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Some contributions on stochastic optimal control for FBSDEs

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In the memory of my father Abdellah.

In the memory of my grandfather K. Ibrahim.

To my beloved mother KARKADI.Aicha.

To all my family and my best friends.

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Contents

Abstract

In this thesis, we are concerned with stochastic optimal control of systems governed by forward-backward stochastic differential equations of mean field type. The first part of this thesis is dedicated to the existence and uniqueness of solutions for systems of forward-backward stochastic differential equation of mean-field type. We use here, the Picard's iteration method.

In the second part, we study the existence of optimal solution of optimal control problem driven by a linear backward stochastic differential equations of mean field type (MF-FBSDE) with non linear cost, where the control domain and the cost functions are assumed convex. The second main result established in this part is a necessary as well as sufficient conditions of optimality satisfied by an optimal control for this kind of stochastic control problem, the proof of this result is based on the convex optimization principle.

In the third part, we consider stochastic control problems for a system of forward backward stochastic differential equations of mean field (MF-FBSDEs) with uncontrolled diffusion. Our interest goes particularity to the questions of existence of optimal relaxed control as well as existence of optimal strict control for a nonlinear MF-FBSDEs. We derive also necessary and sufficient optimality conditions for both relaxed and strict control problems.

The second and the third parts of this thesis are project of paper submitted in international journal.

In the last chapter, we prove the existence of optimal relaxed control as well as optimal strict control for nonlinear MF-FBSDEs with controlled diffusion. This part is published as paper: R.Benbrahim, B.Gherbal, *Existence of Optimal Controls for Forward-Backward Stochastic Differential Equation of Mean-field Type*, Journal of Numerical Mathematics and Stochastic, **9**(1): 33-47, 2017.

Key words: Mean-field, forward backward stochastic differential equation, stochastic control, relaxed control, strict control, existence, necessary and sufficient conditions, tightness, S-topology.

Résumé

Dans cette thèse nous nous sommes intéressés au contrôle stochastique optimal des équations différentielles stochastiques progressives rétrogrades de type champ moyen. Nous présentons dans la première partie, l'existence et l'unicité des solutions pour des systèmes d'équations différentielles stochastiques progressives rétrogrades (EDSPRs) de type champ moyen. Dans ce cas on utilise la méthode d'itération de Picard.

Dans la deuxième partie, nous étudions l'existence d'une solution optimale du problème de contrôle optimal pour des EDSPRs de type champ moyen linéaires avec coût non linéaire, où le domaine de contrôle et les fonctions de coût sont supposés convexes. Un deuxième résultat essentiel dans cette partie, est d'établir les conditions nécessaires et suffisantes d'optimalité satisfaites par le contrôle optimal strict pour ce genre de problème de contrôles stochastiques, la preuve de ce résultat est basée sur le principe d'optimisation convexe.

Dans la troisième partie, nous considérons un problème du contrôle stochastique pour des systèmes d'EDSPRs de type champ moyen. Notre intérêt va en particulier vers les questions d'existence de contrôle optimal relaxé et l'existence de contrôle optimal strict pour les EDSPRs de type champ moyen . Nous établirons aussi des conditions nécessaires et suffisantes d'optimalités pour les deux problèmes de contrôle relaxé et strict.

Les deuxième et troisième parties de cette thèse sont un projet d'article soumis dans un journal international.

Dans le dernier chapitre, nous prouvons l'existence d'un contrôle relaxé optimal et d'un contrôle strict optimal pour les EDSPRs de type champ moyen non linéaires à diffusion contrôlée. Cette partie est publiée sous forme d'un article :

R.Benbrahim, B.Gherbal, *Existence of Optimal Controls for Forward-Backward Stochastic Differential Equation of Mean-field Type*, Journal of Numerical Mathematics and Stochastic, **9**(1): 33-47, 2017.

Mots clés: Champ moyen, équation différentielle stochastique progressive rétrograde, contrôle stochastique, contrôle relaxé, contrôle strict, existence, conditions nécessaires et suffisantes, tension, la S-topologie.

Symbols and Abbreviations

The different symbols and abbreviations used in this thesis

Symbols

$\mathbb{R}$	:	Real numbers.
$\mathbb{R}^n$	:	n-dimensional real Euclidean space.
$\mathbb{R}^{n\times d}$	:	The set of all $(n \times d)$ real matrixes.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$(\mathcal{F}_t)_{t\geq 0}$	:	Filtration.
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$		A filtered probability space.
$\mathbb{E}[x]$	:	Expectation at x.
$\mathbb{E}[x\vert\mathcal{F}_t]$	:	Conditional expectation.
$(W_t)_{t\geq 0}$	:	Brownian motion.
$\delta(da)$	:	The Dirac measure.
μ	:	The relaxed control.
U	:	The set of values taken by the strict control u.
$\mathcal{U}$	:	The set of admissible strict controls.
$\mathbb{V}$	:	The space of positive Radon measures on $[0;1] \times U$.

J(.)	:	The cost function.
u^*	:	Optimal strict control.
μ^*	:	Optimal relaxed control.
H	:	The Hamiltonian.
$\mathcal{R}$	:	The set of relaxed controls.
$CV(.)$	:	The conditional variation

Abbreviations

$SDEs$	:	Stochastic differential equations.
BSDEs	:	Backward stochastic differential equations.
FBSDE	:	Forward-backward stochastic differential equations.
MF-FBSDE	:	Forward-backward stochastic differential equations of mean field type.
càdlàg	:	Right continuous with left limits.
a.s		Almost surely.

Introduction

In 1993 Antonelli in [3], studied the system of forward-backward stochastic differential equations in the first time, and since then it has become very useful in stochastic control problems and mathematical finance. Therefore, certain important problems in mathematical economics and mathematical finance, especially in the optimization problem, can be formulated to be FBSDEs. There are two important approaches to the general stochastic optimal control problem. One is the Bellman dynamic programming principle, which results in the Hamilton-Jacobi-Bellman equation. The other is the maximum princible. This last approach has been established in many papers, see Xu in [34], Wu [33] and Peng and Wu [32]. In 2009, a new kind of backward stochastic differential equations called mean-field BSDEs has been introduced by Buckdahn, Djehiche, Li and Peng [9], which were derived as a limit of some highly dimensional system of FBSDEs, corresponding to a large number of particles. Since that, many authors treated the system of this kind of Mckean-Vlasov type (see [28] and [1]). In this respect we refer the reader also to [8] and [2].

The existence of solution for mean-field FBSDEs systems has been proved by Carmona and Dularue [11]. A maximum principle for fully coupled MF-FBSDEs has been treated by Li and Liu [27], where the control domain is not assumed to be convex. A maximum principle for mean-field FBSDEs with jumps has been investigated by Hafayed [18] and also in Hafayed et al. [19]. See also [20] for systems of MF-FBSDEs driven by Teugels martingales.

In this work, we prove existence of optimal controls for systems governed by the following MF-FBSDEs

$$\begin{cases} dX_t = b(t, X_t, \mathbb{E}[\alpha(X_t)], v_t)dt + \sigma(t, X_t, \mathbb{E}[\beta(X_t)], v_t)dW_t \\ \\ dY_t = -f(t, X_t, \mathbb{E}[\gamma(X_t)], Y_t, \mathbb{E}[\delta(Y_t)], v_t)dt + Z_t dW_t + dN_t \\ X_0 = x_0, Y_T = g(X_T, \mathbb{E}[\lambda(X_T)]), \quad t \in [0, T], \end{cases} \tag{1}$$

where $b, \alpha, \sigma, \beta, f, \gamma, \lambda, g$ and λ are given functions, $(W_t, t \geq 0)$ is a standard Brownian motion, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions. X, Y, Z are square integrable adapted processes and N a square integrable martingale that is orthogonal to W. The control variable v_t, called strict control, is a measurable, $\mathcal{F}_t-$ adapted process with values in a compact metric space U.

We shall consider a functional cost to be minimized, over the set $\mathcal{U}$ of a admissible strict controls, as the following:

$$\begin{aligned} J(v.) := \mathbb{E}\big[& l\left(X_T, \mathbb{E}\left[\theta\left(X_T\right)\right]\right) + k\left(Y_0, \mathbb{E}\left[\rho\left(Y_0\right)\right]\right) \\ & + \int_0^T h\big(t, X_t, \mathbb{E}\left[\varphi\left(X_t\right)\right], Y_t, \mathbb{E}\left[\psi\left(Y_t\right)\right], v_t\big)\, dt\big], \end{aligned} \tag{2}$$

where $l, \theta, k, \rho, h, \varphi$ and ψ are appropriate functions.

The considered system and the cost functional, depend on the state of the system and also on the distribution of the state process, via the expectation of some function of the state. The mean-field FBSDEs (1) called McKean-Vlasov systems are obtained as the mean square limit of an interacting particle system of the form

$$\begin{cases} dX_t^{i,n} = b(t, X_t^{i,n}, \frac{1}{n}\sum\limits_{j=1}^{n} A\left(X_t^{j,n}\right), v_t)dt + \sigma(t, X_t^{i,n}, \frac{1}{n}\sum\limits_{j=1}^{n} B\left(X_t^{j,n}\right), v_t)dW_t \\ \\ dY_t^{i,n} = -f(t, X_t^{i,n}, \frac{1}{n}\sum\limits_{j=1}^{n} C\left(X_t^{j,n}\right), Y_t^{i,n}, \frac{1}{n}\sum\limits_{j=1}^{n} D\left(Y_t^{j,n}\right), v_t)dt + Z_t^{i,n} dW_t^i + dN_t^i, \end{cases} \tag{3}$$

where $\left(W_{\cdot}^{i}, i \geq 0\right)$ is a collection of independent Brownian motion. The system of mean-field FBSDEs (1) occur naturally in the probabilistic analysis of financial optimization and control problems of the McKean-Vlasov type. This kind of approximation result is called "propagation of chaos", which says that when the number of particles (or players) tends to infinity, the equations defining the evolution of the particles could be replaced by a single equation (McKean-Vlasov equation).

The existence of strict optimal controls for stochastic differential equations, follows from the Roxin-type convexity condition (see [13, 16, 26]). Without this condition, a strict optimal control may fail to exist. The idea is then to introduce a new class $\mathcal{R}$ of admissible relaxed control in which, the controller chooses at time t, a probability measure $q_t\left(da\right)$ on the control set U, rather than an element $u_t \in U$.

Fleming [14] derived the first existence result of an optimal relaxed control for SDEs with uncontrolled diffusion coefficient by using compactification techniques. The case of SDEs with controlled diffusion coefficient has been solved by El-Karoui et al. [13], where the optimal relaxed control is shown to be Markovian. See also Haussmann and Lepeltier [21]. Existence of optimal control for FBSDEs has been proved by Bahlali, Gherbal and Mezerdi [5], see also Buckdahn et al [10]. In Bahlali, Gherbal and Mezerdi [6] an existence of optimal control for linear BSDEs has been proved and this result has been extended to a system of linear backward doubly SDEs by Gherbal [15]. For systems of mean-field SDEs, Bahlali et al in [7] proved the existence of optimal controls, where the diffusion coefficient is not controlled.

Our main goal in this thesis is to prove existence of optimal control as will as establish necessary and sufficient optimality conditions for systems governed by FBSDEs of mean-field type. We prove in first, the existence of a strong strict optimal control (that is adapted to the initial filtration) for a control problem governed by linear MF-FBSDEs and establish also necessary and sufficient conditions of optimality for this problem by using the convex optimization principle

. The second main result is to prove the existence of optimal relaxed controls as will as optimal strict controls, for non linear MF-FBSDEs systems with uncontrolled diffusion. Our approch is based on tightness properties of the distributions of the processes defining the control problem and the Skorokhod's selection theorem on the space $\mathbb{D}$ (of càdlàg processes), endowed with the Jakubowski S-topology [24]. Moreover, when the Roxin convexity condition is fulfilled, we prove that the optimal relaxed control is in fact strict. We establish also necessary as well as sufficient optimality conditions for both relaxed and strict control problems by using the convex perturbation method. The third mean result, is to peove existence of optimal controls for systems of non linear MF-FBSDEs with controlled diffusion coefficient. Our results extend in particular those in [5], [6] and [7].

In this thesis, we are interested by the existence of an optimal control where the state equation, as well as the cost function are of mean field type. It is organized as follows:

- In the first chapter (**Existence and uniqueness of solution of Forward-Backward stochastic differential equation of Mean field type**): We present, the existence and uniqueness theorem for solution of MF-FBSDE's where the coefficients were assumed to be Lipschitz.
- In the second chapter (**Existence of an optimal strict control and optimality conditions for linear FBSDEs of mean-field type**): In this chapter, we prove the existence of a strong strict optimal control for a control problem governed by linear MF-FBSDEs and we derive also necessary and sufficients conditions for optimality for this control problem of linear MF-FBSDEs.
- In the third chapter (**Existence of optimal solutions and optimality conditions for optimal control problems of MF-FBSDEs systems with uncontrolled diffusion**): We present and prove the main result concerning the existence of relaxed optimal controls and strict optimal

controls for non linear MF-FBSDEs with uncontrolled diffusion coefficient. We establish also in this chapter necessary as well as sufficient optimality conditions for both relaxed and strict control problems.

- In the fourth chapter (**Existence of optimal solutions for optimal control problems of MF-FBSDEs systems with controlled diffusion**): We prove the existence of optimal controls for systems governed by non linear MF-FBSDEs with controlled diffusion coefficient, by using the weak convergence techniques for the associated MF-FBSDEs on the space of continuous functions and on the space of cádlág functions endowed with the Jakubowski S-topology. Moreover, when the Roxin convexity condition is fulfilled, we get that the set of strict control coincides with that of relaxed control.

CHAPTER 1

Existence and uniqueness of solution of FBSDE of Mean field type

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In this chapter, we presente and prove a theorem of existence and uniqueness of solutions for systems of forward-backward stochastic differential equation of mean-field type, where the coefficient of the system depend not only on the state process, but also on the distribution of the state process, via the expectation of some function of the state. We use the Picard's iteration method.

1.1 Preliminaries

Let (W_t) be a d-dimensional Brownian motion, defined on a probability filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions. We also shall introduce the following two spaces of processes :

$\mathbb{M}^2([0,T];\mathbb{R}^m)$: the set of jointly measurable, processes $\left\{Y_t, t \in \left[0,T\right]\right\}$ with values in $\mathbb{R}^m$ such that Y_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and satisfy

$$\mathbb{E}\Big[\int_0^T |Y_t|^2 \, dt\Big] < \infty.$$

Let $\mathbb{S}^2([0,T];\mathbb{R}^n)$: the set of jointly measurable, processes $\{X_t, t \in [0,T]\}$ with values in $\mathbb{R}^n$ such that X_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and satisfy

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] < \infty.$$

For any positive number $T > 0$, we consider the system of the following froward-backward stochastic differential equations of mean-field type:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, \mathbb{E}[\alpha(X_s)])ds + \int_0^t \sigma(s, X_s, \mathbb{E}[\beta(X_s)])dW_s \\ \\ Y_t = g(X_T, \mathbb{E}[\lambda(X_T)]) + \int_t^T f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)])ds - \int_t^T Z_s dW_s, \end{cases} \tag{1.1}$$

where (**H1.0**):

$$b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d},$$

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

$$g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m,$$

$$\alpha, \beta, \lambda, \gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\delta : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

are a given bounded and continuous functions.

1.2 Existence and uniqueness

To establesh the result of existence and uniqueness of solution for systems of forward-backward stochastic differential equations of mean-field type we need to the following assumptions:

$(\mathbf{H1})$. Assuming that the functions $b, \alpha, \sigma, \beta, f, \gamma, \lambda, g$ and λ satisfy assumption $(\mathbf{H1})$ if there exist two constant k and k_1 such that they satisfy both $(\mathbf{H1.0})$ and the following properties:

$(\mathbf{H1.1})$ for every $t \in [0,T]$, $\forall \left(x_1, x_2, x_1', x_2'\right) \in \mathbb{R}^{4n}, \left(y_1, y_2, y_1', y_2'\right) \in \mathbb{R}^{4m}$,

$$\left|f\left(t, x_1, x_1', y_1, y_1'\right) - f\left(t, x_2, x_2', y_2, y_2'\right)\right| \leq k \ \left(\left|x_1 - x_2\right| + \left|x_1' - x_2'\right| + \left|y_1 - y_2\right| + \left|y_1' - y_2'\right|\right),$$

$$\left|b\left(t, x_1, x_1'\right) - b\left(t, x_2, x_2'\right)\right| \leq k \ \left(\left|x_1 - x_2\right| + \left|x_1' - x_2'\right|\right),$$

$$\left|\sigma\left(t, x_1, x_1'\right) - \sigma\left(t, x_2, x_2'\right)\right| \leq k \ \left(\left|x_1 - x_2\right| + \left|x_1' - x_2'\right|\right),$$

$$\left|\alpha\left(x_1\right) - \alpha\left(x_2\right)\right| \leq k \ \left|x_1 - x_2\right|, \qquad \left|\beta\left(x_1\right) - \beta\left(x_2\right)\right| \leq k \ \left|x_1 - x_2\right|,$$

$$\left|\gamma\left(x_1\right) - \gamma\left(x_2\right)\right| \leq k \ \left|x_1 - x_2\right|, \qquad \left|\lambda\left(x_1\right) - \lambda\left(x_2\right)\right| \leq k \ \left|x_1 - x_2\right|,$$

$$\left|\delta\left(y_1\right) - \delta\left(y_2\right)\right| \leq k \ \left|y_1 - y_2\right|, \qquad \left|g\left(x_1, x_1'\right) - g\left(x_2, x_2'\right)\right| \leq K \ \left(\left|x_1 - x_2\right| + \left|x_1' - x_2'\right|\right).$$

$(\mathbf{H1.2})$ for every $t \in [0,T]$, $\forall \left(x_1, x_2\right) \in \mathbb{R}^{2n}, \left(y_1, y_2\right) \in \mathbb{R}^{2m}$,

$$\left|b\left(t, x_1, x_2\right)\right| + \left|\sigma\left(t, x_1, x_2\right)\right| + \left|g\left(t, x_1, x_2\right)\right| \leq k_1 \ \left(1 + \left|x_1\right| + \left|x_2\right|\right),$$

$$\left|\alpha\left(x_1\right)\right| + \left|\beta\left(x_1\right)\right| + \left|\gamma\left(x_1\right)\right| + \left|\lambda\left(x_1\right)\right| \leq k_1\left(1 + \left|x_1\right|\right),$$

$$\left|f\left(t, x_1, x_2, y_1, y_2\right)\right) \leq k_1\left(1 + \left|x_1\right| + \left|x_2\right| + \left|y_1\right| + \left|y_2\right|\right), \qquad \left|\delta\left(y_1\right)\right| \leq k_1\left(1 + \left|y_1\right|\right).$$

$(\mathbf{H1.3}) : f(.,0,0,0,0) \in \mathbb{M}^2([0,T];\mathbb{R}^m)$.

Theorem 1.2.1 *Under the assumptions* (**H1**). *For any condition initial* $X_0 = x \in \mathbb{L}^2(\Omega, (\mathcal{F}_0), \mathbb{P}, \mathbb{R}^n)$, *the MF-FBSDEs (1.1) has a unique solution* $(X_t, Y_t, Z_t) \in \mathbb{S}^2([0,T], \mathbb{R}^n) \times \mathbb{S}^2([0,T], \mathbb{R}^m) \times \mathbb{M}^2([0,T]; \mathbb{R}^{m\times d})$ *satisfies:*

(i) $(X_t)_{0<t<T}$ *and* $(Y_t)_{0<t<T}$ *are continuous.*

(ii) $\mathbb{E}\left[\sup\limits_{0\leq t\leq T}|X_t|^2 + \sup\limits_{0\leq t\leq T}|Y_t|^2 + \int_0^T ||Z_t^n||^2 dt\right] < \infty.$

Proof.

1-Existence: We first prove the existence of solution, the initial condition $x \in \mathbb{L}^2(\Omega, (\mathcal{F}_0), \mathbb{P}, \mathbb{R}^n)$ is fixed.

Let $(X_t, Y_t, Z_t)_{0<t<T}$ be a possible solution of the problem (1.1). Using Picard's iteration method. Let as define the following sequence $(X_t^n, Y_t^n, Z_t^n)_{n\in\mathbb{N}}$ such that $X^0 = Y^0 = Z^0 = 0$ and $(X_t^{n+1}, Y_t^{n+1}, Z_t^{n+1})$ is the unique solution of the MF-FBSDE (1.1), defined as follows:

$$\begin{cases} X_t^{n+1} = x + \int_0^t b(s, X_s^n, \mathbb{E}[\alpha(X_s^n)])ds + \int_0^t \sigma(s, X_s^n, \mathbb{E}[\beta(X_s^n)])dW_s \\ Y_t^{n+1} = g(X_T^n, \mathbb{E}[\lambda(X_T^n)]) + \int_t^T f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)])ds - \int_t^T Z_s^{n+1} dW_s. \end{cases} \tag{1.2}$$

And such that the stochastic integrals are well defined because it is clear by recurrence that for every $n > 0$,X_t^{n+1} is continuous and adapted, so the process $\sigma(s, X_s^n, \mathbb{E}[\beta(X_s^n)$ is too.

First, we prove the existence of solution of MFSDE in (1.1), for $t \in [0,T]$, first checking by recurrence on n that there exists a constant C_n such that for all $t \in [0,T]$

$$\mathbb{E}\left[|X_t^n|^2\right] \leq C_n. \tag{1.3}$$

Suppose that $\mathbb{E}\left[|X_t^n|^2\right] \leq C_n$, and we show that

$$\mathbb{E}\left[|X_t^{n+1}|^2\right] \leq C_n.$$

We have

$$\left[|X_t^{n+1}|^2\right] = |x + \int_0^t b(s, X_s^n, \mathbb{E}\left[\alpha(X_s^n)\right])ds + \int_0^t \sigma(s, X_s^n, \mathbb{E}\left[\beta(X_s^n)\right])dW_s|^2.$$

Applying the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain

$$|X_t^{n+1}|^2 \leq 3\left(|x|^2 + \left(\int_0^t |b(s, X_s^n, \mathbb{E}\left[\alpha(X_s^n)\right])|ds\right)^2 + \left(\int_0^t ||\sigma(s, X_s^n, \mathbb{E}\left[\beta(X_s^n)\right])||dW_s\right)^2\right).$$

Passing to the expectation, we get

$$\begin{aligned} \mathbb{E}\left[|X_t^{n+1}|^2\right] \leq 3(|x|^2 + \mathbb{E}\left[\left(\int_0^t |b(s, X_s^n, \mathbb{E}\left[\alpha(X_s^n)\right]|ds\right)^2\right] \\ +\mathbb{E}\left[\left(\int_0^t \sigma(s, X_s^n, \mathbb{E}\left[\beta(X_s^n)\right])dW_s\right)^2\right]). \end{aligned} \tag{1.4}$$

By the isometry of Itô's, the theoreme of Fubini and the linear growth condition, we have

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t \sigma(s, X_s^n, \mathbb{E}\left[\beta(X_s^n)\right])dW_s\right)^2\right] &= \mathbb{E}\left[\int_0^t ||\sigma(s, X_s^n, \mathbb{E}\left[\beta(X_s^n)\right])||^2 ds\right] \\ &\leq \mathbb{E}\left[\int_0^t K^2(1+|X_s^n|^2 ds\right] \\ &= \int_0^t K^2\left(1+\mathbb{E}\left[|X_s^n|^2\right]\right)ds. \end{aligned} \tag{1.5}$$

By the inequalities of Cauchy-Schwarz , we have

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t |b(s, X_s^n, \mathbb{E}\left[\alpha(X_s^n)\right])|ds\right)^2\right] &\leq \mathbb{E}\left[\left(\int_0^t ds\right)\left(\int_0^t |b(s, X_s^n, \mathbb{E}\left[\alpha(X_s^n)\right])|^2 ds\right)\right] \\ &\leq T\mathbb{E}\left[\int_0^t K^2(1+|X_s^n|^2)ds\right]. \end{aligned} \tag{1.6}$$

Replacing (1.5) and (1.6) in (1.4), we get

$$\begin{aligned} \mathbb{E}\left[|X_t^{n+1}|^2\right] &\leq 3(|x|^2 + T\mathbb{E}\left[\int_0^t K^2(1+|X_s^n|^2)ds\right] + \int_0^t K^2(1+\mathbb{E}\left[|X_s^n|^2\right])ds) \\ &\leq C + C\int_0^t \mathbb{E}\left[|X_s^n|^2\right]ds, \text{ for } t\in[0,T], \text{ and } C>0. \end{aligned}$$

Then (1.3) is proved.

We will increase by recurrence

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t^{n+1}-X_t^n|^2\right].$$

For every $n>0$, we have

$$\begin{aligned}X_t^{n+1}-X_t^n=&\int_0^t\Big(b(s,X_s^n,\mathbb{E}\left[\alpha(X_s^n)\right])-b(s,X_s^{n-1},\mathbb{E}[\alpha(X_s^{n-1})])\Big)\,ds\\&+\int_0^t\Big(\sigma(s,X_s^n,\mathbb{E}\left[\beta(X_s^n)\right])-\sigma(s,X_s^{n-1},\mathbb{E}[\beta(X_s^{n-1})])\Big)\,dW_s.\end{aligned}$$

Applying Doob's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_s^{n+1}-X_s^n\right|^2\right]\le&\ 2\mathbb{E}\left[\left|\int_0^t\Big(b(s,X_s^n,\mathbb{E}[\alpha(X_s^n)])-b(s,X_s^{n-1},\mathbb{E}[\alpha(X_s^{n-1})])\Big)\,ds\right|^2\right]\\&+2\mathbb{E}\left[\left|\int_0^t\left(\sigma(s,X_s^n,\mathbb{E}[\beta(X_s^n)])-\sigma(s,X_s^{n-1},\mathbb{E}[\beta(X_s^{n-1})])\right)dW_s\right|^2\right].\end{aligned}$$

By the inequalities of Cauchy-Schwarz and Buckholders-Davis-Gundy, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_s^{n+1}-X_s^n\right|^2\right]\ \le&\ \ 2T\mathbb{E}\left[\int_0^t\left|b(s,X_s^n,\mathbb{E}[\alpha(X_s^n)])-b(s,X_s^{n-1},\mathbb{E}[\alpha(X_s^{n-1})])\right|^2ds\right]\\&+2\mathbb{E}\left[\int_0^t\left|\sigma(s,X_s^n,\mathbb{E}[\beta(X_s^n)])-\sigma(s,X_s^{n-1},\mathbb{E}[\beta(X_s^{n-1})])\right|^2ds\right].\end{aligned}$$

Applying the condition of Lipschitz, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_s^{n+1}-X_s^n\right|^2\right]\ \le&\ \ 2Tk\int_0^t\mathbb{E}\left[\left|X_s^n-X_s^{n-1}\right|^2+\left|\mathbb{E}[\alpha(X_s^n)-\alpha(X_s^{n-1})]\right|^2\right]ds\\&+2k\int_0^t\mathbb{E}\left[\left|X_s^n-X_s^{n-1}\right|^2+\left|\mathbb{E}[\beta(X_s^n)-\beta(X_s^{n-1})]\right|^2\right]ds\\\le&\ \ 2k(T+1)\int_0^t\mathbb{E}\left[\left|X_s^n-X_s^{n-1}\right|^2+k\left|X_s^n-X_s^{n-1}\right|^2\right]ds\\\le&\ \ 2k(T+1)(1+k)\int_0^t\mathbb{E}\left[\left|X_s^n-X_s^{n-1}\right|^2\right]ds.\end{aligned}$$

Then

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_s^{n+1}-X_s^n\right|^2\right]\leq C\int_0^t\mathbb{E}\left[\sup_{r\in[0,s]}\left|X_r^n-X_r^{n-1}\right|^2\right]ds. \tag{1.7}$$

We repeat the same method, applying Doob's inequality, to $\left|X_t^n-X_t^{n-1}\right|$, we obtain

$$\begin{aligned}\left|X_s^n-X_s^{n-1}\right|^2\leq 2T\int_0^s&\left|b(r,X_r^{n-1},\mathbb{E}[\alpha(X_r^{n-1})])-b(r,X_r^{n-2},\mathbb{E}[\alpha(X_r^{n-2})])\right|^2dr\\&+2\int_0^s\left|\sigma(r,X_r^{n-1},\mathbb{E}\left[\beta(X_r^{n-1})\right])-\sigma(r,X_r^{n-2},\mathbb{E}[\beta(X_r^{n-2})])\right|^2dr.\end{aligned}$$

Using the fact that b, σ,α and β are Lipshitz functions, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,s]}\left|X_r^n-X_r^{n-1}\right|^2\right]&\leq C\int_0^s\mathbb{E}\left[\left|X_r^{n-1}-X_r^{n-2}\right|^2\right]dr\\&\leq C\int_0^s\mathbb{E}\left[\sup_{k\in[0,r]}\left|X_k^{n-1}-X_k^{n-2}\right|^2\right]dr.\end{aligned} \tag{1.8}$$

Replacing (1.8) in (1.7), we get

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_t^{n+1}-X_t^n\right|^2\right]&\leq C\int_0^t\mathbb{E}\left[\sup_{r\in[0,s]}|X_r^n-X_r^{n-1}|^2\right]ds\\&\leq C\int_0^t\left(C\int_0^s\mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right]dr\right)ds\\&\leq C^2\mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right]\int_0^t\left(\int_0^s dr\right)ds\\&\leq C^2\mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right]\int_0^t sds\\&\leq C^2\mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right]\left[\frac{s^2}{2}\right]_0^t\\&\leq C^2\frac{t^2}{2}\mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right].\end{aligned}$$

In the same way as (1.7) and (1.8), we have

$$\mathbb{E}\left[\sup_{k\in[0,r]}\left|X_k^{n-1}-X_k^{n-2}\right|^2\right]\leq C\int_0^r\mathbb{E}\left[\sup_{l\in[0,k]}\left|X_l^{n-2}-X_l^{n-3}\right|^2\right]dk. \tag{1.9}$$

Replacing (1.8) and (1.9) in (1.7), we oblain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_t^{n+1}-X_t^n\right|^2\right] &\le C\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}|X_r^n-X_r^{n-1}|^2\right]ds \\
&\le C^2\int_0^t\left(\int_0^s \mathbb{E}\left[\sup_{k\in[0,r]}|X_k^{n-1}-X_k^{n-2}|^2\right]dr\right)ds \\
&\le C^3\int_0^t\left(\int_0^s\left(\int_0^r \mathbb{E}\left[\sup_{l\in[0,k]}|X_l^{n-2}-X_l^{n-3}|^2\right]dk\right)dr\right)ds \\
&\le C^3\mathbb{E}\left[\sup_{l\in[0,k]}|X_l^{n-2}-X_l^{n-3}|^2\right]\int_0^t\left(\int_0^s\left(\int_0^r dk\right)dr\right)ds \\
&\le C^3\mathbb{E}\left[\sup_{l\in[0,k]}|X_l^{n-2}-X_l^{n-3}|^2\right]\int_0^t\left(\int_0^s r\,dr\right)ds \\
&\le C^3\mathbb{E}\left[\sup_{l\in[0,k]}|X_l^{n-2}-X_l^{n-3}|^2\right]\int_0^t \frac{s^2}{2}ds \\
&\le C^3\frac{t^3}{3!}\mathbb{E}\left[\sup_{l\in[0,k]}|X_l^{n-2}-X_l^{n-3}|^2\right].
\end{aligned}$$

By recurrence on n, it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right|^2\right] \le \frac{(CT)^{n+1}}{n!}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^1-X_t^0\right|^2\right] \le D\frac{(CT)^n}{n!}. \tag{1.10}$$

By applying Chebyshev's inequality, we have

$$\mathbb{P}\left[\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right| > \frac{1}{2^{n+1}}\right] \le D\frac{(C)^n}{n!}/(\frac{1}{2^{n+1}} = 4D\frac{(4C)^n}{n!}.$$

Which implies that

$$\sum_{n=0}^{\infty}\mathbb{P}\left[\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right| > \frac{1}{2^{n+1}}\right] \le 4D\sum_{n=0}^{\infty}\frac{(4C)^n}{n!} = 4De^{4C} < \infty.$$

Therefore, by the Borel-Cantelli lemma

$$\mathbb{P}\left[\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right| > \frac{1}{2^{n+1}}, \forall n\in\mathbb{N}\right] = 0.$$

Which mean

$$\mathbb{P}\left[\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right|\leq\frac{1}{2^{n+1}},\forall n\in\mathbb{N}\right]=1.$$

Thus, there exists $n_0\in\mathbb{N}$ such that

$$\sup_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right|\leq\frac{1}{2^{n+1}},\ \text{for } n\geq n_0, \tag{1.11}$$

with probability equal to 1. Passing to the sum we get

$$\sup_{t\in[0,T]}\left|X_t^m-X_t^n\right|\leq\sum_{k=m\wedge n-1}^{m\vee n}\sup_{t\in[0,T]}\left|X_t^{k+1}-X_t^k\right|\leq\sum_{k=m\wedge n-1}^{m\vee n}\frac{1}{2^{k+1}}\leq\frac{1}{2^{m\wedge n}}.$$

For $m\wedge n\geq n_0(w)$ or $m\vee n=\max m,k$. Then the process $(X^n)_{n\geq 0}$ is a Cauchy sequence. So there is a continuous process $(X_t)_{t\in[0,T]}$, such as

$$\sup_{t\in[0,T]}\left|X_t^n-X_t\right|\to 0,\ \text{when } n\to\infty,\ \text{with probability } 1. \tag{1.12}$$

So, $\mathbb{P}$-a.s, X_t^n converges to a continuous process X_t. It is very easy to check that X_t is for MF-SDE part in (1.1) by going to the limit in the equation of recurrence for X_t^n. Passing now to solve the second equation of recurrence for Y_t^n. Let's prove that the sequence $(Y_t^n,\int_{\cdot}^T Z_t^n dW_t)$ is Cauchy sequence in the space of Banach.

Applying Itô's formula to $e^{at}|Y_t^{n+1} - Y_t^n|^2$, we get

$$
\begin{aligned}
d(e^{at}|Y_t^{n+1} - Y_t^n|^2) &= ae^{at}(Y_t^{n+1} - Y_t^n)^2 dt + 2e^{at}(Y_t^{n+1} - Y_t^n) d(Y_t^{n+1} - Y_t^n) \\
&+ e^{at} d\left\langle Y^{n+1} - Y^n, Y^{n+1} - Y^n \right\rangle_t \\
&= ae^{at}(Y_t^{n+1} - Y_t^n)^2 dt \\
&- 2\left\langle e^{at}(Y_t^{n+1} - Y_t^n)^2, f(t, X_t^n, \mathbb{E}[\gamma(X_t^n)], Y_t^n, \mathbb{E}[\delta(Y_t^n)]) \right. \\
&\quad \left. - f\left(t, X_t^{n-1}, \mathbb{E}[\gamma\left(X_t^{n-1}\right)], Y_t^{n-1}, \mathbb{E}[\delta\left(Y_t^{n-1}\right)]\right) \right\rangle dt \\
&+ 2\left\langle e^{at}(Y_t^{n+1} - Y_t^n)^2, Z_t^{n+1} - Z_t^n \right\rangle dW_t + e^{at}(Z_t^{n+1} - Z_t^n)^2 dt.
\end{aligned}
$$

Passing to the integral between t and T, we obtain

$$
\begin{aligned}
& e^{aT}\left|g(X_T^n, E[\lambda(X_T^n)]) - g(X_T^{n-1}, E[\lambda(X_T^{n-1})])\right|^2 - e^{at}|Y_t^{n+1} - Y_t^n|^2 \\
&= a\int_t^T e^{as}(Y_s^{n+1} - Y_s^n)^2) ds \\
&- 2\int_t^T \left\langle e^{as}(Y_s^{n+1} - Y_s^n)^2, f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)]) \right. \\
&\quad \left. - f(s, X_s^{n-1}, \mathbb{E}[\gamma\left(X_s^{n-1}\right)], Y_s^{n-1}, \mathbb{E}[\delta\left(Y_s^{n-1}\right)]) \right\rangle dt \\
&+ 2\int_t^T \left\langle e^{as}(Y_s^{n+1} - Y_s^n)^2, Z_s^{n+1} - Z_s^n \right\rangle dW_s + \int_t^T e^{as}||Z_s^{n+1} - Z_s^n||^2 ds.
\end{aligned}
$$

And then,

$$
\begin{aligned}
& e^{at}|Y_t^{n+1} - Y_t^n|^2 + \int_t^T e^{as}||Z_s^{n+1} - Z_s^n||^2 ds \\
&= e^{aT}\left|g(X_T^n, E[\lambda(X_T^n)]) - g(X_T^{n-1}, E[\lambda(X_T^{n-1})])\right|^2 \\
&+ 2\int_t^T \left\langle e^{as}(Y_s^{n+1} - Y_s^n)^2, f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)]) \right. \\
&\quad \left. - f(s, X_s^{n-1}, \mathbb{E}[\gamma\left(X_s^{n-1}\right)], Y_s^{n-1}, \mathbb{E}[\delta\left(Y_s^{n-1}\right)]) \right\rangle dt \\
&- 2\int_t^T \left\langle e^{as}(Y_s^{n+1} - Y_s^n)^2, Z_s^{n+1} - Z_s^n \right\rangle dW_s.
\end{aligned}
$$

Taking expectation and using the Lipshitz condition, we get

$$\begin{aligned}
&\mathbb{E}\left[e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_t^T e^{as}||Z_s^{n+1}-Z_s^n||^2ds\right]\\
&\le K\mathbb{E}\left[e^{at}|X_T^n-X_T^{n-1}|^2\right]-a\mathbb{E}\left[\int_t^T e^{as}|Y_s^{n+1}-Y_s^n|^2ds\right]\\
&+2K\mathbb{E}\left[\int_t^T e^{as}|Y_s^{n+1}-Y_s^n|(|X_s^n-X_s^{n-1}|+|Y_T^s-Y_s^{n-1}|)ds\right].
\end{aligned}$$

From the Yong's formula, for every $\epsilon>0, (2ab)\le\frac{1}{\epsilon^2}a^2+\epsilon^2b^2$, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_t^T e^{as}||Z_s^{n+1}-Z_s^n||^2ds\right]\\
&\le K\mathbb{E}\left[e^{at}|X_T^n-X_T^{n-1}|^2\right]-a\mathbb{E}\left[\int_t^T e^{as}|Y_s^{n+1}-Y_s^n|^2ds\right]\\
&+K^2\epsilon^2\mathbb{E}\left[\int_t^T e^{as}|Y_s^{n+1}-Y_s^n|^2ds\right]\\
&+\frac{1}{\epsilon^2}\mathbb{E}\left[\int_t^T e^{as}(|X_s^n-X_s^{n-1}|+|Y_s^n-Y_s^{n-1}|)^2ds\right].
\end{aligned}$$

Applying the inequality $(a+b)^2\le 2a^2+2b^2$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_t^T e^{as}||Z_s^{n+1}-Z_s^n||^2ds\right]\\
&\le K\mathbb{E}\left[e^{at}|X_T^n-X_T^{n-1}|^2\right]+\frac{2}{\epsilon^2}\mathbb{E}\left[\int_t^T e^{as}|X_s^n-X_s^{n-1}|^2ds\right]\\
&+(K^2\epsilon^2-a)\mathbb{E}\left[\int_t^T e^{as}|Y_s^{n+1}-Y_s^n|^2ds\right]+\frac{2}{\epsilon^2}\mathbb{E}\left[\int_t^T e^{as}|Y_s^n-Y_s^{n-1}|^2ds\right].
\end{aligned}$$

Choosing a and ϵ such that $\frac{2}{\epsilon^2}=\frac{1}{2}$ and $4K^2-a=0$, then

$$\begin{aligned}
&\mathbb{E}\left[e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_t^T e^{as}||Z_s^{n+1}-Z_s^n||^2ds\right]\\
&\le K\mathbb{E}\left[e^{at}|X_T^n-X_T^{n-1}|^2\right]+\frac{1}{2}\mathbb{E}\left[\int_t^T e^{as}(|X_s^n-X_s^{n-1}|^2+|Y_T^s-Y_s^{n-1}|^2)ds\right].
\end{aligned}$$

Then for $t = 0$, we get

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n+1} - Y_t^n|^2\right] + \mathbb{E}\left[\int_0^T e^{as}||Z_s^{n+1} - Z_s^n||^2 ds\right] \tag{1.13}$$
$$\le K\mathbb{E}\left[e^{at}|X_T^n - X_T^{n-1}|^2\right] + \frac{c}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^n - X_t^{n-1}|^2\right]$$
$$+\frac{C}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^n - Y_t^{n-1}|^2 ds\right].$$

We repeat the same method, applying the Itô's formula to $|Y_t^n - Y_t^{n-1}|^2$, we get

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^n - Y_t^{n-1}|^2\right] + \mathbb{E}\left[\int_0^T e^{as}||Z_s^n - Z_s^{n-1}||^2 ds\right]$$
$$\le K\mathbb{E}\left[e^{at}|X_T^{n-1} - X_T^{n-2}|^2\right] + \frac{c}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^{n-1} - X_t^{n-2}|^2\right]$$
$$+\frac{C}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n-1} - Y_t^{n-2}|^2 ds\right].$$

Which implies that

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^n - Y_t^{n-1}|^2\right] \le K\mathbb{E}\left[e^{at}|X_T^{n-1} - X_T^{n-2}|^2\right] \tag{1.14}$$
$$+\frac{c}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^{n-1} - X_t^{n-2}|^2\right] + \frac{C}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n-1} - Y_t^{n-2}|^2 ds\right].$$

Replacing (1.14) in (1.13), we have

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n+1} - Y_t^n|^2\right] + \mathbb{E}\left[\int_0^T e^{as}||Z_s^{n+1} - Z_s^n||^2 ds\right]$$
$$\le K\mathbb{E}\left[e^{at}|X_T^n - X_T^{n-1}|^2\right] + \frac{c}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^n - X_t^{n-1}|^2\right]$$
$$+K'\mathbb{E}\left[e^{at}|X_T^{n-1} - X_T^{n-2}|^2\right] + \frac{C}{2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^{n-1} - X_t^{n-2}|^2\right]$$
$$+\frac{C'}{2^2}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n-1} - Y_t^{n-2}|^2 ds\right].$$

Which implies that

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_0^T e^{as}||Z_s^{n+1}-Z_s^n||^2 ds\right]$$
$$\le K\left(\mathbb{E}\left[e^{at}|X_T^n-X_T^{n-1}|^2\right]+\mathbb{E}\left[e^{at}|X_T^{n-1}-X_T^{n-2}|^2\right]+...+\mathbb{E}\left[e^{at}|X_T^1-X_T^0|^2\right]\right)$$
$$+\frac{C}{2}\left(\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^n-X_t^{n-1}|^2\right]+\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^{n-1}-X_t^{n-2}|^2\right]+..\right.$$
$$\left. ...+\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|X_t^1-X_t^0|^2\right]\right)+\frac{C'}{2^n}\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^1-Y_t^0|^2 ds\right].$$

It follows immediately that

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{at}|Y_t^{n+1}-Y_t^n|^2\right]+\mathbb{E}\left[\int_0^T e^{as}||Z_s^{n+1}-Z_s^n||^2 ds\right]\le\frac{D'}{2^n}$$

Consequently, $(X^n, Y^n, Z^n)_{n\in\mathbb{N}}$ is Cauchy sequence, so convergent. Then, there is a triple stochastic process (X_t, Y_t, Z_t) such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t^n-X_t|\right]\to 0,\ \mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n-Y_t|\right]\to 0 \text{ and } \mathbb{E}\left[\int_0^T ||Z_t^n-Z_t||\right]\to 0,$$

when $n\to\infty$, with a probability equal to 1.

$$X=\lim_{n\to\infty}X_t^n,\ Y=\lim_{n\to\infty}Y_t^n \text{ and } Z=\lim_{n\to\infty}Z_t^n.$$

It is easy to check that (X, Y, Z) is a solution of MF-FBSDE (1.1), it just to passing to the limit in (1.2).

2-Uniqueness: Let us prove the uniqueness of solutions of the system (1.1).

Suppose that (X, Y, Z) and (X', Y', Z') two solutions of the system (1.1) such that $X_0 = X_0^{'} = x$, and for all $0 \le t \le T$,

$$X_t - X'_t = \int_0^t (b(s, X_s, \mathbb{E}[\alpha(X_s)]) - b(s, X'_s, \mathbb{E}[\alpha(X'_s)]))ds$$
$$+ \int_0^t (\sigma(s, X_s, \mathbb{E}[\beta(X_s)]) - \sigma(s, X'_s, \mathbb{E}[\beta(X'_s)]))dW_s,$$

and

$$Y_t - Y'_t = g(X_T, \mathbb{E}[\lambda(X_T)]) - g(X'_T, \mathbb{E}[\lambda(X'_T)])$$
$$+ \int_t^T (f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)]) - f(s, X'_s, \mathbb{E}[\gamma(X'_s)], Y'_s, \mathbb{E}[\delta(Y'_s)]))ds$$
$$- \int_t^T (Z_s - Z'_s)dW_s.$$

Applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we get

$$\mathbb{E}[|X_t - X'_t|^2] \le 2\mathbb{E}[|\int_0^t (b(s, X_s, \mathbb{E}[\alpha(X_s)]) - b(s, X'_s, \mathbb{E}[\alpha(X'_s)]))ds|^2]$$
$$+2\mathbb{E}[|\int_0^t (\sigma(s, X_s, \mathbb{E}[\beta(X_s)]) - \sigma(s, X'_s, \mathbb{E}[\beta(X'_s)]))dW_s|^2].$$

According to Cauchy, Schwarz's inequality and Lipschitz condition, we have

$$\mathbb{E}[|\int_0^t (b(s, X_s, \mathbb{E}[\alpha(X_s)]) - b(s, X'_s, \mathbb{E}[\alpha(X'_s)]))ds|^2]$$
$$\le T\mathbb{E}[\int_0^t |b(s, X_s, \mathbb{E}[\alpha(X_s)]) - b(s, X'_s, \mathbb{E}[\alpha(X'_s)])|^2 ds]$$
$$\le TK^2 \int_0^t \mathbb{E}[|X_s - X'_s|^2]ds.$$

By the isometry of Itô and Lipschitz condition we have,

$$\mathbb{E}[|\int_0^t (\sigma(s, X_s, \mathbb{E}[\beta(X_s)]) - \sigma(s, X'_s, \mathbb{E}[\beta(X'_s)]))dW_s|^2]$$
$$\le \mathbb{E}[\int_0^t ||\sigma(s, X_s, \mathbb{E}[\beta(X_s)]) - \sigma(s, X'_s, \mathbb{E}[\beta(X'_s)])||^2 ds]$$
$$\le K^2 \int_0^t \mathbb{E}[|X_s - X'_s|^2]ds.$$

Then

$$\mathbb{E}[|X_t - X_t'|^2] \leq 2TK^2 \int_0^t \mathbb{E}[|X_s - X_s'|^2]ds + 2K^2 \int_0^t \mathbb{E}[|X_s - X_s'|^2]ds.$$
$$\leq (2TK^2 + 2K^2) \int_0^t \mathbb{E}[|X_s - X_s'|^2]ds.$$

Using the Granwall lemma, we get

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t - X_t'|^2] = 0. \tag{1.15}$$

On the other hand, applying Itô's formula to $|Y_t - Y_t'|^2$, we get

$$d|Y_t - Y_t'|^2 = 2|Y_t - Y_t'|d(Y_t - Y_t') + d\langle Y - Y', Y - Y'\rangle_t.$$

By passage to the integral from t to T and taking the expectation, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq \mathbb{E}\left[|g(X_T, \mathbb{E}[\lambda(X_T)]) - g(X_T', \mathbb{E}[\lambda(X_T')])|^2\right]$$
$$+2\mathbb{E}\left[\int_0^T \langle Y_t - Y_t', f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)]) - f(s, X_s', \mathbb{E}[\gamma(X_s')], Y_s', \mathbb{E}[\delta(Y_s')])\rangle ds.\right.$$

Applying Lipschizt condition, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq K^2\mathbb{E}\left[|X_T - X_T'|^2\right]$$
$$+2K\mathbb{E}\left[\int_0^T |Y_t - Y_t'|(|X_t - X_t'| + |Y_t - Y_t'|ds\right].$$

By $2ab \leq a^2 + b^2$, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq K^2\mathbb{E}\left[|X_T - X_T'|^2\right] + K^2\mathbb{E}\left[\int_0^T |Y_t - Y_t'|^2\right]$$
$$+2\mathbb{E}\left[\int_0^T |X_t - X_t'|^2\right] + 2\mathbb{E}\left[\int_0^T |Y_t - Y_t'|ds\right].$$

Then

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq K^2\mathbb{E}\left[|X_T - X_T'|^2\right]$$
$$+2T\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - X_t'|^2\right] + (K^2+2)\mathbb{E}\left[\int_0^T |Y_t - Y_t'|^2 ds\right].$$

By (1.15), we have $\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - X_t'|^2\right] = 0$, so

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq C\mathbb{E}\left[\int_0^T |Y_t - Y_t'|^2 ds\right]. \tag{1.16}$$

We derive from this inequality, two inequalities

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] \leq C\mathbb{E}\left[\int_0^T |Y_t - Y_t'|^2 ds\right]. \tag{1.17}$$

$$\mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] \leq C\mathbb{E}\left[\int_0^T |Y_t - Y_t'|^2 ds\right]. \tag{1.18}$$

Applying Granwall's inequality in (1.17) gives

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_t'|^2\right] = 0.$$
$$\mathbb{E}\left[\int_0^T ||Z_t - Z_t'||^2 dt\right] = 0.$$

The uniqueness is proved. ■

CHAPTER 2

Existence of an optimal strict control and optimality conditions for linear FBSDEs of mean-field type

CHAPTER 2

Existence of an optimal strict control and optimality conditions for linear FBSDEs of mean-field type

In this chapter, we consider a stochastic control problem governed by a linear forward backward stochastic differential equation of mean field type with non linear functional cost. The system here depend on state and also on the distribution of the state process. The cost functional is also of mean field type. Under the convexity of the domain of control and the cost functions, we prove the existence of strong optimal strict control (which adapted to the initial filtration) by using the strong convergence and Mazur's theorem. We establish also necessary as will as sufficient optimality conditions for this kind of linear control problem by using convex optimization principle.

2.1 Formulation of the problem and assumptions

We consider a stochastic control problem for the following linear FBSDEs of mean-field type

$$\begin{cases} dX_t^u = \big(A.X_t^u + B.\mathbb{E}\left[X_t^u\right] + C.u_t\big)dt + \big(D.X_t^u + E.\mathbb{E}\left[X_t^u\right] + F.u_t\big)dW_t \\ dY_t^u = -\big(\overline{A}.X_t^u + \overline{B}.\mathbb{E}\left[X_t^u\right] + \overline{C}.Y_t^u + \overline{D}.\mathbb{E}\left[Y_t^u\right] + \overline{E}.Z_t^u + \overline{F}.\mathbb{E}\left[Z_t^u\right] + \overline{G}.u_t\big)dt \\ \qquad\qquad +Z_t^u dW_t, \\ X_0^u = x, Y_T^u = \xi, \end{cases} \tag{2.1}$$

and a cost functional:

$$\begin{aligned} J(u.) := \mathbb{E}\big[l\left(X_T^u, \mathbb{E}\left[X_T^u\right]\right) + k\left(Y_0^u, \mathbb{E}\left[Y_0^u\right]\right) \\ + \textstyle\int_0^T h\big(t, X_t^u, \mathbb{E}\left[X_t^u\right], Y_t^u, \mathbb{E}\left[Y_t^u\right], Z_t^u, \mathbb{E}\left[Z_t^u\right], u_t\big)dt\big], \end{aligned} \tag{2.2}$$

where $A, B, C, D, E, F, \overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}, \overline{F}$ and $\overline{G}$ are matrices of suitable sizes.

The solution $(X_\cdot^u, Y_\cdot^u, Z_\cdot^u)$ takes values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m+d}$ and the control variable $u.$ is in

$$\mathcal{U} := \{u. \in \mathbb{M}^2([0,T]; \mathbb{R}^k) \mid u_t \in U, a.e.t \in [0,T], \mathbb{P} - a.s.\},$$

with $U \subseteq \mathbb{R}^k$. Note that we have an additional constraint that a control must be square-integrable just to ensure the existence of solutions of (2.1) under $u.$. The optimal control problem can be stated as follows:

Problem (L): Minimize (2.2) subject to (2.1) over $\mathcal{U}$.

Now we introduce the following assumptions:

$(\mathbf{H2.1})$: The set $U \subseteq \mathbb{R}^k$ is convex and compact, and the functions l, k and h are convex.

2.2 Existence of optimal control

Theorem 2.2.1 *Under* (**H2.1**), *if Problem (L) is finite, then it admits an optimal solution.*

Proof. Let $(X_\cdot^n, Y_\cdot^n, Z_\cdot^n, u_\cdot^n)$ be a minimizing sequence satisfies

$$\lim_{n\to\infty} J\left(u_\cdot^n\right) = \inf_{v_\cdot \in \mathcal{U}} J\left(v_\cdot\right).$$

Since U is a compact set then, the sequence $\left(u_\cdot^n\right)_{n\geq 0}$ is relatively compact.

Thus, there exists a subsequence (which is still labeled by $\left(u_\cdot^n\right)_{n\geq 0}$) such that

$$u_\cdot^n \longrightarrow \widetilde{u}_\cdot, \text{ weakly in } \mathbb{M}^2\left([0,T];\mathbb{R}^k\right).$$

Applying Mazur's theorem, there exists a sequence of convex combinations defined as follows

$$\widehat{u}_\cdot^n = \sum_{k\geq 0} \beta_{kn} u_\cdot^{k+n} \text{ (with } \beta_{kn} \geq 0, \text{ and } \sum_{k\geq 0} \beta_{kn} = 1),$$

satisfies

$$\widehat{u}_\cdot^n \to \widetilde{u}_\cdot \text{ strongly in } \mathbb{M}^2\left([0,T];\mathbb{R}^k\right). \tag{2.3}$$

Since the set $U \subseteq \mathbb{R}^k$ is convex and compact, it follows that $\widetilde{u}_\cdot \in \mathcal{U}$.

Let $(\widehat{X}_\cdot^n, \widehat{Y}_\cdot^n, \widehat{Z}_\cdot^n)$ and $(\widetilde{X}_\cdot, \widetilde{Y}_\cdot, \widetilde{Z}_\cdot)$ be the solutions of the linear MF-FBSDE (2.1), corresponding to $\widehat{u}_\cdot^n$ and $\widetilde{u}_\cdot$ respectively. Then let us prove

$$(\widehat{X}_t^n, \widehat{Y}_t^n) \textit{ converges strongly to } (\widetilde{X}_t, \widetilde{Y}_t) \textit{ in } \mathbb{S}^2\left([0,T];\mathbb{R}^{n+m}\right), \tag{2.4}$$

and

$$\int_0^T \widehat{Z}_s^n dW_s \textit{ converges strongly to } \int_0^T \widetilde{Z}_s dW_s \text{ in } \mathbb{M}^2\left([0,T];\mathbb{R}^{m+d}\right). \tag{2.5}$$

We have

$$|\widehat{X}_t^n - \widetilde{X}_t| \leq |\int_0^t \big(A.(\widehat{X}_s^n - \widetilde{X}_s) + B.(\mathbb{E}[\widehat{X}_s^n] - \mathbb{E}[\widetilde{X}_s]) + C.(\widehat{u}_s^n - \widetilde{u}_s)\big)ds|$$
$$+|\int_0^t \big(D.(\widehat{X}_s^n - \widetilde{X}_s) + E.(\mathbb{E}[\widehat{X}_s^n] - \mathbb{E}[\widetilde{X}_s]) + F.(\widehat{u}_s^n - \widetilde{u}_s)\big)dW_s|,$$

which implies that

$$\sup_{0\leq s\leq t} |\widehat{X}_s^n - \widetilde{X}_s|^2 \leq$$
$$\int_0^t \big(\|A\|^2.(\sup_{0\leq r\leq s} |\widehat{X}_r^n - \widetilde{X}_r|^2) + \|B\|^2.(\sup_{0\leq r\leq s} |\mathbb{E}[\widehat{X}_r^n] - \mathbb{E}[\widetilde{X}_r]|^2) + \|C\|^2.|\widehat{u}_s^n - \widetilde{u}_s|^2\big)ds$$
$$+ \sup_{0\leq s\leq t} (|\int_0^t \big(D.(\widehat{X}_s^n - \widetilde{X}_s) + E.(\mathbb{E}[\widehat{X}_s^n] - \mathbb{E}[\widetilde{X}_s]) + F.(\widehat{u}_s^n - \widetilde{u}_s)\big)dW_s\big|)^2,$$

using the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\big[\sup_{0\leq s\leq T} |\widehat{X}_s^n - \widetilde{X}_s|)^2\big] \leq K_1 \int_0^t \mathbb{E}\big[\sup_{0\leq r\leq s} |\widehat{X}_r^n - \widetilde{X}_r|^2\big]ds + K_1\mathbb{E}\big[\int_0^t |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big].$$

If we set $f(t) = \mathbb{E}\big[\sup_{0\leq s\leq T} |\widehat{X}_s^n - \widetilde{X}_s|^2\big]$, then

$$f(t) \leq K_1 \int_0^t f(s)ds + K_1\mathbb{E}\big[\int_0^t |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big].$$

By applying Gronwall's lemma, there exists a positive constant K such that :

$$\mathbb{E}\big[\sup_{0\leq s\leq T} |\widehat{X}_s^n - \widetilde{X}_s|^2\big] \leq K\mathbb{E}\big[\int_0^t |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big].$$

Since $(\widehat{u}_\cdot^n)$ converges strongly to $\widetilde{u}_\cdot$ in $\mathbb{M}^2([0,T];\mathbb{R}^k)$, we get

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\leq s\leq T} |\widehat{X}_s^n - \widetilde{X}_s|^2\big] = 0. \tag{2.6}$$

On the other hand, applying Itô's formula to $\left|\widehat{Y}_t^n - \widetilde{Y}_t\right|^2$, we obtain

$$|\widehat{Y}_t^n - \widetilde{Y}_t|^2 + \int_t^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds = 2\int_t^T \langle \widehat{Y}_s^n - \widetilde{Y}_s, \overline{A}.(\widehat{X}_s^n - \widetilde{X}_s) + \overline{B}.(\mathbb{E}[\widehat{X}_s^n] - \mathbb{E}[\widetilde{X}_s])$$
$$+\overline{C}.(\widehat{Y}_s^n - \widetilde{Y}_s) + \overline{D}.(\mathbb{E}[\widehat{Y}_s^n] - \mathbb{E}[\widetilde{Y}_s]) + \overline{E}.(\widehat{Z}_s^n - \widetilde{Z}_s) + \overline{F}.(\mathbb{E}[\widehat{Z}_s^n] - \mathbb{E}[\widetilde{Z}_s]) + \overline{G}(\widehat{u}_s^n - \widetilde{u}_s)\rangle ds$$
$$-2\int_t^T \langle \widehat{Y}_s^n - \widetilde{Y}_s, \widehat{Z}_s^n - \widetilde{Z}_s\rangle dW_s.$$

Thus

$$\mathbb{E}\big[\sup_{t\in[0,T]} |\widehat{Y}_t^n - \widetilde{Y}_t|^2\big] + \mathbb{E}\big[\int_0^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds\big]$$
$$\leq 2\mathbb{E}\big[\int_0^T \langle \widehat{Y}_s^n - \widetilde{Y}_s, \overline{A}.(\widehat{X}_s^n - \widetilde{X}_s) + \overline{B}.(\mathbb{E}[\widehat{X}_s^n] - \mathbb{E}[\widetilde{X}_s]) + \overline{C}.(\widehat{Y}_s^n - \widetilde{Y}_s)$$
$$+\overline{D}.(\mathbb{E}[\widehat{Y}_s^n] - \mathbb{E}[\widetilde{Y}_s]) + \overline{E}.(\widehat{Z}_s^n - \widetilde{Z}_s) + \overline{F}.(\mathbb{E}[\widehat{Z}_s^n] - \mathbb{E}[\widetilde{Z}_s]) + \overline{G}.(\widehat{u}_s^n - \widetilde{u}_s)\rangle ds\big].$$

Applying Young's formula, to show that

$$\mathbb{E}\big[\sup_{t\in[0,T]} |\widehat{Y}_t^n - \widetilde{Y}_t|^2\big] + \mathbb{E}\big[\int_0^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds\big] \leq \frac{1}{\alpha}\mathbb{E}\big[\int_0^T |\widehat{Y}_s^n - \widetilde{Y}_s|^2 ds\big]$$
$$+7\alpha K\mathbb{E}\big[\int_0^T \big(|\widehat{X}_s^n - \widetilde{X}_s|^2 + |\widehat{Y}_s^n - \widetilde{Y}_s|^2 + \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 + |\widehat{u}_s^n - \widetilde{u}_s|^2\big) ds\big].$$

Choosing $\alpha = \frac{1}{14K}$, we obtain

$$\mathbb{E}\big[\sup_{t\in[0,T]} |\widehat{Y}_t^n - \widetilde{Y}_t|^2\big] + \frac{1}{2}\mathbb{E}\big[\int_0^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds\big] \leq (14K + \frac{1}{2})\mathbb{E}\big[\int_0^T |\widehat{Y}_t^n - \widetilde{Y}_t|^2 ds\big] \quad (2.7)$$
$$+\frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{X}_s^n - \widetilde{X}_s|^2 ds\big] + \frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big].$$

From the inequality (2.7), we derive two inequalities

$$\mathbb{E}\big[\sup_{t\in[0,T]} |\widehat{Y}_t^n - \widetilde{Y}_t|^2\big] \leq (14K + \frac{1}{2})\mathbb{E}\big[\int_0^T |\widehat{Y}_t^n - \widetilde{Y}_t|^2 ds\big] \quad (2.8)$$
$$+\frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{X}_s^n - \widetilde{X}_s|^2 ds\big] + \frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big],$$

and

$$\frac{1}{2}\mathbb{E}\big[\int_0^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds\big] \leq (14K + \frac{1}{2})\mathbb{E}\big[\int_0^T |\widehat{Y}_t^n - \widetilde{Y}_t|^2 ds\big]$$
$$+\frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{X}_s^n - \widetilde{X}_s|^2 ds\big] + \frac{1}{2}\mathbb{E}\big[\int_0^T |\widehat{u}_s^n - \widetilde{u}_s|^2 ds\big]. \tag{2.9}$$

Applying Gronwall's lemma to (2.8) and by taking limit as $n \to \infty$, and using (2.3) and (2.6), we obtain (2.4). Finally, we deduced directly from (2.3), (2.4) and (2.9) that

$$\mathbb{E}\big[\int_0^T \|\widehat{Z}_s^n - \widetilde{Z}_s\|^2 ds\big] \longrightarrow 0, \text{ as } n \to \infty,$$

which implies (2.5) by applying the isometry of Itô.

Let us prove that $\widetilde{u}_\cdot$ is an optimal control.

Let $(X_\cdot^n, Y_\cdot^n, Z_\cdot^n, u_\cdot^n)$ be a minimizing sequence such that

$$\begin{aligned}\lim_{n\to\infty} J\left(u_\cdot^n\right) &= \lim_{n\to\infty} \mathbb{E}\big[l\left(X_T^n, \mathbb{E}\left[X_T^n\right]\right) + k\left(Y_0^n, \mathbb{E}\left[Y_0^n\right]\right) \\ &\qquad + \textstyle\int_0^T h\big(t, X_t^n, \mathbb{E}\left[X_t^n\right], Y_t^n, \mathbb{E}\left[Y_t^n\right], Z_t^n, \mathbb{E}\left[Z_t^n\right], u_t^n\big) dt\big] \\ &= \inf_{v_\cdot \in \mathcal{U}} J\left(v_\cdot\right).\end{aligned}$$

By the continuity of l, k and h, we have

$$\begin{aligned} J\left(\widetilde{u}_\cdot\right) &= \mathbb{E}\big[l\left(\widetilde{X}_T, \mathbb{E}[\widetilde{X}_T]\right) + k\left(\widetilde{Y}_0, \mathbb{E}[\widetilde{Y}_0]\right) + \textstyle\int_0^T h\big(t, \widetilde{X}_t, \mathbb{E}[\widetilde{X}_t], \widetilde{Y}_t, \mathbb{E}[\widetilde{Y}_t], \widetilde{Z}_t, \mathbb{E}[\widetilde{Z}_t], \widetilde{u}_t\big) dt\big] \\ &= \lim_{n\to\infty} \mathbb{E}\big[l\left(\widehat{X}_T^n, \mathbb{E}[\widehat{X}_T^n]\right) + k\left(\widehat{Y}_0^n, \mathbb{E}[\widehat{Y}_0^n]\right) + \textstyle\int_0^T h\big(t, \widehat{X}_t^n, \mathbb{E}[\widehat{X}_t^n], \widehat{Y}_t^n, \mathbb{E}[\widehat{Y}_t^n], \widehat{Z}_t^n, \mathbb{E}[\widehat{Z}_t^n], \widehat{u}_t^n\big) dt\big].\end{aligned}$$

Since l, k and h are convex, it follows that

$$\begin{aligned}
J(\widetilde{u}.) &\leq \lim_{n\to\infty}\sum_{k\geq 0}\beta_{kn}\mathbb{E}\big[l\left(X_T^{k+n},\mathbb{E}\left[X_T^{k+n}\right]\right)+k\left(Y_0^{k+n},\mathbb{E}\left[Y_0^{k+n}\right]\right)\\
&+\int_0^T h\big(t,X_t^{k+n},\mathbb{E}\left[X_t^{k+n}\right],Y_t^{k+n},\mathbb{E}\left[Y_t^{k+n}\right],Z_t^{k+n},\mathbb{E}\left[Z_t^{k+n}\right],u_t^{k+n}\big)dt\big]\\
&= \lim_{n\to\infty}\sum_{k\geq 0}\beta_{kn}J\left(u_.^{k+n}\right),\\
&\leq \lim_{n\to\infty}\sum_{k\geq 1}\beta_{kn}\underset{1\leq k\leq i(n)}{Max}J\left(u_.^{k+n}\right),\\
&\leq \lim_{n\to\infty}\underset{1\leq k\leq i(n)}{Max}J\left(u_.^{k+n}\right)\sum_{k\geq 1}\beta_{kn},\\
&= \lim_{n\to\infty}J\left(u_.^{n+i(n)}\right),\\
&= \inf_{v.\in\mathcal{U}}J(v.).
\end{aligned}$$

■

2.3 Necessary and sufficient conditions of optimality for a linear MF-FBSDE

Recall that the set U is convex, then the classical way to derive necessary optimality conditions satisfied by the strict optimal control is to use the convex perturbation method.

Let $\widetilde{u}.$ be an optimal strict control and denote by $(\widetilde{X}_t,\widetilde{Y}_t,\widetilde{Z}_t)$ the solution of (2.1) associated with $\widetilde{u}.$. Then, we define the following perturbation (convex perturbation)

$$u_t^{\varepsilon}=\widetilde{u}_t+\varepsilon\left(v_t-\widetilde{u}_t\right),$$

where, $\varepsilon>0$ is sufficiently small and $v.$ is an arbitrary element of $\mathcal{U}$ such that $\mathbb{E}\left[|v.|^2\right]<\infty$.

It is clear that the control $u^{\varepsilon}_{\cdot}$ is admissible and let $(X^{\varepsilon}_t, Y^{\varepsilon}_t, Z^{\varepsilon}_t)$ be the solution of (2.1) controlled by u^{ε}.

By the optimality of $\widetilde{u}.$, the necessary conditions for optimality will be derived from the fact that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J\left(u^{\varepsilon}_{\cdot}\right)-J\left(\widetilde{u}.\right)\right) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J\left(\widetilde{u}.+\varepsilon\left(v.-\widetilde{u}.\right)\right)-J\left(\widetilde{u}.\right)\right) \\ &= \left\langle J'\left(\widetilde{u}.\right), v.-\widetilde{u}.\right\rangle . \end{aligned}$$

The following assumptions will be in force throughout this section.

$(\mathbf{H2.2})$: l, k and h are continuously differentiable with respect to (x, x'), (y, y') and (x, x', y, y', z, z') respectively;

$(\mathbf{H2.3})$: the derivatives of l, k and h with respect to (x, x'), (y, y') and (x, x', y, y', z, z') respectively, are bounded.

To establish a necessary and sufficient conditions for optimality, we use the convex optimization principle (see Ekeland-Temam ([12], prop 2.1, page 35).

Due to convexity of the set U and the fact that J is convex in $\widetilde{u}.$, continuous and Gâteaux-differentiable with continuous derivative J', we can apply the convex optimization principle, to get

$$(\widetilde{u}. \text{ minimize } J) \Leftrightarrow \left\langle J'\left(\widetilde{u}.\right), v.-\widetilde{u}.\right\rangle \geq 0; \forall v. \in \mathcal{U}. \tag{2.10}$$

Now we calculate the Gâteaux derivative of J at a point $\widetilde{u}.$ and in the direction $(v. - \widetilde{u}.)$, we obtain

$$
\begin{aligned}
\langle J'(\widetilde{u}.), v. - \widetilde{u}.\rangle &= \mathbb{E}\big[\langle l_x(\widetilde{X}_T, \mathbb{E}[\widetilde{X}_T]), X_T^v - \widetilde{X}_T\rangle\big] + \mathbb{E}\big[\langle l_{x'}(\widetilde{X}_T, \mathbb{E}[\widetilde{X}_T]), \mathbb{E}[X_T^v - \widetilde{X}_T]\rangle\big] \\
&+\mathbb{E}\big[\langle k_y(\widetilde{Y}_0, \mathbb{E}[\widetilde{Y}_0]), Y_0^v - \widetilde{Y}_0\rangle\big] + \mathbb{E}\big[\langle k_{y'}(\widetilde{Y}_0, \mathbb{E}[\widetilde{Y}_0]), \mathbb{E}[Y_0^v - \widetilde{Y}_0]\rangle\big] \\
&+\mathbb{E}\left[\int_0^T \big(\langle h_x(t, \widetilde{u}_t), X_t^v - \widetilde{X}_t\rangle + \mathbb{E}\big[\langle h_{x'}(t, \widetilde{u}_t), \mathbb{E}[X_t^v - \widetilde{X}_t]\rangle\big]\big) dt\right] \\
&+\mathbb{E}\left[\int_0^T \big(\langle h_y(t, \widetilde{u}_t), Y_t^v - \widetilde{Y}_t\rangle + \mathbb{E}\big[\langle h_{y'}(t, \widetilde{u}_t), \mathbb{E}[Y_t^v - \widetilde{Y}_t]\rangle\big]\big) dt\right] \\
&+\mathbb{E}\left[\int_0^T \big(\langle h_z(t, \widetilde{u}_t), Z_t^v - \widetilde{Z}_t\rangle + \mathbb{E}\big[\langle h_{z'}(t, \widetilde{u}_t), \mathbb{E}[Z_t^v - \widetilde{Z}_t]\rangle\big]\big) dt\right] \\
&+\mathbb{E}\left[\int_0^T \langle h_v(t, \widetilde{u}_t), v_t - \widetilde{u}_t\rangle dt\right].
\end{aligned}
$$

With the notation $h_\delta(t, \widetilde{u}_t) := h_\delta(t, \widetilde{X}_t, \mathbb{E}[\widetilde{X}_t], \widetilde{Y}_t, \mathbb{E}[\widetilde{Y}_t], \widetilde{Z}_t, \mathbb{E}[\widetilde{Z}_t], \widetilde{u}_t)$, with $\delta = x, x', y, y', z, z', v$.

Which implies that

$$
\begin{aligned}
\langle J'(\widetilde{u}.), v. - \widetilde{u}.\rangle &= \mathbb{E}\big[\langle l_x(\widetilde{X}_T, \mathbb{E}[\widetilde{X}_T]) + \mathbb{E}[l_{x'}(\widetilde{X}_T, \mathbb{E}[\widetilde{X}_T])], X_T^v - \widetilde{X}_T\rangle\big] \\
&+\mathbb{E}\big[\langle k_y(\widetilde{Y}_0, \mathbb{E}[\widetilde{Y}_0]) + \mathbb{E}[k_{y'}(\widetilde{Y}_0, \mathbb{E}[\widetilde{Y}_0])], Y_0^v - \widetilde{Y}_0\rangle\big] \\
&+\mathbb{E}\left[\int_0^T \langle h_x(t, \widetilde{u}_t) + \mathbb{E}[h_{x'}(t, \widetilde{u}_t)], X_t^v - \widetilde{X}_t\rangle dt\right] \\
&+\mathbb{E}\left[\int_0^T \langle h_y(t, \widetilde{u}_t) + \mathbb{E}[h_{y'}(t, \widetilde{u}_t)], Y_t^v - \widetilde{Y}_t\rangle dt\right] \\
&+\mathbb{E}\left[\int_0^T \langle h_z(t, \widetilde{u}_t) + \mathbb{E}[h_{z'}(t, \widetilde{u}_t)], Z_t^v - \widetilde{Z}_t\rangle dt\right] \\
&+\mathbb{E}\left[\int_0^T \langle h_v(t, \widetilde{u}_t), v_t - \widetilde{u}_t\rangle dt\right]. \qquad (2.11)
\end{aligned}
$$

The main result in this section is the following

Theorem 2.3.1 *(Necessary and sufficient conditions for optimality). Let $\widetilde{u}.$ be an admissible control with corresponding trajectories $(\widetilde{X}., \widetilde{Y}., \widetilde{Z}.)$. Then $\widetilde{u}.$ is optimal if and only if, there exists a unique triple of*

$\mathcal{F}_t$-adapted process $(P^u_\cdot, \psi^u_\cdot, Q^u_\cdot)$ solution of the following stochastic equations (called adjoint equations),

$$\begin{cases} -dP^u_t = \big(H_x\,(t,u_t) + \mathbb{E}[H_{x'}\,(t,u_t)]\big)dt - \psi^u_t dW_t, \\ \\ dQ^u_t = \big(H_Y\,(t,u_t) + \mathbb{E}[H_{y'}\,(t,u_t)]\big)dt + \big(H_z\,(t,u_t) + \mathbb{E}[H_{z'}\,(t,u_t)]\big)dW_t, \\ \\ P^u_T = l_x\,(X^u_T, \mathbb{E}[X^u_T]) + \mathbb{E}[l_{x'}\,(X^u_T, \mathbb{E}[X^u_T])], \\ \\ Q^u_0 = k_y\,(Y^u_0, \mathbb{E}[Y^u_0]) + \mathbb{E}[k_{y'}\,(Y^u_0, \mathbb{E}[Y^u_0])], \end{cases} \tag{2.12}$$

such that

$$\langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle \geq 0,\ \forall v_t \in \mathcal{U},\ a.e,\ as, \tag{2.13}$$

where $H_\delta(t,\widetilde{u}_t) := H_\delta(t,\widetilde{X}_t,\mathbb{E}[\widetilde{X}_t],\widetilde{Y}_t,\mathbb{E}[\widetilde{Y}_t],\widetilde{Z}_t,\mathbb{E}[\widetilde{Z}_t],\widetilde{u}_t,\widetilde{P}_t,\widetilde{\psi}_t,\widetilde{Q}_t)$, and the Hamiltonian function is defined by

$$\begin{aligned} H\,(t,x,\mathbb{E}[x],y,\mathbb{E}[y],z,\mathbb{E}[z],v,P,\psi,Q) &:= \langle P, Ax + B\mathbb{E}[x] + Cv\rangle \\ &+ \langle \psi, Dx + E\mathbb{E}[x] + Fv\rangle + h\,(t,x,\mathbb{E}[x],y,\mathbb{E}[y],z,\mathbb{E}[z],v) \\ &+ \left\langle Q, \overline{A}x + \overline{B}\mathbb{E}[x] + \overline{C}y + \overline{D}\mathbb{E}[y] + \overline{E}z + \overline{F}\mathbb{E}[z] + \overline{G}v\right\rangle. \end{aligned}$$

Proof. The adjoint equations (2.12) can be rewritten as follows

$$\begin{cases} -dP^u_t = \big(P^u_t A + \psi^u_t D + Q^u_t\overline{A} + h_x(t,u_t) + \mathbb{E}[P^u_t B + \psi^u_t E + Q^u_t\overline{B} + h_{x'}(t,u_t)]\big)dt \\ \qquad\qquad -\psi^u_t dW_t, \\ dQ^u_t = \big(Q^u_t\overline{C} + h_y(t,u_t) + \mathbb{E}[\big(Q^u_t\overline{D} + h_{y'}(t,u_t)]\big)dt + \big(Q^u_t\overline{E} + h_z(t,u_t) \\ \qquad\qquad +\mathbb{E}[\big(Q^u_t\overline{F} + h_{z'}(t,u_t)]\big)dW_t, \\ P^u_T = l_x\,(X^u_T, \mathbb{E}[X^u_T]) + \mathbb{E}[l_{x'}\,(X^u_T, \mathbb{E}[X^u_T])], \\ Q^u_0 = k_y\,(Y^u_0, \mathbb{E}[Y^u_0]) + \mathbb{E}[k_{y'}\,(Y^u_0, \mathbb{E}[Y^u_0])], \end{cases}$$

Therefore after recalling also (2.12), the equality (2.11) becomes

$$\begin{aligned}
\langle J'(\widetilde{u}.), v. - \widetilde{u}.\rangle &= \mathbb{E}\big[\langle \widetilde{P}_T, X_T^v - \widetilde{X}_T\rangle\big] + \mathbb{E}\big[\langle \widetilde{Q}_0, Y_0^v - \widetilde{Y}_0\rangle\big] \\
&+\mathbb{E}\big[\int_0^T \langle h_x(t,\widetilde{u}_t) + \mathbb{E}[h_{x'}(t,\widetilde{u}_t)], X_t^v - \widetilde{X}_t\rangle dt\big] + \mathbb{E}\big[\int_0^T \langle h_y(t,\widetilde{u}_t) + \mathbb{E}[h_{y'}(t,\widetilde{u}_t)], Y_t^v - \widetilde{Y}_t\rangle dt\big] \\
&+\mathbb{E}\big[\int_0^T \langle h_z(t,\widetilde{u}_t) + \mathbb{E}[h_{z'}(t,\widetilde{u}_t)], Z_t^v - \widetilde{Z}_t\rangle dt\big] + \mathbb{E}\big[\int_0^T \langle h_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle dt\big]. \qquad (2.14)
\end{aligned}$$

Applying integration by part to $\langle \widetilde{P}_t, X_t^v - \widetilde{X}_t\rangle$ and $\langle \widetilde{Q}_t, Y_t^v - \widetilde{Y}_t\rangle$, passing to integral on $[0,T]$ and take the expectations to deduce

$$\begin{aligned}
\mathbb{E}\big[\langle \widetilde{P}_T, X_T^v - \widetilde{X}_T\rangle\big] &= -\mathbb{E}\big[\int_0^T \langle \widetilde{Q}_t\overline{A} + h_x(t,\widetilde{u}_t) + \mathbb{E}[\widetilde{Q}_t\overline{B} + h_{x'}(t,\widetilde{u}_t)], X_t^v - \widetilde{X}_t\rangle dt\big] \\
&+\mathbb{E}\big[\int_0^T \langle \widetilde{P}_t, C(v_t - \widetilde{u}_t)\rangle dt\big] + \mathbb{E}\big[\int_0^T \langle \widetilde{\psi}_t, F(v_t - \widetilde{u}_t)\rangle dt\big], \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\big[\langle \widetilde{Q}_0, Y_0^v - \widetilde{Y}_0\rangle\big] &= -\mathbb{E}\big[\int_0^T \langle h_y(t,\widetilde{u}_t) + \mathbb{E}[h_{y'}(t,\widetilde{u}_t)], Y_t^v - \widetilde{Y}_t\rangle dt\big] \qquad (2.16) \\
&+\mathbb{E}\big[\int_0^T \langle \widetilde{Q}_t, \overline{A}(X_t^v - \widetilde{X}_t) + \overline{B}(\mathbb{E}[X_t^v] - \mathbb{E}[\widetilde{X}_t)] + \overline{G}(v_t - \widetilde{u}_t)\rangle dt\big] \\
&-\mathbb{E}\big[\int_0^T \langle h_z(t,\widetilde{u}_t) + \mathbb{E}[h_{z'}(t,\widetilde{u}_t)], Z_t^v - \widetilde{Z}_t\rangle dt\big].
\end{aligned}$$

Combining (2.15), (2.16) and (2.14), we obtain

$$\langle J'(\widetilde{u}.), v. - \widetilde{u}.\rangle = \mathbb{E}\big[\int_0^T \langle \widetilde{P}_t C + \widetilde{\psi}_t F + \widetilde{Q}_t\overline{G} + h_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t)\rangle dt\big].$$

On the other hand, let us calculate the Gâteaux derivative of H at a point $\widetilde{u}$ in the direction $(v - \widetilde{u})$, we get

$$\begin{aligned}
\mathbb{E}\big[\int_0^T \langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle dt\big] &= \mathbb{E}\big[\int_0^T \langle \widetilde{P}_t C + \widetilde{\psi}_t F + \widetilde{Q}_t\overline{G} + h_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t)\rangle dt\big] \\
&= \langle J'(\widetilde{u}.), v. - \widetilde{u}.\rangle. \qquad (2.17)
\end{aligned}$$

Combines (2.17) and (2.11), we obtain

$$(\widetilde{u}.\text{ minimize } J) \Leftrightarrow \mathbb{E}\big[\int_0^T \langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle dt\big] \geq 0, \forall v. \in \mathcal{U}.$$

It follows that

$$\mathbb{E}\big[\langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle\big] \geq 0,\ dt - a.e.$$

Now, let Θ be an arbitrary element of the σ-algebra $\mathcal{F}_t$, and set

$$\mu_t = v_t \mathbf{1}_\Theta + \widetilde{u}_t \mathbf{1}_{\Omega-\Theta}.$$

It is clear that $\mu.$ is an element of $\mathcal{U}$.

Applying the above inequality with $v.$, we show that

$$\mathbb{E}\big[\mathbf{1}_\Theta \langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle\big] \geq 0,\ \forall\Theta \in \mathcal{F}_t.$$

Which implies that

$$\mathbb{E}\big[\langle H_v(t,\widetilde{u}_t), v_t - \widetilde{u}_t\rangle /\ \mathcal{F}_t\big] \geq 0.$$

Since the quantity inside the conditional expectation is $\mathcal{F}_t$-measurable, so the result is proved. ■

CHAPTER 3

Existence of optimal solutions and optimality conditions for optimal control problems of MF-FBSDEs systems with uncontrolled diffusion

CHAPTER 3

Existence of optimal solutions and optimality conditions for optimal control problems of MF-FBSDEs systems with uncontrolled diffusion

In this chapter, we study the existence of optimal control for systems, governed by non linear forward-backward stochastic differential equations of mean field type, we prove the existence of optimal relaxed control for this system of MF-FBSDEs. The proof of the first main result is based on tightness results of the distributions of the processes defining the control problem and the Skorokhod representation theorem on the Skorokhod space, equipped with the S-topology of Jakubowski [24] . Furthermore, when the Roxin convexity condition is fulfilled, we prove that the optimal relaxed control is in fact strict. The second main result in this chapter is to establish necessary as well as sufficient optimality conditions for both relaxed and strict control problems for system of non linear MF-FBSDE.

3.1 Statement of the problems and assumptions

3.1.1 Strict control problem

We study the existence of strict optimal controls for systems governed by the following FBSDE of mean-field type

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, \mathbb{E}[\alpha(X_s)], u_s)ds + \int_0^t \sigma(s, X_s, \mathbb{E}[\beta(X_s)])dW_s \\ \\ Y_t = g(X_T, \mathbb{E}[\lambda(X_T)]) + \int_t^T f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)], u_s)ds \\ \qquad - \int_t^T Z_s dW_s - (N_T - N_t), \end{cases} \tag{3.1}$$

and the expected cost on the time interval $[0, T]$ is given by

$$J(u.) := \mathbb{E}\big[l(X_T, \mathbb{E}[\theta(X_T)]) + k(Y_0, \mathbb{E}[\rho(Y_0)]) \\ + \int_0^T h\big(t, X_t, \mathbb{E}[\varphi(X_t)], Y_t, \mathbb{E}[\psi(Y_t)], u_t\big)\, dt\big], \tag{3.2}$$

where u_t is a strict control, $(W_t, t \geq 0)$ is a d-dimensional Brownian motion defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and N is a square integrable martingale that is orthogonal to W.

Our objective is to minimize the cost function (3.2), over the set of admissible controls, which are a $\mathcal{F}_t$-measurable processes valued in a compact metric space $U \subset \mathbb{R}^k$.

It should be noted that the probability space and the Brownian motion may change with the control u. Therefore, we need to have another definition of the admissible control, gives as follows:

Definition 3.1.1 *A 6-tuple* $u. = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W., v.\right)$ *is called* ω*-admissible strict control, and* (X_t, Y_t, Z_t) *a* ω*-admissible triple if:*

i)-$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions;*

ii)-W_t *is an d-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$;

iii)-v_t *is an* $\mathcal{F}_t$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in the action space* U*;*

iv)-(X_t, Y_t, Z_t) *is the unique solution of the MF-FBSDE (3.1) on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *under* v_t*.*

The set of all ω*-admissible controls is denoted by* $\mathcal{U}^{\omega}$*.*

Our stochastic optimal control problem under the weak formulation can be stated as follows:

Minimize (3.2) over $\mathcal{U}^{\omega}$. We say that the ω-admissible control u^* is ω-optimal control, if it satisfies

$$J\left(u^*_{\cdot}\right) = \inf_{u_{\cdot} \in \mathcal{U}^{\omega}} J\left(u_{\cdot}\right). \tag{3.3}$$

3.1.2 Relaxed control problem

To proof the existence of optimal solution of our strict control problem $\{(3.1), (3.2), (3.3)\}$ one typically seeks a certain compactness structure. The weak formulation enables us to find the compactness of the image measure of some processes involved on a certain functional space. However, because the control v is measurable only in t and there is no convenient compactness property on the space of measurable functions, we need to embed it in a larger space with proper compactness. The idea is then to replace the U-valued process v_t with $\mathbb{P}(U)$-valued process (μ_t), where $\mathbb{P}(U)$ is the space of probability measures equipped with the topology of weak convergence. These measure valued control are called relaxed control. If $\mu_t(da) = \delta_{v_t}(da)$ is a Dirac measure charging v_t for each t, then we get a strict control problem as a special case of the relaxed one.

We denote by $\mathbb{V}$ the space of positive Radon measures on $[0,T]\times U$, whose projections on $[0,T]$ coincide with Lebesgue measure dt. Equipped with the topology of stable convergence of measures, $\mathbb{V}$ is a compact metrizable space, (see Jacod and Mémin [23]).

The system in this case, is then driven by the following MF-FBSDE

$$\begin{cases} X_t = x + \int_0^t \int_U b(s, X_s, \mathbb{E}[\alpha(X_s)], u)\mu_s(du)\, ds + \int_0^t \sigma(s, X_s, \mathbb{E}[\beta(X_s)])dW_s \\ \\ Y_t = g(X_T, \mathbb{E}[\lambda(X_T)]) + \int_t^T \int_U f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)], u)\mu_s(du)\, ds \\ \qquad\qquad - \int_t^T Z_s dW_s - (N_T - N_t). \end{cases} \tag{3.4}$$

Because of the the possibility of change of the probability space and the Brownian motion, the definition of admissible relaxed control is given by:

Definition 3.1.2 *A 6-tuple* $q. = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W., \mu.\right)$ *is called* ω*-admissible relaxed control, and* (X_t, Y_t, Z_t) *a* ω*-admissible triple if:*

i)-$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions;*

ii)-W_t *is an* d*-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$;

iii)-μ_t *is* $\mathcal{F}_t$*-progressively measurable and such that for each* t, $1_{]0,t]} \cdot \pi$ *is* $\mathcal{F}_t$*- measurable, taking values in* $\mathbb{V}$*;*

iv)-(X_t, Y_t, Z_t) *is the unique solution of the MF-FBSDE (3.4) on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *under* μ_t*.*

The set of all admissible relaxed controls is denoted by $\mathcal{R}$*.*

Accordingly, the cost functional to be minimized over the set $\mathcal{R}$ of admissible relaxed control, well be given by:

$$\begin{aligned} J(\mu.) := \mathbb{E}\big[& l\left(X_T, \mathbb{E}\left[\theta\left(X_T\right)\right]\right) + k\left(Y_0, \mathbb{E}\left[\rho\left(Y_0\right)\right]\right) \\ & + \int_0^T h\big(t, X_t, \mathbb{E}\left[\varphi\left(X_t\right)\right], Y_t, \mathbb{E}\left[\psi\left(Y_t\right)\right], u\big)\, \mu_t(du)\, dt\big]. \end{aligned} \tag{3.5}$$

A relaxed control $q_\cdot^*$ is called optimal if it satisfies

$$J\left(q_\cdot^*\right) = \inf_{q_\cdot \in \mathcal{R}} J\left(q\right). \tag{3.6}$$

3.1.3 Notation and assumptions

We now introduce the following spaces of processes:

$\mathbb{M}^2([0,T];\mathbb{R}^m)$: the set of jointly measurable, processes $\left\{Y_t, t \in \left[0,T\right]\right\}$ with values in $\mathbb{R}^m$ such that Y_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and satisfy

$$\mathbb{E}\big[\int_0^T \left|Y_t\right|^2 dt\big] < \infty.$$

Let $\mathbb{S}^2([0,T];\mathbb{R}^n)$: the set of jointly measurable, processes $\left\{X_t, t \in \left[0,T\right]\right\}$ with values in $\mathbb{R}^n$ such that X_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and satisfy

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|X_t\right|^2\right] < \infty.$$

$\mathbb{C}\left([0,T];\mathbb{R}^n\right)$: the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$, equipped with the topology of uniform convergence.

$\mathbb{D}\left([0,T];\mathbb{R}^m\right)$: the Skorokhod space of càdlàg functions from $[0,T]$ to $\mathbb{R}^m$, that is functions which are continuous from the right with left hand limits, equipped with the S-topology of Jakubowski (see [24], [25]).

Let us assume the following conditions

(**H3**.**1**) Assume that the functions

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n, \\ \sigma &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d}, \\ f &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R}^m, \\ g &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m, \\ \alpha, \beta, \lambda, \gamma &: \mathbb{R}^n \to \mathbb{R}^n, \\ \delta &: \mathbb{R}^m \to \mathbb{R}^m, \end{aligned}$$

are bounded and continuous. Moreover, assume that there exist a constant $K > 0$, such that for every $(x_1, x_2, x_1', x_2') \in \mathbb{R}^{4n}, (y_1, y_2, y_1', y_2')\, \mathbb{R}^{4m}$,

$$\begin{aligned} &|f(t, x_1, x_2, y_1, y_2, u) - f(t, x_1', x_2', y_1', y_2', u)| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'| + |y_1 - y_1'| + |y_2 - y_2'|)\,, \\ &|b(t, x_1, x_2, u) - b(t, x_1', x_2', u)| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'|), \\ &|\sigma(t, x_1, x_2) - \sigma(t, x_1', x_2')| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'|)\,, \\ &|\alpha(x_1) - \alpha(x_1')| \leq K\ |x_1 - x_1'|\,, \qquad |\beta(x_1) - \beta(x_1')| \leq K\ |x_1 - x_1'|\,, \\ &|\gamma(x_1) - \gamma(x_1')| \leq K\ |x_1 - x_1'|\,, \qquad |\lambda(x_1) - \lambda(x_1')| \leq K\ |x_1 - x_1'|\,, \\ &|\delta(y_1) - \delta(y_1')| \leq K\ |y_1 - y_1'|\,, \qquad |g(x_1, x_2) - g(x_1', x_2')| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'|)\,. \end{aligned}$$

(**H3.2**) Assume that the functions

$$h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R},$$

$$l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

$$k : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$$

$$\theta, \varphi : \mathbb{R}^n \to \mathbb{R}^n,$$

$$\rho, \psi : \mathbb{R}^m \to \mathbb{R}^m,$$

are bounded and continuous and there exist a constant $K > 0$, such that for every $(x_1, x_2, x_1', x_2') \in \mathbb{R}^{4n}, (y_1, y_2, y_1', y_2')\, \mathbb{R}^{4m}$,

$$|h(t, x_1, x_2, y_1, y_2, u) - h(t, x_1', x_2', y_1', y_2', u)| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'| + |y_1 - y_1'| + |y_2 - y_2'|)\,.$$

3.2 Existence of optimal relaxed controls

Our results in this paper extends those of [5], [6] and [7] to a systems governed by FBSDE of mean-field type.

Theorem 3.2.1 *Under conditions* $(\mathbf{H3.1}) - (\mathbf{H3.2})$, *the relaxed control problem* $\{(3.4), (3.5), (3.6)\}$ *has an optimal solution.*

To prove this theorem, we need some auxiliary results on the tightness of the distributions of the processes defining the control problem.

Let $q_{\cdot}^{n}=\left(\Omega^{n},\mathcal{F}^{n},\left(\mathcal{F}_{t}^{n}\right)_{t\geq0},\mathbb{P}^{n},W_{\cdot}^{n},\mu_{\cdot}^{n}\right)$ be a minimizing sequence, that is $\lim\limits_{n\rightarrow\infty}J\left(q_{\cdot}^{n}\right)=\inf\limits_{q_{\cdot}\in\mathcal{R}}J\left(q_{\cdot}\right).$

Let (X^{n},Y^{n},Z^{n}) be the unique solution of the following MF-FBSDE associated with μ^{n}

$$\begin{cases} X_{t}^{n}=x+\int_{0}^{t}\int_{U}b\left(s,X_{s}^{n},\mathbb{E}\left[\alpha\left(X_{s}^{n}\right)\right],u\right)\mu_{s}^{n}\left(du\right)ds+\int_{0}^{t}\sigma\left(s,X_{s}^{n},\mathbb{E}\left[\beta\left(X_{s}^{n}\right)\right]\right)dW_{s}^{n}, \\ \\ Y_{t}^{n}=g\left(X_{T}^{n},\mathbb{E}\left[\lambda\left(X_{T}^{n}\right)\right]\right)+\int_{t}^{T}\int_{U}f\left(s,X_{s}^{n},\mathbb{E}\left[\gamma\left(X_{s}^{n}\right)\right],Y_{s}^{n},\mathbb{E}\left[\delta\left(Y_{s}^{n}\right)\right],u\right)\mu_{s}^{n}\left(du\right)ds \\ \qquad\qquad -\int_{t}^{T}Z_{s}^{n}dW_{s}^{n}. \end{cases} \tag{3.7}$$

Lemma 3.2.2 *Let (X^{n},Y^{n},Z^{n}) be the unique solution of the system (3.7). There exists a positive constant K such that*

$$\sup_{n}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_{t}^{n}\right|^{2}+\sup_{0\leq t\leq T}\left|Y_{t}^{n}\right|^{2}+\int_{0}^{T}\left\|Z_{t}^{n}\right\|^{2}dt\right]\leq K. \tag{3.8}$$

Proof. Let us show that

$$\sup_{n}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_{t}^{n}\right|^{2}\right]<+\infty.$$

We have

$$\begin{aligned}\mathbb{E}\left[\left|X_{t}^{n}\right|^{2}\right]\leq3\mathbb{E}\left[\left|x\right|^{2}\right]&+3\mathbb{E}\left[\left|\int_{0}^{t}\int_{U}b\left(s,X_{s}^{n},\mathbb{E}\left[\alpha\left(X_{s}^{n}\right)\right],u\right)\mu_{s}^{n}\left(du\right)ds\right|^{2}\right]\\&+3\mathbb{E}\left[\left|\int_{0}^{t}\sigma\left(s,X_{s}^{n},\mathbb{E}\left[\beta\left(X_{s}^{n}\right)\right]\right)dW_{s}^{n}\right|^{2}\right].\end{aligned}$$

Using isometry of Itô, Cauchy-Schwarz inequality, the boundedness of b,σ and Burkholder-Davis-Gundy's inequality, there exists a constant K which does not depend on n such that

$$\sup_{n}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_{t}^{n}\right|^{2}\right]<K.$$

On the other hand, applying Itô's formula to $|Y_t^n|^2$, we obtain

$$\begin{aligned}\mathbb{E}\left[|Y_t^n|^2+\int_t^T\|Z_s^n\|^2ds\right]&=\mathbb{E}\left[|g(X_T^n,\mathbb{E}\left[\lambda\left(X_T^n\right)\right])|^2\right]\\&\quad+2\mathbb{E}\left[\int_t^T\int_U\langle Y_s^n,f\left(s,X_s^n,\mathbb{E}\left[\gamma\left(X_s^n\right)\right],Y_s^n,\mathbb{E}\left[\delta\left(Y_s^n\right)\right],u\right)\rangle\mu_s^n(du)ds\right]\\&\leq\mathbb{E}\left[|g(X_T^n,\mathbb{E}\left[\lambda\left(X_T^n\right)\right])|^2\right]+\mathbb{E}\left[\int_t^T|Y_s^n|^2ds\right]\\&\quad+\mathbb{E}\left[\int_t^T\int_U\left|f\left(s,X_s^n,\mathbb{E}\left[\gamma\left(X_s^n\right)\right],Y_s^n,\mathbb{E}\left[\delta\left(Y_s^n\right)\right],u\right)\right|^2\mu_s^n(du)ds\right].\end{aligned}$$

Using the boundedness of g and f and by Gronwall's lemma, it follows that

$$\sup_n\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2+\int_0^T\|Z_s^n\|^2ds\right]<+\infty.$$

■

Lemma 3.2.3 *The sequence of distributions of $(X^n,W^n,Y^n,\int_0^{\cdot}Z_s^ndW_s^n)$ is tight on the space $\Delta:=\mathbb{C}\left([0,T];\mathbb{R}^n\right)\times\mathbb{C}\left([0,T];\mathbb{R}^d\right)\times\mathbb{D}\left([0,T];\mathbb{R}^m\right)\times\mathbb{D}\left([0,T];\mathbb{R}^{m\times d}\right)$ endowed with the topology of uniform convergence for the first and second factor and endowed with the S-topology of Jakubowski (see[24]) for the third and forth factor.*

Proof. According to Kolmogorov's theorem (see Ikeda and Watanabe [22] page 18), we need to verify that

$$\mathbb{E}\left[|X_t^n-X_s^n|^4\right]\leq K_1\left|t-s\right|^2,$$

$$\mathbb{E}\left[|W_t^n-W_s^n|^4\right]\leq K_2\left|t-s\right|^2,$$

for some constants K_1 and K_2 independent from n.

We have

$$\mathbb{E}\left[|X_t^n - X_s^n|^4\right] \leq C\mathbb{E}\left[\left|\int_s^t \int_U b\left(s, X_s^n, \mathbb{E}\left[\alpha\left(X_s^n\right)\right], u\right) \mu_s^n\left(du\right) ds\right|^4\right]$$
$$+ C\mathbb{E}\left[\left|\int_s^t \sigma\left(s, X_s^n, \mathbb{E}\left[\beta\left(X_s^n\right)\right]\right) dW_s^n\right|^4\right].$$

Using Burkholder-Davis-Gundy's inequality to the martingale part and the boundedness of b and σ, we obtain

$$\mathbb{E}\left[|X_t^n - X_s^n|^4\right] \leq C\mathbb{E}\left[\left(\int_s^t \int_U \left|b\left(s, X_s^n, \mathbb{E}\left[\alpha\left(X_s^n\right)\right], u\right)\right|^2 \mu_s^n\left(du\right) ds\right)^2\right]$$
$$+ C\mathbb{E}\Big[\left(\int_s^t \left|\sigma\left(s, X_s^n, \mathbb{E}\left[\beta\left(X_s^n\right)\right]\right)\right|^2 ds\right)^2\Big]$$
$$\leq K_1 \left|t - s\right|^2.$$

The second inequality by the same method.

Let us prove that $(Y_\cdot^n, \int_0^\cdot Z_s^n dW_s^n)$ is tight on the space $\mathbb{D}\left([0,T]; \mathbb{R}^m\right) \times \mathbb{D}\left([0,T]; \mathbb{R}^{m\times d}\right)$.

Let $0 = t_0 < t_1 < ... < t_n = T$. We define the conditional variation by

$$CV\left(Y_\cdot^n\right) := \sup \mathbb{E}\left[\sum_i \left|\mathbb{E}\left(Y_{t_{i+1}}^n - Y_{t_i}^n\right) \mid \mathcal{F}_{t_i}^{W^n}\right|\right],$$

where the supremum is taken over all partitions of the interval $[0,T]$. By the same method given in [31], we get

$$CV\left(Y_\cdot^n\right) \leq K\mathbb{E}\left[\int_0^T \int_U \left|f\left(s, X_s^n, \mathbb{E}\left[\gamma\left(X_s^n\right)\right], Y_s^n, \mathbb{E}\left[\delta\left(Y_s^n\right)\right], u\right)\right| \mu_s^n\left(du\right) ds\right],$$

where K is a constant depending only on t. Hence combining conditions (**H3.1**) and Lemma (3.2.2), we deduce that

$$\sup_n \left[CV\left(Y_\cdot^n\right) + \sup_{0\leq t\leq T} \mathbb{E}\left[|Y_t^n|\right] + \sup_{0\leq t\leq T} \mathbb{E}\left[\left|\int_0^T Z_s^n dW_s^n\right|\right]\right] < +\infty.$$

Thus the Meyer-Zheng tightness criteria is fulfilled (see [30]), then the sequences $Y^n_\cdot$ and $\int_0^\cdot Z^n_s dW^n_s$ are tight. ■

Lemma 3.2.4 *The familly of distributions of the relaxed control $(\mu^n)_n$ is tight in $\mathbb{V}$.*

Proof. Since $[0,T]\times U$ is compact, then by applying Prokhorov's theorem, the space $\mathbb{V}$ of probability measures on $[0,T]\times U$ is then compact. Since $(\mu^n)_n$ valued in the compact space $\mathbb{V}$, then the familly of distributions associated to $(\mu^n)_n$ is tight. ■

3.2.1 Proof of theorem 3.2.1

Let $(q^n_\cdot)_{n\geq 0}$ be a minimizing sequence and (X^n, Y^n, Z^n) be the unique solution of the mean-field FBSDE (3.7). Using Lemmas 3.2.3 and 3.2.4, it follows that the sequence of processes $\pi^n := \left(\mu^n, X^n, W^n, Y^n, \int_0^\cdot Z^n_s dW^n_s\right)$ is tight on the space $\mathbb{V}\times\Delta$. Then by the Skorokhod representation theorem, there exists a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, a sequences $\tilde{\pi}^n = \left(\tilde{\mu}^n, \tilde{X}^n, \tilde{W}^n, \tilde{Y}^n, \int_0^\cdot \tilde{Z}^n_s d\tilde{W}^n_s\right)$ and $\tilde{\pi} = \left(\tilde{\mu}, \tilde{X}, \tilde{W}, \tilde{Y}, \int_0^\cdot \tilde{Z}_s d\tilde{W}_s\right)$ defined on this space and a countable subset $\mathcal{D}$ of $[0,T]$ such that on $\mathcal{D}^c$, we have

(a1) for each $n\in\mathbb{N}$, law$(\pi^n)\equiv$ law$(\tilde{\pi}^n)$,

(a2) there exists a subsequence $(\tilde{\pi}^{n_k})$ of $(\tilde{\pi}^n)$, still denoted $(\tilde{\pi}^n)$, which converges to $\tilde{\pi}$, $\tilde{\mathbb{P}}$-a.s. on the space $\mathbb{V}\times\Delta$,

(a3) $(\tilde{Y}^n, \int_0^\cdot \tilde{Z}^n_s d\tilde{W}^n_s)$ converges to the càdlàg processes $(\tilde{Y}, \int_0^\cdot \tilde{Z}_s d\tilde{W}_s)$, $dt\times\tilde{\mathbb{P}}$-a.s. Also $\tilde{Y}^n_T \to \tilde{Y}_T$, $\tilde{\mathbb{P}}$-a.s.

(a4) $\sup\limits_{0\leq t\leq T}\left|\tilde{X}^n_t - \tilde{X}_t\right| \to 0, \quad \tilde{\mathbb{P}}$-a.s.

(a5) $(\tilde{\mu}^n)$ converges in the stable topology to $\tilde{\mu}$, $\tilde{\mathbb{P}}$-a.s.

According to property (a1), we have

$$\begin{cases} \tilde{X}_t^n = x + \int_0^t \int_U b\left(s, \tilde{X}_s^n, \mathbb{E}[\alpha(\tilde{X}_s^n)], u\right) \tilde{\mu}_s^n(du)\, ds + \int_0^t \sigma\left(s, \tilde{X}_s^n, \mathbb{E}[\beta(\tilde{X}_s^n)]\right) d\tilde{W}_s^n, \\ \tilde{Y}_t^n = g\left(\tilde{X}_T^n, \mathbb{E}[\lambda(\tilde{X}_T^n)]\right) + \int_t^T \int_U f\left(s, \tilde{X}_s^n, \mathbb{E}[\gamma(\tilde{X}_s^n)], \tilde{Y}_s^n, \mathbb{E}[\delta(\tilde{Y}_s^n)], u\right) \tilde{\mu}_s^n(du)\, ds \\ \qquad\qquad - \left(\tilde{N}_T^n - \tilde{N}_t^n\right), \end{cases} \tag{3.9}$$

where $\tilde{N}_t^n = \int_0^t \tilde{Z}_s^n d\tilde{W}_s^n$.

Using properties $(a2), (a3), (a4), (a5)$, $under$ $(\mathbf{H3.1})$ and passing to the limit in the MF-FBSDE (3.9), one can show that there exists a countable set $\mathcal{D} \subset [0,\ T)$ such that

$$\begin{cases} \tilde{X}_t = x + \int_0^t \int_U b\left(s, \tilde{X}_s, \mathbb{E}[\alpha(\tilde{X}_s)], u\right) \tilde{\mu}_s(du)\, ds + \int_0^t \sigma\left(s, \tilde{X}_s, \mathbb{E}[\beta(\tilde{X}_s)]\right) d\tilde{W}_s, t > 0 \\ \\ \tilde{Y}_t = g\left(\tilde{X}_T, \mathbb{E}[\lambda(\tilde{X}_T)]\right) + \int_t^T \int_U f\left(s, \tilde{X}_s, \mathbb{E}[\gamma(\tilde{X}_s)], \tilde{Y}_s, \mathbb{E}[\delta(\tilde{Y}_s)], u\right) \tilde{\mu}_s(du)\, ds \\ \qquad\qquad - \left(\tilde{N}_T - \tilde{N}_t\right), t \in [0,T] \diagdown \mathcal{D}. \end{cases}$$

Since $\tilde{Y}$ and $\tilde{N}$ are càdlàg, then one can get for every $t \in [0,T]$

$$\tilde{Y}_t = g\left(\tilde{X}_T, \mathbb{E}[\lambda(\tilde{X}_T)]\right) + \int_t^T \int_U f\left(s, \tilde{X}_s, \mathbb{E}[\gamma(\tilde{X}_s)], \tilde{Y}_s, \mathbb{E}[\delta(\tilde{Y}_s)], u\right) \tilde{\mu}_s(du)\, ds$$
$$- \left(\tilde{N}_T - \tilde{N}_t\right).$$

Now, let $\tilde{\mathcal{F}}_s = \mathcal{F}_s^{\tilde{X},\tilde{Y},\tilde{\mu}}$, the minimal admissible and complete filtration generated by $(\tilde{X}_r,\ \tilde{Y}_r, \tilde{\mu}_r,\ r \leq s)$. One can show easely that $\tilde{N}$ is a $\tilde{\mathcal{F}}_s$-martingale. Therefore by the martingale decomposition theorem, there exist a process $\tilde{Z} \in \mathbb{M}^2([0,T]\,; \mathbb{R}^{m\times d})$ such that

$$\tilde{N}_t = \int_0^t \tilde{Z}_s d\tilde{W}_s + \tilde{M}_t, \text{ and } \left\langle \tilde{M}, \tilde{W} \right\rangle_t = 0,$$

which implies that

$$\tilde{Y}_t = g\left(\tilde{X}_T, \mathbb{E}[\lambda(\tilde{X}_T)]\right) + \int_t^T \int_U f\left(s, \tilde{X}_s, \mathbb{E}[\gamma(\tilde{X}_s)], \tilde{Y}_s, \mathbb{E}[\delta(\tilde{Y}_s)], u\right) \tilde{\mu}_s(du)\, ds$$
$$- \int_t^T \tilde{Z}_s d\tilde{W}_s - \left(\tilde{M}_T - \tilde{M}_t\right),$$

To finish the proof of our result (Theorem 3.2.1), it remains to proof that $\tilde{\mu}$ is an optimal relaxed control (which minimize the cost functional J over the set $\mathcal{R}$ of admissible relaxed control).

Using properties (a1)-(a5), we get

$$\begin{aligned}
\inf_{q.\in\mathcal{R}} J(q.) &= \lim_{n\to\infty} J(q.^n) = \lim_{n\to\infty} J(\tilde{q}.^n), \\
&= \lim_{n\to\infty} \mathbb{E}\Big[l\left(X_T^n, \mathbb{E}[\theta(X_T^n)]\right) + k\left(Y_0^n, \mathbb{E}[\rho(Y_0^n)]\right) \\
&\qquad + \int_0^T \int_U h\left(t, X_t^n, \mathbb{E}[\varphi(X_t^n)], Y_t^n, \mathbb{E}[\psi(Y_t^n)], u\right) \mu_t^n(du)\, dt \Big] \\
&= \lim_{n\to\infty} \mathbb{E}\left[l\left(\tilde{X}_T^n, \mathbb{E}[\theta(\tilde{X}_T^n)]\right) + k\left(\tilde{Y}_0^n, \mathbb{E}[\rho(\tilde{Y}_0^n)]\right) \right. \\
&\qquad \left. + \int_0^T \int_U h\left(t, \tilde{X}_t^n, \mathbb{E}[\varphi(\tilde{X}_t^n)], \tilde{Y}_t^n, \mathbb{E}[\psi(\tilde{Y}_t^n)], u\right) \tilde{\mu}_t^n(du)\, dt \right] \\
&= \mathbb{E}\left[l\left(\tilde{X}_T, \mathbb{E}[\theta(\tilde{X}_T)]\right) + k\left(\tilde{Y}_0, \mathbb{E}[\rho(\tilde{Y}_0)]\right) \right. \\
&\qquad \left. + \int_0^T \int_U h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], u\right) \tilde{\mu}_t(du)\, dt \right] \\
&= J(\tilde{q}.),
\end{aligned}$$

then theorem (3.2.1) is proved.

3.3 Existence of optimal strict control

To prove existence of optimal solution to the strict control problem $\{(3.1), (3.2), (3.3)\}$, we need the Roxin's condition (see Yong and Zhou, [35] p. 69), given by

$(\mathbf{H3.3})$: (Roxin-type convexity condition): The set

$$(b, f, h)(t, x, x', y, y', U) := \{b_i(t, x, x', u), f_j(t, x, x', y, y', u)$$

$$, h(t, x, x', y, y', u) \diagdown u \in U, i = 1, \cdots, n, j = 1, \cdots, m\},$$

is convex and closed in $\mathbb{R}^{n+m+1}$.

Proposition 3.3.1 *Assume that* ($\mathbf{H3.1}$)-($\mathbf{H3.2}$) *and* ($\mathbf{H3.3}$) *hold. Then, the optimal relaxed control* $\tilde{\mu}_t$ *has the form of a Dirac measure charging a strict control* $\tilde{u}_t$, *(i.e,* $\tilde{\mu}_t\,(du) = \delta_{\tilde{u}_t}\,(du)$*).*

Proof. We put

$$\int_U b\left(t, \tilde{X}_t, \mathbb{E}[\alpha(\tilde{X}_t)], u\right) \tilde{\mu}_t\,(du) := \tilde{b}\,(t,w) \in b\,(t,x,x',U)\,,$$
$$\int_U f\left(t, \tilde{X}_t, \mathbb{E}[\gamma(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\delta(\tilde{Y}_t)], u\right) \tilde{\mu}_t\,(du) := \tilde{f}\,(t,w) \in f\,(t,x,x',y,y',U)\,,$$
$$\int_U h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], u\right) \tilde{\mu}_t\,(du) := \tilde{h}\,(t,w) \in h\,(t,x,x',y,y',U)\,.$$

Under ($\mathbf{H3.3}$) and the measurable selection theorem (see Li-Yong [29] p. 102, Corollary 2.26), there is a U-valued, $\mathcal{F}^{\tilde{X},\tilde{Y},\tilde{\mu}}$-adapted process $\tilde{v}$, such that for every $t \in [0,\ T] \setminus \mathcal{D}$ and $w \in \tilde{\Omega}$,

$$\left(\tilde{f},\tilde{h}\right)(t,w) = (f,h)\left(t, \tilde{X}\,(t,w)\,, \tilde{X}'\,(t,w)\,, \tilde{Y}\,(t,w)\,, \tilde{Y}'\,(t,w)\,, \tilde{v}\,(t,w)\right),$$
$$\tilde{b}\,(t,w) = b\left(t, \tilde{X}\,(t,w)\,, \tilde{X}'\,(t,w)\,, \tilde{v}\,(t,w)\right).$$

Hence, for every $t \in [0,\ T] \setminus \mathcal{D}$ and $w \in \tilde{\Omega}$, we have

$$\int_U b\left(t, \tilde{X}_t, \mathbb{E}[\alpha(\tilde{X}_t)], u\right) \tilde{\mu}_t\,(du) = b\left(t, \tilde{X}_t, \mathbb{E}[\alpha(\tilde{X}_t)], \tilde{v}_t\right),$$
$$\int_U f\left(t, \tilde{X}_t, \mathbb{E}[\gamma(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\delta(\tilde{Y}_t)], u\right) \tilde{\mu}_t\,(du) = f\left(t, \tilde{X}_t, \mathbb{E}[\gamma(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\delta(\tilde{Y}_t)], \tilde{v}_t\right),$$
$$\int_U h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], u\right) \tilde{\mu}_t\,(du) = h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], \tilde{v}_t\right).$$

Since $\tilde{X}$ is continuous and $(\tilde{Y}_\cdot, \int_0^\cdot \tilde{Z}_t d\tilde{W}_t)$ is cadlàg, then the process $(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)$ satisfies for each $t \in [0,T]$, the following system of MF-FBSDE

$$\begin{cases} \tilde{X}_t = x + \int_0^t b\left(s, \tilde{X}_s, \mathbb{E}[\alpha(\tilde{X}_s)], \tilde{v}_s\right) ds + \int_0^t \sigma\left(s, \tilde{X}_s, \mathbb{E}[\beta(\tilde{X}_s)]\right) d\tilde{W}_s \\ \tilde{Y}_t = g\left(\tilde{X}_T, \mathbb{E}[\lambda(\tilde{X}_T)]\right) + \int_t^T f\left(s, \tilde{X}_s, \mathbb{E}[\gamma(\tilde{X}_s)], \tilde{Y}_s, \mathbb{E}[\delta(\tilde{Y}_s)], \tilde{v}_s\right) ds \\ \qquad - \int_t^T \tilde{Z}_s d\tilde{W}_s - \left(\tilde{M}_T - \tilde{M}_t\right). \end{cases}$$

Moreover,

$$J(\tilde{q}) = \mathbb{E}\left[l\left(\tilde{X}_T, \mathbb{E}[\theta(\tilde{X}_T)]\right) + k\left(\tilde{Y}_0, \mathbb{E}[\rho(\tilde{Y}_0)]\right) + \int_0^T \int_U h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], u\right) \tilde{\mu}_t\,(du)\,dt\right]$$

$$= \mathbb{E}\left[l\left(\tilde{X}_T, \mathbb{E}[\theta(\tilde{X}_T)]\right) + k\left(\tilde{Y}_0, \mathbb{E}[\rho(\tilde{Y}_0)]\right) + \int_0^T h\left(t, \tilde{X}_t, \mathbb{E}[\varphi(\tilde{X}_t)], \tilde{Y}_t, \mathbb{E}[\psi(\tilde{Y}_t)], \tilde{v}_t\right) dt\right]$$

$$= J(\tilde{u}),$$

where $\tilde{u} = \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{v}\right)$. Which ends the proof. ■

3.4 Necessary and sufficient optimality conditions for relaxed and strict control problems

In this section, we establish necessary as well as sufficient optimality conditions for both relaxed and strict control problems.

3.4.1 Necessary and sufficient optimality conditions for relaxed control

We start by establish necessary and sufficient optimality conditions for existence of optimal relaxed control. To simplify the calculations, let $\alpha = \lambda = \beta = \gamma = \theta = \varphi = 1_{\mathbb{R}^n}, \delta = \rho = \psi = 1_{\mathbb{R}^m}$ and the system (3.4) becomes

$$\begin{cases} X_t^\mu = x + \int_0^t \int_U b(s, X_s^\mu, \mathbb{E}[X_s^\mu], u)\mu_s\,(du)\,ds + \int_0^t \sigma(s, X_s^\mu, \mathbb{E}[X_s^\mu])dW_s \\ Y_t^\mu = g(X_T^\mu, \mathbb{E}[X_T^\mu]) + \int_t^T \int_U f(s, X_s^\mu, \mathbb{E}[X_s^\mu], Y_s^\mu, \mathbb{E}[Y_s^\mu], u)\mu_s\,(du)\,ds \\ \qquad\qquad - \int_t^T Z_s^\mu dW_s, \end{cases} \tag{3.10}$$

and the functional cost to be minimize over the set of relaxed controls $\mathcal{R}$ is given by

$$\begin{aligned} J(\mu.) := \mathbb{E}\big[l\left(X_T^{\mu}, \mathbb{E}\left[X_T^{\mu}\right]\right) + k\left(Y_0^{\mu}, \mathbb{E}\left[Y_0^{\mu}\right]\right) \\ + \int_0^T \int_U h\big(t, X_t^{\mu}, \mathbb{E}\left[X_t^{\mu}\right], Y_t^{\mu}, \mathbb{E}\left[Y_t^{\mu}\right], u\big) \mu_t\left(du\right) dt \big]. \end{aligned} \tag{3.11}$$

We say that a relaxed control $q.$ is an optimal control if

$$J(q.) = \inf_{\mu. \in \mathcal{R}} J(\mu.). \tag{3.12}$$

Recall that the set of relaxed controls is convex, then to establish necessary optimality condition we use the convex perturbation method. Let $q.$ be an optimal relaxed control with associated trajectories (X_t^q, Y_t^q, Z_t^q) solution of the MF-FBSDEs (3.10). Then, we can define a perturbed relaxed control by

$$q_t^{\varepsilon} = q_t + \varepsilon(\mu_t - q_t),$$

where $\varepsilon > 0$ is sufficiently small and $\mu.$ is an arbitrary element of $\mathcal{R}$. Denote by $(X_t^{\varepsilon}, Y_t^{\varepsilon}, Z_t^{\varepsilon})$ the solution of (3.10) corresponding to $q_\cdot^{\varepsilon}$.

We shall consider in this section the following assumptions.

- (**H3.4**) (Regularity conditions)

$$\begin{cases} (i) \text{ the mappings } b, g, \sigma, l \text{ are bounded and continuously differentiable with respect to } (x, x'), \\ \quad \text{and the functions } f, h \text{ and } k \text{ are continuously differentiable with respect to } (x, x', y, y') \\ \quad \text{and } (y, y'), \text{ respectively,} \\ (ii) \text{ the derivatives of } b, g, \sigma, f \text{ with respect to the above arguments are continuous and bounded,} \\ (iii) \text{ the derivatives of } h \text{ are bounded by } C(1 + |x| + |x'| + |y| + |y'|), \\ (iv) \text{ the derivatives of } l \text{ and } k \text{ are bounded by } C\left(1 + |x| + |x'|\right) \text{ and } C\left(1 + |y| + |y'|\right) \text{ respectively,} \end{cases}$$

for some positive constant C.

3.4.1.1 Estimates

Using the optimality of $q.$, the variational inequality will be derived from the following inequality

$$0 \leq J(q^{\varepsilon}) - J(q).$$

For this end, we need some results.

Proposition 3.4.1 *Under assumptions* $(\mathbf{H3.1}) - (\mathbf{H3.2})$, *we have*

$$\lim_{\varepsilon \to 0} \left(\sup_{0 \leq t \leq T} \mathbb{E}\left[|X_t^{\varepsilon} - X_t^q|^2 \right] \right) = 0, \tag{3.13}$$

$$\lim_{\varepsilon \to 0} \left(\sup_{0 \leq t \leq T} \mathbb{E}\left[|Y_t^{\varepsilon} - Y_t^q|^2 \right] \right) = 0, \tag{3.14}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^T \|Z_t^{\varepsilon} - Z_t^q\|^2 dt \right] = 0. \tag{3.15}$$

Proof. We calculate $(X_t^{\varepsilon} - X_t^q)$,

$$\begin{aligned} X_t^{\varepsilon} - X_t^q = &\int_0^t \left[\int_U b\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right], u\right) q_s^{\varepsilon}(du) - \int_U b\left(s, X_s^q, \mathbb{E}\left[X_s^q\right], u\right) q_s(du) \right] ds \\ &+ \int_0^t \left[\sigma\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right]\right) - \sigma\left(s, X_s^q, \mathbb{E}\left[X_s^q\right]\right) \right] ds. \end{aligned}$$

Using the definition of q_t^{ε} and taking expectation to get

$$\begin{aligned} \mathbb{E}\left[|X_t^{\varepsilon} - X_t^q|^2 \right] \leq\ & C\mathbb{E}\Big[\int_0^t \left| \int_U b\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right], u\right) q_s(du) - \int_U b\left(s, X_s^q, \mathbb{E}\left[X_s^q\right], u\right) q_s(du) \right|^2 ds \Big] \\ &+ C\varepsilon^2 \mathbb{E}\Big[\int_0^t \left| \int_U b\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right], u\right) \mu_s(du) - \int_U b\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right], u\right) q_s(du) \right|^2 ds \Big] \\ &+ C\mathbb{E}\Big[\int_0^t \left| \sigma\left(s, X_s^{\varepsilon}, \mathbb{E}\left[X_s^{\varepsilon}\right]\right) - \sigma\left(s, X_s^q, \mathbb{E}\left[X_s^q\right]\right) \right|^2 ds \Big]. \end{aligned}$$

Using the fact that b and σ are uniformly Lipschitz with respect to x, x', to obtain

$$\mathbb{E}\left[|X_t^{\varepsilon} - X_t^q|^2 \right] \leq C\mathbb{E}\Big[\int_0^t |X_t^{\varepsilon} - X_t^q|^2 ds \Big] + C\varepsilon^2.$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality, we can show (3.13).

On the other hand, applying Itô's formula to $(Y_t^\varepsilon - Y_t^q)^2$ and taking expectation to get

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] = \mathbb{E}\left[|g(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - g(X_T^q, \mathbb{E}[X_T^q])|^2\right]$$
$$+2\mathbb{E}\Big[\int_t^T \langle Y_s^\varepsilon - Y_s^q, \int_U f\left(s, X_s^\varepsilon, \mathbb{E}\left[X_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], u\right) q_s^\varepsilon(du)$$
$$- \int_U f\left(s, X_s^q, \mathbb{E}\left[X_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], u\right) q_s(du)\rangle ds\Big].$$

Thus

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq \mathbb{E}\left[|g(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - g(X_T^q, \mathbb{E}[X_T^q])|^2\right]$$
$$+\mathbb{E}\left[\int_0^s |Y_t^\varepsilon - Y_s^q|^2 ds\right] + \mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, X_s^\varepsilon, \mathbb{E}\left[X_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], u\right) q_s^\varepsilon(du)$$
$$- \int_U f\left(s, X_s^q, \mathbb{E}\left[X_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], u\right) q_s(du)\Big|^2 ds\Big].$$

Using the definition of q_t^ε, we obtain

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq \mathbb{E}\left[|g(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - g(X_T^q, \mathbb{E}[X_T^q])|^2\right]$$
$$+\mathbb{E}\left[\int_0^t |Y_s^\varepsilon - Y_s^q|^2 ds\right] + C\varepsilon^2\mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, X_s^\varepsilon, \mathbb{E}\left[X_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], u\right) \mu_s(du)$$
$$- \int_U f\left(s, X_s^\varepsilon, \mathbb{E}\left[X_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], u\right) q_s(du)\Big|^2 ds\Big]$$
$$+C\mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, X_s^\varepsilon, \mathbb{E}\left[X_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], u\right) q_s(du)$$
$$- \int_U f\left(s, X_s^q, \mathbb{E}\left[X_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], u\right) q_s(du)\Big|^2 ds\Big].$$

Since f and g are uniformly Lipschitz with respect to their arguments, we have

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq K\mathbb{E}\left[\int_0^t |Y_s^\varepsilon - Y_s^q|^2 ds\right] + \Pi_t^\varepsilon, \tag{3.16}$$

where

$$\Pi_t^\varepsilon = 2C\mathbb{E}\left[|X_T^\varepsilon - X_T^q|^2\right] + 2C\mathbb{E}\Big[\int_0^t |X_s^\varepsilon - X_s^q|^2 ds\Big] + C\varepsilon^2.$$

From (3.12) we can show that

$$\lim_{\varepsilon\to 0}\Pi_t^{\varepsilon}=0. \tag{3.17}$$

We derive from the inequality (3.16), two inequalities

$$\mathbb{E}\left[|Y_t^{\varepsilon}-Y_t^{q}|^2\right]\leq K\mathbb{E}\left[\int_0^t|Y_s^{\varepsilon}-Y_s^{q}|^2ds\right]+\Pi_t^{\varepsilon}, \tag{3.18}$$

and

$$\mathbb{E}\left[\int_t^T\|Z_s^{\varepsilon}-Z_s^{q}\|^2ds\right]\leq K\mathbb{E}\left[\int_0^t|Y_s^{\varepsilon}-Y_s^{q}|^2ds\right]+\Pi_t^{\varepsilon}. \tag{3.19}$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality in (3.17) and using (3.13) and (3.17) to get (3.14). Finally (3.15) derived from (3.19), (3.14) and (3.17). ■

Proposition 3.4.2 *Let* $\left(\widehat{X}_t,\widehat{Y}_t,\widehat{Z}_t\right)$ *, be the solution of the following variational equations of MF-FBSDE (3.10)*

$$\begin{cases} d\widehat{X}_t=\int_U b_x\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)q_t(du)\widehat{X}_tdt+\mathbb{E}\left[\int_U b_{x'}\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)q_t(du)\mathbb{E}[\widehat{X}_t]\right]dt \\ \qquad +\left(\sigma_x\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)\widehat{X}_t+\mathbb{E}\left[\sigma_{x'}\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)\mathbb{E}[\widehat{X}_t]\right]\right)dW_t \\ \qquad +\left(\int_U b\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)q_t(du)-\int_U b\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],u\right)\mu_t(du)\right)dt \\ \\ d\widehat{Y}_t=-(\int_U f_x(t,\eta_t^q,u)q_t(du)\widehat{X}_t+\mathbb{E}\left[\int_U f_{x'}(t,\eta_t^q,u)q_t(du)\mathbb{E}[\widehat{X}_t]\right] \\ \qquad\qquad +\int_U f_y(t,\eta_t^q,u)q_t(du)\widehat{Y}_t+\mathbb{E}\left[\int_U f_{y'}(t,\eta_t^q,u)q_t(du)\mathbb{E}[\widehat{Y}_t]\right])dt \\ \qquad -(\int_U f\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],Y_t^q,\mathbb{E}\left[Y_t^q\right],u\right)q_t(du)-\int_U f\left(t,X_t^q,\mathbb{E}\left[X_t^q\right],Y_t^q,\mathbb{E}\left[Y_t^q\right],u\right)\mu_t(du))dt \\ \qquad +\widehat{Z}_tdW_t, \\ \\ \widehat{X}_0=0,\widehat{Y}_T=g_x\left(X_T^q,\mathbb{E}[X_T^q]\right)\widehat{X}_T+\mathbb{E}\left[g_{x'}\left(X_T^q,\mathbb{E}[X_T^q]\right)\mathbb{E}[\widehat{X}_T]\right], \end{cases} \tag{3.20}$$

where $(t,\eta_t^q,u):=(t,X_t^q,\mathbb{E}\left[X_t^q\right],Y_t^q,\mathbb{E}\left[Y_t^q\right],u)$. *We have the following estimates*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\frac{1}{\varepsilon}(X_t^\varepsilon-X_t^q)-\widehat{X}_t\right|^2\right]=0, \tag{3.21}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\frac{1}{\varepsilon}(Y_t^\varepsilon-Y_t^q)-\widehat{Y}_t\right|^2\right]=0, \tag{3.22}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\frac{1}{\varepsilon}(Z_t^\varepsilon-Z_t^q)-\widehat{Z}_t\right|^2\right]=0. \tag{3.23}$$

Proof. For simplicity, denote by

$$\mathcal{X}_t^\varepsilon=\frac{1}{\varepsilon}(X_t^\varepsilon-X_t^q)-\widehat{X}_t,\mathcal{Y}_t^\varepsilon=\frac{1}{\varepsilon}(Y_t^\varepsilon-Y_t^q)-\widehat{Y}_t,\mathcal{Z}_t^\varepsilon=\frac{1}{\varepsilon}(Z_t^\varepsilon-Z_t^q)-\widehat{Z}_t. \tag{3.24}$$

i) Let us prove (3.21). From (3.10), (3.20) and notations (3.24), we have

$$\begin{aligned}
\mathcal{X}_t^\varepsilon &= \frac{1}{\varepsilon}\int_0^t\left[\int_U b\left(s,X_s^\varepsilon,\mathbb{E}\left[X_s^\varepsilon\right],u\right)q_s^\varepsilon(du)-\int_U b\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s^\varepsilon(du)\right]ds\\
&+\frac{1}{\varepsilon}\int_0^t\left[\int_U b\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s^\varepsilon(du)-\int_U b\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s(du)\right]ds\\
&+\frac{1}{\varepsilon}\int_0^t\left[\sigma\left(s,X_s^\varepsilon,\mathbb{E}\left[X_s^\varepsilon\right]\right)-\sigma\left(s,X_s^q,\mathbb{E}\left[X_s^q\right]\right)\right]dW_s\\
&-\int_0^t\int_U b_x\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s(du)\widehat{X}_s ds-\int_0^t\mathbb{E}\left[\int_U b_{x'}\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s(du)\mathbb{E}[\widehat{X}_s]\right]ds\\
&-\int_0^t\left(\sigma_x\left(s,X_s^q,\mathbb{E}\left[X_s^q\right]\right)\widehat{X}_s+\mathbb{E}\left[\sigma_{x'}\left(s,X_s^q,\mathbb{E}\left[X_s^q\right]\right)\mathbb{E}[\widehat{X}_s]\right]\right)dW_s\\
&-\int_0^t\left(\int_U b\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)q_s(du)-\int_U b\left(s,X_s^q,\mathbb{E}\left[X_s^q\right],u\right)\mu_s(du)\right)ds.
\end{aligned}$$

Using the definition of q_s^ε and taking expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{X}_t^\varepsilon|^2\right] &\le C\mathbb{E}\left[\int_0^t\int_0^1\int_U|b_x\left(s,\Lambda_s^\varepsilon,u\right)\mathcal{X}_t^\varepsilon|^2 q_s(du)d\lambda ds\right]\\
&+C\mathbb{E}\left[\int_0^t\int_0^1\int_U|\mathbb{E}\left[b_{x'}\left(s,\Lambda_s^\varepsilon,u\right)\mathbb{E}[\mathcal{X}_t^\varepsilon]\right]|^2 q_s(du)d\lambda ds\right]\\
&+C\mathbb{E}\left[\int_0^t\int_0^1|\sigma_x\left(s,\Lambda_s^\varepsilon\right)\mathcal{X}_t^\varepsilon|^2 d\lambda ds\right]\\
&+C\mathbb{E}\left[\int_0^t\int_0^1|\mathbb{E}\left[\sigma_{x'}\left(s,\Lambda_s^\varepsilon\right)\mathbb{E}[\mathcal{X}_t^\varepsilon]\right]|^2 d\lambda ds\right]+C\mathbb{E}\left[|\Phi_t^\varepsilon|^2\right],
\end{aligned}$$

where $(t, \Lambda_s^\varepsilon, u) := \Big(t, X_s^q + \lambda\varepsilon(\mathcal{X}_s^\varepsilon + \widehat{X}_s), \mathbb{E}[X_s^q + \lambda\varepsilon(\mathcal{X}_s^\varepsilon + \widehat{X}_s)], u\Big)$, and

$$\begin{aligned}
\Phi_t^\varepsilon &= \int_0^t \int_0^1 \int_U b_x\left(s, \Lambda_s^\varepsilon, u\right)\left(X_s^\varepsilon - X_s^q\right)\mu_s(du)d\lambda ds \\
&+ \int_0^t \int_0^1 \int_U \mathbb{E}\left[b_{x'}\left(s, \Lambda_s^\varepsilon, u\right)\mathbb{E}[X_s^\varepsilon - X_s^q]\right]\mu_s(du)d\lambda ds \\
&+ \int_0^t \int_0^1 \left(\sigma_x\left(s, \Lambda_s^\varepsilon\right)\left(X_t^\varepsilon - X_s^q\right) + \mathbb{E}\left[\sigma_{x'}\left(s, \Lambda_s^\varepsilon\right)\mathbb{E}[X_s^\varepsilon - X_s^q]\right]\right)d\lambda dW_s \\
&+ \int_0^t \int_0^1 \int_U \Big(b_x\left(s, \Lambda_s^\varepsilon, u\right)\widehat{X}_s + \mathbb{E}\Big[b_{x'}\left(s, \Lambda_s^\varepsilon, u\right)\mathbb{E}[\widehat{X}_s]\Big]\Big) q_s(du)d\lambda ds \\
&+ \int_0^t \int_0^1 \Big(\sigma_x\left(s, \Lambda_s^\varepsilon\right)\widehat{X}_t + \mathbb{E}\Big[\sigma_{x'}\left(s, \Lambda_s^\varepsilon\right)\mathbb{E}[\widehat{X}_s]\Big]\Big) d\lambda dW_s \\
&- \int_0^t \int_U b_x\left(s, X_s^q, \mathbb{E}[X_s^q], u\right)\widehat{X}_s q_s(du)ds \\
&- \int_0^t \int_U \mathbb{E}\Big[b_{x'}\left(s, X_s^q, \mathbb{E}[X_s^q], u\right)\mathbb{E}[\widehat{X}_s]\Big] q_s(du)ds \\
&- \int_0^t \int_U \Big(\sigma_x\left(s, X_s^q, \mathbb{E}[X_s^q]\right)\widehat{X}_s + \mathbb{E}\Big[\sigma_{x'}\left(s, X_s^q, \mathbb{E}[X_s^q]\right)\mathbb{E}[\widehat{X}_s]\Big]\Big) dW_s,
\end{aligned}$$

using the fact that $b_x, b_{x'}, \sigma_x, \sigma_{x'}$ are continuous and bounded to get

$$\mathbb{E}\left[|\mathcal{X}_t^\varepsilon|^2\right] \leq C\mathbb{E}\left[\int_0^t |\mathcal{X}_s^\varepsilon|^2 ds\right] + C\mathbb{E}\left[|\Phi_t^\varepsilon|^2\right], \tag{3.25}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\Phi_t^\varepsilon|^2\right] = 0. \tag{3.26}$$

By using (3.26), Granwall's lemma and Burkholder-Davis-Gundy inequality in (3.25), one can show (3.21).

ii) Let us prove (3.22) and (3.23). We put

$(t, \Delta_s^\varepsilon, u) := \big(t, X_s^q + \lambda\varepsilon(\mathcal{X}_s^\varepsilon + \widehat{X}_s), \mathbb{E}[X_s^q + \lambda\varepsilon(\mathcal{X}_s^\varepsilon + \widehat{X}_s)], Y_s^q + \lambda\varepsilon(\mathcal{Y}_s^\varepsilon + \widehat{Y}_s), \mathbb{E}[Y_s^q + \lambda\varepsilon(\mathcal{Y}_s^\varepsilon + \widehat{Y}_s)], u\big).$

From (3.20) and (3.24) we have

$$\begin{cases} d\mathcal{Y}_t^\varepsilon = -\int_0^1 \left(\int_U f_y\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathcal{Y}_t^\varepsilon + \mathbb{E}\left[\int_U f_{y'}\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathbb{E}[\mathcal{Y}_t^\varepsilon]\right] + \Psi_t^\varepsilon\right) d\lambda dt + \mathcal{Z}_t^\varepsilon dW_t \\ \\ \mathcal{Y}_T^\varepsilon = \frac{1}{\varepsilon}\left(g\left(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]\right) - g\left(X_T^q, \mathbb{E}[X_T^q]\right)\right) - \left(g_{x'}\left(X_T^q, \mathbb{E}[X_T^q]\right)\widehat{X}_T + \mathbb{E}\left[g_{x'}\left(X_T^q, \mathbb{E}[X_T^q]\right)\mathbb{E}[\widehat{X}_T]\right]\right), \end{cases} \tag{3.27}$$

where

$$\begin{aligned} \Psi_t^\varepsilon = & \int_U f_x\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathcal{X}_t^\varepsilon + \mathbb{E}\left[\int_U f_{x'}\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathbb{E}[\mathcal{X}_t^\varepsilon]\right] \\ & + \int_U f_x\left(t, \Delta_t^\varepsilon, u\right) \mu_t(du)\left(X_t^\varepsilon - X_t^q\right) + \mathbb{E}\left[\int_U f_{x'}\left(t, \Delta_t^\varepsilon, u\right) \mu_t(du)\mathbb{E}[(X_t^\varepsilon - X_t^q)]\right] \\ & + \int_U f_y\left(t, \Delta_t^\varepsilon, u\right) \mu_t(du)\left(Y_t^\varepsilon - Y_t^q\right) + \mathbb{E}\left[\int_U f_{y'}\left(t, \Delta_t^\varepsilon, u\right) \mu_t(du)\mathbb{E}[(Y_t^\varepsilon - Y_t^q)]\right] \\ & -\Big(\int_U f_x\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\left(X_t^\varepsilon - X_t^q\right) + \mathbb{E}\left[\int_U f_{x'}\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathbb{E}[(X_t^\varepsilon - X_t^q)]\right] \\ & - \int_U f_y\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\left(Y_t^\varepsilon - Y_t^q\right) + \mathbb{E}\left[\int_U f_{y'}\left(t, \Delta_t^\varepsilon, u\right) q_t(du)\mathbb{E}[(Y_t^\varepsilon - Y_t^q)]\right]\Big). \end{aligned}$$

Using the fact that the derivatives $f_x, f_{x'}, f_y, f_{y'}$ are continuous and bounded and from (3.13),(3.14) and (3.21), we show

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\Psi_t^\varepsilon|^2\right] = 0. \tag{3.28}$$

Applying Itô's formula to $|\mathcal{Y}_t^\varepsilon|^2$ we obtain

$$\begin{aligned} \mathbb{E}\left[|\mathcal{Y}_t^\varepsilon|^2\right] + \mathbb{E}\left[\int_t^T \|\mathcal{Z}_s^\varepsilon\|^2 ds\right] = & \mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right] + 2\mathbb{E}\left[\int_t^T\int_0^1 \langle \mathcal{Y}_s^\varepsilon, \int_U f_y\left(s, \Delta_s^\varepsilon, u\right) q_s(du)\mathcal{Y}_s^\varepsilon\right. \\ & \left. + \mathbb{E}\left[\int_U f_{y'}\left(s, \Delta_s^\varepsilon, u\right) q_s(du)\mathbb{E}[\mathcal{Y}_s^\varepsilon]\right] + \Psi_s^\varepsilon\rangle d\lambda ds\right]. \end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\left[|\mathcal{Y}_t^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T\|\mathcal{Z}_s^\varepsilon\|^2ds\right]&\leq\mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T|\mathcal{Y}_s^\varepsilon|^2ds\right]\\&+3\mathbb{E}\left[\int_t^T\int_0^1\left(\left|\int_U f_y\left(s,\Delta_s^\varepsilon,u\right)q_s(du)\mathcal{Y}_s^\varepsilon\right|^2\right.\right.\\&\left.\left.+\left|\mathbb{E}\left[\int_U f_y\left(s,\Delta_s^\varepsilon,u\right)q_s(du)\mathbb{E}[\mathcal{Y}_s^\varepsilon]\right]\right|^2+|\Psi_s^\varepsilon|^2\right)d\lambda ds\right].\end{aligned}$$

Since $f_y, f_{y'}$ are bounded we get

$$\mathbb{E}\left[|\mathcal{Y}_t^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T\|\mathcal{Z}_s^\varepsilon\|^2ds\right]\leq C\mathbb{E}\left[\int_t^T|\mathcal{Y}_s^\varepsilon|^2ds\right]+\left(\mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right]+C\mathbb{E}\left[\int_t^T\int_0^1|\Psi_s^\varepsilon|^2\,d\lambda ds\right]\right).$$

We derive from this inequality two inequalities

$$\mathbb{E}\left[|\mathcal{Y}_t^\varepsilon|^2\right]\leq C\mathbb{E}\left[\int_t^T|\mathcal{Y}_s^\varepsilon|^2ds\right]+\left(\mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right]+C\mathbb{E}\left[\int_t^T\int_0^1|\Psi_s^\varepsilon|^2\,d\lambda ds\right]\right).\tag{3.29}$$

$$\mathbb{E}\left[\int_t^T\|\mathcal{Z}_s^\varepsilon\|^2ds\right]\leq C\mathbb{E}\left[\int_t^T|\mathcal{Y}_s^\varepsilon|^2ds\right]+\left(\mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right]+C\mathbb{E}\left[\int_t^T\int_0^1|\Psi_s^\varepsilon|^2\,d\lambda ds\right]\right).\tag{3.30}$$

On the other hand we have

$$\begin{aligned}\mathbb{E}\left[|\mathcal{Y}_T^\varepsilon|^2\right]&=\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left(g\left(X_T^\varepsilon,\mathbb{E}[X_T^\varepsilon]\right)-g\left(X_T^q,\mathbb{E}[X_T^q]\right)\right)\right.\right.\\&\left.\left.-\left(g_{x'}\left(X_T^q,\mathbb{E}[X_T^q]\right)\widehat{X}_T+\mathbb{E}\left[g_{x'}\left(X_T^q,\mathbb{E}[X_T^q]\right)\mathbb{E}[\widehat{X}_T]\right]\right)\right|^2\right]\\&\leq 2\mathbb{E}\left[\int_0^1\left|\left(g_x\left(\Lambda_T^\varepsilon\right)-g_x\left(X_T^q,\mathbb{E}[X_T^q]\right)\right)\widehat{X}_T\right|^2d\lambda\right]\\&+2\mathbb{E}\left[\int_0^1\left|\mathbb{E}\left[\left(g_{x'}\left(\Lambda_T^\varepsilon\right)-g_{x'}\left(X_T^q,\mathbb{E}[X_T^q]\right)\right)\mathbb{E}\left[\widehat{X}_T\right]\right]\right|^2d\lambda\right]\\&+2\mathbb{E}\left[\int_0^1\left|g_x\left(\Lambda_T^\varepsilon\right)\mathcal{X}_T^\varepsilon+\mathbb{E}\left[g_{x'}\left(\Lambda_T^\varepsilon\right)\mathbb{E}\left[\mathcal{X}_T^\varepsilon\right]\right]\right|^2d\lambda\right].\end{aligned}$$

Since g_x, g_x' are continuous and bounded, using (3.21) to get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\mathcal{Y}_t^\varepsilon|^2\right]=0.\tag{3.31}$$

Now, applying Gronwall's lemma, Burkholder-Davis-Gundy inequality and using (3.28) and (3.31) to obtain (3.22) and from (3.22), (3.28) and (3.31) we get (3.23). ∎

Proposition 3.4.3 (Variational inequality) *Let (H3.1) hold. Let $q.$ be an optimal relaxed control with associated trajectories $(X_t^{q.}, Y_t^{q.}, Z_t^{q.})$. Then, for any element $\mu.$ of $\mathcal{R}$, we have*

$$\begin{aligned}
0 \leq & \mathbb{E}\left[l_x(X_T^q, \mathbb{E}[X_T^q])\widehat{X}_T\right] + \mathbb{E}\left[l_{x'}(X_T^q, \mathbb{E}\left[X_T^q\right])\mathbb{E}[\widehat{X}_T]\right] \\
& +\mathbb{E}\left[k_y(Y_0^q, \mathbb{E}[Y_0^q])\widehat{Y}_0\right] + \mathbb{E}\left[k_{y'}(Y_0^q, \mathbb{E}[Y_0^q])\mathbb{E}[\widehat{Y}_0]\right] \\
& +\mathbb{E}\big[\int_0^T \int_U \big(h_x(t, \Delta_t^q, u)\widehat{X}_t + \mathbb{E}\left[h_{x'}(t, \Delta_t^q, u)\mathbb{E}[\widehat{X}_t]\right] \\
& +h_y(t, \Delta_t^q, u)\widehat{Y}_t + \mathbb{E}\left[h_{y'}(t, \Delta_t^q, u)\mathbb{E}[\widehat{Y}_t]\right]\big)q_t(da)dt\big] \\
& +\mathbb{E}\big[\int_0^T \big(\int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)\mu_t(du) \\
& -\int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(da)\big)dt\big].
\end{aligned} \tag{3.32}$$

Proof. From the optimality of $q.$ we have

$$\begin{aligned}
0 \leq & \mathbb{E}\left[l(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - l(X_T^q, \mathbb{E}[X_T^q])\right] + \mathbb{E}\left[k(Y_0^\varepsilon, \mathbb{E}[Y_0^\varepsilon]) - k(Y_0^q, \mathbb{E}[Y_0^q])\right] \\
& +\mathbb{E}\big[\int_0^T \big(\int_U h(t, X_t^\varepsilon, \mathbb{E}[X_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], u)q_t^\varepsilon(du) - \int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(da)\big)dt\big].
\end{aligned}$$

Using the definition of q_t^ε we get

$$\begin{aligned}
0 \leq & \mathbb{E}\left[l(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - l(X_T^q, \mathbb{E}[X_T^q])\right] + \mathbb{E}\left[k(Y_0^\varepsilon, \mathbb{E}[Y_0^\varepsilon]) - k(Y_0^q, \mathbb{E}[Y_0^q])\right] \\
& +\varepsilon\,\mathbb{E}\big[\int_0^T \big(\int_U h(t, X_t^\varepsilon, \mathbb{E}[X_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], u)\mu_t(du) - \int_U h(t, X_t^\varepsilon, \mathbb{E}[X_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], u)q_t(da)\big)dt\big] \\
& +\mathbb{E}\big[\int_0^T \int_U \big(h(t, X_t^\varepsilon, \mathbb{E}[X_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], u) - h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)\big)q_t(da)dt\big].
\end{aligned}$$

Thus

$$
\begin{aligned}
0 \leq \mathbb{E}\left[\int_0^1 \left(l_x(\Lambda_T^\varepsilon)\widehat{X}_T + \mathbb{E}\left[l_{x'}(\Lambda_T^\varepsilon)\mathbb{E}[\widehat{X}_T]\right]\right) d\lambda\right] + \mathbb{E}\left[\int_0^1 \left(k_y(\Lambda_0^\varepsilon)\widehat{Y}_0 + \mathbb{E}\left[k_{y'}(\Lambda_0^\varepsilon)\mathbb{E}[\widehat{Y}_0]\right]\right)\right] \\
+\mathbb{E}\Big[\int_0^T\int_0^1\int_U \big(h_x(t,\Delta_t^\varepsilon,u)\widehat{X}_t + \mathbb{E}\left[h_{x'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\widehat{X}_t]\right] \\
+h_y(t,\Delta_t^\varepsilon,u)\widehat{Y}_t + \mathbb{E}\left[h_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\widehat{Y}_t]\right]\big)q_t(da)d\lambda dt\Big] \\
+\mathbb{E}\Big[\int_0^T \big(\int_U h(t,X_t^\varepsilon,\mathbb{E}[X_t^\varepsilon],Y_t^\varepsilon,\mathbb{E}[Y_t^\varepsilon],u) - h(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],u)\big)q_t(da)dt\Big] + \Xi_t^\varepsilon,
\end{aligned}
\tag{3.33}
$$

where Ξ_t^ε is given by

$$
\begin{aligned}
\Xi_t^\varepsilon = \mathbb{E}\left[\int_0^1 (l_x(\Lambda_T^\varepsilon)\mathcal{X}_T^\varepsilon + \mathbb{E}\left[l_{x'}(\Lambda_T^\varepsilon)\mathbb{E}[\mathcal{X}_T^\varepsilon]\right]) d\lambda\right] + \mathbb{E}\left[\int_0^1 (k_y(\Lambda_0^\varepsilon)\mathcal{Y}_0^\varepsilon + \mathbb{E}\left[k_{y'}(\Lambda_0^\varepsilon)\mathbb{E}[\mathcal{Y}_0^\varepsilon]\right]) d\lambda\right] \\
+\mathbb{E}\Big[\int_0^T\int_0^1\int_U \big(h_x(t,\Delta_t^\varepsilon,u)(X_t^\varepsilon - X_t^q) + \mathbb{E}\left[h_{x'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[(X_t^\varepsilon - X_t^q)]\right] \\
+h_y(t,\Delta_t^\varepsilon,u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}\left[h_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[(Y_t^\varepsilon - Y_t^q)]\right]\big)\mu_t(da)d\lambda dt\Big] \\
-\mathbb{E}\Big[\int_0^T\int_0^1\int_U \big(h_x(t,\Delta_t^\varepsilon,u)(X_t^\varepsilon - X_t^q) + \mathbb{E}\left[h_{x'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[(X_t^\varepsilon - X_t^q)]\right] \\
+h_y(t,\Delta_t^\varepsilon,u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}\left[h_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[(Y_t^\varepsilon - Y_t^q)]\right]\big)q_t(da)d\lambda dt\Big] \\
+\mathbb{E}\Big[\int_0^T\int_0^1\int_U \big(h_x(t,\Delta_t^\varepsilon,u)\mathcal{X}_t^\varepsilon + \mathbb{E}\left[h_{x'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\mathcal{X}_t^\varepsilon]\right] + h_y(t,\Delta_t^\varepsilon,u)\mathcal{Y}_t^\varepsilon \\
+\mathbb{E}\left[h_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\mathcal{Y}_t^\varepsilon]\right]\big)q_t(da)d\lambda dt\Big].
\end{aligned}
$$

Since the derivatives $l_x, l_{x'}, k_y, k_{y'}, h_x, h_{x'}, h_y, h_{y'}$ are continuous and bounded, then by using (3.13), (3.14), (3.21), (3.22) and the Cauchy-Schwartz inequality we show that

$$
\lim_{\varepsilon\to 0}\mathbb{E}\left[|\Xi_t^\varepsilon|^2\right] = 0.
$$

Then let ε go to 0 in (3.33), we get the variational inequality. ■

3.4.1.2 Necessary optimality conditions for relaxed control

Let us introduce the adjoint equations of the MF-FBSDE (3.10) and then gives the maximum principle.

Define the Hamiltonian H from $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times U\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m$ to $\mathbb{R}$ by

$$\begin{aligned}H(t,x,\mathbb{E}[x],y,\mathbb{E}[y],\mu,P,K,Q) &:= -P\int_U b(t,x,\mathbb{E}[x],u)\mu(du) + \ K\sigma(t,x,\mathbb{E}[x])\\ &+Q\int_U f(t,x,\mathbb{E}[x],y,\mathbb{E}[y],u)\mu(du) + \int_U h(t,x,\mathbb{E}[x],y,\mathbb{E}[y],u)\mu(du).\end{aligned} \tag{3.34}$$

Theorem 3.4.4 *(Necessary optimality conditions for relaxed control) Assume that* $(\mathbf{H3.1})$-$(\mathbf{H3.4})$ *hold. Let* $q. \in \mathcal{R}$ *an optimal relaxed control. Let* (X^q, Y^q, Z^q) *be the associated solution of MF-FBSDE (3.10). Then there exists a unique solution* (P^q, K^q, Q^q) *of the following adjoint equations of MF-FBSDE (3.10):*

$$\begin{cases} dP_t^q = -\big(H_x(t,\phi_t^q) + \mathbb{E}\big[H_{x'}(t,\phi_t^q)\big]\big)dt + K_t^q dW_t,\\ dQ_t^q = \big(H_y(t,\phi_t^q) + \mathbb{E}\big[H_{y'}(t,\phi_t^q)\big]\big)dt + \big(H_z(t,\phi_t^q) + \mathbb{E}\big[H_{z'}(t,\phi_t^q)\big]\big)dW_t,\\ \\ Q_0^q = k_y(Y_0^q,\mathbb{E}[Y_0^q]) + \mathbb{E}\big[k_{y'}(Y_0^q,\mathbb{E}[Y_0^q])\big],\\ P_T^q = l_x(X_T^q,\mathbb{E}[X_T^q]) + \mathbb{E}\big[l_{x'}(X_T^q,\mathbb{E}[X_T^q])\big] + g_x(X_T^q,\mathbb{E}[X_T^q])Q_T^q + \mathbb{E}\big[g_{x'}(X_T^q,\mathbb{E}[X_T^q])\mathbb{E}[Q_T^q]\big], \end{cases} \tag{3.35}$$

such that

$$\begin{aligned}&H(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],q_t,P_t^q,K_t^q,Q_t^q)\\ &\quad\le H(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],\mu_t,P_t^q,K_t^q,Q_t^q),\ a.e.\,t,\ \mathbb{P}-a.s.,\ \forall\mu\in\mathbb{P}(U),\end{aligned} \tag{3.36}$$

where $(t,\phi_t^q) := (t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],q_t,P_t^q,K_t^q,Q_t^q)$.

Proof. By the values of Q_0^q and P_T^q in (3.35), the inequality variational (3.32) becomes

$$
\begin{aligned}
0 \leq \mathbb{E}\left[\langle P_T^q, \widehat{X}_T\rangle\right] &- \mathbb{E}\left[g_x(X_T^q, \mathbb{E}[X_T^q])Q_T^q + \mathbb{E}\big[g_{x'}(X_T^q, \mathbb{E}[X_T^q])\mathbb{E}[Q_T^q]\big]\right] \\
+\mathbb{E}\left[\langle Q_0^q, \widehat{Y}_0\rangle\right] &+ \mathbb{E}\Big[\int_0^T \int_U \big(h_x(t, \Delta_t^q, u)\widehat{X}_t + \mathbb{E}\left[h_{x'}(t, \Delta_t^q, u)\mathbb{E}[\widehat{X}_t]\right] \\
&\qquad + h_y(t, \Delta_t^q, u)\widehat{Y}_t + \mathbb{E}\left[h_{y'}(t, \Delta_t^q, u)\mathbb{E}[\widehat{Y}_t]\right]\big) q_t(da)dt\Big] \\
+\mathbb{E}\Big[\int_0^T \Big(\int_U & h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)\mu_t(du) \\
&- \int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(da)\Big)dt\Big]. \qquad (3.37)
\end{aligned}
$$

Now applying Itô's formula to compute $\langle P_t^q, \widehat{X}_t\rangle$ and $\langle Q_t^q, \widehat{Y}_t\rangle$ and taking the expectations we derive

$$
\begin{aligned}
\mathbb{E}\big[\langle P_T^q, \widehat{X}_T\rangle\big] = &- \mathbb{E}\Big[\int_0^T \langle Q_t^q \int_U \big(f_x(t, \Delta_t^q, u) + \mathbb{E}\big[f_{x'}(t, \Delta_t^q, u)\big]\big)q_t(du) \\
&+ \int_U \big(h_x(t, \Delta_t^q, u) + \mathbb{E}\big[h_{x'}(t, \Delta_t^q, u)\big]\big)q_t(du), \widehat{X}_t\rangle dt\Big] \\
&+ \mathbb{E}\Big[\int_0^T P_t^q\Big(\int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du) - \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)\mu_t(du)\Big)dt\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
-\mathbb{E}\big[\langle Q_0^q, \widehat{Y}_0\rangle\big] = \mathbb{E}\big[\langle Q_T^q, \widehat{Y}_T\rangle\big] &+ \mathbb{E}\Big[\int_0^T \langle Q_t^q, \int_U \big(f_x(t, \Delta_t^q, u)\widehat{X}_t + \mathbb{E}\big[f_{x'}(t, \Delta_t^q, u)\mathbb{E}[\widehat{X}_t]\big]\big)q_t(du)\rangle dt\Big] \\
&- \mathbb{E}\Big[\int_0^T \langle \int_U \big(h_y(t, \Delta_t^q, u) + \mathbb{E}\big[h_{y'}(t, \Delta_t^q, u)\big]\big)q_t(du), \widehat{Y}_t\rangle dt\Big] \\
&+ \mathbb{E}\Big[\int_0^T Q_t^q\Big(\int_U f(t, \Delta_t^q, u)q_t(du) - \int_U f(t, \Delta_t^q, u)\mu_t(du)\Big)dt\Big].
\end{aligned}
$$

Then for every $\mu \in \mathcal{R}$, the inequality (3.37) becomes

$$
0 \leq \mathbb{E}\Big[\int_0^T \big(H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], q_t, P_t^q, K_t^q, Q_t^q) - H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], \mu_t, P_t^q, K_t^q, Q_t^q)\big)dt\Big].
$$

Therefore inequality (3.36) follows by a standard arguments. ■

3.4.1.3 Sufficient optimality conditions for relaxed control

In this subsection we study when the necessary conditions for optimality in Theorem 3.4.4 become sufficient as well.

Theorem 3.4.5 *(Sufficient optimality conditions for relaxed control) Assume that* $(\mathbf{H3.1})$-$(\mathbf{H3.4})$ *hold. Given* $q. \in \mathcal{R}$*, let* (X^q, Y^q, Z^q) *and* (P^q, K^q, Q^q) *be the corresponding solutions of the MF-FBSDEs (3.10) and (3.35) respectively. Suppose that* l, k, h *and the function* $H(t, \cdot, \cdot, \cdot, \cdot, q_t, P_t^q, K_t^q, Q_t^q)$ *are convex. Then* $(X^q, Y^q, Z^q, q.)$ *is an optimal solution of the control problem* $\{(3.10), 3.11), (3.12)\}$ *if it satisfies (3.36).*

Proof. Let $q. \in \mathcal{R}$ be arbitrary (candidate to be optimal), and let (X^q, Y^q, Z^q) denote the trajectory associated to $q.$. For any $\mu. \in \mathcal{R}$ with associated trajectory (X^μ, Y^μ, Z^μ), we have

$$J(\mu.) - J(q.) = \mathbb{E}\big[l(X_T^\mu, \mathbb{E}[X_T^\mu]) - l(X_T^q, \mathbb{E}[X_T^q])\big] + \mathbb{E}\big[k(Y_0^\mu, \mathbb{E}[Y_0^\mu]) - k(Y_0^q, \mathbb{E}[Y_0^q])\big]$$
$$+ \mathbb{E}\Big[\int_0^T \Big(\int_U h(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], u)\mu_t(du) - \int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(du)\Big)dt\Big].$$

Since l and k are convex, we get

$$l(X_T^\mu, \mathbb{E}[X_T^\mu]) - l(X_T^q, \mathbb{E}[X_T^q]) \geq \langle l_x(X_T^q, \mathbb{E}[X_T^q]), X_T^\mu - X_T^q\rangle + \mathbb{E}\big[\langle l_{x'}(X_T^q, \mathbb{E}[X_T^q]), \mathbb{E}[X_T^\mu - X_T^q]\rangle\big],$$

$$k(Y_0^\mu, \mathbb{E}[Y_0^\mu]) - k(Y_0^q, \mathbb{E}[Y_0^q]) \geq \langle k_y(Y_0^q, \mathbb{E}[Y_0^q]), Y_0^\mu - Y_0^q\rangle + \mathbb{E}\big[\langle k_{y'}(Y_0^q, \mathbb{E}[Y_0^q]), \mathbb{E}[Y_0^\mu - Y_0^q]\rangle\big].$$

Thus

$$J(\mu.) - J(q.) \geq \langle l_x(X_T^q, \mathbb{E}[X_T^q]), X_T^\mu - X_T^q\rangle + \mathbb{E}\big[\langle l_{x'}(X_T^q, \mathbb{E}[X_T^q]), \mathbb{E}[X_T^\mu - X_T^q]\rangle\big]$$
$$+ \langle k_y(Y_0^q, \mathbb{E}[Y_0^q]), Y_0^\mu - Y_0^q\rangle + \mathbb{E}\big[\langle k_{y'}(Y_0^q, \mathbb{E}[Y_0^q]), \mathbb{E}[Y_0^\mu - Y_0^q]\rangle\big]$$
$$+ \mathbb{E}\Big[\int_0^T \Big(\int_U h(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], u)\mu_t(du) - \int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(du)\Big)dt\Big].$$

Therefore after recalling also (3.35) one gets

$$J(\mu.) - J(q.) \geq \mathbb{E}\big[\langle P_T^q, X_T^\mu - X_T^q\rangle\big] - \mathbb{E}\big[\langle g_x(X_T^q, \mathbb{E}[X_T^q])Q_T^q + \mathbb{E}\big[g_{x'}(X_T^q, \mathbb{E}[X_T^q])\mathbb{E}[Q_T^q]\big], X_T^\mu - X_T^q\rangle\big]$$
$$+ \mathbb{E}\big[\langle Q_0^q, Y_0^\mu - Y_0^q\rangle\big] + \mathbb{E}\Big[\int_0^T \Big(\int_U h(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], u)\mu_t(du)$$
$$- \int_U h(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(du)\Big)dt\Big]. \tag{3.38}$$

Applying Itô's formula to $\langle P_t^q, X_t^\mu - X_t^q\rangle$ and $\langle Q_t^q, Y_t^\mu - Y_t^q\rangle$, and take the expectations to obtain

$$\mathbb{E}\big[\,\langle P_T^q, X_T^\mu - X_T^q\rangle\,\big] = \mathbb{E}\Big[\int_0^T \langle P_t^q, \int_U b(t, X_t^\mu, \mathbb{E}[X_t^\mu], u)\mu_t(du) - \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du)\rangle dt\Big]$$
$$+\mathbb{E}\Big[\int_0^T \langle K_t^q, \sigma(t, X_t^\mu, \mathbb{E}[X_t^\mu]) - \sigma(t, X_t^q, \mathbb{E}[X_t^q])\rangle\, dt\Big]$$
$$-\mathbb{E}\Big[\int_0^T \langle H_x(t, \phi_t^q) + \mathbb{E}[H_{x'}(t, \phi_t^q)], X_t^\mu - X_t^q\rangle dt\Big], \tag{3.39}$$

and

$$\mathbb{E}\big[\,\langle Q_0^q, Y_0^\mu - Y_0^q\rangle\,\big] = \mathbb{E}\big[\,\langle Q_T^q, Y_T^\mu - Y_T^q\rangle\,\big] - \mathbb{E}\Big[\int_0^T \langle H_y(t, \phi_t^q) + \mathbb{E}[H_{y'}(t, \phi_t^q)], Y_t^\mu - Y_t^q\rangle dt\Big]$$
$$+ \mathbb{E}\Big[\int_0^T \langle Q_t^q, \int_U f(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], u)\mu_t(du)$$
$$- \int_U f(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], u)q_t(du)\rangle dt\Big]. \tag{3.40}$$

By using (3.39) and (3.40) in inequality (3.38) and apply the fact that (3.34), we get

$$J(\mu.) - J(q.) \geq \mathbb{E}\Big[\int_0^T \big(H(t, \phi_t^\mu) - H(t, \phi_t^q)\big)dt\Big] - \mathbb{E}\Big[\int_0^T \langle H_x(t, \phi_t^q) + \mathbb{E}[H_{x'}(t, \phi_t^q)], X_t^\mu - X_t^q\rangle dt\Big]$$
$$- \mathbb{E}\Big[\int_0^T \langle H_y(t, \phi_t^q) + \mathbb{E}[H_{y'}(t, \phi_t^q)], Y_t^\mu - Y_t^q\rangle dt\Big]. \tag{3.41}$$

Since g is convex we have

$$\mathbb{E}\big[\,\big\langle Q_T^q\big(g_x(X_T^q, \mathbb{E}[X_T^q]) + \mathbb{E}\big[g_{x'}(X_T^q, \mathbb{E}[X_T^q])\big]\big), X_T^\mu - X_T^q\big\rangle\,\big]$$
$$\leq \mathbb{E}\big[Q_T^q\big(g(X_T^\mu, \mathbb{E}[X_T^\mu]) - g(X_T^q, \mathbb{E}[X_T^q])\big)\big]$$
$$= \mathbb{E}\big[Y_T^\mu - Y_T^q\big].$$

On the other hand, by the convexity of $H(t,x,x',y,y',q,P,K,Q)$ in (x,x',y,y') and its linearity in q, then by using the clarke generalized gradient of H evaluated at (x,x',y,y'), we obtain

$$H(t,\phi_t^{\mu})-H(t,\phi_t^{q})\geq H_x(t,\phi_t^{q})\big(X_t^{\mu}-X_t^{q}\big)+\mathbb{E}\big[H_{x'}(t,\phi_t^{q})\mathbb{E}[X_t^{\mu}-X_t^{q}]\big]+H_y(t,\phi_t^{q})\big(Y_t^{\mu}-Y_t^{q}\big)$$
$$+\mathbb{E}\big[H_{y'}(t,\phi_t^{q})\mathbb{E}[Y_t^{\mu}-Y_t^{q}]\big].$$

Therefore, applying this inequality in (3.41) gives

$$J(\mu.)-J(q.)\geq 0,\forall\mu\in\mathcal{R}.$$

The theorem is proved. ■

3.4.2 Necessary and sufficient optimality conditions for strict control

In this part, we shall derive necessary and sufficient optimality condition for strict control problem and shows that it follows from the relaxed one. This strict control problem is governed by the following MF-FBSDE

$$\begin{cases} X_t^v=x+\int_0^t b(s,X_s^v,\mathbb{E}[X_s^v],v_s)ds+\int_0^t\sigma(s,X_s^v,\mathbb{E}[X_s^v])dW_s \\ \\ Y_t^v=g(X_T^v,\mathbb{E}[X_T^v])+\int_t^T f(s,X_s^v,\mathbb{E}[X_s^v],Y_s^v,\mathbb{E}[Y_s^v],v_s)ds-\int_t^T Z_s^v dW_s, \end{cases} \tag{3.42}$$

and the functional cost to be minimize over the set of admissible strict controls $\mathcal{U}^w$ is given by

$$J(v.):=\mathbb{E}\big[l\left(X_T^v,\mathbb{E}\left[X_T^v\right]\right)+k\left(Y_0^v,\mathbb{E}\left[Y_0^v\right]\right)+\int_0^T h\big(t,X_t^v,\mathbb{E}\left[X_t^v\right],Y_t^v,\mathbb{E}\left[Y_t^v\right],v_t\big)dt\big]. \tag{3.43}$$

We say that a strict control $u.$ is an optimal control if

$$J(u.)=\inf_{v.\in\mathcal{U}^w}J(v.). \tag{3.44}$$

We denote by

$$\mathcal{R}_\delta = \{\mu_. \in \mathcal{R} / \;\; \mu = \delta_v : v \in \mathcal{U}^w\},$$

the set of all relaxed controls in the form of Dirac measure charging a strict control. Denote by $\mathbb{P}(U_\delta)$ the action set of all relaxed control $\mathcal{R}_\delta$.

3.4.2.1 Necessary optimality conditions for strict control

Define the Hamiltonian $\mathcal{H}$ in the strict control problem from $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^m$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{H}(t,x,\mathbb{E}[x],y,\mathbb{E}[y],v,P,K,Q) &:= -Pb(t,x,\mathbb{E}[x],v) + \;K\sigma(t,x,\mathbb{E}[x]) \\ &+Qf(t,x,\mathbb{E}[x],y,\mathbb{E}[y],v) + h(t,x,\mathbb{E}[x],y,\mathbb{E}[y],v). \end{aligned} \tag{3.45}$$

Theorem 3.4.6 *(Necessary optimality conditions for strict control) Let $u_. \in \mathcal{U}^w$ an optimal strict control. Let (X^u, Y^u, Z^u) be the associated solution of MF-FBSDE (3.42). Then there exists a unique solution (P^u, K^u, Q^u) of the following adjoint equations of MF-FBSDE (3.42):*

$$\begin{cases} dP_t^u = -\big(\mathcal{H}_x(t,\phi_t^u) + \mathbb{E}\big[\mathcal{H}_{x'}(t,\phi_t^u)\big]\big)dt + K_t^u dW_t, \\ dQ_t^u = \big(\mathcal{H}_y(t,\phi_t^u) + \mathbb{E}\big[\mathcal{H}_{y'}(t,\phi_t^u)\big]\big)dt + \big(\mathcal{H}_z(t,\phi_t^u) + \mathbb{E}\big[\mathcal{H}_{z'}(t,\phi_t^u)\big]\big)dW_t, \\ \\ Q_0^u = k_y(Y_0^u, \mathbb{E}[Y_0^u]) + \mathbb{E}\big[k_{y'}(Y_0^u, \mathbb{E}[Y_0^u])\big], \\ P_T^u = l_x(X_T^u, \mathbb{E}[X_T^u]) + \mathbb{E}\big[l_{x'}(X_T^u, \mathbb{E}[X_T^u])\big] + g_x(X_T^u, \mathbb{E}[X_T^u])Q_T^u + \mathbb{E}\big[g_{x'}(X_T^u, \mathbb{E}[X_T^u])\mathbb{E}[Q_T^u]\big], \end{cases} \tag{3.46}$$

such that

$$\mathcal{H}(t, X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], u_t, P_t^u, K_t^u, Q_t^u) \tag{3.47}$$
$$\leq \mathcal{H}(t, X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], v_t, P_t^u, K_t^u, Q_t^u), \ a.e.\, t, \ \mathbb{P}-a.s., \ \forall v \in \mathcal{U}^w,$$

where $\phi_t^u := (X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], u_t, P_t^u, K_t^u, Q_t^u)$.

Proof. Note that the strict control $u.$ embedded into the space $\mathbb{V}$ in the sense that $u.$ is corresponding with the Dirac measure $\lambda_{u.}(dt, da) = \delta_{u.}(du)$ with the propriety: For any bounded and uniformly continuous function $\varphi(t, x, x', y, y', u)$ we have

$$\varphi(t, x, x', y, y', u_t) = \int_U \varphi(t, x, x', y, y', u)\delta_{u_t}(du) := \overline{\varphi}(t, x, x', y, y', \lambda_u). \tag{3.48}$$

Hence by the necessary optimality condition for relaxed controls (Theorem 3.4.4), there exist a unique solution (P_t^q, K_t^q, Q_t^q) of (3.35) such that

$$H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], q_t, P_t^q, K_t^q, Q_t^q)$$
$$\leq H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], \mu_t, P_t^q, K_t^q, Q_t^q), \ a.e.\, t, \ \mathbb{P}-a.s., \ \forall \mu \in \mathcal{R},$$

and since $\mathcal{R}_\delta \subset \mathcal{R}$ we have

$$H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], q_t, P_t^q, K_t^q, Q_t^q) \tag{3.49}$$
$$\leq H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], \mu_t, P_t^q, K_t^q, Q_t^q), \ a.e.\, t, \ \mathbb{P}-a.s., \ \forall \mu \in \mathcal{R}_\delta.$$

Using the fact that if $\mu \in \mathcal{R}_\delta$, then there exist $v_t \in U_\delta \subset U$ such that $\mu = \delta_v$, and we have proved in section 3.2 (Proposition 3.3.1) that the optimal relaxed control $q_t(du) = \delta_{u_t}(du)$ with u_t an optimal strict control, then we can show that

$$
\begin{aligned}
&(X_t^q, Y_t^q, Z_t^q) = (X_t^u, Y_t^u, Z_t^u), \qquad (X_t^\mu, Y_t^\mu, Z_t^\mu) = (X_t^v, Y_t^v, Z_t^v),\\
&(P_t^q, K_t^q, Q_t^q) = (P_t^u, K_t^u, Q_t^u), \qquad (P_t^\mu, K_t^\mu, Q_t^\mu) = (P_t^v, K_t^v, Q_t^v), \\
&H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], q_t, P_t^q, K_t^q, Q_t^q) = \mathcal{H}(t, X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], u_t, P_t^u, K_t^u, Q_t^u),\\
&H(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], \mu_t, P_t^\mu, K_t^\mu, Q_t^\mu) = \mathcal{H}(t, X_t^v, \mathbb{E}[X_t^v], Y_t^v, \mathbb{E}[Y_t^v], v_t, P_t^v, K_t^v, Q_t^v).
\end{aligned}
\tag{3.50}
$$

Using (3.49) and (3.50) we get (3.47). The proof is completed. ■

3.4.2.2 Sufficient optimality conditions for strict control

We shall try to shows if the necessary optimality conditions for strict control (3.47) becomes sufficient.

Theorem 3.4.7 *(Sufficient optimality conditions for strict control) Assume that the function l, k, h and $\mathcal{H}(t, \cdot, \cdot, \cdot, \cdot, u_t, P_t^u, K_t^u, Q_t^u)$ are convex. Then $(X_\cdot^u, Y_\cdot^u, Z_\cdot^u, u_\cdot)$ is an optimal solution of the strict control problem $\{(3.42), (3.43), (3.44)\}$ if it satisfies (3.47).*

Proof. Let u_t be an arbitrary element of U_δ such that the necessary optimality conditions for strict control (3.47) hold, i.e.

$$
\begin{aligned}
&\mathcal{H}(t, X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], u_t, P_t^u, K_t^u, Q_t^u)\\
&\quad \leq \mathcal{H}(t, X_t^u, \mathbb{E}[X_t^u], Y_t^u, \mathbb{E}[Y_t^u], v_t, P_t^u, K_t^u, Q_t^u), \ a.e.\, t, \ \mathbb{P}-a.s., \ \forall v \in \mathcal{U}_\delta,
\end{aligned}
$$

and by applying the embedding mentioned in (3.48), one can show that

$$
\begin{aligned}
&H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], q_t, P_t^q, K_t^q, Q_t^q)\\
&\quad \leq H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], \mu_t, P_t^q, K_t^q, Q_t^q), \ a.e.\, t, \ \mathbb{P}-a.s., \ \forall \mu \in \mathcal{R}_\delta.
\end{aligned}
$$

Thus by sufficient optimality conditions for relaxed control (Theorem 3.4.5) we have

$$J(q.) = \inf_{\mu. \in \mathcal{R}_\delta} J(\mu.),$$

and from Proposition 3.3.1, we have proved that the optimal relaxed control is a Dirac measure charging in optimal strict control ($q_t(du) = \delta_{u_t}(du)$), then we can show that

$$J(u.) = \inf_{v. \in \mathcal{U}^w} J(v.).$$

The prove is completed. ■

CHAPTER 4

Existence of optimal solutions for optimal control problems of MF-FBSDEs systems with controlled diffusion

CHAPTER 4

Existence of optimal solutions for optimal control problems of MF-FBSDEs systems with controlled diffusion

In this chapter, we prove the existence of optimal controls for systems governed by forward-backward stochastic differential equations of mean-field type (MF-FBSDEs) with controlled diffusion, in which the coefficients depend not only on the state process, but also on the distribution of the state process, via the expectation of some function of the state. Moreover the cost functional is also of mean-field type. We prove this result of existence by using the weak convergence techniques for the associated MF-FBSDEs on the space of continuous functions and on the space of cádlág functions endowed with the Jakubowski S-topology. Moreover, when the Roxin convexity condition is fulfilled, we get that the set of strict control coincides with that of relaxed control.

4.1 Statement of the problems and assumptions

4.1.1 Strict control problem

We study the existence of strict optimal controls for systems governed by the following FBSDE of mean-field type

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, \mathbb{E}[\alpha(X_s)], u_s)ds + \int_0^t \sigma(s, X_s, \mathbb{E}[\beta(X_s)], u_s)dW_s \\ \\ Y_t = g(X_T, \mathbb{E}[\lambda(X_T)]) + \int_t^T f(s, X_s, \mathbb{E}[\gamma(X_s)], Y_s, \mathbb{E}[\delta(Y_s)], u_s)ds - \int_t^T Z_s dW_s - (M_T - M_t), \end{cases} \tag{4.1}$$

where b, α, σ, $\beta, f, \gamma, \delta, g$ and λ are given functions, $(W_t, t \geq 0)$ is a d-dimensional Brownian motion, defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ satisfying the usual conditions. X, Y, Z are square integrable adapted processes and M a square integrable martingale that is orthogonal to W. The control variable u_t, called strict control, is a measurable, $\mathcal{F}_t-$ adapted process with values in a compact metric space U.

The expected cost on the time interval $[0, T]$ is given by

$$\begin{aligned} J(u.) := \mathbb{E}\big[& l(X_T, \mathbb{E}[\theta(X_T)]) + k(Y_0, \mathbb{E}[\rho(Y_0)]) \\ & + \int_0^T h\big(t, X_t, \mathbb{E}[\varphi(X_t)], Y_t, \mathbb{E}\left[\psi\left(Y_t\right)\right], u_t\big)\, dt\big], \end{aligned} \tag{4.2}$$

where $l, \theta, k, \rho, h, \varphi$ and ψ are appropriate functions.

Our objective is to minimize the cost function (4.2), over the set of admissible controls.

It should be noted that the probability space and the Brownian motion may change with the control u. Therefore, we need to have another definition of the admissible control, gives as follows:

Definition 4.1.1 *A 6-tuple* $\upsilon_\cdot = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W_\cdot, u_\cdot\right)$ *is called* ω*-admissible strict control, and* (X_t, Y_t, Z_t) *a* ω*-admissible triple if:*

i)-$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions;*

ii)-W_t *is an d-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$;

iii)-u_t *is an* $\mathcal{F}_t$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in the action space* U*;*

iv)-(X_t, Y_t, Z_t) *is the unique solution of the MF-FBSDE (4.1) on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ *under* u_t*.*

The set of all ω*-admissible controls is denoted by* $\mathcal{U}^\omega$*.*

Our stochastic optimal control problem under the weak formulation can be stated as follows:

Minimize (4.2) over $\mathcal{U}^\omega$. We say that the ω-admissible control υ^* is ω-optimal control, if it satisfies

$$J\left(\upsilon_\cdot^*\right) = \inf_{\upsilon_\cdot \in \mathcal{U}^\omega} J\left(\upsilon\right). \tag{4.3}$$

4.1.2 Assumptions

Let us assume the following conditions (**H4.1**) Assume that the functions

$$b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},$$

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R}^m,$$

$$g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m,$$

$$\alpha, \beta, \lambda, \gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\delta : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

are bounded and continuous. Moreover assume that there exists $K > 0$, such that for every $(x_1, x_2, x'_1, x'_2) \in \mathbb{R}^{4n}, (y_1, y_2, y'_1, y'_2) \, \mathbb{R}^{4m}$,

$$|f(t, x_1, x_2, y_1, y_2, u) - f(t, x'_1, x'_2, y'_1, y'_2, u)| \leq K \ (|x_1 - x'_1| + |x_2 - x'_2| + |y_1 - y'_1| + |y_2 - y'_2|),$$

$$|b(t, x_1, x_2, u) - b(t, x'_1, x'_2, u)| \leq K \ (|x_1 - x'_1| + |x_2 - x'_2|),$$

$$|\sigma(t, x_1, x_2, u) - \sigma(t, x'_1, x'_2, u)| \leq K \ (|x_1 - x'_1| + |x_2 - x'_2|).$$

Also, the functions $\alpha, \beta, \gamma, \lambda$ are uniformly Lipschitz in x and δ is uniformly Lipschitz in y.

(**H4.2**) Assume that the functions

$$l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$$k : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},$$

$$h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R},$$

$$\varphi, \theta : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\psi, \rho : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

are bounded and continuous. Moreover assume that

$$|h(t, x_1, x_2, y_1, y_2, u) - h(t, x'_1, x'_2, y'_1, y'_2, u)| \leq K \ (|x_1 - x'_1| + |x_2 - x'_2| + |y_1 - y'_1| + |y_2 - y'_2|).$$

(**H4.3**) (U, d) is a compact metric space.

(**H4.4**) (Roxin-type convexity condition): The set

$$\left(b, \sigma\sigma^T, f, h\right)(t, x, x', y, y', U) := \{b^i(t, x, x', u), \left(\sigma\sigma^T\right)^{ij}(t, x, x', u),$$

$$f^j(t, x, x', y, y', u), h(t, x, x', y, y', u) \diagdown u \in U, i = 1, \cdots, n, j = 1, \cdots, m\},$$

is convex and closed in $\mathbb{R}^{n+m+1}$.

To proof the existence of optimal solution of our strict control problem $\{(4.1), (4.2), (4.3)\}$, we need a certain structure of compactness. The weak formulation allows us to find the compactness of the image measure of some processes involved on a certain functional space. However, because the control u is measurable only in t and there is no convenient compactness property on the space of measurable functions, we need to embed it in a larger space with proper compactness.

We denote by $\mathbb{V}$ the space of positive Radon measures μ on $[0, T] \times U$ such that

$$\mu([0, s] \times U) = s, \forall \in [0, T]. \tag{4.4}$$

Equipped with the topology of stable convergence of measures, $\mathbb{V}$ is a compact metrizable space, (see Jacod and Mémin [23]). On the other hand, by (4.4), μ can be represented as

$\mu(dt, du) = \mu(t, du)dt$, where $\mu(t, du)$ is a probability measure on U for almost all t and is determined uniquely except on a t-null set. In this context, any U-valued measurable process $u.$ may be embedded into $\mathbb{V}$ in which $u.$ corresponds to the Dirac measure $\delta_{u.}(dt, da)$ as follows: for any bounded and uniformly continuous functions ϖ we have

$$\varpi(t, x, u_t) = \int_U \varpi(t, x, u)\delta_{u.}(t, da) := \widehat{\varpi}(t, x, \delta_{u.}). \tag{4.5}$$

4.2 Existence of optimal controls

Our results in this paper extends those of [5] and [7] to a systems governed by FBSDE of mean-field type and with controlled diffusion coefficient.

Theorem 4.2.1 *Under conditions* $(\mathbf{H4.1})$-$(\mathbf{H4.4})$, *the strict control problem* $\{(4.1), (4.2), (4.3)\}$ *has an* *optimal solution.*

To prove this theorem, we need some auxiliary results on the tightness of the distributions of the processes defining the control problem.

Let $\upsilon^n_\cdot = \left(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t\geq 0}, \mathbb{P}^n, W^n_\cdot, u^n\right)$ be a minimizing sequence, that is $\lim\limits_{n\to\infty} J(\upsilon^n_\cdot) = \inf\limits_{\upsilon.\in\mathcal{U}^\omega} J(\upsilon.)$.

Let (X^n, Y^n, Z^n) be the unique solution of the following MF-FBSDE

$$\begin{cases} X^n_t = x + \int_0^t b\left(s, X^n_s, \mathbb{E}\left[\alpha\left(X^n_s\right)\right], u^n_s\right) ds + \int_0^t \sigma\left(s, X^n_s, \mathbb{E}\left[\beta\left(X^n_s\right)\right], u^n_s\right) dW^n_s, \\ \\ Y^n_t = g\left(X^n_T, \mathbb{E}\left[\lambda\left(X^n_T\right)\right]\right) + \int_t^T f\left(s, X^n_s, \mathbb{E}\left[\gamma\left(X^n_s\right)\right], Y^n_s, \mathbb{E}\left[\delta\left(Y^n_s\right)\right], u^n_s\right) ds \\ \qquad\qquad - \int_t^T Z^n_s dW^n_s, \end{cases} \tag{4.6}$$

Lemma 4.2.2 *Let (X^n, Y^n, Z^n) be the unique solution of the system (4.6). There exists a positive constant C such that*

$$\sup_n \mathbb{E}\left(\sup_{0\leq t\leq T} |X^n_t|^2 + \sup_{0\leq t\leq T} |Y^n_t|^2 + \int_t^T ||Z^n_s||^2 ds\right) \leq C. \tag{4.7}$$

Proof. Its easily to show that

$$\sup_n \mathbb{E}\left[\sup_{0\leq t\leq T} |X^n_t|^2\right] < +\infty.$$

By using the boundedness of b and σ and using Burkholder-Davis-Gundy's inequality.

On the other hand, applying Itô's formula to $|Y^n_t|^2$, we obtain

$$\begin{aligned} \mathbb{E}\left[|Y^n_t|^2 + \int_t^T ||Z^n_s||^2 ds\right] &= \mathbb{E}\left[|g(X^n_T, \mathbb{E}\left[\lambda\left(X^n_T\right)\right])|^2\right] \\ &\quad + 2\mathbb{E}\left[\int_t^T \langle Y^n_s, f\left(s, X^n_s, \mathbb{E}\left[\gamma\left(X^n_s\right)\right], Y^n_s, \mathbb{E}\left[\delta\left(Y^n_s\right)\right], u^n_s\right)\rangle ds\right] \\ \leq \mathbb{E}\left[|g(X^n_T, \mathbb{E}\left[\lambda\left(X^n_T\right)\right])|^2\right] &+ \mathbb{E}\left[\int_t^T |Y^n_s|^2 ds\right] \\ &\quad + \mathbb{E}\left[\int_t^T \left|f\left(s, X^n_s, \mathbb{E}\left[\gamma\left(X^n_s\right)\right], Y^n_s, \mathbb{E}\left[\delta\left(Y^n_s\right)\right], u^n_s\right)\right|^2 ds\right]. \end{aligned}$$

Using the boundedness of g and f and by Gronwall's lemma, it follows that

$$\sup_n \mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2+\int_t^T ||Z_s^n||^2 ds\right]<+\infty.$$

■

Lemma 4.2.3 *The sequence of distributions of the processes $(X^n, W^n, Y^n, \int_0^{\cdot} Z_s^n dW_s^n)$ is tight on the space $\Gamma := \mathbb{C}([0,T];\mathbb{R}^n)\times\mathbb{C}\left([0,T];\mathbb{R}^d\right)\times\mathbb{D}([0,T];\mathbb{R}^m)\times\mathbb{D}\left([0,T];\mathbb{R}^{m\times d}\right)$ endowed with the topology of uniform convergence for the first and second factor and endowed with the S-topology of Jakubowski (see[24]) for the third and fourth factor.*

Proof. According to Kolmogorov's theorem (see Ikeda and Watanabe [22] page 18), we need to verify that

$$\mathbb{E}\left[|X_t^n - X_s^n|^4\right] \le K_1 |t-s|^2 ,$$
$$\mathbb{E}\left[|W_t^n - W_s^n|^4\right] \le K_2 |t-s|^2 ,$$

for some constants K_1 and K_2 independent from n.

We have

$$\mathbb{E}\left[|X_t^n - X_s^n|^4\right] \le C\mathbb{E}\left[\left|\int_s^t b\left(s, X_s^n, \mathbb{E}\left[\alpha\left(X_s^n\right)\right], u_s^n\right) ds\right|^4\right] + C\mathbb{E}\left[\left|\int_s^t \sigma\left(s, X_s^n, \mathbb{E}\left[\beta\left(X_s^n\right)\right], u_s^n\right) dW_s^n\right|^4\right].$$

Using Burkholder-Davis-Gundy's inequality to the martingale part and the boundedness of b and σ, we obtain

$$\begin{aligned}\mathbb{E}\left[|X_t^n - X_s^n|^4\right] &\leq C\mathbb{E}\left[\left(\int_s^t |b(s, X_s^n, \mathbb{E}[\alpha(X_s^n)], u_s^n)|^2 ds\right)^2\right] \\ &\quad + C\mathbb{E}\Big[\left(\int_s^t |\sigma(s, X_s^n, \mathbb{E}[\beta(X_s^n)], u_s^n)|^2 ds\right)^2\Big] \\ &\leq K_1 |t-s|^2 .\end{aligned}$$

The second inequality is obvious.

Let us prove that $(Y_\cdot^n, \int_0^\cdot Z_s^n dW_s^n)$ is tight on the space $\mathbb{D}([0,T];\mathbb{R}^m) \times \mathbb{D}\left([0,T];\mathbb{R}^{m\times d}\right)$.

Let $0 = t_0 < t_1 < ... < t_n = T$. We define the conditional variation by

$$CV(Y_\cdot^n) := \sup \mathbb{E}\left[\sum_i \left|\mathbb{E}\left(Y_{t_{i+1}}^n - Y_{t_i}^n\right) \diagup \mathcal{F}_{t_i}^{W^n}\right|\right],$$

where the supremum is taken over all partitions of the interval $[0,T]$. By the same method given in [31], we get

$$CV(Y_\cdot^n) \leq C\mathbb{E}\left[\int_0^T |f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)], u_s^n)| ds\right],$$

where C is a constant depending only on t. Hence combining conditions (**H4.1**) and Lemma 4.2.2, we deduce that

$$\sup_n \left[CV(Y_\cdot^n) + \sup_{0\leq t\leq T} \mathbb{E}[|Y_t^n|] + \sup_{0\leq t\leq T} \mathbb{E}\left[\left|\int_0^t Z_s^n dW_s^n\right|\right]\right] < +\infty.$$

Thus the Meyer-Zheng tightness criteria is fulfilled (see [30]), then the sequences $Y_\cdot^n$ and $\int_0^\cdot Z_s^n dW_s^n$ are tight. ∎

Lemma 4.2.4 *The familly of distributions of the relaxed control $\left(\delta_{u_\cdot^n}\right)_n$ is tight in $\mathbb{V}$.*

Proof. Since $[0,T]\times U$ is compact, then by applying Prokhorov's theorem, the space $\mathbb{V}$ of probability measures on $[0,T]\times U$ is then compact. Since $\left(\delta_{u^n_\cdot}\right)_n$ valued in the compact space $\mathbb{V}$, then the familly of distributions associated to $\left(\delta_{u^n_\cdot}\right)_n$ is tight. ■

4.2.1 Proof of theorem 4.2.1

Proof. Using Lemmas 4.2.3 and 4.2.4, it follows that the sequence of processes $\eta^n := \left(\delta_{u^n_\cdot}, X^n_\cdot, W^n_\cdot, Y^n_\cdot, \int_0^\cdot Z^n_s dW^n_s\right)$ is tight on the space $\mathbb{V}\times\Gamma$. Then by the Skorokhod representation theorem, there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$, a sequences $\widetilde{\eta}^n = \left(\widetilde{\pi}^n, \widetilde{X}^n, \widetilde{W}^n, \widetilde{Y}^n, \int_0^\cdot \widetilde{Z}^n_s d\widetilde{W}^n_s\right)$ and $\widetilde{\eta} = \left(\widetilde{\pi}, \widetilde{X}, \widetilde{W}, \widetilde{Y}, \int_0^\cdot \widetilde{Z}_s d\widetilde{W}_s\right)$ defined on this space and a countable subset D of $[0,T]$ such that on D^c, we have

(i) for each $n\in\mathbb{N}$, law$(\eta^n) \equiv$ law$(\widetilde{\eta}^n)$,

(ii) there exists a subsequence $(\widetilde{\eta}^{n_k})$ of $(\widetilde{\eta}^n)$, still denoted $(\tilde{\eta}^n)$, which converges to $\widetilde{\eta}, \widetilde{\mathbb{P}}$-a.s. on the space $\mathbb{V}\times\Gamma$,

(iii) $(\widetilde{Y}^n, \int_0^\cdot \widetilde{Z}^n_s d\widetilde{W}^n_s)$ converges to the càdlàg processes $(\widetilde{Y}, \int_0^\cdot \widetilde{Z}_s d\widetilde{W}_s)$, $dt\times\widetilde{\mathbb{P}}$-a.s. Also $\widetilde{Y}^n_T \to \widetilde{Y}_T$. $\widetilde{\mathbb{P}}$

(iv) $\sup\limits_{0\leq t\leq T}\left|\widetilde{X}^n_t - \widetilde{X}_t\right| \to 0, \quad \widetilde{\mathbb{P}}$-a.s.

(v) $(\widetilde{\pi}^n)$ converges in the stable topology to $\widetilde{\pi}, \widetilde{\mathbb{P}}$-a.s.

Set

$$\begin{cases} \widetilde{\mathcal{F}}^n_t := \left(\sigma(\widetilde{W}^n_s, \widetilde{X}^n_s, \widetilde{Y}^n_s, s\leq t) \vee (\widetilde{\pi}^n)^{-1}(\mathcal{B}_t(\mathbb{V}))\right), \\ \\ \widetilde{\mathcal{F}}_t := \left(\sigma(\widetilde{W}_s, \widetilde{X}_s, \widetilde{Y}_s, s\leq t) \vee (\widetilde{\pi})^{-1}(\mathcal{B}_t(\mathbb{V}))\right), \end{cases}$$

where $\mathcal{B}_t(\mathbb{V})$ is defined by

$$\mathcal{B}_t(\mathbb{V}) := \sigma\left(\{\pi\in\mathbb{V} \mid \pi(\phi^s)\in B\} : s\in[0,t], B\in\mathcal{B}(\mathbb{R})\right),$$

$\pi \in \mathbb{V}$ is a linear functional on $\mathcal{C}([0,T] \times U)$ in the way:

$$\pi(\phi) := \int_0^T \int_U \phi(t,u)\pi(dt,du), \forall \phi \in \mathcal{C}([0,T] \times U),$$

and $\phi^t(s,u) := \phi(s \wedge t, u)$.

According to property (i), we have the following MF-FBSDE on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t^n\}_{t\geq 0}, \widetilde{\mathbb{P}}\right)$

$$\begin{cases} \widetilde{X}_t^n = x + \int_0^t \int_U b(s, \widetilde{X}_s^n, \mathbb{E}[\alpha(\widetilde{X}_s^n)], u)\widetilde{\pi}_s^n\,(du)\,ds + \int_0^t \int_U \sigma(s, \widetilde{X}_s^n, \mathbb{E}[\beta(\widetilde{X}_s^n)], u)\widetilde{\pi}_s^n\,(du)\,d\widetilde{W}_s^n, \\ \\ \widetilde{Y}_t^n = g(\widetilde{X}_T^n, \mathbb{E}[\lambda(\widetilde{X}_T^n)]) + \int_t^T \int_U f(s, \widetilde{X}_s^n, \mathbb{E}[\gamma(\widetilde{X}_s^n)], \widetilde{Y}_s^n, \mathbb{E}[\delta(\widetilde{Y}_s^n)], u)\widetilde{\pi}_s^n\,(du)\,ds \\ \qquad\qquad -(\widetilde{N}_T^n - \widetilde{N}_t^n), \end{cases} \tag{4.8}$$

and by notation (4.5), we have

$$\begin{cases} \widetilde{X}_t^n = x + \int_0^t \widehat{b}(s, \widetilde{X}_s^n, \mathbb{E}[\alpha(\widetilde{X}_s^n)], \widetilde{\pi}_s^n)ds + \int_0^t \widehat{\sigma}(s, \widetilde{X}_s^n, \mathbb{E}[\beta(\widetilde{X}_s^n)], \widetilde{\pi}_s^n)d\widetilde{W}_s^n, \\ \\ \widetilde{Y}_t^n = g(\widetilde{X}_T^n, \mathbb{E}[\lambda(\widetilde{X}_T^n)]) + \int_t^T \widehat{f}(s, \widetilde{X}_s^n, \mathbb{E}[\gamma(\widetilde{X}_s^n)], \widetilde{Y}_s^n, \mathbb{E}[\delta(\widetilde{Y}_s^n)], \widetilde{\pi}_s^n)ds \\ \qquad\qquad -(\widetilde{N}_T^n - \widetilde{N}_t^n), \end{cases} \tag{4.9}$$

where $\widetilde{N}_t^n = \int_0^t \widetilde{Z}_s^n d\widetilde{W}_s^n$.

Since $\widetilde{W}_\cdot^n$ is an $\{\widetilde{\mathcal{F}}_t^n\}_{t\geq 0}$-Brownian motion, all the integrals in (4.8) and in (4.9) are well-defined.

Using properties (ii), (iii), (iv),(v), under (**H**4.1)-(**H**4.4) and passing to the limit in the MF-FBSDE (4.9), one can show that there exists a countable set $D \subset [0,\ T)$ such that

$$\begin{cases} \widetilde{X}_t = x + \widetilde{B}(t) + \widetilde{\Sigma}(t), t > 0 \\ \\ \widetilde{Y}_t = g(\widetilde{X}_T, \mathbb{E}[\lambda(\widetilde{X}_T)]) + (\widetilde{F}(T) - \widetilde{F}(t)) - (\widetilde{N}_T - \widetilde{N}_t), t \in [0,T] \diagdown D, \end{cases} \tag{4.10}$$

Since $\widetilde{Y}$ and $\widetilde{N}$ are càdlàg, then one can get for every $t \in [0,T]$

$$\widetilde{Y}_t = g(\widetilde{X}_T, \mathbb{E}[\lambda(\widetilde{X}_T)]) + (\widetilde{F}(T) - \widetilde{F}(t)) - (\widetilde{N}_T - \widetilde{N}_t).$$

Also we have

$$\begin{aligned}\inf_{u.\in\mathcal{U}^\omega} J(u.) = \lim_{n\to\infty} J\left(\delta_{u^n_\cdot}\right) = \lim_{n\to\infty} J(\widetilde{\pi}^n_\cdot)\\ := \lim_{n\to\infty} \widetilde{\mathbb{E}}\big[l(\widetilde{X}^n_T, \widetilde{\mathbb{E}}[\theta(\widetilde{X}^n_T)]) + k(\widetilde{Y}^n_0, \widetilde{\mathbb{E}}[\rho(\widetilde{Y}^n_0)])\\ + \int_0^T \widehat{h}\big(t, \widetilde{X}^n_t, \widetilde{\mathbb{E}}[\varphi(\widetilde{X}^n_t)], \widetilde{Y}^n_t, \widetilde{\mathbb{E}}[\psi(\widetilde{Y}^n_t)], \widetilde{\pi}^n_t)dt\big],\\ = \widetilde{\mathbb{E}}\big[l(\widetilde{X}_T, \widetilde{\mathbb{E}}[\theta(\widetilde{X}_T)]) + k(\widetilde{Y}_0, \widetilde{\mathbb{E}}[\rho(\widetilde{Y}_0)]) + \widetilde{H}(T)\big],\end{aligned} \tag{4.11}$$

The rest of the proof is inspired from Yong and Zhou [35]. Let us consider the sequence

$a^n(s) := \widehat{\sigma}\widehat{\sigma}^T(s, \widetilde{X}^n_s, \mathbb{E}[\beta(\widetilde{X}^n_s)], \widetilde{\pi}^n_s), s \in [0, T]$. Setting

$$\begin{cases} b^i_n(s) := \widehat{b}^i(s, \widetilde{X}^n_s, \mathbb{E}[\alpha(\widetilde{X}^n_s)], \widetilde{\pi}^n_s), i = 1, \cdots, n,\\ a^{ik}_n(s) := \widehat{\sigma}\widehat{\sigma}^T(s, \widetilde{X}^n_s, \mathbb{E}[\beta(\widetilde{X}^n_s)], \widetilde{\pi}^n_s), i = 1, \cdots, n, k = 1, \cdots, d,\\ f^j_n(s) := \widehat{f}^j(s, \widetilde{X}^n_s, \mathbb{E}[\gamma(\widetilde{X}^n_s)], \widetilde{Y}^n_s, \mathbb{E}[\delta(\widetilde{Y}^n_s)], \widetilde{\pi}^n_s), j = 1, \cdots, m,\\ h_n(s) := \widehat{h}(s, \widetilde{X}^n_s, \mathbb{E}[\varphi(\widetilde{X}^n_s)], \widetilde{Y}^n_s, \mathbb{E}[\psi(\widetilde{Y}^n_s)], \widetilde{\pi}^n_s),\end{cases}$$

Since $b^i_n \to b^i, i = 1, \cdots, n$, $f^j_n \to f^j, j = 1, \cdots, m$, $h_n \to h$, $a^{ik}_n \to a^{ik}$ weakly, and

$$\begin{aligned} & b^i(s,\omega) \in b^i(s, \widetilde{X}_s(\omega), \mathbb{E}[\alpha(\widetilde{X}_s(\omega))], U),\\ & (\sigma\sigma^T)^{ik}(s,\omega) \in (\sigma\sigma^T)^{ik}(s, \widetilde{X}_s(\omega), \mathbb{E}[\beta(\widetilde{X}_s(\omega))], U),\\ & f^j(s,\omega) \in f^j(s, \widetilde{X}_s(\omega), \mathbb{E}[\gamma(\widetilde{X}_s(\omega))], \widetilde{Y}_s(\omega), \mathbb{E}[\delta(\widetilde{Y}_s(\omega))], U),\\ & h(s,\omega) \in h(s, \widetilde{X}_s(\omega), \mathbb{E}[\varphi(\widetilde{X}_s(\omega))], \widetilde{Y}_s(\omega), \mathbb{E}[\psi(\widetilde{Y}_s(\omega))], U),\\ & \forall (s,\omega) \in [0,T] \times \widetilde{\Omega}, i, k = 1, \cdots, n, j = 1, \cdots, m.\end{aligned} \tag{4.12}$$

From (4.10), (**H4.4**) and a measurable selection theorem (see Li-Yong [29], p. 102, Corollary 2.26), there is a U-valued, $\tilde{\mathcal{F}}_t$-adapted process $\tilde{u}.$ such that

$$b(s,\omega) = b(s, \widetilde{X}_s(\omega), \mathbb{E}[\alpha(\widetilde{X}_s(\omega))], \widetilde{u}_s(\omega)), \tag{4.13}$$

$$\sigma\sigma^T(s,\omega) = \sigma\sigma^T(s, \widetilde{X}_s(\omega), \mathbb{E}[\beta(\widetilde{X}_s(\omega))], \widetilde{u}_s(\omega)),$$

$$f(s,\omega) = f(s, \widetilde{X}_s(\omega), \mathbb{E}[\gamma(\widetilde{X}_s(\omega))], \widetilde{Y}_s(\omega), \mathbb{E}[\delta(\widetilde{Y}_s(\omega))], \widetilde{u}_s(\omega)),$$

$$h(s,\omega) = h(s, \widetilde{X}_s(\omega), \mathbb{E}[\varphi(\widetilde{X}_s(\omega))], \widetilde{Y}_s(\omega), \mathbb{E}[\psi(\widetilde{Y}_s(\omega))], \widetilde{u}_s(\omega)).$$

Since $\widetilde{\Sigma}(t)$ given in (4.10), is an $\widetilde{\mathcal{F}}_t$-martingale, we have

$$\langle \widetilde{\Sigma}^n \rangle(t) = \int_0^t \widehat{\sigma}\widehat{\sigma}^T(s, \widetilde{X}_s^n, \mathbb{E}[\beta(\widetilde{X}_s^n)], \widetilde{\pi}_s^n) ds \equiv \int_0^t a^n(s) ds,$$

where $\langle \widetilde{\Sigma}^n \rangle$ is the quadratic variation of $\widetilde{\Sigma}^n$. Thus $\widetilde{\Sigma}^n(\widetilde{\Sigma}^n)^T(t) - \int_0^t a^n(s)ds$ is an $\widetilde{\mathcal{F}}_t$-martingale and from the fact that

$$\int_s^t a^n(r)dr \text{ converges weakly to } \int_s^t \sigma\sigma^T(r, \widetilde{X}_r, \mathbb{E}[\alpha(\widetilde{X}_r)], \widetilde{u}_r)dr,$$

we can show that $\widetilde{\Sigma}(\widetilde{\Sigma})^T(t) - \int_0^t \sigma\sigma^T(r, \widetilde{X}_r, \mathbb{E}[\alpha(\widetilde{X}_r)], \widetilde{u}_r)dr$ is an $\widetilde{\mathcal{F}}_t$-martingale. Which implies

$$\langle \widetilde{\Sigma} \rangle(t) = \int_0^t \widehat{\sigma}\widehat{\sigma}^T(s, \widetilde{X}_s, \mathbb{E}[\beta(\widetilde{X}_s)], \widetilde{u}_s) ds.$$

By a martingale representation theorem, there is an extension space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}})$ of $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}}\right)$ on which lives an d-dimensional $\overline{\mathcal{F}}_t$-Brownian motion $\overline{W}_t$ such that

$$\widetilde{\Sigma}(t) = \int_0^t \sigma(s, \widetilde{X}_s, \mathbb{E}[\beta(\widetilde{X}_s)], \widetilde{u}_s) d\overline{W}_s. \tag{4.14}$$

Similarly, one can show that

$$\widetilde{B}(t) = \int_0^t b(s, \widetilde{X}_s, \mathbb{E}[\alpha(\widetilde{X}_s)], \widetilde{u}_s) ds, \tag{4.15}$$

$$\widetilde{F}(t) = \int_0^t f(s, \widetilde{X}_s, \mathbb{E}[\gamma(\widetilde{X}_s)], \widetilde{Y}_s, \mathbb{E}[\delta(\widetilde{Y}_s)], \widetilde{u}_s) ds,$$

$$\widetilde{H}(t) = \int_0^t h(s, \widetilde{X}_s, \mathbb{E}[\varphi(\widetilde{X}_s)], \widetilde{Y}_s, \mathbb{E}[\psi(\widetilde{Y}_s)], \widetilde{u}_s) ds.$$

Also, since $\widetilde{N}$ is a $\widetilde{\mathcal{F}}_s$-martingale. Therefore by the martingale decomposition theorem, there exist a process $\widetilde{Z} \in \mathbb{M}^2([0,T]\,;\mathbb{R}^{m\times d})$ such that

$$\widetilde{N}_t = \int_0^t \widetilde{Z}_s d\overline{W}_s + \widetilde{M}_t, \text{ and } \left\langle \widetilde{M}, \overline{W} \right\rangle_t = 0. \tag{4.16}$$

Putting the values of $\widetilde{\Sigma}(t), \widetilde{B}(t), \widetilde{F}(t), \widetilde{N}_t, \widetilde{H}(t)$ (from (4.14),(4.15),(4.16)) into (4.10) and (4.11), we get

$$\begin{cases} \widetilde{X}_t := x + \int_0^t b(s, \widetilde{X}_s, \mathbb{E}[\alpha(\widetilde{X}_s)], \widetilde{u}_s)ds + \int_0^t \sigma(s, \widetilde{X}_s, \mathbb{E}[\beta(\widetilde{X}_s)], \widetilde{u}_s)d\overline{W}_s, t \geq 0, \\ \\ \widetilde{Y}_t := g(\widetilde{X}_T, \mathbb{E}[\lambda(\widetilde{X}_T)]) + \int_t^T f(s, \widetilde{X}_s, \mathbb{E}[\gamma(\widetilde{X}_s)], \widetilde{Y}_s, \mathbb{E}[\delta(\widetilde{Y}_s)], \widetilde{u}_s)ds - \int_t^T \widetilde{Z}_s d\overline{W}_s \\ \qquad -(\widetilde{M}_T - \widetilde{M}_t), t \in [0,T]\,, \end{cases}$$

and

$$\begin{aligned} \inf_{u.\in\mathcal{U}^\omega} J\,(u.) &= \widetilde{\mathbb{E}}\big[l(\widetilde{X}_T, \widetilde{\mathbb{E}}[\theta(\widetilde{X}_T)]) + k(\widetilde{Y}_0, \widetilde{\mathbb{E}}[\rho(\widetilde{Y}_0)]) + \int_0^T \widehat{h}\big(t, \widetilde{X}_t, \widetilde{\mathbb{E}}[\varphi(\widetilde{X}_t)], \widetilde{Y}_t, \widetilde{\mathbb{E}}[\psi(\widetilde{Y}_t)], \widetilde{u}_t)dt\big] \\ &= J\,(\widetilde{v}.)\,. \end{aligned}$$

By the Definition 4.1.1 , we arrive that $\widetilde{v}. := \big(\overline{\Omega}, \overline{\mathcal{F}}, \{\overline{\mathcal{F}}_t\}_{t\geq 0}, \overline{\mathbb{P}}, \overline{W}., \widetilde{u}.\big)$ is an ω-optimal control. ■

Appendix: S-topology

The S-topology has been introduced by Jakubowski [24], as a topology defined on the Skorokhod space of càdlàg functions $\mathbb{D}\left([0,T];\mathbb{R}^k\right)$. This topology is weaker than the Skorokhod topology and the tightness criteria are easier to establish. This criteria is the same as that of the Meyer and Zheng topology [30].

Let $N^{a,b}(Y)$ denotes the number of up-crossing of the function $Y \in \mathbb{D}([0;T];\mathbb{R}^m)$ in a given level $a < b$. We recall some facts about the S-topology.

Proposition A.1. *(A criteria for S-tight). A sequence $(Y^n)_{n>0}$ is S-tight if and only if it is relatively compact on the S-topology.*

Proposition A.2. *Let $(Y^n)_{n>0}$ be a family of stochastic processes in $\mathbb{D}([0;T];R^m)$. Then this family is tight for the S-topology if and only if $(\|Y^n\|)_n$ and $\left(N^{a,b}(Y^n)\right)_n$ are tight for each $a < b$.*

We recall (see Meyer & Zheng [30] and Jakubowski [24],[25]) that for a familly $(Y^n)_n$ of quasi-martingales on the probability space $\left(\Omega,\{\mathcal{F}_t\}_{0\leq t\leq T},\mathbb{P}\right)$, the following condition insures the tightness of the familly $(Y^n)_n$ on the space $\mathbb{D}\left([0,T];\mathbb{R}^m\right)$ endowed with the S-topology

$$\sup_n\left(\sup_{0\leq t\leq T}\mathbb{E}\left|Y_t^n\right| + CV\left(Y^n\right)\right) < +\infty,$$

where, for a quasi-martingale Y^n, $CV\left(Y^n\right)$ stands for the conditional variation of Y on $[0,T]$, and is defined by

$$CV\left(Y^{n}\right)=\sup\mathbb{E}\left[\sum_{i}\left|\mathbb{E}\left(Y_{t_{i}+1}^{n}-Y_{t_{i}}^{n}\right)\mid\mathcal{F}_{t_{i}}^{n}\right|\right],$$

where the supremum is taken over all partitions of $[0,T]$.

The process Y is call *quasimartingale* if $CV(Y)<+\infty$. When Y is a $\mathcal{F}_t$-martingale, $CV(Y)=0$.

Proposition A.3. *(The a.s. Skorokhod representation). Let $(\mathbb{D},S)$ be a topological space on which there exists a countable family of S-continuous functions separating points in Y. Let $\{Y^n\}_{n\in\mathbb{N}}$ be a uniformly tight sequence of laws on $\mathbb{D}$. In every subsequence $\{Y^{n_k}\}$ one can find a further subsequence $\{Y^{n_{k_l}}\}$ and stochastic processes $\left\{Y^l\right\}$ defined on $\left([0,T],\mathcal{B}_{[0,T]},l\right)$ such that*

$$Y^{l}\sim Y^{n_{k_{l}}},\ l=1,2,... \tag{1}$$

for each $w\in[0,T]$

$$Y^{l}\left(w\right)\underset{S}{\rightarrow}Y^{0}\left(w\right),\ \text{as}\ l\rightarrow+\infty, \tag{2}$$

and for each $\varepsilon>0$, there exists an S-compact subset $K_\varepsilon\subset D$ such that

$$\mathbb{P}\left(\left\{w\in[0,T]:Y^{l}\left(w\right)\in K_{\varepsilon},l=1,2,...\right\}\right)>1-\varepsilon. \tag{3}$$

One can say that (2) and (3) describe "the almost sure convergence in compacts" and that (1), (2) and (3) define the strong a.s. Skorokhod representation for subsequences ("strong" because of condition (3)).

Proposition A.4. Let $(Y^n,\,M^n)$ be a multidimensional process in $\mathbb{D}([0,\,T];\,\mathbb{R}^m)$ converging to $(Y,\,M)$ in the S-topology. Let $(\mathcal{F}_t^{Y^n})_{t\geq0}$ (resp. $(\mathcal{F}_t^{Y})_{t\geq0}$) be the minimal complete admissible filtration for Y^n (resp.Y). We assume that $\sup_n \mathbb{E}\left[\sup_{0\leq t\leq T}|M_t^n|^2\right] < C_T \ \forall T > 0$, M^n is a $\mathcal{F}^{Y^n}$-martingale and M is a $\mathcal{F}^Y$-adapted. Then M is a $\mathcal{F}^Y$-martingale.

Proposition A.5. *Let $(Y^n)_{n>0}$ be a sequence of processes converging weakly in $\mathbb{D}([0, T]; R^m)$ to Y. We assume that $\sup_n \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2\right] < +\infty$. Hence, for any $t \geq 0$, $\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] < +\infty$.*

Conclusion

In this thesis, we have proved the existence of optimal solutions of an optimal control problem . In particular, the problems are governed by non linear forward-backward stochastic differential equations of mean-field type (MF-FBSDEs), in which the coefficients of the system depend not only on the state process, but also on the distribution of the state process. Moreover, the cost functional is also of mean-field type. The basic idea behind the proof of this result is the relaxed control, which is needed in order to provide some compact structure. Moreover, under the Roxin's convexity condition, the set of strict controls coincides with that of relaxed controls. An open question is to prove this result, where the generator depends on the second backward variable Z. There is a serious difficulty in this case, this difficulty consists in finding a natural assumption ensuring the tightness of the second variable Z.

When the control problem is governed by a linear MF-FBSDEs as in chapter 2, we have not needed to use the same techniques as nonlinear case, but we have proved the existence of a strong optimal strict control which is adapted to the initial filtration by using the convexity of the cost function and the Mazur's theorem. In this case the generator depending explicitly upon the second backward variable Z.

A special case is that in which both l, k and h are convex quadratic functions. The control problem (Problem (L) in chapter 2) is then reduced to a stochastic linear quadratic optimal control problem.

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