q}_3)$, $(\widetilde{\pi}_2, \widetilde{\pi}_3)$ and $\left(V^\theta, l, L\right)$ become

$$\begin{cases} d\widetilde{p}_2(t) = -H_x^\theta(t)\,dt + (\widetilde{q}_2(t) - \theta l_2(t)\widetilde{p}_2)\,dW^\theta(t) + \int_\Gamma \{\widetilde{\pi}_2(t,\lambda) - \theta L_2(t,\lambda)\widetilde{p}_2(t)\}\widetilde{N}^\theta(d\lambda,dt), \\ d\widetilde{p}_3(t) = -H_y^\theta(t)\,dt - H_z^\theta(t)\,dW^\theta(t) - \int_\Gamma \nabla H_r^\theta(t)\widetilde{N}^\theta(d\lambda,dt), \\ dV^\theta(t) = \theta l(t)V^\theta(t)\,dW(t) + \theta V^\theta(t)\int_\Gamma L(t,\lambda)\widetilde{N}(d\lambda,dt), \\ V^\theta(T) = A^\theta(T), \\ \widetilde{p}_2(T) = \Phi_x(x(T)),\ \widetilde{p}_3(0) = \Psi_y(y(0)). \end{cases}$$

The solution $\left(\widetilde{p},\widetilde{q},\widetilde{\pi},V^{\theta},l,L\right)$ of the system (3.16) is unique, such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{p}\left(t\right)\right|^{2}+\sup_{0\leq t\leq T}\left|V^{\theta}\left(t\right)\right|^{2}+\int_{0}^{T}\left(\left|\widetilde{q}\left(t\right)\right|^{2}+\left|l\left(t\right)\right|^{2}\right.\right.$$
$$\left.\left.+\int_{\Gamma}\left(\left|\widetilde{\pi}\left(t,\lambda\right)\right|^{2}+\left|L\left(t,\lambda\right)\right|^{2}\right)m\left(d\lambda\right)\right)dt\right]<\infty,$$

where

$$\begin{aligned}H^{\theta}\left(t\right) &:= H^{\theta}\left(t,x\left(t\right),y\left(t\right),z\left(t\right),r^{u}\left(t,\lambda\right),\widetilde{p}_{2}\left(t\right),\widetilde{q}_{2}\left(t\right),\right.\\ &\quad\left.\widetilde{p}_{3}\left(t\right),\widetilde{\pi}_{2}\left(t,\lambda\right),V^{\theta}\left(t\right),l\left(t\right),L\left(t,\lambda\right)\right)\\ &= f\left(t\right)+b\left(t\right)\widetilde{p}_{2}+\sigma\left(t\right)\widetilde{q}_{2}+\left(g\left(t\right)+z\left(t\right)\theta l\left(t\right)\right)\widetilde{p}_{3}\\ &\quad-\int_{\Gamma}\left\{\gamma\left(t-,\lambda\right)\widetilde{\pi}_{2}\left(t,\lambda\right)-\left(g\left(t\right)+r\left(t,\lambda\right)L\left(t,\lambda\right)\right)\widetilde{p}_{3}\right\}m\left(d\lambda\right).\end{aligned}$$

The proof is completed. ■

Theorem 3.2 *(Risk-Sensitive necessary optimality conditions): We assume that* $(\mathbf{H}_4)$ *holds, if* $\left(x^{u}\left(.\right),y^{u}\left(.\right),z^{u}\left(.\right),r^{u}\left(.,.\right),u\left(.\right)\right)$ *is an optimal solution of the risk-sensitive control problem* $\left\{(3.1),(3.2),(3.3)\right\}$, *then there exist* $\mathcal{F}_t$*-adapted processes* $\left(V^{\theta}\left(t\right),l\left(t\right),L\left(t,\lambda\right)\right)$, *and* $\left(\widetilde{p}_{2}\left(t\right),\widetilde{q}_{2}\left(t\right)\right),\left(\widetilde{p}_{3}\left(t\right)\right)$, $\left(\widetilde{\pi}_{2}\left(t,.\right)\right)$ *that satisfy* (3.16), (3.17) *such that*

$$H_{v}^{\theta}\left(t\right)\left(v_{t}-u_{t}\right)\geq 0, \tag{3.23}$$

for all $u\in\mathcal{U}$, *almost every* $0\leq t\leq T$ *and* $\mathbb{P}$*-almost surely.*

Proof. The Hamiltonian $\widetilde{H}^{\theta}$ associated with (3.4), is given by

$$\begin{aligned}&\widetilde{H}^{\theta}\left(t,\xi^{u}\left(t\right),x^{u}\left(t\right),y^{u}\left(t\right),z^{u}\left(t\right),r^{u}\left(t,.\right),\overrightarrow{p}^{u}\left(t\right),\overrightarrow{q}^{u}\left(t\right),\overrightarrow{\pi}^{u}\left(t,.\right)\right)\\ &=\left\{\theta V^{\theta}\left(t\right)\right\}H^{\theta}\left(t,x^{u}\left(t\right),y^{u}\left(t\right),z^{u}\left(t\right),r_{t}^{u}\left(t,.\right),\widetilde{p}_{2}\left(t\right),\widetilde{q}_{2}\left(t\right),\widetilde{p}_{3}\left(t\right)\right.\\ &\left.,\widetilde{\pi}_{2}\left(t,.\right),V^{\theta}\left(t\right),l_{2}\left(t\right),l_{3}\left(t\right),L_{2}\left(t,.\right),L_{3}\left(t,.\right)\right),\end{aligned}$$

and H^{θ} is the risk-sensitive Hamiltonian given by (3.18). To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $\left(\widetilde{p}_{2},\widetilde{q}_{2}\right)$, $\left(\widetilde{p}_{3},\widetilde{q}_{3}\right)$, $\left(\widetilde{\pi}_{2},\widetilde{\pi}_{3}\right)$and $\left(V^{\theta},l,L\right)$, which solve (3.16). Hence, since $V^{\theta}>0$, the variational inequality (3.6) translates into $H_{v}^{\theta}\left(t\right)\geq 0$, for all $u\in\mathcal{U}$, almost every $0\leq t\leq T$ and $\mathbb{P}$-almost surely. ■

3.3 Risk sensitive sufficient optimality conditions

This section is concerned with a study of the necessary condition of optimality (3.23) when it becomes sufficient.

Theorem 3.3 *(Risk neural sufficient optimality conditions)Assume that* $\Phi(.)$ *and* $\Psi(.)$ *are convex and for all* $(x, y, z, r, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times U$ *the function* $\widetilde{H}^{\theta}(., x, y, z, , r, v, ., ., .)$ *is concave, and for any* $v \in U$ *such that* $\mathbb{E}\left|v\right|^2 < \infty$. *Then,* u *is an optimal control of the problem* $\{(3.1), (3.2), (3.3)\}$, *if it satisfies* (3.6).

Proof. Let u be an admissible control (candidate to be optimal) for any $v \in U$, we have

$$\begin{aligned} J^{\theta}(v) - J^{\theta}(u) = \quad & \mathbb{E}\left[\exp\left\{\theta\Psi\left(y^{v}(0)\right) + \theta\Phi\left(x^{v}(T)\right) + \theta\xi^{v}(T)\right\}\right] \\ & -\mathbb{E}\left[\exp\left\{\theta\Psi\left(y^{u}(0)\right) + \theta\Phi\left(x^{u}(T)\right) + \theta\xi^{u}(T)\right\}\right]. \end{aligned}$$

Since Ψ and Φ are convex, and applying Taylor's expansion, we get

$$\begin{aligned} & J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\theta A_T\left(\xi^{v}(T) - \xi^{u}(T)\right)\right] + \mathbb{E}\left[\theta\Phi_x\left(x^{u}(T)\right)A_T\left(x^{v}(T) - x^{u}(T)\right)\right] \\ & +\mathbb{E}\left[\theta\Psi_y\left(y^{u}(0)\right)A_T\left(y^{v}(0) - y^{u}(0)\right)\right]. \end{aligned}$$

According to (3.5), we remark that $p_1(T) = \theta A_T$, $p_2(T) = \theta\Phi_x\left(x^{u}(T)\right)A_T$ and $p_3(0) = \theta\Psi_y\left(y^{u}(0)\right)A_T$, then

$$\begin{aligned} J^{\theta}(v) - J^{\theta}(u) \geq \quad & \mathbb{E}\left[p_1(T)\left(\xi_T^{v} - \xi_T^{u}\right)\right] + \mathbb{E}\left[p_2(T)\left(x^{v}(T) - x^{u}(T)\right)\right] \\ & +\mathbb{E}\left[p_3(0)\left(y^{v}(0) - y^{u}(0)\right)\right]. \end{aligned} \tag{3.24}$$

We apply Itô's formula to $p_1(t)\left(\xi^{v}(t) - \xi^{u}(t)\right)$,

$$\begin{aligned} d\left(p_1(t)\left(\xi^{v}(t) - \xi^{u}(t)\right)\right) = \quad & \left(\xi^{v}(t) - \xi^{u}(t)\right)dp_1(t) + p_1(t)\,d\left(\xi^{v}(t) - \xi^{u}(t)\right) \\ & + \left\langle\left(\xi^{v} - \xi^{u}\right), p_1\right\rangle_t dt + \int_{\Gamma}\left\langle\left(\xi^{v} - \xi^{u}\right), p_1\right\rangle_t m(d\lambda)\,dt \end{aligned},$$

We apply expectation, we get

$$\begin{aligned}\mathbb{E}\left[p_1(T)\left(\xi^v(T)-\xi^u(T)\right)\right] &= \mathbb{E}\left[\int_0^T \left(f\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_t\right) - \right.\right. \\ &\left. f\left(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_t\right)\right) p_1(t)\, dt\Big].\end{aligned} \tag{3.25}$$

And we apply also Itô's formula to $p_2(t)\left(x^v(t)-x^u(t)\right)$

$$\begin{aligned}d\left(p_2(t)\left(x^v(t)-x^u(t)\right)\right) = &\left(x^v(t)-x^u(t)\right) dp_2(t) + p_2(t)\, d\left(x^v(t)-x^u(t)\right) \\ &+ \left\langle x^v - x^u, p_2\right\rangle_t dt + \int_\Gamma \left\langle x^v - x^u, p_2\right\rangle_t m(d\lambda)\, dt,\end{aligned}$$

We apply expectation, we get

$$\begin{aligned}&\mathbb{E}\left[p_2(T)\left(x^v(T)-x^u(T)\right)\right] = \\ &\mathbb{E}\left[\int_0^T -\left(f_x(t)\, p_1 + b_x(t)\, p_2 + \sigma_x(t)\, q_2\right.\right. \\ &\left.\left. + g_x(t)\, p_3 + \int_\Gamma \gamma_x(t-,\lambda)\, \pi_2(\lambda,t)\, m(d\lambda)\right)\left(x_t^v - x_t^u\right) dt\right] \\ &+\mathbb{E}\left[\int_0^T \left(b\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_t\right) - \right.\right. \\ &\left. b\left(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_t\right)\right) p_2(t)\, dt\Big] \\ &+\mathbb{E}\left[\int_0^T \left(\sigma\left(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_t\right) - \right.\right. \\ &\left. \sigma\left(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_t\right)\right) q_2(t)\, dt\Big] \\ &+\mathbb{E}\left[\int_0^T \int_\Gamma \left(\gamma\left(t, x^v(t-), y^v(t-), z^v(t-), r^v(t-,\lambda), v_{t-}\right) - \right.\right. \\ &\left. \gamma\left(t, x^u(t-), y^u(t-), z^u(t-), r^u(t-,\lambda), u_{t-}\right)\right) \pi(t,\lambda)\, m(d\lambda)\, dt\Big],\end{aligned} \tag{3.26}$$

We apply also Itô's formula to $p_3(t)\left(y^v(t)-y^u(t)\right)$

$$\begin{aligned}d\left(p_3(t)\left(y^v(t)-y^u(t)\right)\right) = &\left(y^v(t)-y^u(t)\right) dp_3(t) + p_3(t)\, d\left(y^v(t)-y^u(t)\right) \\ &+ \left\langle y^v - y^u, p_3\right\rangle_t dt + \int_\Gamma \left\langle y^v - y^u, p_3\right\rangle_t m(d\lambda)\, dt\end{aligned},$$

We apply expectation, We get

$$\begin{aligned}
&\mathbb{E}\left[p_3(0)\left(y^v(0)-y^u(0)\right)\right]= \\
&\mathbb{E}\left[\int_0^T \left(g\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v_t\right)\right.\right. \\
&\left.-g\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t\right)\right)p_3(t)\,dt] \\
&-\mathbb{E}\left[\int_0^T \left(f_y(t)p_1(t)+b_y(t)p_2(t)+g_y(t)p_3(t)+\sigma_y(t)q_2(t)\right.\right. \\
&\left.\left.+\int_\Gamma \gamma_y(t-,\lambda)\pi_2(t,\lambda)m(d\lambda)\right)\left(y^v(t)-y^u(t)\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T \left(f_z(t)p_1(t)+b_z(t)p_2(t)+g_z(t)p_3(t)+\sigma_z(t)q_2(t)\right.\right. \\
&\left.\left.+\int_\Gamma \gamma_z(t-,\lambda)\pi_2(t,\lambda)m(d\lambda)\right)\left(z^v(t)-z^u(t)\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T\int_\Gamma \left(f_r(t)p_1(t)+b_r(t)p_2(t)+g_r(t)p_3(t)+\sigma_r(t)q_2(t)\right.\right. \\
&\left.+\gamma_r(t-,\lambda)\pi_2(t,\lambda)\right)\left(r_t^v(\lambda)-r_t^u(\lambda)\right)m(d\lambda)\,dt].
\end{aligned} \tag{3.27}$$

By replacing (3.25) ,(3.26) and (3.27) into (3.24) , we have

$$\begin{aligned}
&J^\theta(v)-J^\theta(u) \\
&\geq \mathbb{E}\left[\int_0^T \left(\widetilde{H}^\theta\left(t,x^v(t),y^v(t),z^v(t),r^v(t,.),v_t,p(t),q(t),\pi(t,.)\right)-\right.\right. \\
&\left.\left.\widetilde{H}^\theta\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t,p^u(t),q^u(t),\pi(t,.)\right)\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T \widetilde{H}_x^\theta\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t,p^u(t),q^u(t),\pi(t,.)\right)\left(x^v(t)-x^u(t)\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T \widetilde{H}_y^\theta\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t,p^u(t),q^u(t),\pi(t,.)\right)\left(y^v(t)-y^u(t)\right)dt\right] \\
&-E\left[\int_0^T \widetilde{H}_z^\theta\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t,p^u(t),q^u(t),\pi(t,.)\right)\left(z^v(t)-z^u(t)\right)dt\right] \\
&-E\left[\int_0^T \nabla\widetilde{H}_r^\theta\left(t,x^u(t),y^u(t),z^u(t),r^u(t,.),u_t,p^u(t),q^u(t),\pi(t,.)\right)\right. \\
&\left.\left(r^v(t,.)-r^u(t,.)\right)dt\right].
\end{aligned}$$

Since the Hamiltonian $\widetilde{H}^{\theta}$ is concave with respect to (x,y,z,r,v), we have

$$\mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t,x^u\left(t\right),y^u\left(t\right),z^u\left(t\right),r^u\left(t,.\right),u_t,p^u\left(t\right),q^u\left(t\right),\pi\left(t,.\right)\right)\left(v_t-u_t\right)dt\right]$$
$$\leq \mathbb{E}\left[\int_0^T\left(\widetilde{H}^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),v_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)-\right.\right.$$
$$\left.\left.\widetilde{H}^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),u_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\right)dt\right]$$
$$+\mathbb{E}\left[\int_0^T \widetilde{H}_x^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),v_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\left(x^v\left(t\right)-x^u\left(t\right)\right)dt\right]$$
$$+\mathbb{E}\left[\int_0^T \widetilde{H}_y^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),v_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\left(y^v\left(t\right)-y^u\left(t\right)\right)dt\right]$$
$$+E\left[\int_0^T \widetilde{H}_z^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),v_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\left(z^v\left(t\right)-z^u\left(t\right)\right)dt\right]$$
$$+E\left[\int_0^T \nabla\widetilde{H}_r^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),v_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\right.$$
$$\left.\left(r^v\left(t,.\right)-r^u\left(t,.\right)\right)dt\right].$$

Then

$$J^{\theta}\left(v\right)-J^{\theta}\left(u\right)$$
$$\geq \mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t,x^v\left(t\right),y^v\left(t\right),z^v\left(t\right),r^v\left(t,.\right),u_t,p^u\left(t\right),q^u\left(t\right),\pi^u\left(t,.\right)\right)\left(v_t-u_t\right)dt\right].$$

In virtue of the necessary condition of optimality (3.6) the last inequality implies that $J^{\theta}\left(v\right)-J^{\theta}\left(u\right)\geq 0$. Then, the theorem is improved. ■

Theorem 3.4 *(Risk sensitive sufficient optimality conditions)Assume that* $\Phi(.)$ *and* $\Psi(.)$ *are convex and for all* $(x,y,z,r,v)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Gamma\times U$ *the function* $H^{\theta}(.,x,y,z,,r,v,.,.,.)$ *is concave, and for any* $v\in U$ *such that* $\mathbb{E}\left|v\right|^2<\infty$. *Then, u is an optimal control of the problem* $\{(3.1),(3.2),(3.3)\}$, *if it satisfies* (3.23).

3.4 Example: Mean-Variance (Cash-flow)

Now we return to the problem of optimal portfolio stated in the motivating example, and apply the risk sensitive necessary optimality condition (Theorem 3.2).

Our state dynamics is

$$\begin{cases} dx(t) = (\rho v(t) - cx(t))\, dt + \sigma v(t)\, dW(t) + \int_\Gamma v(t)(1 + r(t,\lambda))\, \widetilde{N}(d\lambda, dt), \\ x(0) = m_0 = d, \end{cases} \tag{3.28}$$

and

$$\begin{cases} dy(t) = (\rho v(t) - cx(t) + \lambda y(t))\, dt + z(t)\, dW(t) + \int_\Gamma r(t,\lambda)\, \widetilde{N}(d\lambda, dt), \\ y(T) = 0 = a. \end{cases} \tag{3.29}$$

The cost functional is

$$J^\theta(v(.)) = \exp\left\{\theta \widetilde{J}^\theta(v(.))\right\},$$

where $\widetilde{J}$ is the neutral cost functional given by the following expected with an exponential form see section 1.2.3

$$\widetilde{J}^\theta(v(.)) = \frac{\theta}{2}\mathbb{E}(\Psi_T - a)^2 + \mathbb{E}(\Psi_T) + o(\theta^2), \tag{3.30}$$

Where $\Psi_T = (x_T + y_0)$. The investor wants to minimize (3.30) subject to (3.28) and (3.29) by taking $v(.)$ over $\mathcal{U}$, the mean–variance portfolio selection problem is to find $u(t)$ which minimize

$$\mathbb{V}ar(\Psi_T) = \mathbb{E}(x_T + y_0 - a)^2$$

The Hamiltonian function (3.18) gets the form

$$\begin{aligned} H^\theta(t) &:= H^\theta(t, x(t), y(t), z(t), r(t,\lambda), \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), \widetilde{\pi}_2(t,.), l(t), L(t,.), v_t) \\ &= f(t) + b(t)\widetilde{p}_2(t) + \sigma(t)\widetilde{q}_2(t) + \{g(t) - \theta l(t) z(t)\}\widetilde{p}_3(t) \\ &+ \int_\Gamma \{\gamma(t-,\lambda)\widetilde{\pi}_2(t,\lambda) - (g(t) - \theta L(t,\lambda) r(t,\lambda))\widetilde{p}_3(t)\} m(d\lambda) \\ &= (\rho v(t) - cx(t))\widetilde{p}_2(t) + \sigma v(t)\widetilde{q}_2(t) + \{(\rho v(t) - cx(t) + \lambda y(t)) - \theta l(t) z(t)\}\widetilde{p}_3(t) \\ &- \int_\Gamma \{v(t)(1 + r(t,\lambda))\widetilde{\pi}_2(t,\lambda) - ((\rho v(t) - cx(t) + \lambda y(t)) - \theta L(t,\lambda) r(t,\lambda)) \\ &\widetilde{p}_3(t)\} m(d\lambda). \end{aligned}$$

Then, to get the optimal control, the derivative of the above Hamiltonian with respect to the control process gives us

$$\begin{aligned} H_u^\theta(t) &:= H_u^\theta\left(t, x(t), y(t), z(t), r(t,.), \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), \widetilde{\pi}_2(t,.), l(t), L(t,.), v_t\right) \\ &= \rho\widetilde{p}_2(t) + \sigma\widetilde{q}_2(t) + \int_\Gamma (1 + r(t,\lambda))\,\widetilde{\pi}_2(t,\lambda)\, m(d\lambda) \\ &= 0 \end{aligned} \tag{3.31}$$

Let $(x^u(t), u(t))$ be an optimal pair, the adjoint equation (3.21), is given by

$$\begin{cases} d\widetilde{p}_2^u(t) &= c\left(\widetilde{p}_2^u(t) + c\widetilde{p}_3^u(t)\right)dt + \left(\widetilde{q}_2^u(t) - \theta l_2(t)\,\widetilde{p}_2^u(t)\right)dW^\theta(t) \\ &\quad + \displaystyle\int_\Gamma \left(\widetilde{\pi}_2(t,\lambda) - \theta L_2(t,\lambda)\,\widetilde{p}_2^u(t)\right)\widetilde{N}^\theta(d\lambda, dt), \\ \widetilde{p}_2^u(T) &= 1 + \theta\left(x_T - y_0 - a\right). \end{cases}$$

By using of 3.19, we get

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left\{\left(c + \theta^2 l^2(t) + \int_\Gamma \theta^2 L^2(t,\lambda)\, m(d\lambda)\right)\widetilde{p}_2^u(t) + c\widetilde{p}_3^u(t) - \theta l(t)\,\widetilde{q}_2^u \right. \\ &\quad \left. - \int_\Gamma \theta L(t,\lambda)\,\widetilde{\pi}_2^u(t,\lambda)\, m(d\lambda)\right\}dt + \left(\widetilde{q}_2^u(t) - \theta l_2(t)\,\widetilde{p}_2^u(t)\right)dW(t) \\ &\quad + \displaystyle\int_\Gamma \left(\widetilde{\pi}_2(t,\lambda) - \theta L_2(t,\lambda)\,\widetilde{p}_2^u(t)\right)\widetilde{N}(d\lambda, dt), \\ \widetilde{p}_2^u(T) &= 1 + \theta\left(x_T - y_0 - a\right). \end{cases} \tag{3.32}$$

Therefore, an optimal solution $(x_t^u, \widetilde{p}_2^u(t), u_t)$ can be obtained by solving the system FBSDE with jumps diffusion (3.28) and (3.32), unfortunately, in such system is difficult to find the explicit solution, to this end we use the similar technique as in [41] see also [40], we conjecture the solution to (3.28) and (3.32) is related by

$$\widetilde{p}_2^u(t) = A(t)\, x^u(t) + B(t), \tag{3.33}$$

for some deterministic differentiable functions $A(t)$ and $B(t)$. Applying Itô's formula to

(3.33) , we get

$$\begin{cases} d\widetilde{p}_2^u(t) = & \left[\dot{A}(t)\,x^u(t) + A(t)(\rho u_t - c x^u(t)) + \dot{B}(t)\right] dt + A(t)\,\sigma u_t dW(t) \\ & + \int_\Gamma A(t)\,(1 + r(t,\lambda))\,u_t \widetilde{N}(d\lambda, dt), \\ d\widetilde{p}_2^u(T) = & A(T)\,x^u(T) + B(T). \end{cases} \tag{3.34}$$

On the other hand, by substituting (3.33) into (3.32) , and denote by

$$\begin{aligned} \widetilde{q}_3^u(t) &= (\theta l_2(t)\,\widetilde{p}_2^u(t) - \widetilde{q}_2^u(t)) \\ \widetilde{\pi}_3(t,.) &= \widetilde{\pi}_2(t,.) - \theta L_2(t,.)\,\widetilde{p}_2^u(t). \end{aligned} \tag{3.35}$$

By using the Girsanov's transformation in (3.32) , as in section 2 lemma (3.1), we obtain

$$\begin{cases} d\widetilde{p}_2^u(t) = & \left\{\left(c + \theta^2 l^2(t) + \int_\Gamma \theta^2 L^2(t,\lambda)\, m(d\lambda)\right) \widetilde{p}_2^u(t) + c\widetilde{p}_3^u(t) - \theta l(t)\,\widetilde{q}_3^u(t)\right\} dt \\ & + \widetilde{q}_3^u(t)\,dW(t) + \int_\Gamma \widetilde{\pi}_3(t,\lambda)\,\widetilde{N}(d\lambda, dt), \\ \widetilde{p}_2^u(T) = & 1 + \theta(x_T - y_0 - a). \end{cases} . \tag{3.36}$$

By equating the coefficients and the final conditions of (3.36) with (3.34) , we have

$$\begin{aligned} \widetilde{\pi}_3(t,\lambda) = & \quad A(t)\,(1 + r(t,.))\,u_t, \\ \widetilde{q}_3^u(t) = & \quad \sigma u_t A(t), \\ A(T) = & \quad \theta, \\ B(T) = & \quad 1 - \theta(y_0 + a). \end{aligned} \tag{3.37}$$

By identifying (3.35) with (3.37) , we can rewrite

$$\widetilde{q}_2^u(t) = \theta l_2(t)\,(A(t)\,x^u(t) + B(t)) + \sigma u_t A(t),$$

and

$$\widetilde{\pi}_2^u(t,.) = \theta L_2(t,.)\,(A(t)\,x^u(t) + B(t)) + r(t,.)\,u_t A(t),$$

then replacing the both equations (3.37), and the last equations of $\widetilde{q}_2^u(t)$ and $\widetilde{\pi}_2^u(t,.)$ into

(3.31), we have,

$$\begin{aligned}&\rho\left(A(t)x^{u}\left(t\right)+B(t)\right)+\rho\widetilde{p}_{3}\left(t\right)+\sigma\theta l\left(t\right)\left(A(t)x^{u}\left(t\right)+B(t)\right)+\sigma^{2}A(t)u_{t}\\&+\int_{\Gamma}\left\{\left(1+r\left(t,\lambda\right)\right)\theta L\left(t,\lambda\right)\left(A(t)x^{u}\left(t\right)+B(t)\right)+\left(1+r\left(t,\lambda\right)\right)^{2}A(t)u_{t}-\rho\widetilde{p}_{3}\left(t\right)\right\}m\left(d\lambda\right)\\&=0,\end{aligned}$$

then we get,

$$u\left(t,x_{t}\right)=-\frac{\left(\rho+\sigma\theta l\left(t\right)+\int_{\Gamma}\left(1+r\left(t,\lambda\right)\right)\theta L\left(t,\lambda\right)m\left(d\lambda\right)\right)\left(A(t)x^{u}\left(t\right)+B(t)\right)+\rho\widetilde{p}_{3}\left(t\right)}{A(t)G\left(t\right)},\tag{3.38}$$

where $G\left(t\right)=\sigma^{2}-\int_{\Gamma}\left(1+r\left(t,\lambda\right)\right)^{2}m\left(d\lambda\right).$

In the other side, we have from (3.34) and (3.36) . Then

$$u_{t}=-\frac{\overset{\bullet}{A}\left(t\right)x^{u}\left(t\right)-2cA(t)x^{u}\left(t\right)-cB(t)+\overset{\bullet}{B}\left(t\right)-c\widetilde{p}_{3}^{u}\left(t\right)}{A\left(t\right)\left(\rho+\sigma\theta l\left(t\right)+\int_{\Gamma}\left(1+r\left(t,\lambda\right)\right)\theta L\left(t,\lambda\right)m\left(d\lambda\right)\right)}.\tag{3.39}$$

From (3.38) and (3.39), we have

$$\begin{cases}\overset{\bullet}{A}\left(t\right)=\left\{2c+\dfrac{\left(\rho+\sigma\theta l\left(t\right)+\int_{\Gamma}\left(1+r\left(t,\lambda\right)\right)\theta L\left(t,\lambda\right)m\left(d\lambda\right)\right)^{2}}{G\left(t\right)}\right\}A(t),\\A(T)=\theta.\end{cases}\tag{3.40}$$

and

$$\begin{cases}\overset{\bullet}{B}\left(t\right)=\left\{c+\dfrac{\left(\rho+\sigma\theta l\left(t\right)+\int_{\Gamma}\left(1+r\left(t,\lambda\right)\right)\theta L\left(t,\lambda\right)m\left(d\lambda\right)\right)^{2}}{G\left(t\right)}\right\}B(t)+c\widetilde{p}_{3}^{u}\left(t\right),\\B(T)=1-\theta\left(y_{0}+a\right).\end{cases}\tag{3.41}$$

Then the explicit solutions of (3.40) , and (3.41) have the form

$$\begin{cases} A(t) = \theta \exp \int_t^T \left\{ 2c + \frac{\left(\rho + \sigma\theta l\,(s) + \int_\Gamma (1 + r\,(s,\lambda))\,\theta L\,(s,\lambda)\,m\,(d\lambda)\right)^2}{G\,(s)} \right\} ds, \\ B(t) = (1 - \theta\,(y_0 + a)) \exp \int_t^T \left[\left\{ c + \frac{\left(\rho + \sigma\theta l\,(s) + \int_\Gamma (1 + r\,(s,\lambda))\,\theta L\,(s,\lambda)\,m\,(d\lambda)\right)^2}{G\,(s)} \right\} \right. \\ \left. B(s) + c\widetilde{p}_3^u\,(s) \right] ds. \end{cases} \tag{3.42}$$

Remark 3.1 *It's very important to remark that the solution of the function* $B\,(t)$ *in the form* (3.42) *is depend to the solution of* $\widetilde{p}_3\,(t)$ *. If we put* $\widetilde{p}_3\,(t) = \psi\,(t)\,y\,(t) + \varphi\,(t)$ *, for smooth deterministic functions* ψ*, and* φ*, by using the similar technique as an optimal solution in the last paragraph, to the triplet* $(y^u\,(t)\,, \widetilde{p}_3^u\,(t)\,, u\,(t))$*. Then the solutions of* ψ*, and* φ *yield respectively the equations*

$$\begin{cases} \dot{\psi}\,(t) = \rho^2\psi^2\,(t) - (2\lambda\sigma^2 A(t) - \theta^2 l^2(t))\,\psi\,(t)\,, \\ \dot{\varphi}\,(t) = (\rho\psi\,(t) + \theta^2 l^2(t) - \lambda)\varphi\,(t) + K(t), \\ \psi\,(0) = \theta, \text{ and } \varphi\,(0) = 1 - \theta\,(y_0 - a)\,. \end{cases} \tag{3.43}$$

The main result in this section, can be given in the form of maximum principle of mean variance problem with risk sensitive performance.

Theorem 3.5 *We assume that the pair* $(A\,(t)\,, B(t))$ *has unique solution given by* (3.42)*, the pair* $(\varphi\,(t)\,, \psi(t))$ *has also the explicit solution of the system* (3.43)*. Then the optimal control of the problem* (3.28)*,* (3.29) *and* (3.30) *has the state feedback form*

$$u\,(t, x_t, y_t, r_t\,(.)) = \\ -\frac{\left(\rho + \sigma\theta l\,(t) + \int_\Gamma (1 + r\,(t,\lambda))\,\theta L\,(t,\lambda)\,m\,(d\lambda)\right)(A(t)x^u\,(t) + B(t)) + \rho\,(\psi\,(t)\,y^u\,(t) + \varphi\,(t))}{A(t)G\,(t)}.$$

Conclusion and Perspectives

This thesis contains two main results. The first one is the necessary optimality condition for the systems of fully coupled FBSDE, fully coupled FBSDE with jump under risk sensitive performance, which are mentioned respectively by the following theorems $(2.2, 3.2)$ using an almost similar scheme as in Djechiche et al [11]. The second main result, suggests sufficient optimality conditions of the above systems, and mentioned respectively by the following theorems $(2.3, 3.3)$. The proofs is based on the convexity conditions of the Hamiltonians functions, the initial and terminal terms of the performance functions.

- If we put in our first example of [20] $A_t = B_t = C_t = D_t = 0$, we get the same result as in [8], and the sufficient optimality conditions are similar to those in the paper of Chala [9].

- Our results in Kallout and Chala [20], can be compared with the maximum principle obtained by Djechiche et al [11], and we note here that our study is the result's extension of Chala [8], [9].

- In our results of Kallout and Chala [21], we study the generale case - we add the jumps diffusion term to our system in Kallout and Chala [20]-. This result discussed as a third resullt's generalization of Chala [8] [9].

- On the other hand, in the case where the system is governed by mean field type, we may take the existing paper established by Djechiche et al [11]. We have generalized the result of Kallout and Chala [20] into the fully coupled stochastic differential equation governed by mean field type, and this problem to be thoroughly addressed in our future paper, and will be compared with [24].

- Remarkably, the risk sensitive control problem studied by Lim and Zhou in [23] is different from ours.

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