REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITE MOHAMED KHIDER BISKRA
FACULTE DES SCIENCESEXACTES DES SCIENCES DE LA NATURE ET DE LA VIE
DEPARTEMENT DE MATHEMATIQUES

Thèse présentée en vue de l'obtention du Diplôme de :

Doctorat en Mathématiques

Option : Probabilités

Par

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Titre :

Sur les conditions nécessaires et suffisantes d'optimalité pour une classe des contrôles stochastiques mixed de type champ-moyen et leurs applications aux finances.

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06/2021

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This work is dedicated to my parents, my family, my wife, my kids, my brothers and sisters, my brothers and sisters in law and to all those who encouraged me.

Acknowledgments

I would like to express my deepest gratitude to my advisor **Professor Mokhtar Hafayed** this work would have not been possible without his help and support, I learnt a lot from him, and I will keep in touch with him so I learn more and more.

My sincere thanks to **Professor Boulakhras Gherbal** , **Professor Abdelmouman Tiaiba,** and **Professor Khalil Saadi** because they agreed to spend their times for reading and evaluating my thesis.
I thank all my colleagues of the Mathematics Department at the University Bachir El Ibrahimi, Bordj Bou Arreridj, Algeria and in particular my friends Nazih Eddine Belkacem and Bilal Ghermoul.And all the staff of the Mathematics Department at the University of Biskra, especially my friends Salah Eddine Benziouche, Yahia Djebrane and Nabil Khalfallah.

Résumé

Cette thèse de doctorat s'inscrit dans le cadre de l'analyse stochastique dont le thème central est : les conditions nécessaires et suffisantes sous forme du maximum stochastique de type champ moyen d'optimalité avec information partielle et ces applications. L'objectif de ce travail est d'étudier des problèmes d'optimisation stochastique. Il s'agira ensuite de faire le point sur les conditions nécessaires et suffisantes d'optimalité avec information partielle pour un systeme gouverné par des équations différentielles stochastiques de type champ moyen. Cette thèse s'articule autour de trois chapitres :

Le premier chapitre est essentiellement un rappel. Le candidat présente quelques concepts et résultats qui lui permettent d'aborder son travail ; tels que les processus stochastiques, l'espérance conditionnelle, les martingales, les formules d'Ito, les classes de contrôles stochastiques, . . . etc.

Dans le deuxième chapitre, on a établi et on a prouvé les conditions nécessaires et suffisantes de presque optimalité d'order ε^{λ} vérifiées par un contrôle optimal stochastique, pour un systeme différentiel gouverné par des équations différentielles stochastiques EDSs. Le domaine de contrôle stochastique est supposé convexe. La méthode utilisée est basée sur le lemme d'Ekeland. Les résultats obtenus dans le chapitre 2, sont tous nouveaux et font l'objet d'un premier article intitulé :

Boukaf Samira & Mokhtar Hafayed, & Ghebouli Messaoud : "A study on optimal control problem with ε^{λ}-error bound for stochastic systems with application to linear quadratic problem", International Journal of Dynamics and Control, Springer DOI : 10.1007/s40435-015-0178-x (2017) Volume 5, Issue 2, pp 297–305 (2017).

Dans le troisième chapitre, on a démontré le principe du maximum stochastique sous l'information partielle, où le systeme est gouverné par des équations différentielles stochastiques progressives rétrogrades avec un processus de Lévy. Aussi, comme application,

on a traité un problème d'optimization en finance. Les résultats obtenus dans le chapitre 3 sont tous nouveaux et font l'objet d'un deuxième article intitulé :

Mokhtar Hafayed, & Ghebouli Messaoud & Samira Boukaf & Yan Shi : "Partial information optimal control of mean-field forward-backward stochastic system driven by Teugels martingales with applications" (2016) DOI 10.1016/j.neucom. 2016.03.002. Neurocomputing, Vol 200 pages 11–21 (2016).

Abstract

This thesis is concerned with stochastic control of mean-field type. The central theme is the necessary and sufficient conditions in the form of the Pontryagin's stochastic maximum of the mean-field type for optimality with partial information and some applications. Recently, the main purpose of this thesis is to derive a set of necessary as well as sufficient conditions of optimality with partial information, where the system is governed by stochastic differential equations of the mean field type. This thesis is structured around three chapters :

The first chapter is essentially a reminder. we presents some concepts and results that allow us to prove our results, such as stochastic processes, conditional expectation, martingales, Ito formulas, class of stochastic control, etc. In the second chapter, we have proved the necessary and sufficient conditions of near-optimality of order ε^{λ} satisfied by an optimal stochastic control, where the system is governed by stochastic differential equations EDSs. The stochastic control domain is assumed to be convex. The method used is based on the Ekeland lemma. The results obtained in Chapter 2 are all new and are the subject of a first article entitled :

Boukaf Samira & Mokhtar Hafayed and Ghebouli Messaoud : A study on optimal control problem with ε^{λ}-error bound for stochastic systems with application to linear quadratic problem, International Journal of Dynamics and Control, Springer DOI : 10.1007 / s40435-015-0178-x (2017), Volume 5, Issue 2, pp 297–305 (2017).

In the third chapter, we have proved the stochastic maximum principle under partial information, where the system is governed by forward backward stochastic differential equations (FBSDEs) deriven by Lévy process. These results have been applied to solve an optimization problem in finance. Moreover, as an application, we study a partial information mean-variance portfolio selection problem, driven by Teugels martingales associated with Gamma process, where the explicit optimal portfolio strategy is derived in feedback

form. The results obtained in Chapter 3 are all new and are the subject of a second article entitled :

Mokhtar Hafayed, & Ghebouli Messaoud & Samira Boukaf & Yan Shi : Partial information optimal control of mean-field forward-backward stochastic system driven by Teugels martingales with applications (2016) DOI 10.1016/j.neucom. 2016.03.002. Neurocomputing, Vol 200 pages 11–21 (2016)..

Symbols and Acronyms

- **a.e**. almost everywhere
- **a.s.** almost surely
- **càdlàg** continu à droite, limite à gauche
- **cf**. compare (abbreviation of Latin confer)
- **e.g**. for example (abbreviation of Latin exempli gratia)
- **i.e,**. that is (abbreviation of Latin id est)
- **HJB** *The Hamilton-Jacobi-Bellman equation*
- **SDE** : Stochastic differential equations.
- **BSDE :** Backward stochastic differential equation.
- **FBSDEs** : Forward-backward stochastic differential equations.
- **FBSDEJs** : Forward-Backward stochastic differential equations with jumps.
- **PDE** : Partial differential equation.
- **ODE** : Ordinary differential equation.
- $\mathbb{R}$: Real numbers.
- $\mathbb{R}_+$: Nonnegative real numbers.
- $\mathbb{N}$: Natural numbers.
- $\dfrac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0,T]$.
- $E(\cdot)$, $E(\cdot \mid G)$ Expectation ; conditional expectation
- $\sigma(A)$: $\sigma-$algebra generated by A.
- I_A : Indicator function of the set A.
- $\mathcal{F}^X$: The filtration generated by the process X.
- $W(\cdot), B(\cdot)$: Brownian motions
- $\mathcal{F}_t^B$ the natural filtration generated by the brownian motion $B(\cdot)$,
- $\mathrm{F}_1 \vee \mathrm{F}_2$ denotes the σ-field generated by $\mathrm{F}_1 \cup \mathrm{F}_2$.

Chapitre 1

Introduction

We can look to optimal control theory as a set of tools that help us to optimize a dynamical system evolving over a time with respect to differential equations. It is modeled as a vector u which is called the control. The optimized control should minimize an application which is called the cost function. There are two main methods to do that :

1. The Pontryagin Maximum Principal.
2. The Bellman's Dynamic Programming.

1.1 Stochastic processes

1.1.1 Filtration

Definition 1.1.1 A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in[0,T]}$ of $\sigma-$ fields of $\mathcal{F}$: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leqslant s \leqslant t < T$. $\mathcal{F}_s$ is interpreted as the information known at time t.

1.1.2 Stochastic process

Definition 1.1.2 A stochastic process is a collection of random variables $\{X_t, t \in I\}$ Usually t represents time, and I is the set of indices.

1.1.3 Brownian Motion

Definition 1.1.3 A random process $\{W(t), t \in [0, +\infty)\}$ is called a standard Brownian motion if

1. $W(0) = 0$;
2. for all $0 \leqslant t_1 < t_2,\ W(t_2) - W(t_1) \backsim N(0, t_2 - t_1)$;
3. $W(t)$ has independant increments. That is, for all $0 \leqslant t_1 < t_2 < t_3 <, \cdots, t_n$, the random variables $W(t_2) - W(t_1)$, $W(t_3) - W(t_2), \cdots, W(t_n) - W(t_n - 1)$ are independent;
4. $W(t)$ has continuos sample paths;

1.1.4 Levy processes

Definition 1.1.3 A process $(X(t))_{t \geqslant 0}$ defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is said to be a Levy process if the following conditions hold :

a) The trajectories of X are a.s right continuous with aleft limit.

b) $\mathbb{P}(X_0 = 0) = 1$

c) for all $0 \leqslant s \leqslant t,\ X_t - X_s$; has the same distribution as X_{t-s} ;

d) $0 \leqslant s \leqslant t,\ X_t - X_s$ is independent of $(X_u, u \leqslant s)$;

1.2 Formulations of stochastic optimal control problems

In this section, we present two mathematical tools (strong and weak formulations) of stochastic optimal control problems .

1.2.1 Strong formulation

If $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ is a filtred probability space on which a Brownian motion $W(t)$ is defined, consider the following controlled stochastic differential equation :

$$\begin{cases} dx(t) &= f(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t), \\ x(0) &= x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$f : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^{n\times d},$$

and $x(\cdot)$ is the variable of state.

The function $u(\cdot)$ represents the controller's action. The controller at time t has some information which is respresented by the filtration u but is not sure about the future of trajectory which let him unable to take any decision before this time.

The control $u(\cdot)$ is an element of the set

$$\mathcal{U}[0,T] = \{u : [0,T] \times \Omega \longrightarrow A \text{ such that } u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\in[0,T]} - \text{adapted}\}.$$

We introduce the cost functional as follows

$$J(u(\cdot)) \doteq E\left[\int_0^T l(t,x(t),u(t))dt + g(x(T))\right], \tag{1.2}$$

where

$$l : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R},$$
$$g : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Definition 1.1. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be given satisfying the usual conditions and let $W(t)$ be a given d-dimensional standard $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion. A control $u(\cdot)$ is called an admissible control, and $(x(\cdot), u(\cdot))$ an admissible pair, if

Définition 1.2.1 **i)** $u(\cdot) \in \mathcal{U}[0,T]$;

ii) $x(\cdot)$ *is the unique solution of equation (2.1);*

iii) $l(\cdot, x(\cdot), u(\cdot)) \in \mathbb{L}^1_{\mathcal{F}}([0,T];\mathbb{R})$ *and* $g(x(T)) \in \mathbb{L}^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$.

1.2.2 Weak formulation

When it comes to solving stochastic optimal control the weak formulation mathematical aspect, and On the contrary to the strong formulation the filtred probalility space on which we define the Brownian motion are not all fixed.

Definition 1.2. A 6-tuple $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W(\cdot), u(\cdot)\right)$ is called weak-admissible control and $(x(\cdot), u(\cdot))$ an weak admissible pair, if

Définition 1.2.2 **1.** $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions ;*

2. $W(\cdot)$ *is an d-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$;

3. $u(\cdot)$ *is an* $\{\mathcal{F}_t\}_{t\in[0,T]}-$*adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in* U *;*

4. $x(\cdot)$ *is the unique solution of equation (2.1),*

5. $l(\cdot, x(\cdot), u(\cdot)) \in \mathbb{L}^1_{\mathcal{F}}([0,T];\mathbb{R})$ *and* $g(x(T)) \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R})$.

The set of all weak admissible controls is denoted by $\mathcal{U}^w([0,T])$. Sometimes, might write $u(\cdot)) \in \mathcal{U}^w([0,T])$ instead of $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W(\cdot), u(\cdot)\right) \in \mathcal{U}^w([0,T])$.
Our stochastic optimal control problem under weak formulation can be formulated as follows :

Problem 1.2. The objective is to minimize the cost functional given by equation (3.2) over the of admissible controls $\mathcal{U}^w([0,T])$. Namely, one seeks $\pi^*(\cdot) = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W(\cdot), u(\cdot)\right) \in \mathcal{U}^w([0,T])$ such that

$$J(\pi^*(\cdot)) = \inf_{\pi(\cdot)\in\mathcal{U}^w([0,T])} J(\pi(\cdot)).$$

1.3 Methods to solving optimal control problem

Two fundamental methods for studing optimal control are Bellman's dynamic programming method and Pontryagin's maximum principle.

1.3.1 The Dynamic Programming Principle.

In Bellman's dynamic programming the solution to an optimization problem is given by A sequence of sub-optimal solutions by minimizing or maximizing the Hamiltonian or generalized Hamiltonian in the HJB equation.

The Bellman principle. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions, $T > 0$ a finite time, and W a d-dimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$.

We consider the state stochastic differential equation

$$dx(s) \quad = \quad f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dW(s), \;\; \mathrm{s} \in [0, T] \tag{1.3}$$

The control $u = u(s)_{0\leq s\leq T}$ is a progressively measurable process valued in the control set U, a subset of $\mathbb{R}^k$, satisfies a square integrability condition. We denote by $\mathcal{U}\left([t, T]\right)$ the set of control processes u.

Conditions. To ensure the existence of the solution to SDE-(**??**), the Borelian functions

$$f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$$

$$\sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$$

satisfy the following conditions :

$$|f(t, x, u) - f(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq C\,|x - y|\,,$$

$$|f(t,x,u)| + |\sigma(t,x,u)| \leq C\left[1+|x|\right],$$

for some constant $C > 0$. We define the gain function as follows :

$$J(t,x,u) \quad = \quad E\left[\int_t^T l(s,x(s),u(s))ds + g(x\,(T))\right], \tag{1.4}$$

where

$$l : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R},$$

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},$$

Some restrictions on f and g are necessary so that the expectation quoted above is well defined. The aim is to maximize the gain function. We define the value function as follows :

$$V(t,x) \quad = \sup_{u\in\mathcal{U}([t,T])} J(t,x,u), \tag{1.5}$$

where $x(t) = x$ is the initial state given at time t. For an initial state (t,x), we say that $u^* \in \mathcal{U}\,([t,T])$ is an optimal control if

$$V(t,x) = J(t,x,u^*).$$

Theorem 1.1. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be given. Then we have

$$V(t,x) = \sup_{u\in\mathcal{U}([t,T])} E\left[\int_t^{t+h} l(s,x(s),u(s))dt + V(t+h,x(t+h))\right], \text{ for } t \leq t+h \leq T. \tag{1.6}$$

Proof. The proof of the dynamic programming principle is a classical one, and we refer the reader to Yong and Zhou [98].

The Hamilton-Jacobi-Bellman equation The HJB equation is the infinitesimal version of the dynamic programming principle. It is formally derived by assuming that the value function is $C^{1,2}([0,T]\times\mathbb{R}^n)$, applying Itô's formula to $V(s,x^{t,x}(s))$ between $s=t$ and $s=t+h$, and then sending h to zero into (1.5). The classical HJB equation associated to the stochastic control problem (1.5) is

$$-V_t(t,x)-\sup_{u\in U}\left[\mathcal{L}^u V(t,x)+l(t,x,u)\right] \;=\; 0,\;\text{ on } [0,T]\times\mathbb{R}^n, \tag{1.7}$$

where $\mathcal{L}^u$ is the second-order infinitesimal generator associated to the diffusion x with control u

$$\mathcal{L}^u V \;=\; f(x,u).D_xV+\frac{1}{2}tr\left(\sigma(x,u)\,\sigma^{\intercal}(x,u)\,D_x^2V\right).$$

This partial differential equation (PDE) is often written also as :

$$-V_t(t,x)-H(t,x,D_xV(t,x),D_x^2V(t,x)) \;=\; 0,\quad \forall (t,x)\in[0,T]\times\mathbb{R}^n, \tag{1.8}$$

where for $(t,x,\Psi,Q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}_n$ ($\mathcal{S}_n$ is the set of symmetric $n\times n$ matrices) :

$$H(t,x,\Psi,Q) \;=\; \sup_{u\in U}\left[f(t,x,u).\Psi+\frac{1}{2}tr\left(\sigma\sigma^{\intercal}(t,x,u)\,Q\right)+l(t,x,u)\right]. \tag{1.9}$$

The function H is sometimes called Hamiltonian of the associated control problem, and the PDE (1.7) or (1.8) is the dynamic programming or HJB equation.
There is also an a priori terminal condition :

$$V(T,x) \;=\; g(x),\;\forall x\in\mathbb{R}^n,$$

which results from the very definition of the value function V.

The classical verification approach The classical verification approach consists in finding a smooth solution to the HJB equation, and to check that this candidate, under suitable sufficient conditions, coincides with the value function. This result is usually called a verification theorem and provides as a byproduct an optimal control. It relies mainly on

Itô's formula. The assertions of a verification theorem may slightly vary from problem to problem, depending on the required sufficient technical conditions. These conditions should actually be adapted to the context of the considered problem. , a verification theorem is generally stated as follows :

Theorem 1.2. Let W be a $C^{1,2}$ function on $[0,T]\times\mathbb{R}^n$ and continuous in T, with suitable growth condition. Suppose that for all $(t,x)\in[0,T]\times\mathbb{R}^n$, there exists $u^*(t,x)$ mesurable, valued in U such that W solves the HJB equation :

$$\begin{aligned} 0 &= -W_t(t,x) - \sup_{u\in U}\left[\mathcal{L}^u W(t,x) + l(t,x,u)\right] \\ &= -W_t(t,x) - \mathcal{L}^{u^*(t,x)}W(t,x) - l(t,x,u^*(t,x)), \text{ on } [0,T]\times\mathbb{R}^n, \end{aligned}$$

together with the terminal condition $W(T,\cdot) = g$ on $\mathbb{R}^n$, and the stochastic differential equation :

$$dx(s) = f(s,x(s),u^*(s,x(s)))ds + \sigma(s,x(s),u^*(s,x(s)))dW(t),$$

admits a unique solution x^*, given an initial condition $x(t) = x$. Then, $W = V$ and $u^*(s,x^*)$ is an optimal control for $V(t,x)$.

A proof of this verification theorem can be found in book, by Yong & Zhou [98].

1.3.2 The Pontryagin's maximum principle

Kushner [50, 51] was one of the leading mathematicians to work on stochastic maximum principle. A more develloping works have been done by a lot of research workers a mong them Bensoussan [8], Peng [76]. The Maximum Principle was first set to teal with the deterministic control problems in 1956 and we can look to it as a generalisation of the calculus of variations.

The maximum principle. The idea is to perturbe the solution and use the Taylor expansion at the state trajectory and tending the perturbation to zero :

$$\begin{cases} dx(t) &= f(t,x(t),u(t))dt,\ t\in[0,T], \\ x(0) &= x_0, \end{cases} \tag{1.10}$$

where

$$f:[0,T]\times\mathbb{R}\times\mathcal{A}\longrightarrow\mathbb{R},$$

the cost function to be minimized shoud be as follows :

$$J(u(\cdot)) \;=\; \int_0^T l(t,x(t),u(t))+g(x(T)), \tag{1.11}$$

such that

$$l:[0,T]\times\mathbb{R}\times\mathcal{A}\longrightarrow\mathbb{R},$$

$$g:\mathbb{R}\longrightarrow\mathbb{R}.$$

Where l and g are affecting the running cost and the terminal cost respectively.

We now assume that there exists a control $u^*(t)$ which is optimal, i.e.

$$J(u^*(\cdot)) \;=\; \inf_u J(u(\cdot)).$$

If $x^*(t)$ is a solution to (1.10) and $u^*(t)$ is its optimal control by perturbing the optimal control by using the spike variation methode :

$$u^\varepsilon(t)=\begin{cases} v & \text{for } \tau-\varepsilon\le t\le\tau, \\ u^*(t) & \text{otherwise.} \end{cases} \tag{1.12}$$

$x^\varepsilon(t)$ is the solution of (1.10) with respect $u^\varepsilon(t)$ we consider that $x^*(t)$ is equal $x^\varepsilon(t)$ up to $t=\tau-\varepsilon$ and that

$$\begin{aligned} x^{\varepsilon}(\tau)-x^{*}(\tau) &= (f(\tau,x^{\varepsilon}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)))\varepsilon+o(\varepsilon) \\ &= (f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)))\varepsilon+o(\varepsilon), \end{aligned} \tag{1.13}$$

where the second equality holds since $x^{\varepsilon}(\tau)-x^{*}(\tau)$ is of order ε. We look at the Taylor expansion of the state with respect to ε. Let

$$z(t)=\frac{\partial}{\partial\varepsilon}x^{\varepsilon}(t)\mid_{\varepsilon=0},$$

i.e. the Taylor expansion of $x^{\varepsilon}(t)$ is

$$x^{\varepsilon}(t) = x^{*}(t)+z(t)\varepsilon+o(\varepsilon). \tag{1.14}$$

Then, by (1.13)

$$z(\tau) = f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)). \tag{1.15}$$

By deriving and using the chain rule :

$$\begin{aligned} dz(t) &= \frac{\partial}{\partial\varepsilon}dx^{\varepsilon}(t)\mid_{\varepsilon=0} \\ &= \frac{\partial}{\partial\varepsilon}f(t,x^{\varepsilon}(t),u^{\varepsilon}(t))dt\mid_{\varepsilon=0} \\ &= f_x(t,x^{\varepsilon}(t),u^{\varepsilon}(t))\frac{\partial}{\partial\varepsilon}x^{\varepsilon}(t)dt\mid_{\varepsilon=0} \\ &= f_x(t,x^{*}(t),u^{*}(t))z(t)dt, \end{aligned}$$

if we let $l=0$ this implies that :

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial\varepsilon}J(u^{\varepsilon})\bigg|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}g(x^{\varepsilon}(T))\mid_{\varepsilon=0} \\ &= g_x(x^{\varepsilon}(T))\frac{\partial}{\partial\varepsilon}x^{\varepsilon}(T)\mid_{\varepsilon=0} \\ &= g_x(x^{*}(T))\,z(T). \end{aligned}$$

Let's now define the adjoint equation

$$\begin{cases} d\Psi(t) &= -f_x(t, x^*(t), u^*(t))\Psi(t)dt, t \in [0, T], \\ \Psi(T) &= g_x(x^*(T)). \end{cases}$$

Then it follows that

$$d(\Psi(t)z(t)) = 0,$$

i.e. $\Psi(t)z(t)) =$ constant. By the terminal condition for the adjoint equation we have

$$\Psi(t)z(t) = g_x(x^*(T))z(T) \geq 0, \text{ for all } 0 \leq t \leq T.$$

In particular, by (1.15)

$$\Psi(\tau)\left(f(\tau, x^*(\tau), v) - f(\tau, x^*(\tau), u^*(\tau))\right) \geq 0.$$

Since τ was chosen arbitrarily, this is equivalent to

$$\Psi(t)f(t, x^*(t), u^*(t)) = \inf_{v \in \mathcal{U}} \Psi(t)f(t, x^*(t), v), \text{ for all } 0 \leq t \leq T.$$

By an iterative substitution we obtain

$$H(t, x^*(t), u^*(t), \Psi(t)) = \inf_{v} H(t, x^*(t), v, \Psi(t)) \text{ for all } 0 \leq t \leq T, \tag{1.16}$$

where H is the so-called Hamiltonian (sometimes defined with a minus sign which turns the minimum condition above into a maximum condition) :

$$H(x, u, \Psi) = l(x, u) + \Psi f(x, u),$$

and the adjoint equation is given by

$$\begin{cases} d\Psi(t) &= -(l_x(t,x^*(t),u^*(t)) + f_x(t,x^*(t),u^*(t))\Psi(t))dt, \\ \Psi(T) &= g_x(x^*(T)). \end{cases} \tag{1.17}$$

Equations (1.16) and (1.17) characterize the Hamiltonian system .

Stochastic maximum principle : A stochastic optimal control is the natural extention of the deterministic one by interchanging an ODE with an SDE :

$$dx(t) = f(t,x(t),u(t))dt + \sigma(t,x(t))dW(t), t \in [0,T], \tag{1.18}$$

where f and σ are deterministic functions and the last term is an Itô integral with respect to a Brownian motion W defined on a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.
More generally, the diffusion coefficient σ may has an explicit dependence on the control : $t \in [0,T]$.

$$dx(t) = f(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t), \tag{1.19}$$

The cost function for the stochastic case is the expected value of the cost function (1.11), i.e. we want to minimize

$$J(u(\cdot)) = E\left[\int_0^T l(t,x(t),u(t)) + g(x(T))\right].$$

For the case (1.18) the adjoint equation is given by the following Backward SDE :

$$\begin{cases} -d\Psi(t) &= \{f_x(t,x^*(t),u^*(t))\Psi(t) + \sigma_x(t,x^*(t))Q(t) \\ & \quad +(l_x(t,x^*(t),u^*(t))\}dt - Q(t)dW(t), \\ \Psi(T) &= g_x(x^*(T)). \end{cases} \tag{1.20}$$

A solution to this backward SDE is a pair $(\Psi(t), Q(t))$ which satisfies (1.20). The Hamiltonian is

$$H(x,u,\Psi(t),Q(t)) \quad = \quad l(t,x,u)+\Psi(t)f(t,x,u)+Q(t)\sigma(t,x),$$

and the maximum principle reads for all $0 \leq t \leq T$,

$$H(t,x^*(t),u^*(t),\Psi(t),Q(t)) \quad = \quad \inf_{u\in\mathcal{U}} H(t,x^*(t),u,\Psi(t),Q(t)) \ \ \mathbb{P}-\text{a.s.} \tag{1.21}$$

Noting that there is also third case : if the state is given by (1.19) but the action space $\mathcal{A}$ is assumed to be convex, it is possible to derive the maximum principle in a local form. This is accomplished by using a convex perturbation of the control instead of a spike variation, see Bensoussan 1983 [8]. The necessary condition for optimality is then given by the following : for all $0 \leq t \leq T$

$$E\int_0^T H_u(t,x^*(t),u^*(t),\Psi^*(t),Q^*(t))\,(u-u^*(t)) \geq 0.$$

1.4 Some classes of stochastic controls

Let $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$ be a complete filtred probability space.

1. Admissible control An admissible control is $\mathcal{F}_t$-adapted process $u(t)$ with values in a borelian $A \subset \mathbb{R}^n$

$$\mathcal{U} := \{u(\cdot) : [0,T]\times\Omega \to A : u(t) \text{ is } \mathcal{F}_t\text{-adapted}\}. \tag{1.22}$$

2. Optimal control The optimal control problem consists to minimize a cost functional $J(u)$ over the set of admissible control $\mathcal{U}$. We say that the control $u^*(\cdot)$ is an optimal control if

$$J(u^*(t)) \leq J(u(t)), \text{ for all } u(\cdot)\in\mathcal{U}.$$

3. Near-optimal control Let $\varepsilon > 0$, a control is a near-optimal control (or ε-optimal) if for all control $u(\cdot) \in \mathcal{U}$ we have

$$J(u^{\varepsilon}(t)) \leq J(u(t)) + \varepsilon. \tag{1.23}$$

See for some applications.

4. Singular control An admissible control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$adapted processes, such that $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0_-) = 0$. Since $d\xi(t)$ may be singular with respect to Lebesgue measure dt, we call $\xi(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part.

5. Feedback control : We say that $u(\cdot)$ is a feedback control if $u(\cdot)$ depends on the state variable $X(\cdot)$. If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then $u(\cdot)$ is a feedback control if $u(\cdot)$ is $\mathcal{F}_t^X-$adapted.

6. Impulsive control. Impulse control : Here one is allowed to reset the trajectory at stopping times (τ_i) from $X_{\tau_{i-}}$ (the value immediately before i) to a new (non-anticipative) value X_{τ_i}, resp., with an associated cost $L\left(X_{\tau_{i-}}, X_{\tau_i}\right)$. The aim of the controlled is to minimizes the cost functional :

$$\begin{aligned} &E \int_0^T \exp\left[-\int_0^t C(X(s), u(s))ds\right] K(X(t), u(t)) \\ &+ \sum_{\tau_i < T} \exp\left[-\int_0^{\tau_i} C(X(s), u(s))ds\right] g(X_\tau, X_{\tau_{i-}}) \\ &+ \exp\left[-\int_0^{\tau_i} C(X(s), u(s))ds\right] h(X(T)). \end{aligned}$$

7. Ergodic control Some stochastic systems may exhibit over a long period a stationary behavior characterized by an invariant measure. This measure, if it does exists, is obtained by the average of the states over a long time. An ergodic control problem consists in optimizing over the long term some criterion taking into account this invariant measure. (See Pham [77], Borkar [10]). The cost functional is given by

$$\lim \sup_{T \to +\infty} \frac{1}{T} E \int_0^T f(x(t), u(t)) dt.$$

8. Robust control In the problems formulated above, the dynamics of the control system is assumed to be known and fixed. Robust control theory is a method to measure the performance changes of a control system with changing system parameters. This is of course important in engineering systems, and it has recently been used in finance in relation with the theory of risk measure. Indeed, it is proved that a coherent risk measure for an uncertain payoff $x(T)$ at time T is represented by :

$$\rho(-X(t)) = \sup_{Q \in S} E^Q(X(T)),$$

where S is a set of absolutly continuous probability measures with respect to the original probability P.

9. Partial observation control problem It is assumed so far that the controller completely observes the state system. In many real applications, he is only able to observe partially the state via other variables and there is noise in the observation system. For example in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. We are facing a partial observation control problem. This may be formulated in a general form as follows : we have a controlled signal (unobserved) process governed by the following SDE :

$$dx(t) = f(t, x(t), y(t), u(t)) dt + \sigma(t, x(t), y(t), u(t)) dW(t),$$

and

$$dy\,(t) = g\,(t, x\,(t)\,, y\,(t)\,, u\,(t))\,dt + h\,(t, x\,(t)\,, y\,(t)\,, u\,(t))\,dB\,(t)\,,$$

where $B\,(t)$ is another Brownian motion, eventually correlated with $W\,(t)$. The control $u(t)$ is adapted with respect to the filtration generated by the observation F_t^Y and the functional to optimize is :

$$J\,(u\,(\cdot)) = E\left[h\,(x\,(T)\,, y(T)) + \int_0^T g\,(t, x\,(t)\,, y(t), u\,(t))\,dt\right].$$

10. Random horizon In classicla problem, the time horizon is fixed until a deterministic terminal time T. In some real applications, the time horizon may be random, the cost functional is given by the following :

$$J\,(u\,(\cdot)) = E\left[h\,(x\,(\tau)) + \int_0^\tau g\,(t, x\,(t)\,, y(t), u\,(t))\,dt\right],$$

where τ s a finite random time.

11. Relaxed control The idea is then to compactify the space of controls $\mathcal{U}$ by extending the definition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t\,(du)\,dt$, where μ_t is a probability measure, is the closure under weak* topology of the measures $\delta_{u(t)}(du)dt$ corresponding to usual, or strict, controls. This notion of relaxed control is introduced for deterministic optimal control problems in Young (*Young, L.C. Lectures on the calculus of variations and optimal control theory, W.B. Saunders Co., 1969.*) (See Borkar [10]).

CHAPITRE II

A study on optimal control problem with $\varepsilon^{\lambda}-$error bound for stochastic systems with applications to linear quadratic problem

Chapitre 2

A study on optimal control problem with $\varepsilon^\lambda-$error bound for stochastic systems with applications to linear quadratic problem

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Abstract. In this part, we study near-optimal stochastic control problem with $\varepsilon^\lambda-$error bound for systems governed by nonlinear controlled Itô stochastic differential equations (SDEs in short). The control is allowed to enter into both drift and diffusion coefficients and the control domain need be convex. The proof of our main result is based on Ekeland's variational principle and some approximation arguments on the state variable and adjoint process with respect to the control variable. Finally, as an example, the linear quadratic control problem is given to illustrate our theoretical results.

AMS Subject Classification : *93E20, 60H10.*

Keywords : Stochastic control with $\varepsilon^\lambda-$error bound, Weak maximum principle, Necessary and sufficient of conditions of near-optimality, Ekeland's variational principle, Convex

perturbation.

2.1 Introduction

Stochastic near-optimization is as sensible and important as optimization for both theory and applications. In this work, we consider stochastic control problem with $\varepsilon^{\lambda}-$error bound for systems driven by non linear controlled SDEs of the form

$$\begin{cases} dx\left(t\right)=f\left(w,t,x\left(t\right),u\left(t\right)\right)dt+\sigma\left(w,t,x\left(t\right),u\left(t\right)\right)dW\left(t\right), \\ x(0)=\xi, \end{cases} \tag{2.1}$$

where $\left(W\left(t\right)\right)_{t\in[0,T]}$ is a standard $n-$dimensional Brownian motion defined on the filtered probability space $\left(\Omega,\mathcal{F},\left(\mathcal{F}_t\right)_{t\in[0,T]},P\right)$. The filtration $\mathcal{F}_t$ is a canonical filtration of $W\left(t\right)$ augmented by P-null sets. The initial condition ξ is an $\mathcal{F}_0$-measurable random variable. We associate to this state equation the following cost functional

$$J\left(u\left(\cdot\right)\right)=E\left[h\left(x\left(T\right)\right)+\int_0^T g\left(w,t,x\left(t\right),u\left(t\right)\right)dt\right], \tag{2.2}$$

and the value function is defined as

$$V=\inf_{u(\cdot)\in\mathcal{U}}\left\{J\left(u\left(\cdot\right)\right)\right\}. \tag{2.3}$$

The maximum principle has been and remains an important tool in many situations in which optimal control plays a role. Near-optimization is as sensible and important as optimization for both theory and applications. The theory of stochastic near-optimization was introduced by Zhou [100]. Various kinds of near-optimal stochastic control problems have been investigated in [31, 28, 32, 37, 43, 44, 88, 99, 48]. The necessary and sufficient conditions of near-optimal mean-field singular stochastic control have been studied in Hafayed and Abbas [31]. The necessary and sufficient conditions for near-optimality for mean-field jump diffusions with applications have been derived by Hafayed, Abba and Abbas [28]. Near-optimality necessary and sufficient conditions for singular controls in jump diffusion processes have been investigated in Hafayed and Abbas [32]. In Hafayed, Veverka and Abbas [37], the authors extended Zhou's maximum principle of near-optimality [100] to singular stochastic control. The near-optimal stochastic control problem for jump diffusions has been investigated by Hafayed, Abbas and Veverka [43]. The near-optimality necessary and sufficient conditions for classical controlled FBSDEJs with applications to finance have been investigated in Hafayed, Veverka and Abbas [44]. Stochastic maximum principle of near-optimal control of fully coupled forward-backward stochastic differential equation has been investigated in Tang [88]. Near-optimal stochastic control problem for linear general controlled FBSDEs has been studied in Zhang, Huang and Li [99]. The near-optimal control problem for recursive stochastic problem has been studied in Hui, Huang, Li and Wang [48].

It is shown that the near-optimal controls in stochastic control problems, as the alternative to the exact optimal ones, are of great importance for both the theoretical analysis and practical application purposes due to its nice structure and broad-range availability as well as feasibility. The near-optimal controls in stochastic control problems are more available than the exact optimal ones, in the sense that the near-optimal controls always exist, while the exact optimal stochastic controls may not even exist in many situations. Moreover, since there are many near-optimal controls, it is possible to select among them appropriate ones that are easier for analysis and implementation. This justifies the use of near-optimal stochastic controls, which exist under minimal hypothesis and are sufficient in most practical cases.

Motivated by the arguments above and inspired by [100, 31, 28, 32, 43, 99], our purpose in this work is to derive a first-order necessary and sufficient conditions for any near-optimal stochastic control with $\varepsilon^{\lambda}-$error bound, where the diffusion coefficient can contain a control variable, and the control domain is necessarily convex. The proof of our main result is based on Ekeland's variational principle [21] and some approximation arguments on the state variable and adjoint process with respect to the control variable. As an applications, a linear quadratic control problem is discussed.

The rest of the chapter is organized as follows. In the second section we present the assumptions and the formulation of the problem. The necessary conditions for any near-optimal stochastic control is given in the third section. The sufficient conditions are given in the fourth section. An application to the linear quadratic control problem is given in the last section.

2.2 Assumptions and Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a fixed filtered probability space satisfying the usual conditions, in which a $n-$dimentional Brownian motion $W(t)$ is defined. We list some notations that will be used throughout this work. Any element $x \in \mathbb{R}^d$ will be identified to a column vector with i^{th} component, and the norm $|x| = \sum_{i=1}^{d} |x_i|$. We denote $\mathcal{A}^*$ the transpose of any vector or matrix $\mathcal{A}$. We denote $sgn(\cdot)$ the sign function. For a function Ψ, we denote by Ψ_x the gradient or Jacobian of a scalar function Ψ with respect to the variable x. We denote by $\mathbb{L}^2_{\mathcal{F}}([0,T], \mathbb{R}^n)$ the Hilbert space of $\mathcal{F}_t-$adapted processes $(x(t))$ such that $E\int_0^T |x(t)|^2 dt < +\infty$.

Throughout this work we assume the following.

Let $\sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$, $f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $g : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $h : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$, are Borel measurable functions such that $\forall (w,t,x,y,u) \in \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$.

Assumption (H1) f, σ, g and h are continuously differentiable with respect to x, u, dominated by $C(1+|x|)$, and their derivatives are bounded functions.

Assumption (H2) $\left|\frac{\partial \rho}{\partial u}(w,t,x,u) - \frac{\partial \rho}{\partial u}(w,t,y,v)\right| \leq C(|x-y|^\beta + |u-v|^\beta)$, for $\rho := f$, σ

and $\beta \in (0,1)$.

Assumption (H3) The derivatives $\frac{\partial f}{\partial x}, \frac{\partial \sigma}{\partial x}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial u}$ are Lipschitz in x, u and h_x is Lipschitz in x.

Definition 1.2.1.. Let $T > 0$ be a fixed strictly positive real number and $\mathbb{U}$ be a nonempty compact convex subset of $\mathbb{R}^m$. An admissible control is defined as a function $u(\cdot) : [0,T] \times \Omega \longrightarrow \mathbb{U}$ which is $\mathcal{F}_t$-predictable, such that the SDE-(2.1) has a unique solution and write $u(\cdot) \in \mathcal{U}$. The set $\mathcal{U}$ is called the set of admissible controls.

From assumption (H1), the SDE-(2.1) has a unique strong solution given by

$$x(t) = \xi + \int_0^t f(w, s, x(s), u(s))\, ds + \int_0^t \sigma(w, s, x(s), u(s))\, dW(s).$$

The criteria to be minimized over the set of admissible controls given in (3.2) is well defined.

We introduce the adjoint equation for our control problem (2.1)-(3.2) as follows

$$\begin{cases} -dp(t) = [\frac{\partial f^*}{\partial x}(w,t,x(t),u(t))\, p(t) + \frac{\partial \sigma^*}{\partial x}(w,t,x(t),u(t))\, q(t) \\ \qquad\qquad + \frac{\partial g}{\partial x}(w,t,x(t),u(t))]dt - q(t)\, dW(t), \\ p(T) = h_x(x(T)), \end{cases} \tag{2.4}$$

and the Hamiltonian associated with our control problem (2.1)-(3.2) is given as

$$H(t,x,u,p(t),q(t)) = p(t)\, f(t,x,u) + q(t)\, \sigma(t,x,u) + g(t,x,u). \tag{2.5}$$

To simplify our notation, we suppress "w" in $f(w,t,x(t),u(t))$ and write $f(t,x(t),u(t))$ for $f(w,t,x(t),u(t))$. Similarly for the functions f, σ, g, h.

We aim at using Ekeland's variational principle [21] to establish necessary conditions of ε-optimality satisfied by a sequence of ε-optimal controls.

Lemma 1.2.1. (Ekeland's Lemma [21]) *Let $(E,\ d)$ be a complete metric space and f :*

$E \to \overline{\mathbb{R}}$ be a lower semi-continuous and bounded from below. If for each $\varepsilon > 0$, there exists $u^{\varepsilon} \in E$ satisfies $f(u^{\varepsilon}(\cdot)) \leq \inf_{u(\cdot)\in E}(f(u(\cdot))) + \varepsilon$. Then for any $\delta > 0$, there exists $u^{\delta}(\cdot) \in E$ such that

(1) $f\left(u^{\delta}(\cdot)\right) \leq f(u^{\varepsilon}(\cdot))$.

(2) $d\left(u^{\delta}(\cdot), u^{\varepsilon}(\cdot)\right) \leq \delta$.

(3) $f\left(u^{\delta}(\cdot)\right) \leq f(u(\cdot)) + \frac{\varepsilon}{\delta} d\left(u^{\delta}(\cdot), u(\cdot)\right)$, for all $u(\cdot) \in E$.

To apply Ekeland's variational principle to our problem, we must define a distance d on the space of admissible controls such that $(\mathcal{U}, d)$ becomes a complete metric space. For any $u(\cdot)$, $v(\cdot) \in \mathcal{U}$ we lay

$$d(u(\cdot), v(\cdot)) = \left[E \int_0^T |u(t) - v(t)|^2 \, dt\right]^{\frac{1}{2}}. \tag{2.6}$$

2.3 Stochastic maximum principle with $\varepsilon^{\lambda}-$error bound

Our goal in this section is to derive necessary conditions with $\varepsilon^{\lambda}-$error bound for SDEs with controlled diffusion coefficient, where the control domain is necessarily convex. We give the definition of ε-optimal control as given in [100].

Definition 1.3.1. For a given $\varepsilon > 0$ the admissible control $u^{\varepsilon}(\cdot)$ is ε-optimal if

$$|J(u^{\varepsilon}(\cdot)) - V| \leq \mathcal{Q}(\varepsilon),$$

where $\mathcal{Q}$ is a function of ε satisfying $\lim_{\varepsilon\to 0} \mathcal{Q}(\varepsilon) = 0$. The estimater $\mathcal{Q}(\varepsilon)$ is called an error bound. If $\mathcal{Q}(\varepsilon) = C\varepsilon^{\delta}$ for some $\delta > 0$ independent of the constant C, then $u^{\varepsilon}(\cdot)$ is called ε-optimal control with order ε^{δ}. If $\mathcal{Q}(\varepsilon) = \varepsilon$, the admissible control $u^{\varepsilon}(\cdot)$ called $\varepsilon-$optimal.

Now we are able to state and prove the Pontryagin's maximum principle of ε-optimality for our control problem, which is the main result in this section.

Theorem 1.3.1. *Assume that (H1), (H2) and (H3) hold. For any $\lambda \in \left[0, \frac{1}{2}\right)$, there exists* *a positive constant* $C = C(\lambda)$ *such that for each* $\varepsilon > 0$ *and any* ε*-optimal control* $u^{\varepsilon}(\cdot)$ *there exists a constant* $C > 0$ *such that for all* $u \in \mathbb{U}$

$$E\int_0^T \frac{\partial H}{\partial u}\left(t, x^{\varepsilon}(t), u^{\varepsilon}(t), p^{\varepsilon}(t), q^{\varepsilon}(t)\right)\left(u(t) - u^{\varepsilon}(t)\right)dt \geq -C\varepsilon^{\lambda},\ dt - a.e., \tag{2.7}$$

where $x^{\varepsilon}(\cdot)$ *denotes the solution of the state equation (2.1) and the pair* $\left(p^{\varepsilon}(\cdot), q^{\varepsilon}(\cdot)\right)$ *is the solution of the adjoint equation (3.4) associated with* u^{ε}.

To prove the above Theorem, we need the following auxiliary results on the variation of the state and adjoint processes with respect to the control variable.

Lemma 1.3.2. *Let* $x^{u}(t)$ *and* $x^{v}(t)$ *be the solution of the state equation (2.1) associated with* $u(\cdot)$ and $v(\cdot)$ *respectively. Then there exists a positive constant* C *such that, for* $\alpha > 0:$

$$E\left[\sup_{0\leq t\leq T}\left|x^{u}(t) - x^{v}(t)\right|^{\alpha}\right] \leq Cd^{\frac{\alpha}{2}}\left(u(\cdot), v(\cdot)\right).$$

Proof.

First we assume $\alpha \geq 2$. Using Hölder's and Burkholder-Davis-Gundy inequalities, we obtain

$$E\left[\left|x^{u}(t) - x^{v}(t)\right|^{\alpha}\right] \leq E\Big|\textstyle\int_0^t (f(s, x^{u}(s), u(s)) - f(s, x^{v}(s), v(s)))ds$$

$$+\textstyle\int_0^t \left(\sigma(s, x^{u}(s), u(s)) - \sigma(s, x^{v}(s), v(s))\right)dW(s)\Big|^{\alpha}$$

$$\leq CE\textstyle\int_0^t \left|f(s, x^{u}(s), u(s)) - f(s, x^{v}(s), v(s))\right|^{\alpha} ds$$

$$+CE\textstyle\int_0^t \left|\sigma(s, x^{u}(s), u(s)) - \sigma(s, x^{v}(s), v(s))\right|^{\alpha} ds$$

by adding and subtracting $f(s, x^{v}(s), u(s))$, $\sigma(s, x^{v}(s), u(s))$ and applying the Lipschitz continuity of the coefficients f and σ it holds that

$$E\left[|x^u(t)-x^v(t)|^\alpha\right] \leq CE\int_0^t |x^u(s)-x^v(s)|^\alpha\, ds + CE\int_0^t |u(s)-v(s)|^\alpha\, ds$$
$$\leq CE\int_0^t |x^u(s)-x^v(s)|^\alpha\, ds + C\left[E\int_0^t |u(s)-v(s)|^2\, ds\right]^{\frac{\alpha}{2}},$$

using Gronwall's inequality, we get the desired inequality.

Now, we assume $0<\alpha<2$. Since $\frac{2}{\alpha}>1$, then by using Hölder's inequality and the above result, we have

$$E\left[|x^u(t)-x^v(t)|^\alpha\right] \leq \left[E\,|x^u(t)-x^v(t)|^2\right]^{\frac{\alpha}{2}} \leq Cd^{\frac{\alpha}{2}}(u(\cdot),v(\cdot)).$$

This completes the proof of Lemma 1.3.2. □

Lemma 1.3.3. *Let $(p^u(t), q^u(t))$ and $(p^v(t), q^v(t))$ be two adjoint processes corresponding to u and v respectively. Then we have the following estimate : for any $\alpha \geq 1$*

$$E\int_0^T (|p^u(t)-p^v(t)|^\alpha + |q^u(t)-q^v(t)|^\alpha)dt \leq Cd^\alpha(u(\cdot),v(\cdot)).$$

Proof. First we denote by $p(t)=(p^u(t)-p^v(t))$ and $q(t)=(q^u(t)-q^v(t))$, then $(\tilde{p}(t),\tilde{q}(t))$ satisfies the following backward stochastic differential equation :

$$\begin{cases} -d\tilde{p}(t) = [\frac{\partial f^*}{\partial x}(t,x^u(t),u(t))\,\tilde{p}(t) + \frac{\partial \sigma^*}{\partial x}(t,x^u(t),u(t))\,\tilde{q}(t) \\ \qquad + G(t)]dt - \tilde{q}(t)\,dW(t), \\ \tilde{p}(t) = h_x(x^u(T)) - h_x(x^v(T)), \end{cases}$$

where the process $G(t)$ is given by

$$G(t) = (\tfrac{\partial f}{\partial x}(t, x^u(t), u(t)) - \tfrac{\partial f}{\partial x}(t, x^v(t), v(t)))p^v(t)$$
$$+(\tfrac{\partial \sigma}{\partial x}(t, x^u(t), u(t)) - \tfrac{\partial \sigma}{\partial x}(t, x^v(t), v(t)))q^v(t)$$
$$+(\tfrac{\partial g}{\partial x}(t, x^u(t), u(t)) - \tfrac{\partial g}{\partial x}(t, x^v(t), v(t))).$$

Let η be the solution of the following linear SDE

$$\begin{cases} d\eta_t = [\frac{\partial f}{\partial x}(t, x^u(t), u(t))\,\eta_t + |\tilde{p}(t)|^{\alpha-1} sgn(\tilde{p}(t))]dt \\ +[\frac{\partial \sigma}{\partial x}(t, x^u(t), u(t))\,\eta_t + |\tilde{q}(t)|^{\alpha-1} sgn(\tilde{q}(t))]dW(t), \\ \eta_0 = 0, \end{cases} \tag{2.8}$$

where $sgn(y) \equiv (sgn(y_1), sgn(y_2), ..., sgn(y_n))^*$ for any vector $y = (y_1, y_2, .., y_n)^*$. It is worth mentioning that since $\frac{\partial f}{\partial x}$ and $\frac{\partial \sigma}{\partial x}$ are bounded and the fact that

$$E\int_0^T \left\{ \left| |\tilde{p}(t)|^{\alpha-1} sgn(\tilde{p}(t)) \right|^2 + \left| |\tilde{q}(t)|^{\alpha-1} sgn(\tilde{q}(t)) \right|^2 \right\} dt < \infty,$$

then the SDE (2.8) has a unique strong solution. Let $\gamma \geq 2$ such that $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$ then we get

$$\begin{aligned} &E\left\{\sup_{t\leq T} |\eta_t|^\gamma\right\} \leq CE\int_0^T \left\{|\tilde{p}(t)|^{\alpha\gamma-\gamma} + |\tilde{q}(t)|^{\alpha\gamma-\gamma}\right\} dt \\ &= CE\int_0^T \{|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha\} dt. \end{aligned} \tag{2.9}$$

Now applying Itô's formula to $p(t)\,\eta_t$ on $[0, T]$ and taking expectations, we obtain

$$E(\tilde{p}(t)\,\eta_T - \tilde{p}(0)\eta_0) = E\int_0^T -G(t)\,\eta_t dt + E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt,$$

using the fact that $\eta_0 = 0$ we can easily show that

$$E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt = E\int_0^T G(t)\,\eta_t dt + E(\tilde{p}(T)\,\eta_T)$$
$$= E\int_0^T G(t)\,\eta_t dt + E\left[(h_x(x^u(T)) - h_x(x^v(T)))\,\eta_T\right],$$

by applying Hölder's inequality to the right hand side, it holds that

$$E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt \leq \left[E\int_0^T |G(t)|^\alpha dt\right]^{\frac{1}{\alpha}} \left[E\int_0^T |\eta_t|^\gamma dt\right]^{\frac{1}{\gamma}}$$
$$+ \left[E\,|h_x(x^u(T)) - h_x(x^v(T))|^\alpha\right]^{\frac{1}{\alpha}} \left[E\,|\eta_T|^\gamma\right]^{\frac{1}{\gamma}}$$

using inequality (2.9), it holds that

$$E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt \leq C\left[E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt\right]^{\frac{1}{\gamma}} \left\{\left[E\int_0^T |G(t)|^\alpha dt\right]^{\frac{1}{\alpha}}\right.$$
$$\left. + C\left[E\,|h_x(x^u(T)) - h_x(x^v(T))|^\alpha\right]^{\frac{1}{\alpha}}\right\},$$

which implies that

$$\left[E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt\right]^{1-\frac{1}{\gamma}} \leq C\left[E\int_0^T |G(t)|^\alpha dt\right]^{\frac{1}{\alpha}}$$
$$+ C\left[E\,|h_x(x^u(T)) - h_x(x^v(T))|^\alpha\right]^{\frac{1}{\alpha}},$$

thus

$$E\int_0^T (|\tilde{p}(t)|^\alpha + |\tilde{q}(t)|^\alpha)dt \leq CE\int_0^T |G(t)|^\alpha dt$$
$$+ CE\,|h_x(x^u(T)) - h_x(x^v(T))|^\alpha.$$

Since h_x is Lipschitz in x and due to Lemma 1.3.2, we have

$$E\left\{|h_x(x^u(T)) - h_x(x^v(T))|^\alpha\right\} \leq Cd^\alpha(u(\cdot), v(\cdot)). \tag{2.10}$$

We proceed to estimate the first term on the right hand side, then we have

$$\begin{aligned}
E\int_0^T |G(t)|^\alpha dt \leq E\int_0^T &\left\{\left|\frac{\partial f}{\partial x}(t, x^u(t), u(t)) - \frac{\partial f}{\partial x}(t, x^v(t), v(t))\right| |p^v(t)|\right. \\
&+ \left|\frac{\partial \sigma}{\partial x}(t, x^v(t), u(t)) - \frac{\partial \sigma}{\partial x}(t, x^v(t), v(t))\right| |q^v(t)| + \left|\frac{\partial g}{\partial x}(t, x^u(t), u(t))\right. \\
&- \left.\left.\left|\frac{\partial g}{\partial x}(t, x^v(t), v(t))\right|\right|\right\}^\alpha dt
\end{aligned}$$

$$\begin{aligned}
&\leq CE\int_0^T \left|\frac{\partial f}{\partial x}(t, x^u(t), u(t)) - \frac{\partial f}{\partial x}(t, x^v(t), v(t))\right|^\alpha |p^v(t)|^\alpha dt \\
&+ CE\int_0^T \left|\frac{\partial \sigma}{\partial x}(t, x^u(t), u(t)) - \frac{\partial \sigma}{\partial x}(t, x^v(t), v(t))\right|^\alpha |q^v(t)|^\alpha dt \\
&+ CE\int_0^T \left|\frac{\partial g}{\partial x}(t, x^u(t), u(t)) - \frac{\partial g}{\partial x}(t, x^v(t), v(t))\right|^\alpha dt \\
&= \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.
\end{aligned}$$

Using the bounded of $p^v(t)$ and Hölder's inequality with $\frac{1}{2/(2-\alpha)} + \frac{1}{2/\alpha} = 1$ we have

$$\begin{aligned}
\mathbb{I}_1 \leq C &\left[E\int_0^T \left|\frac{\partial f}{\partial x}(t, x^u(t), u(t)) - \frac{\partial f}{\partial x}(t, x^v(t), v(t))\right|^{\frac{2\alpha}{2-\alpha}} dt\right]^{1-\frac{\alpha}{2}} \\
&\times \left[E\int_0^T |p^v(t)|^2 dt\right]^{\frac{\alpha}{2}},
\end{aligned}$$

adding and subtracting $\frac{\partial f}{\partial x}(t, x^u, v)$, then by using the Lipschitz continuity on $\frac{\partial f}{\partial x}(t, x^u, v)$ in x and u (Assumption (H3)) and Lemma 1.3.2, we have

$$\mathbb{I}_1 \leq Cd^\alpha(u(\cdot), v(\cdot)).$$

Using similar argument developed above, we can prove $\mathbb{I}_2 + \mathbb{I}_3 \leq Cd^{\alpha}\left(u(\cdot), v\left(\cdot\right)\right).$ Then we conclude

$$E\int_0^T \left|G\left(t\right)\right|^{\alpha} dt \leq Cd^{\alpha}\left(u\left(\cdot\right), v\left(\cdot\right)\right). \tag{2.11}$$

Finally, combining (2.10) and (2.11), the proof of Lemma 1.3.3 is complete □

Lemma 1.3.4 *(Maximum principle for ε-optimality). For each $\varepsilon > 0$ there exists $\overline{u}^{\varepsilon}\left(\cdot\right) \in \mathcal{U}$* processes $\overline{p}^{\varepsilon}(t)$ and $\overline{q}^{\varepsilon}(t)$ *such that, $\forall u\left(\cdot\right) \in \mathcal{U}$*

$$E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u\left(t\right) - \overline{u}^{\varepsilon}\left(t\right)\right) dt \geq -C\varepsilon^{\lambda},\ dt - a.e. \tag{2.12}$$

Proof. Applying Ekeland's variational principle with $\delta = \varepsilon^{1/2}$ there exists an admissible control $\overline{u}^{\varepsilon}$ such that

(i) $d\left(\overline{u}^{\varepsilon}\left(\cdot\right), u^{\varepsilon}\left(\cdot\right)\right) \leq \varepsilon^{1/2}$,

(ii) $\overline{J}\left(\overline{u}^{\varepsilon}\left(\cdot\right)\right) \leq \overline{J}\left(u\left(\cdot\right)\right)$, for any $u\left(\cdot\right) \in \mathcal{U}$ where

$$\bar{J}\left(u\left(\cdot\right)\right) := J\left(u\left(\cdot\right)\right) + \varepsilon^{1/2} d\left(\overline{u}^{\varepsilon}\left(\cdot\right), u^{\varepsilon}\left(\cdot\right)\right). \tag{2.13}$$

Notice that $\overline{u}^{\varepsilon}\left(\cdot\right)$ which is ε-optimal for the initial cost J is optimal for the new cost $\overline{J}$ defined by (2.13).

Let us denote $u^{\varepsilon,\theta}\left(\cdot\right)$ a perturbed control given by $\overline{u}^{\varepsilon,\theta}(t) = \overline{u}^{\varepsilon}(t) + \theta\left(v\left(t\right) - \overline{u}^{\varepsilon}\left(t\right)\right).$ By using the fact that

(i) $\overline{J}\left(\overline{u}^{\varepsilon}\left(\cdot\right)\right) \leq \overline{J}(u^{\varepsilon,\theta}\left(\cdot\right))$, (ii) $d(\overline{u}^{\varepsilon}\left(\cdot\right), u^{\varepsilon,\theta}\left(\cdot\right)) \leq C\theta$, we get

$$J(u^{\varepsilon,\theta}\left(\cdot\right)) - J(\overline{u}^{\varepsilon}\left(\cdot\right)) \geq -\varepsilon^{1/2} d(u^{\varepsilon}\left(\cdot\right), u^{\varepsilon,\theta}\left(\cdot\right)) \geq -C\varepsilon^{1/2}\theta. \tag{2.14}$$

Dividing (2.14) by θ and sending θ to zero we get

$$\frac{d}{d\theta}\left(J(u^{\varepsilon,\theta}\left(t\right))\right)\Big|_{\theta=0} \geq -C\varepsilon^{\frac{1}{2}} \geq -C\varepsilon^{\lambda}. \tag{2.15}$$

Arguing as in [8] for the left hand side of inequality (2.15), the desired result follows □

Proof of Theorem 1.3.1.

First, for each $\varepsilon > 0$ by using Lemma 1.3.4, there exists $\overline{u}^{\varepsilon}(\cdot)$ and $\mathcal{F}_t-$adapted processes $\overline{p}^{\varepsilon}(t)$ and $\overline{q}^{\varepsilon}(t)$ such that, $\forall u(\cdot) \in \mathcal{U}$:

$$E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u(t) - \overline{u}^{\varepsilon}(t)\right) dt \geq -C\varepsilon^{\lambda},\ dt - a.e.$$

Now, to prove (3.11) it remains to estimate the following difference :

$$\begin{aligned}
&E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u(t) - \overline{u}^{\varepsilon}(t)\right)dt \\
&- E\int_0^T \frac{\partial H}{\partial u}\left(t, x^{\varepsilon}, u^{\varepsilon}, p^{\varepsilon}(t), q^{\varepsilon}(t)\right)\left(u(t) - u^{\varepsilon}(t)\right)dt.
\end{aligned}$$

First, by adding and subtracting $E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u(t) - u^{\varepsilon}(t)\right)dt$, we have

$$\begin{aligned}
&E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u(t) - \overline{u}^{\varepsilon}(t)\right)dt \\
&- E\int_0^T \frac{\partial H}{\partial u}\left(t, x^{\varepsilon}, u^{\varepsilon}, p^{\varepsilon}(t), q^{\varepsilon}(t)\right)\left(u(t) - u^{\varepsilon}(t)\right)dt \\
&\leq E\int_0^T \frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right)\left(u^{\varepsilon}(t) - \overline{u}^{\varepsilon}(t)\right)dt \\
&+ E\int_0^T \Big(\frac{\partial H}{\partial u}\left(t, \overline{x}^{\varepsilon}, \overline{u}^{\varepsilon}, \overline{p}^{\varepsilon}(t), \overline{q}^{\varepsilon}(t)\right) - \frac{\partial H}{\partial u}\left(t, x^{\varepsilon}, u^{\varepsilon}, p^{\varepsilon}(t), q^{\varepsilon}(t)\right)\Big) \\
&\times \left(u(t) - u^{\varepsilon}(t)\right)dt \\
&= \mathbb{I}_1 + \mathbb{I}_2,
\end{aligned}$$

by using Schwarz inequality and the bounded of $\frac{\partial H}{\partial u}$ in integral sense, we get

$$\begin{aligned}\mathbb{I}_1 &\leq E\int_0^T \left|\frac{\partial H}{\partial u}\left(t,\overline{x}^{\varepsilon},\overline{u}^{\varepsilon},\overline{p}^{\varepsilon}\left(t\right),\overline{q}^{\varepsilon}\left(t\right)\right)\right| \left|\left(u^{\varepsilon}\left(t\right)-\overline{u}_t^{\varepsilon}\right)\right| dt\\ &\leq \left[E\left\{\int_0^T \left|\frac{\partial H}{\partial u}\left(t,\overline{x}^{\varepsilon},\overline{u}^{\varepsilon},\overline{p}^{\varepsilon}\left(t\right),\overline{q}^{\varepsilon}\left(t\right)\right)\right|^2 dt\right\}\right]^{\frac{1}{2}} \left[E\left\{\int_0^T \left|\left(u^{\varepsilon}\left(t\right)-\overline{u}_t^{\varepsilon}\right)\right|^2 dt\right\}\right]^{\frac{1}{2}}\\ &\leq Cd\left(u^{\varepsilon}(\cdot),\overline{u}^{\varepsilon}(\cdot)\right)\leq C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Let us turn to the second term, it holds that

$$\begin{aligned}\mathbb{I}_2 &= E\int_0^T \left(\frac{\partial H}{\partial u}\left(t,\overline{x}^{\varepsilon},\overline{u}^{\varepsilon},\overline{p}^{\varepsilon}\left(t\right),\overline{q}^{\varepsilon}\left(t\right)\right)-\frac{\partial H}{\partial u}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}\left(t\right),q^{\varepsilon}\left(t\right)\right)\right)\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)dt\\ &= E\int_0^T [\overline{p}^{\varepsilon}(t)\frac{\partial f}{\partial u}\left(t,\overline{x}^{\varepsilon}(t),\overline{u}^{\varepsilon}(t)\right)-p^{\varepsilon}\left(t\right)\frac{\partial f}{\partial u}\left(t,x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)]\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)dt\\ &+E\int_0^T [\overline{q}^{\varepsilon}(t)\frac{\partial \sigma}{\partial u}\left(t,\overline{x}^{\varepsilon}\left(t\right),\overline{u}^{\varepsilon}(t)\right)-q^{\varepsilon}\left(t\right)\frac{\partial \sigma}{\partial u}\left(t,x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)]\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)dt\\ &+E\int_0^T [\frac{\partial g}{\partial u}\left(t,\overline{x}^{\varepsilon}\left(t\right),\overline{u}^{\varepsilon}(t)\right)-\frac{\partial g}{\partial u}\left(t,x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)]\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)dt\\ &= \mathbb{J}_1+\mathbb{J}_2+\mathbb{J}_3.\end{aligned}$$

We estimate the first term on the right hand side $\mathbb{J}_1$ by adding and subtracting $p^{\varepsilon}\left(t\right)\frac{\partial f}{\partial u}\left(t,\overline{x}^{\varepsilon}(t),\overline{u}^{\varepsilon}(t)\right)$ then we have

$$\begin{aligned}&\mathbb{J}_1 \leq E\textstyle\int_0^T \left|\overline{p}^{\varepsilon}(t)-p^{\varepsilon}\left(t\right)\right| \left|\frac{\partial f}{\partial u}\left(t,\overline{x}^{\varepsilon}(t),\overline{u}^{\varepsilon}(t)\right)\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)\right| dt\\ &+E\textstyle\int_0^T \left|\frac{\partial f}{\partial u}\left(t,\overline{x}^{\varepsilon}(t),\overline{u}^{\varepsilon}(t)\right)-\frac{\partial f}{\partial u}\left(t,x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)\right)p^{\varepsilon}\left(t\right)\left(u\left(t\right)-u^{\varepsilon}\left(t\right)\right)\right| dt.\end{aligned}$$

First, by adding and subtracting $\frac{\partial f}{\partial u}\left(t, x^{\varepsilon}\left(t\right), \overline{u}^{\varepsilon}(t)\right)$ it holds that

$$\mathbb{J}_1 \leq E\int_0^T \left|\overline{p}^{\varepsilon}(t) - p^{\varepsilon}\left(t\right)\right| \left|\frac{\partial f}{\partial u}\left(t, \overline{x}_t^{\varepsilon}, \overline{u}_t^{\varepsilon}\right)\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right| dt$$

$$+E\int_0^T \left|\frac{\partial f}{\partial u}\left(t, \overline{x}^{\varepsilon}(t), \overline{u}^{\varepsilon}(t)\right) - \frac{\partial f}{\partial u}\left(t, x^{\varepsilon}\left(t\right), \overline{u}^{\varepsilon}(t)\right)\right| \left|p^{\varepsilon}\left(t\right)\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right| dt$$

$$+E\int_0^T \left|\frac{\partial f}{\partial u}\left(t, x^{\varepsilon}\left(t\right), \overline{u}^{\varepsilon}(t)\right) - \frac{\partial f}{\partial u}\left(t, x^{\varepsilon}\left(t\right), u^{\varepsilon}\left(t\right)\right)\right| \left|p^{\varepsilon}\left(t\right)\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right| dt$$

$$= \mathbb{J}_1^1 + \mathbb{J}_1^2 + \mathbb{J}_1^3.$$

Using Hölder inequality, the bounded of $\frac{\partial f}{\partial u}$, Lemma 1.3.2 and integral properties of admissible controls, we obtain, for $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$,

$$\mathbb{J}_1^1 \leq \left[E\left\{\int_0^T \left|\frac{\partial f}{\partial u}\left(t, \overline{x}^{\varepsilon}(t), \overline{u}^{\varepsilon}(t)\right)\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right|^{\gamma} dt\right\}\right]^{\frac{1}{\gamma}} \left[E\left\{\int_0^T \left|\overline{p}^{\varepsilon}(t) - p^{\varepsilon}\left(t\right)\right|^{\alpha} dt\right\}\right]^{\frac{1}{\alpha}}$$
$$\leq C\left(E\left\{\int_0^T \left|\overline{p}^{\varepsilon}(t) - p^{\varepsilon}\left(t\right)\right|^{\alpha} dt\right\}\right)^{\frac{1}{\alpha}}$$
$$\leq C\left(d^{\alpha}\left(\overline{u}^{\varepsilon}(\cdot), u^{\varepsilon}(\cdot)\right)\right)^{\frac{1}{\alpha}} \leq C\varepsilon^{\frac{1}{2}}.$$

To estimate the second term $\mathbb{J}_1^2$ we use assumption (H2), then we have

$$\mathbb{J}_1^2 \leq CE\int_0^T \left|\overline{x}^{\varepsilon}(t) - x^{\varepsilon}\left(t\right)\right|^{\beta} \left|p^{\varepsilon}\left(t\right)\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right| dt,$$

using Hölder inequality, where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$ then a simple computations gets

$$\mathbb{J}_1^2 \leq C\left(E\int_0^T \left|\overline{x}^{\varepsilon}(t) - x^{\varepsilon}\left(t\right)\right|^{\alpha\beta} \left|p^{\varepsilon}\left(t\right)\right|^{\alpha}\right)^{\frac{1}{\alpha}} \left(E\int_0^T \left|\left(u\left(t\right) - u^{\varepsilon}\left(t\right)\right)\right|^{\gamma} dt\right)^{\frac{1}{\gamma}}$$
$$\leq C\left(E\int_0^T \left|\overline{x}^{\varepsilon}(t) - x^{\varepsilon}\left(t\right)\right|^{\alpha\beta} \left|p^{\varepsilon}\left(t\right)\right|^{\alpha}\right)^{\frac{1}{\alpha}},$$

applying Hölder inequality for $\frac{1}{2/(2-\alpha)}+\frac{1}{2/\alpha}=1$ it holds that

$$\begin{aligned}\mathbb{J}_1^2 &\leq C\left[\left(E\int_0^T \left|\overline{x}^{\varepsilon}(t)-x^{\varepsilon}(t)\right|^{\frac{2\alpha\beta}{2-\alpha}}\right)^{\frac{2-\alpha}{\alpha}}\times\left(E\int_0^T \left|p^{\varepsilon}(t)\right|^{\alpha\cdot\frac{2}{\alpha}}\right)^{\frac{\alpha}{2}}\right]^{\frac{1}{\alpha}} \\ &\leq C\left(d^{\frac{2\alpha\beta}{2-\alpha}}\left(u^{\varepsilon}(\cdot),\overline{u}^{\varepsilon}(\cdot)\right)\right)^{\frac{2-\alpha}{2}\cdot\frac{1}{\alpha}}\leq C\varepsilon^{\lambda}.\end{aligned}$$

Next by applying assumption (H2) and Hölder inequality then we can proceed to estimate $\mathbb{J}_1^3$ as follows

$$\begin{aligned}\mathbb{J}_1^3 &\leq CE\int_0^T \left|\overline{u}^{\varepsilon}(t)-u^{\varepsilon}(t)\right|^{\beta}\left|p^{\varepsilon}(t)\right|\left|\left(u(t)-u^{\varepsilon}(t)\right)\right|dt \\ &\leq C\left(E\int_0^T \left|\overline{u}^{\varepsilon}(t)-u^{\varepsilon}(t)\right|^{\alpha\beta}\left|p^{\varepsilon}(t)\right|^{\alpha}dt\right)^{\frac{1}{\alpha}}\left(E\int_0^T \left|\left(u(t)-u^{\varepsilon}(t)\right)\right|^{\gamma}dt\right)^{\frac{1}{\gamma}} \\ &\leq C\left(\left(E\int_0^T \left|\overline{u}^{\varepsilon}(t)-u^{\varepsilon}(t)\right|^{\frac{2\alpha\beta}{2-\alpha}}dt\right)^{\frac{2-\alpha}{2}}\left(E\int_0^T \left|p^{\varepsilon}(t)\right|^{\alpha\frac{2}{\alpha}}dt\right)^{\frac{2}{\alpha}}\right)^{\frac{1}{\alpha}} \\ &\leq C\varepsilon^{\beta}.\end{aligned}$$

Using similar arguments developed above for $\mathbb{J}_2$ and $\mathbb{J}_3$, then a simple computations we can prove that $\mathbb{I}_1\leq C\varepsilon^{\lambda}$. Applying similar method developed above for $\mathbb{I}_2$ and $\mathbb{I}_3$ we conclude

$$\begin{aligned}&E\int_0^T \frac{\partial H}{\partial u}\left(t,\overline{x}^{\varepsilon},\overline{u}^{\varepsilon},\overline{p}^{\varepsilon}(t),\overline{q}^{\varepsilon}(t)\right)\left(u(t)-\overline{u}^{\varepsilon}(t)\right)dt \\ &-E\int_0^T \frac{\partial H}{\partial u}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(u(t)-u^{\varepsilon}(t)\right)dt\leq C\varepsilon^{\lambda}.\end{aligned}\tag{2.16}$$

Finally combining (2.12) and (2.16) the proof of Theorem 1.3.1 is complete. □

2.4 Sufficient conditions for ε-optimality

In this section, we will prove that under an additional hypothesis, the ε-maximum condition on the Hamiltonian function is a sufficient condition for ε-optimality.

Theorem 1.4.2. Assume that $H\left(t,\cdot,\cdot,p^{\varepsilon}(\cdot),q^{\varepsilon}(\cdot)\right)$ is convex for $a.e.\ t\in[0,T]$, $P-a.s$, and h is convex. Let $(u^{\varepsilon}(\cdot), x^{\varepsilon}(\cdot))$ be a ε-optimal solution of the control problem (2.1)-(3.2) and $(p^{\varepsilon}(t), q^{\varepsilon}(t))$ be the solution of the adjoint equation associated with $u^{\varepsilon}(\cdot)$. If for some $\varepsilon > 0$ and for any $u(\cdot) \in \mathcal{U}$:

$$E\int_0^T \frac{\partial H}{\partial u}\left(t, x^{\varepsilon}, u^{\varepsilon}, p^{\varepsilon}(t), q^{\varepsilon}(t)\right)\left(u(t) - u^{\varepsilon}(t)\right)dt \geq -C\varepsilon^{\lambda}, \tag{2.17}$$

then $u^{\varepsilon}(\cdot)$ is an ε-optimal control of order ε^{λ}, i.e.,

$$J\left(u^{\varepsilon}(\cdot)\right) \leq \inf_{v(\cdot)\in\mathcal{U}} J\left(v(\cdot)\right) + C\varepsilon^{\lambda},$$

where C is a positive constant independent from ε.

Proof. Let $u^{\varepsilon}(\cdot)$ be an arbitrary element of $\mathcal{U}$ (candidate to be $\varepsilon-$optimal) and $x^{\varepsilon}(\cdot)$ is the corresponding trajectory. For any $v(\cdot) \in \mathcal{U}$ and its corresponding trajectory $x^{v}(\cdot)$, we have

$$J\left(u^{\varepsilon}(\cdot)\right) - J\left(v(\cdot)\right) = E\int_0^T \left(g\left(t, x^{\varepsilon}(t), u^{\varepsilon}(t)\right) - g\left(t, x^{v}(t), v(t)\right)\right)dt$$

$$+E\left[h\left(x^{\varepsilon}(T)\right) - h\left(x^{v}(T)\right)\right].$$

Since h is convex, we have

$$\begin{aligned}&J\left(u^{\varepsilon}(\cdot)\right) - J\left(v(\cdot)\right)\\&\leq E\left[h_x\left(x^{\varepsilon}(T)\right)\left(x^{\varepsilon}(T) - x^{v}(T)\right)\right] + E\int_0^T \left(g\left(t, x^{\varepsilon}(t), u^{\varepsilon}(t)\right) - g\left(t, x^{v}(t), v(t)\right)\right)dt,\end{aligned}$$

replacing $h_x\left(x^{\varepsilon}(T)\right)$ with its value, see (3.4) we have

$$\begin{aligned}&J\left(u^{\varepsilon}(\cdot)\right) - J\left(v(\cdot)\right) \leq E\left[p^{\varepsilon}(T)\left(x^{\varepsilon}(T) - x^{v}(T)\right)\right]\\&+E\int_0^T \left(g\left(t, x^{\varepsilon}(t), u^{\varepsilon}(t)\right) - g\left(t, x^{v}(t), v(t)\right)\right)dt.\end{aligned} \tag{2.18}$$

On the other hand, by applying Itô's formula to $p^{\varepsilon}(T)\left(x^{\varepsilon}(T) - x^{v}(T)\right)$, and by taking expectation, we obtain

$$\begin{aligned}
&E\left[p^{\varepsilon}(T)\left(x^{\varepsilon}(T)-x^{v}(T)\right)\right] \\
&=E\int_0^T(H\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right)-H\left(t,x^{v}(t),v,p^{\varepsilon}(t),q^{\varepsilon}(t)\right))dt \\
&-E\int_0^T\frac{\partial H}{\partial x}\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(x^{\varepsilon}(t)-x^{v}(t)\right)dt \\
&-E\int_0^T\left(g\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t)\right)-g\left(t,x^{v}(t),v(t)\right)\right)dt,
\end{aligned} \tag{2.19}$$

then by combining (2.18) and (2.19) we have

$$\begin{aligned}
&J\left(u^{\varepsilon}(\cdot)\right)-J\left(v(\cdot)\right)\leq E\int_0^T(H\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)-H\left(t,x^{v},v,p^{\varepsilon}(t),q^{\varepsilon}(t)\right))dt \\
&-E\int_0^T\frac{\partial H}{\partial x}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(x^{\varepsilon}(t)-x^{v}(t)\right)dt.
\end{aligned} \tag{2.20}$$

Since H is convex in (x,u) we obtain

$$\begin{aligned}
&H\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)-H\left(t,x^{v},v,p^{\varepsilon}(t),q^{\varepsilon}(t)\right) \\
&\leq\frac{\partial H}{\partial x}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(x^{\varepsilon}(t)-x^{v}(t)\right) \\
&+\frac{\partial H}{\partial u}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(u^{\varepsilon}(t)-v(t)\right),
\end{aligned}$$

then by using the necessary optimality conditions (2.17), it follows that

$$\begin{aligned}
&C\varepsilon^{\lambda}\geq H\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)-H\left(t,x^{v},v,p^{\varepsilon}(t),q^{\varepsilon}(t)\right) \\
&-\frac{\partial H}{\partial x}\left(t,x^{\varepsilon},u^{\varepsilon},p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(x^{\varepsilon}(t)-x^{v}(t)\right).
\end{aligned} \tag{2.21}$$

Finally combining (2.20) and (2.21) the desired result follows. □

2.5 Application : linear quadratic control problem

In this section, we consider a linear quadratic control problem as a particular case of our control problem. First, we restrict ourselves to the one dimensional case. We assume that $T = 1$ and the convex control domain be $U = [0,1]$, $f(t, x(t), u(t)) = -u(t)$, $\sigma(t, x(t), u(t)) = u(t)$, $g(t, x(t), u(t)) = \frac{1}{2}u^2(t)$ and $h(x(t)) = x(t)$.
Consider the following stochastic control problem

$$\begin{cases} dx(t) = -u(t)\,dt + u(t)\,dW(t), \\ x(0) = \frac{1}{2}, \end{cases} \tag{2.22}$$

and the cost functional being

$$J(u(\cdot)) = E\left\{x(1) + \int_0^1 \tfrac{1}{2}u^2(t)\,dt\right\}. \tag{2.23}$$

The Hamiltonian function gets the form

$$H(t, x, u, p(t), q(t)) = (q(t) - p(t))\,u + \frac{1}{2}u^2, \tag{2.24}$$

and the corresponding adjoint equation is given as follows

$$-dp(t) = q(t)\,dW(t),\ \ p(1) = 1. \tag{2.25}$$

It is clear that $(p(t), q(t)) = (1, 0)$ is the only unique adapted solution to (2.25). Moreover, the Hamiltonian function has the form

$$H(t, x, u, p(t), q(t)) = -u + \frac{1}{2}u^2. \tag{2.26}$$

If the admissible control $u^{\varepsilon}(\cdot)$ is $\varepsilon-$optimal in the sense that $J(u^{\varepsilon}(\cdot)) \leq \inf_{u(\cdot)\in\mathcal{U}} J(u(\cdot)) +$

ε, then by applying Theorem 1.3.1, we obtain for any $u \in [0,1]$.

$$E\int_0^1 \left(u^{\varepsilon}(t)-1\right)\left(u(t)-u^{\varepsilon}(t)\right)dt \geq -C\varepsilon^{\lambda}. \tag{2.27}$$

For example, a simple computation shows that the admissible control $u^{\varepsilon}(t) = 1-\varepsilon$, satisfies the above inequality, where $\varepsilon > 0$ is sufficiently small. Conversely, for the sufficient part, let $u^{\varepsilon}(t) = 1-\varepsilon$ which satisfy (3.51) candidate to be ε-optimal. Since H is convex in u and by using *Theorem 1.4.2* it follows that $u^{\varepsilon}(t)$ satisfies inequality (3.51), which means that $u^{\varepsilon}(\cdot)$ is ε-optimal for our control problem (3.45)-(3.46), and its corresponding trajectory is

$$x^{\varepsilon}(t) = \frac{1}{2} - (1-\varepsilon)t + (1-\varepsilon)W(t).$$

2.6 Concluding remarks and future research

In this chapter, necessary and sufficient conditions for near-optimal control with $\varepsilon^{\lambda}-$error bound for SDEs have been established. Linear quadratic control problem has been studied to illustrate our theoretical results. If we assume that $\varepsilon = 0$, our maximum principle (*Theorem 1.3.1*) reduces to maximum principle of optimality developed in Benssoussan [8].

An open questions are to establish necessary and sufficient conditions for near-optimality with $\varepsilon^{\lambda}-$error bound for SDEs with impulse control, Linear quadratic stochastic control with $\varepsilon^{\lambda}-$error bound for SDEs with impulse and SDEs with random jumps. We will work for this interesting issue in the future research.

CHAPITRE-III
Partial information optimal control of mean-field forward-backward stochastic system driven by Teugels martingales with applications

Chapitre 3

Partial information optimal control of mean-field forward-backward stochastic system driven by Teugels martingales with applications

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Abstract. In this chapitre, we consider the partial information optimal control of a mean-field forward-backward stochastic systems (FBSDEs), driven by orthogonal Teugels martingales associated with some Lévy processes having moments of all orders, and an independent Brownian motion. We establish necessary and sufficient conditions of optimality by applying convex variation method and duality techniques. As an application, we study a partial information mean-variance portfolio selection problem, driven by Teugels martingales associated with Gamma process, where the explicit optimal portfolio strategy is derived in feedback form.

Keywords. Stochastic process. Optimal control. Teugels martingales. Lévy processes. Partial information. Mean-field forward-backward stochastic system. Convex variation method. feedback control.

AMS Subject Classification : 60H10, 93E20.

3.1 Introduction

We study partial information optimal control for mean-field forward-backward stochastic differential equation (MF-FBSDEs), driven by Teugels martingales associated with some Lévy processes having moments of all orders and an independent Brownian motion of the form :

$$\begin{aligned}
dx^v(t) &= b(t, x^v(t), E(x^v(t)), v(t))dt \\
&\quad + \sum_{j=1}^{d} \sigma^j(t, x^v(t), E(x^v(t)))dW^j(t) \\
&\quad + \sum_{j=1}^{\infty} g^j(t, x^v(t_-), E(x^v(t_-)), v(t))dH^j(t) \\
dy^v(t) &= -f(t, x^v(t), E(x^v(t)), y^v(t), E(y^v(t)), \\
&\quad z^v(t), E(z^v(t)), q^v(t), E(q^v(t)), v(t))dt \\
&\quad + \sum_{j=1}^{d} z^{v,j}(t)dW^j(t) + \sum_{j=1}^{\infty} q^{v,j}(t_-)dH^j(t) \\
x^v(0) &= x_0, \; y^v(T) = h(x^v(T), E(x^v(T))),
\end{aligned} \tag{3.1}$$

where $H(t) = (H^j(t))_{j\geq 1}$ are pairwise strongly orthonormal Teugels martingales, associated with some Lévy processes having moments of all orders, $W(\cdot)$ is a standard d-dimensional Brownian motion, and b, f, σ, g and h are given maps.

The criteria to be minimized associated with the state equation (2.1) is defined by

$$
\begin{aligned}
& J\left(v(\cdot)\right) \\
& := E\left\{\int_0^T \ell(t, x^v(t), E(x^v(t)), y^v(t),\right. \\
& z^v(t), E(z^v(t)), q^v(t), E(q^v(t)), v(t))dt \\
& \left. + \, \phi(x^v(T), E(x^v(T))) + \varphi(y^v(0), E(y^v(0)))\right\},
\end{aligned} \tag{3.2}
$$

where ℓ, ϕ, and φ are an appropriate functions.

Noting that the Teugels martingales $H(t) = (H^j(t))_{j\geq 1}$ are a natural martingales, which generate the Hilbert space of square integrable martingales, with respect to the natural filtration of a Lévy process having moments of all orders. The MF-FBSDE associated with Lévy processes (2.1) occur naturally in the probabilistic analysis of financial optimization problems.

Partial information or incomplete information means that the information available to the controller is possibly less than the whole information. That is, any admissible control is adapted to a subfiltration $(\mathcal{G}_t)_t$ of $(\mathcal{F}_t)_t$. This kind of problem, which has potential applications in mathematical finance and mathematical economics, arises naturally, because it may fail to obtain an admissible control with full information in real world applications.

The stochastic systems related to Lévy processes have been investigated by many authors. For example, [62, 64, 68, 87, 89, 70, 71, 9, 55, 35, 29]. In Meng and Tang [62] the authors investigated the general stochastic optimal control problem for the stochastic systems driven by Teugels martingales and an independent multi-dimensional Brownian motion and recently, they prove the corresponding stochastic maximum principle. Optimal control problem for a backward stochastic control systems associated with Lévy processes under partial information has been studied in Meng, Zhang and Tang [64]. The stochastic linear-quadratic problem with Lévy processes was studied by Mitsui and Tabata [68] and Tang and Wu [87]. Optimal control of BSDEs driven by Teugels martingales have been studied in Tang and Zhang [89]. Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance have been investigated in Nualart and Schoutens [70]. A predictable representations for Lévy processes with recent examples have been studied in Nualart and Schoutens [71], where the authors proved a martingale representation theorem for Lévy processes satisfying some exponential moment condition. We refer the readers to [70, 71, 9] for Lévy processes, Teugels martingales with some examples. Maximum principle for anticipated recursive stochastic optimal control problem with delay and Lévy processes has been proved in Li and Wu [55]. Partial information maximum principle for mean-field SDEs, driven by Teugels martingales associated with Lévy processes has been proved in Hafayed, Abbas and Abba [35]. Necessary and sufficient conditions for optimal singular control for mean-field systems driven by Teugels martingales have been established in Hafayed, Abba and Abbas [29].

Maximum principle for forward-backward stochastic control system with random jumps and application to finance has been obtained in Shi and Wu [?]. The general stochastic maximum principle for fully coupled controlled FBSDEs has been studied by Yong [97] and Wu [93]. Partial information maximum principle for BSDEs with application has been studied in Huang, Wang and Xiong [47]. A maximum principle for optimal control problem of fully coupled forward-backward stochastic systems with partial information has been studied by Meng [63]. A good account and an extensive list of references on stochastic optimal control for FBSDEs can be found in [97, 93, 58].

Mean-field maximum principle for optimal stochastic control has been investigated by many authors, see for instance, [31, 24, 26, 12, 5, 54, 80, 79]. Second order necessary and sufficient conditions of near-optimal singular control for mean-field SDE have been established in Hafayed and Abbas [31] The maximum principle for optimal control of mean-field FBSDEJs has been studied in Hafayed [24]. Singular optimal control for MF-FBSDEs and applications to finance have been investigated in Hafayed [26]. A general maximum principle was introduced for a class of stochastic control problems involving SDEs of mean-field type in [12]. Under the conditions that the control domains are convex, a various local maximum principle have been studied in [5, 54]. Necessary and sufficient conditions for controlled jump diffusion with recent application in bicriteria mean-variance portfolio selection problem have been proved in Shen and Siu [80]. Recently, maximum principle for mean-field jump-diffusions stochastic delay differential equations and its applicationt to finance have been investigated in Yang, Meng and Shi [79]. Under partial information, mean-field type stochastic maximum principle for optimal control has been investigated in Wang, Zhang and Zhang [91]. A novel approach to feedback control of fuzzy stochastic systems has been developed in Su, Wu, Shi, and Song [86]. A study on feedback control of markovian jump has been proved in Wu, Su and Shi [92].

Our purpose in this paper is to establish necessary as well as sufficient conditions for optimal stochastic control of systems governed by MF-FBSDEs, driven by Teugels martingale associated with Lévy processes having moments of all orders, where the coefficient of the system and the performance functional depend not only on the state process but also its marginal law of the state process through its expected value. As an application, we study a partial information mean-variance portfolio selection problem driven by Teugels martingales associated with Gamma process as Lévy process of bounded variation.

The rest of this paper is organized as follows. In section 2, we formulate the control problem and describe the assumptions of the model. In section 3 and 4, we establish the necessary and sufficient conditions of optimality. To illustrate our theoretical result, an application to finance is given in the section 5. Finally, Section 6 concludes the paper.

3.2 Problem Formulation and Preliminaries

In this paper, we study stochastic optimal control problems of MF-FBSDEs associated with Lévy processes. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a fixed filtered probability space equipped with a $P-$completed right continuous filtration on which a $d-$dimensional Brownian motion

$W(\cdot) := (W(t))_t$ is defined. Let $L(\cdot) := \{(L(t)) : t \in [0,T]\}$ be a $\mathbb{R}$-valued Lévy process, independent of the Brownian motion $W(\cdot)$, of the form $L(t) = bt + \lambda(t)$, where $\lambda(t)$ is a pure jump process. Assume that the Lévy measure $\mu(d\theta)$ corresponding to the Lévy process $\lambda(t)$ satisfies :

1. There exist $\gamma > 0$ such that for every $\delta > 0 : \int_{(-\delta,\delta)} \exp(\gamma |\theta|)\, \mu(d\theta) < \infty$.
2. $\int_{\mathbb{R}} (1 \wedge \theta^2) \mu(d\theta) < \infty$.

It is known that the law of $L(t)$ is infinitely divisible with characteristic function of the form $E(\exp(izL(t))) = (\varphi(z))^t$ where $\varphi(z)$ is the characteristic function of $L(t)$. We assume that $\mathcal{F}_t$ is $P-$augmentation of the natural filtration $(\mathcal{F}_t^{(W,L)})_{t\in[0,T]}$ defined as follows

$$\mathcal{F}_t^{(W,L)} := \mathcal{F}_t^W \vee \sigma\{L(s) : 0 \leq s \leq t\} \vee \mathcal{F}_0,$$

where $\mathcal{F}_t^W := \sigma\{W(s) : 0 \leq s \leq t\}$, $\mathcal{F}_0$ denotes the totality of $P-$null sets, and $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Let $\mathcal{G}_t$ be a subfiltration of $\mathcal{F}_t : t \in [0,T]$

Definition 1.1. Let $T > 0$ be a fixed strictly positive real number and $\mathbb{U}$ be a nonempty compact convex subset of $\mathbb{R}^k$. An admissible control is defined as a function $v(\cdot) : [0,T] \times \Omega \to \mathbb{U}$ which is $\mathcal{F}_t$-predictable, $E\int_0^T |v(t)|^2\, dt < \infty$ such that the equation (2.1) has a unique solution. We denote $\mathcal{U}_{\mathcal{G}}([0,T])$.

An admissible control $v^*(\cdot)$ is called optimal if it satisfies

$$J(v^*(\cdot)) := \inf_{v(\cdot)\in\mathcal{U}_{\mathcal{G}}([0,T])} J(v(\cdot)). \tag{3.3}$$

The jumps of $x^v(t)$ caused by the Lévy martingales is the power jump processes defined by

$$\begin{cases} L_{(k)}(t) := \sum_{0<\tau\leq t} (\Delta L(\tau))_k : k > 1 \\ L_{(1)}(t) := L(t), \end{cases}$$

where $\Delta L(t) = L(t) - L(t_-)$, and $L(t_-) = \lim_{s\to t, s<t} L(s), t > 0$. Moreover, we define the continuous part of $L_{(k)}(t)$ by

$$L_{(k)}^{(c)}(t) := L_{(k)}(t) - \sum_{0<\tau\leq t} (\Delta L(\tau))_k : k > 1,$$

i.e., the process obtained by removing the jumps of $L(t)$. If we define

$$N_{(k)}(t) := L_{(k)}(t) - E\left\{L_{(k)}(t)\right\} : k \geq 1.$$

Now, the jumps of $x^v(t)$, and $y^v(t)$ caused by the Lévy martingales $\Delta_L x^v(t)$ is defined by

$$\Delta_L x^v(t) := g(t, x^v(t_-), E[x^v(t_-)], v(t))\Delta L(t).$$
$$\Delta_L y^v(t) := \sum_{j=1}^{\infty} q^{v,j}(t_-)\Delta L^j(t).$$

The family of Teugels martingales $(H^j(\cdot))_{j\geq 1}$ is defined by

$$H^j(t) := \sum_{1<k\leq j} \alpha_{jk} N_k(t),$$

where the coefficients α_{jk} associated with the orthonormalization of the polynomials $\{1, x, x^2, ...\}$ with respect to the measure $m(dx) = x^2\mu(dx) + \sigma^2\delta_0(dx)$. The Teugels martingales $(H^j(\cdot))_{j\geq 1}$ are pathwise strongly orthogonal and their predictable quadratic variation processes are given by $\langle H^i(t), H^j(t)\rangle = \delta_{ij}t$. We refer the readers to Nualart and Schoutens [71] for some other relevant results and practical examples of Lévy processes and Teugles martingales.

For convenience, we will use the following notation in this paper.

— l^2 : denotes the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ such that

$$\|x\| := \left[\sum_{n=1}^{\infty} x_n\right]^2 < \infty,$$

and $l^2(\mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued $(f_n)_{n\geq 1}$ such that

$$\|f\|_{l^2(\mathbb{R}^n)} := \left[\sum_{n=1}^{\infty} \|f_n\|_{\mathbb{R}^n}^2\right]^{\frac{1}{2}} < \infty.$$

— $\mathbb{L}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)$ denotes the Banach space of $\mathcal{F}_t-$predictable processes $f = \{f_n(t,w) : (t,w) \in [0,T$

such that

$$\|f\|_{\mathbb{L}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)} := E\left(\int_0^T \sum_{n=1}^{\infty} \|f_n\|_{\mathbb{R}^n}^2\, dt\right)^{\frac{1}{2}} < \infty.$$

— $\mathbb{M}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)$ denotes the space of all $\mathbb{R}^n-$valued and $\mathcal{F}_t-$adapted processes $f = \{f(t,w) : (t,w) \in [0,T] \times \Omega\}$ such that

$$\|f\|_{\mathbb{M}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)} := E\left(\int_0^T \|f(t)\|_{\mathbb{R}^n}^2\, dt\right)^{\frac{1}{2}} < \infty.$$

— $\mathbb{S}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)$ denotes the Banach space of $\mathcal{F}_t-$adapted and *cadlag* processes $f = \{f(t,w) : (t,w) \in [0,T] \times \Omega\}$ such that

$$\|f\|_{\mathbb{S}_{\mathcal{F}}^2([0,T];\mathbb{R}^n)} := E(\sup_{0\leq t\leq T} \|f\|_{\mathbb{R}^n})^{\frac{1}{2}} < \infty.$$

— $\mathbb{L}^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ the Banach space of $\mathbb{R}^n$-valued, square integrable random variables on $(\Omega, \mathcal{F}, P)$.

— $\mathcal{M}^{n\times m}(\mathbb{R})$ denotes the space of $n \times m$ real matrices.

— For a differentiable function f, we denote by $f_x(t)$ its gradient with respect to the variable x.

— $1_A(\cdot)$ denotes the indicator function on the set A.

In this paper, we assume : for $\sigma \equiv (\sigma^j)_{j=1}^{d}$ and $g \equiv (g^j)_{j=1}^{\infty}$

$$
\begin{aligned}
& b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \longrightarrow \mathbb{R}^n, \\
& \sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \longrightarrow \mathcal{M}^{n\times d}(\mathbb{R}), \\
& g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \longrightarrow l^2\left(\mathbb{R}^n\right), \\
& f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \\
& \quad \times \mathbb{R}^{n\times d} \times l^2\left(\mathbb{R}^n\right) \times \mathbb{U} \longrightarrow \mathbb{R}^n, \\
& \ell : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \\
& \quad \times \mathbb{R}^{n\times d} \times l^2\left(\mathbb{R}^n\right) \times \mathbb{U} \longrightarrow \mathbb{R}, \\
& h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}. \\
& \phi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}. \\
& \varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}.
\end{aligned}
$$

Assumptions (H1) The functions $b, \sigma, g, f, \ell, h, \phi, \varphi$ are continuously differentiable in their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, q, \widetilde{q}, v)$. The terminal value $y_T \in l^2_{\mathcal{F}}\left([0,T]\,;\mathbb{R}^n\right)$ and

$$
\begin{aligned}
& |\ell| \leq C(1 + |x|^2 + |\widetilde{x}|^2 + |y|^2 + |\widetilde{y}|^2 \\
& \qquad + |z|^2 + |\widetilde{z}|^2 + |q|^2 + |\widetilde{q}|^2 + |v|^2). \\
& |\phi| \leq C(1 + |x|^2 + |\widetilde{x}|^2), \\
& |\varphi| \leq C(1 + |y|^2 + |\widetilde{y}|^2),
\end{aligned}
$$

Assumptions (H2) *(1)* The derivatives of b, σ, g and f with respect to their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, q, \widetilde{q}, v)$ are continuous and bounded.

(2) The derivatives $\varphi_y, \varphi_{\widetilde{y}}$ are bounded by $C\left(1 + |y| + |\widetilde{y}|\right)$, the derivatives ϕ_x, $\phi_{\widetilde{x}}$, h_x, and

$h_{\widetilde{x}}$are bounded by $C\left(1+|x|+|\widetilde{x}|\right)$ and

$$\begin{aligned}&|\ell_x|+|\ell_{\widetilde{x}}|+|\ell_y|+|\ell_{\widetilde{y}}|+|\ell_z|+|\ell_{\widetilde{z}}|+|\ell_q|+|\ell_{\widetilde{q}}|\\ &\leq C(1+|x|+|\widetilde{x}|+|y|+|\widetilde{y}|+|z|+|\widetilde{z}|+|q|+|\widetilde{q}|+|v|).\end{aligned}$$

(3) For all $t\in[0,T]$, $b(\cdot,0,0,0)\in\mathbb{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^n\right)$, $f(\cdot,0,0,0,0,0,0,0,0,0)\in\mathbb{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^n\right)$, $g(\cdot,0,0,0)\in\mathbb{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^n\right)$, $\sigma(\cdot,0,0,0)\in\mathbb{M}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right)$.

Under assumptions (H1)∼(H2), with *Lemma 2.1* in Meng and Tang [62] and *Lemma 2.3* in Tang and Zhang [89], Eq-(2.1) admits a unique strong solution $\left(x^v\left(\cdot\right),y^v\left(\cdot\right),z^v\left(\cdot\right),q^v(\cdot)\right)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{M}^{n\times d}(\mathbb{R})\times l^2\left(\mathbb{R}^n\right)$ such that

$$\begin{aligned}x^v(t)=x_0&+\int_0^t b(s,x^v(s),E(x^v(s)),v(s))ds\\ &+\sum_{j=1}^{d}\int_0^t\sigma^j(s,x^v(s),E(x^v(s)))dW^j(s)\\ &+\sum_{j=1}^{\infty}\int_0^t g^j(t,x^v(s_-),E(x^v(s_-)),v(t))dH^j(s),\end{aligned}$$

and for $t\in[0,T]$

$$\begin{aligned}y^v(t)=y_T-\int_t^T &f(s,x^v(s),E(x^v(s)),y^v(s),\\ &E(y^v(s)),z^v(s),E(z^v(s)),q^v\left(t\right),v(s))ds\\ &+\sum_{j=1}^{d}\int_t^T z^{v,j}(s)dW^j(s)+\sum_{j=1}^{\infty}\int_t^T q^{v,j}(s)dH^j(s),\end{aligned}$$

First-order adjoint process. Let us introduce the adjoint equations involved in the stochastic maximum principle for the control problem (2.1)-(3.2). For simplicity of notations, we will still use $f_x(t):=f_x(t,x^v(\cdot),E(x^v(\cdot)),v(\cdot))$, *etc.* So for any admissible control $v(\cdot)\in\mathcal{U}_{\mathcal{G}}\left([0,T]\right)$ and the corresponding state trajectory $\left(x^v\left(\cdot\right),y^v\left(\cdot\right),z^v\left(\cdot\right),q^v(\cdot)\right):=$

$(x(\cdot), y(\cdot), z(\cdot), q(\cdot))$, we consider the following adjoint equations of mean-field type :

$$\begin{cases} -d\Phi^{v}(t) = \{b_{x}(t)\Phi^{v}(t) + E[b_{\widetilde{x}}(t)\Phi^{v}(t)] \\ +\sum_{j=1}^{d}\sigma_{x}^{j}(t)Q^{v,j}(t) + E[\sum_{j=1}^{d}\sigma_{\widetilde{x}}^{j}(t)Q^{v,j}(t)] \\ +\sum_{j=1}^{\infty}g_{x}^{j}(t)G^{v,j}(t) + E[\sum_{j=1}^{\infty}g_{\widetilde{x}}^{j}(t)G^{v,j}(t)] \\ -f_{x}(t)K^{v}(t) - E[f_{\widetilde{x}}(t)K^{v}(t)] \\ +\ell_{x}(t) + E[\ell_{\widetilde{x}}(t)]\}dt \\ -\sum_{j=1}^{d}Q^{v,j}(t)dW^{j}(t) - \sum_{j=1}^{\infty}G^{v,j}(t)dH^{j}(t), \\ dK^{v}(t) = [f_{y}(t)K^{v}(t) + E[f_{\widetilde{y}}(t)K^{v}(t)] \\ -\ell_{y}(t) - E[\ell_{\widetilde{y}}(t)]]dt \\ +\sum_{j=1}^{d}[f_{z^{j}}(t)K^{v}(t) + E[f_{\widetilde{z}^{j}}(t)K^{v}(t)] \\ -\ell_{z^{j}}(t) - E[\ell_{\widetilde{z}^{j}}(t)]]dW^{j}(t) \\ +\sum_{j=1}^{\infty}[f_{q^{j}}(t)K^{v}(t) + E[f_{\widetilde{q}^{j}}(t)K^{v}(t)] \\ -\ell_{q^{j}}(t) - E[\ell_{\widetilde{q}^{j}}(t)]]dH^{j}(t) \\ \Phi^{v}(T) = -\left[h_{x}(T)K^{v}(T) + E(h_{\widetilde{x}}(T)K^{v}(T))\right] \\ +\phi_{x}(x(T)) + E\left[\phi_{\widetilde{x}}(x(T))\right], \\ K^{v}(0) = -\{\varphi_{y}(y(0), E[(y(0))) \\ +E\left[\varphi_{\widetilde{y}}(y(0), E(y(0)))\right]\}. \end{cases} \tag{3.4}$$

We define the Hamiltonian function $\mathcal{H} : [0,T] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$ $\times l^{2}(\mathbb{R}^{n}) \times l^{2}(\mathbb{R}^{n}) \times \mathbb{U} \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n\times d} \times l^{2}(\mathbb{R}^{n}) \to \mathbb{R}$, associated with the stochastic control problem (2.1)-(3.2) as follows

$$\begin{aligned}
&\mathcal{H}\left(t,x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},q,\widetilde{q},v,\Phi(\cdot),Q(\cdot),G(\cdot),K(\cdot)\right)\\
&:=\Phi(t)b\left(t,x,\widetilde{x},v\right)+\textstyle\sum_{j=1}^{d}Q^{j}(t)\sigma^{j}\left(t,x,\widetilde{x},v\right)\\
&+\textstyle\sum_{j=1}^{\infty}G^{j}(t)g^{j}\left(t,x,\widetilde{x},v\right)-K(t)f(t,x,\tilde{x},y,\tilde{y},q,\tilde{q},z,\tilde{z},v)\\
&+\ell(t,x,\tilde{x},y,\tilde{y},q,\tilde{q},z,\tilde{z},v).
\end{aligned}\tag{3.5}$$

If we denote by $\mathcal{H}(t):=H\left(t,x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},q,\widetilde{q},v,\Phi(\cdot),Q(\cdot),G(\cdot),K(\cdot)\right)$, then the adjoint equation (3.4) can be rewritten as the following stochastic Hamiltonian system :

$$\begin{cases}
-d\Phi^{v}(t)=\left[\mathcal{H}_{x}(t)+E[\mathcal{H}_{\widetilde{x}}(t)]\right]dt-\sum_{j=1}^{d}Q^{v,j}(t)dW^{j}(t)\\
\qquad\qquad -\sum_{j=1}^{\infty}G^{v,j}(t)dH^{j}(t),\\
dK^{v}(t)=-(\mathcal{H}_{y}(t)+E[\mathcal{H}_{\widetilde{y}}(t)])dt\\
\qquad\qquad -\sum_{j=1}^{d}(\mathcal{H}^{j}_{z^{j}}(t)+E[\mathcal{H}^{j}_{\widetilde{z}^{j}}(t)])dW^{j}(t)\\
\qquad\qquad -\sum_{j=1}^{\infty}(\mathcal{H}^{j}_{q^{j}}(t)+E[\mathcal{H}^{j}_{\widetilde{q}^{j}}(t)])dH^{j}(t),\\
\Phi^{v}(T)=-\left[h_{x}(T)K^{v}(T)+E(h_{\widetilde{x}}(T)K(T))\right]\\
\qquad\qquad +\phi_{x}(T)+E\left[\phi_{\widetilde{x}}(T)\right],\\
K^{v}(0)=-\varphi_{y}(y(0),E[y(0)])+E\left[\varphi_{\widetilde{y}}(y(0),E(y(0)))\right],
\end{cases}\tag{3.6}$$

where

$$\mathcal{H}^{j}(t):=H\left(t,x,\widetilde{x},y,\widetilde{y},z^{j},\widetilde{z}^{j},q^{j},\widetilde{q}^{j},v,\Phi(\cdot),Q(\cdot),G(\cdot),K(\cdot)\right).\tag{3.7}$$

It is well known fact that under assumptions (H1) and (H2), the adjoint equations (3.4) or (3.6) admits a unique solution such that $(\Phi(t),Q(t),G(t),K(t))\in\mathbb{S}^{2}_{\mathcal{F}}([0,T];\mathbb{R}^{n})\times\mathbb{L}^{2}_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d})\times l^{2}_{\mathcal{F}}([0,T];\mathbb{R}^{n})\times\mathbb{S}^{2}_{\mathcal{F}}([0,T];\mathbb{R}^{n})$. The solution $(\Phi(t),Q(t),G(t),K(t))$ to the above MF-FBSDEs (3.6) and (3.4) is called the first-order adjoint process.

3.3 Necessary conditions for optimal control of MF-FBSDEs with Teugels martingale

In this section, we establish a maximum principle of optimal control, where the control system is governed by MF-FBSDEs driven by orthogonal Teugels martingales associated with some Lévy processes having moments of all orders. In addition to the assumptions in Section 2, we need to make the following assumptions :

Assumptions (H3)

(1) For all t, r such that $0 \leq t \leq t+r \leq T$, all $i = 1, ..., k$ and all bounded $\mathcal{G}_t-$measurable $\alpha = \alpha(w)$, the control $\beta(t) = (0, ..., 0, \beta_i(t), 0, ..., 0) \in \mathbb{U}$, with

$$\beta_i(s) = \alpha_i I_{[t,t+r]}(s),\ s \in [0, T],$$

belong to $\mathcal{U}_{\mathcal{G}}([0, T])$.

(2) For all $v(\cdot), \beta(\cdot) \in \mathcal{U}_{\mathcal{G}}([0, T])$ with $\beta(\cdot)$ bounded, there exist $\delta > 0$ such that $v(\cdot) + \varepsilon\beta(\cdot) \in \mathcal{U}_{\mathcal{G}}([0, T])$ for all $\varepsilon \in (-\delta, \delta)$.

Now, for a given $v(\cdot)$, $\beta(\cdot) \in \mathcal{U}_{\mathcal{G}}([0, T])$ with β bounded, we define

$$\begin{aligned} X_1(t) = X_1^{v^*,\beta}(t) &:= \frac{d}{d\varepsilon} x^{v^*+\varepsilon\beta}(t) \mid_{\varepsilon=0}, \\ Y_1(t) = Y_1^{v^*,\beta}(t) &:= \frac{d}{d\varepsilon} y^{v^*+\varepsilon\beta}(t) \mid_{\varepsilon=0}, \\ Z_1(t) = Z_1^{v^*,\beta}(t) &:= \frac{d}{d\varepsilon} z^{v^*+\varepsilon\beta}(t) \mid_{\varepsilon=0}, \\ Q_1(t) = Q_1^{v^*,\beta}(t) &:= \frac{d}{d\varepsilon} q^{v^*+\varepsilon\beta}(t) \mid_{\varepsilon=0}, \end{aligned} \tag{3.8}$$

Note that the process $(X_1(\cdot), Y_1(\cdot), Z_1(\cdot), Q_1(\cdot))$ satisfies the following MF-FBSDEs, namely variational equations, driven by both Brownian motion and Teugels martingales

$$
\begin{cases}
dX_1(t) = [b_x(t)X_1(t) + b_{\widetilde{x}}(t)E(X_1(t)) + b_v(t)\beta(t)]\,dt \\
+\sum_{j=1}^{d}[\sigma_x^j(t)X_1(t) + \sigma_{\widetilde{x}}^j(t)E(X_1(t)) + \sigma_v^j(t)\beta(t)]dW^j(t) \\
+\sum_{j=1}^{\infty}[g_x^j(t)X_1(t) + g_{\widetilde{x}}^j(t)E(X_1(t)) + g_v^j(t)\beta(t)]dH^j(t), \\
dY_1(t) = [f_x(t)Y_1(t) + f_{\widetilde{x}}(t)E(Y_1(t)) + f_v(t)\beta(t)]dt \\
+\sum_{j=1}^{d} Z_1^{;j}(t)dW(t) + \sum_{j=1}^{\infty} Q_1^{;j}(t)dH(t), \\
X_1(0) = 0,\ Y_1(T) = [h_x(x(T), E(x(T))) \\
+E(h_{\widetilde{x}}(x(T), E(x(T))))]X_1(T).
\end{cases}
\tag{3.9}
$$

The main result of this section is stated in the following theorem.

Theorem 1. *Let $v^*(\cdot)$ be a local minimum for the cost functional J over $\mathcal{U}_{\mathcal{G}}([0,T])$, in the sense that for all bounded $\beta(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])$, there exist $\delta > 0$ such that $(v^*(\cdot) + \varepsilon\beta(\cdot)) \in \mathcal{U}_{\mathcal{G}}([0,T])$ for all $\varepsilon \in (-\delta, \delta)$ and*

$$
\Psi(\varepsilon) := J(v^*(\cdot) + \varepsilon\beta(\cdot)),\ \text{for all } \varepsilon \in (-\delta, \delta),
\tag{3.10}
$$

is minimal at $\varepsilon = 0$.

Let $(x^*(\cdot), y^*(t), z^*(t), q^*(t))$ be the solution of the MF-FBSDEs-(2.1) *corresponding to $v^*(\cdot)$. Let assumptions (H1)-(H3) hold. Then there exists a unique adapted process $(\Phi^*(\cdot), Q^*(\cdot), G^*(\cdot), K^*($ solution of adjoint equation (3.4) corresponding to $v^*(\cdot)$, such that $v^*(\cdot)$ is a stationary point for $E[\mathcal{H} \mid \mathcal{G}_t]$ in the sense that for almost all $t \in [0,T]$, we have*

$$
E\left[\mathcal{H}_v(t, \psi^*(t), E(\psi^*(t)), v^*(t), \Phi^*(t), Q^*(t), G^*(t),\ K^*(t)) \mid \mathcal{G}_t\right] = 0,\ a.e.,\ t \in [0,T],
\tag{3.11}
$$

where $(\psi^*(t), E(\psi^*(t))) := (x^*(t), E(x^*(t)), y^*(t), E(y^*(t)), z^*(t), E(z^*(t)), q^*(t), E(q^*(t)))$.

To prove *Theorem 1*, we need the following Lemma, which deals with the duality relations between $\Phi^*(t)$, $X_1(t)$, and $K^*(t)$, $Y_1(t)$.

Lemma 1. By applying Itô's formula to $\Phi^*(t)X_1(t)$, $K^*(t)Y_1(t)$ and take expectation, we get

$$
\begin{aligned}
&E\left(\Phi^*(T)X_1(T)\right) + E\left(K^*(T)Y_1(T)\right) \\
&= -E\left\{\left[\varphi_y\left(0\right) + E\left(\varphi_{\widetilde{y}}\left(0\right)\right)\right]Y_1(0)\right\} - E\int_0^T \left\{X_1(t)[\ell_x(t) + E(\ell_{\widetilde{x}}(t)]\right. \\
&+ Y_1(t)[\ell_y(t) + E(\ell_{\widetilde{y}}(t)] + Z_1(t)[\ell_z(t) + E(\ell_{\widetilde{z}}(t)] + Q_1(t)[\ell_q(t) + E(\ell_{\widetilde{q}}(t)] \\
&\left.+\ell_v(t)\beta(t)\right\}dt + E\int_0^T \mathcal{H}_v(t)\beta(t)dt
\end{aligned}
$$

Proof. By applying Itô's formula to $\Phi^*(t)X_1(t)$ and take expectation, we get

$$
\begin{aligned}
&E(\Phi^*(T)X_1(T)) \\
&= E\int_0^T \Phi^*(t)dX_1(t) + E\int_0^T X_1(t)d\Phi^*(t) \\
&+E\int_0^T \sum_{j=1}^d Q^{j*}(t)[\sigma_x^j(t)X_1(t) + \sigma_{\widetilde{x}}^j(t)E\left(X_1(t)\right) \\
&+\sigma_v^j(t)\beta(t)]dt + E\int_0^T \sum_{j=1}^\infty G^{j*}(t)[g_x^j(t)X_1(t) + g_{\widetilde{x}}^j(t)E\left(X_1(t)\right) \\
&+g_v^j(t)\beta(t)]dt \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
I_1 &= E\int_0^T \Phi^*(t)dX_1(t) \\
&= E\int_0^T \Phi^*(t)[b_x\left(t\right)X_1(t) + b_{\widetilde{x}}\left(t\right)E(X_1(t)) \\
&\quad + b_v(t)\beta(t)]dt \\
&= E\int_0^T \Phi^*(t)b_x\left(t\right)X_1(t) + E\int_0^T \Phi^*(t)b_{\widetilde{x}}\left(t\right)E(X_1(t)) \\
&+E\int_0^T \Phi^*(t)b_v(t)\beta(t))dt.
\end{aligned}
\tag{3.13}
$$

By simple computations, we get

$$
\begin{aligned}
I_2 &= E\int_0^T X_1(t)d\Phi^*(t) \\
&= -E\int_0^T X_1(t)\left\{b_x(t)\,\Phi^*(t) + E\left(b_{\widetilde{x}}(t)\,\Phi^*(t)\right)\right. \\
&+ \sum_{j=1}^d (\sigma_x^j(t)\,Q^{j*}(t) + E(\sigma_{\widetilde{x}}^j(t)\,Q^{j*}(t))) \\
&+ \sum_{j=1}^\infty (g_x^j(t)\,G^{j*}(t) + E[g_{\widetilde{x}}^j(t)\,G^{j*}(t)]) \\
&\left. - f_x(t)K^v(t) - E(f_{\widetilde{x}}(t)K^v(t)) + \ell_x(t) + E[\ell_{\widetilde{x}}(t)]\right\} dt
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
I_3 &= E\int_0^T \sum_{j=1}^d Q^{j*}(t)[\sigma_x^j(t)X_1(t) + \sigma_{\widetilde{x}}^j(t)E\left(X_1(t)\right) + \sigma_v^j(t)\beta(t)]dt \\
&= E\int_0^T \sum_{j=1}^d Q^{j*}(t)\sigma_x^j(t)X_1(t)dt \\
&+ E\int_0^T \sum_{j=1}^d Q^{j*}(t)\sigma_{\widetilde{x}}^j(t)E\left(X_1(t)\right)dt + E\int_0^T \sum_{j=1}^d Q^{j*}(t)\sigma_v^j(t)\beta(t)dt,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
I_4 &= E\int_0^T \sum_{j=1}^d G^{j*}(t)[g_x^j(t)X_1(t) + g_{\widetilde{x}}^j(t)E\left(X_1(t)\right) + g_v^j(t)\beta(t)]dt \\
&= E\int_0^T \sum_{j=1}^d G^{j*}(t)g_x^j(t)X_1(t)dt + E\int_0^T \sum_{j=1}^d G^{j*}(t)g_{\widetilde{x}}^j(t)E\left(X_1(t)\right)dt \\
&+ E\int_0^T \sum_{j=1}^d G^{j*}(t)g_v^j(t)\beta(t)dt.
\end{aligned}
\tag{3.16}
$$

Combining (3.12)∼(3.16), we get

$$
\begin{cases}
E(\Phi^*(T)X_1(T)) \\
= E\int_0^T \Phi^*(t)b_v(t)\beta(t))dt + E\int_0^T \sum_{j=1}^d Q^{j*}(t)\sigma_v^j(t)\beta(t)dt \\
+ E\int_0^T \sum_{j=1}^\infty G^{j*}(t)g_v^j(t)\beta(t)dt - E\int_0^T X_1(t)(\ell_x(t) + E\left(\ell_{\widetilde{x}}(t)\right))dt \\
- E\int_0^T X_1(t)[f_x(t)\,K(t))dt - E(f_x(t)\,K(t))]dt
\end{cases}
\tag{3.17}
$$

Similarly, by applying *Itô's formula* to $K^*(t)Y_1(t)$ and take expectation, we get

$$\begin{aligned}
&E\left(K^*(T)Y_1(T)\right) \\
&= -E\left\{\left[\varphi_y\left(y(0),E(y(0))\right)+E\left(\varphi_{\widetilde{y}}\left(y(0),E(y(0))\right)\right)\right]Y_1(0)\right\} \\
&+E\int_0^T\left\{K^*(t)f_x(t)X_1(t)+K^*(t)f_{\widetilde{x}}(t)E\left(X_1(t)\right)\right. \\
&+K^*(t)f_v(t)\beta(t))-Y_1(t)[\ell_y(t)+E(\ell_{\widetilde{y}}(t))] \\
&\left.-Z_1(t)[\ell_z(t)+E(\ell_{\widetilde{z}}(t))]-Q_1(t)[\ell_q(t)+E(\ell_{\widetilde{q}}(t)]\right\}dt
\end{aligned} \tag{3.18}$$

combining (3.17) and (3.18), the desired result (**??**) follows. This completes the proof of *Lemma 1.* □

Proof of Theorem 1. From (3.10), we have

$$\begin{aligned}
0 &= \tfrac{d}{d\varepsilon}\Psi(\varepsilon)\mid_{\varepsilon=0}=\tfrac{d}{d\varepsilon}J(v^*(t)+\varepsilon\beta(t))\mid_{\varepsilon=0} \\
&= E\left[\int_0^T\ell_x(t)X_1(t)+\ell_{\widetilde{x}}(t)E(X_1(t))+\ell_y(t)Y_1(t)+\ell_{\widetilde{y}}(t)E(Y_1(t))\right. \\
&+\ell_z(t)Z_1(t)+\ell_{\widetilde{z}}(t)E(Z_1(t))+\ell_q(t)Q_1(t)+\ell_{\widetilde{q}}(t)E(Q_1(t)) \\
&+\left.\int_0^T\ell_v(t)\beta(t)\right]dt+E[\phi_x(x^*(T),E(x^*(T)))X_1(T) \\
&+\phi_{\widetilde{x}}(x^*(T),E(x^*(T)))E(X_1(T))] \\
&+E[\varphi_y(y^*(0),E(y^*(0)))Y_1(0)+\varphi_{\widetilde{y}}(y^*(0),E(y^*(0)))E(Y_1(0))].
\end{aligned} \tag{3.19}$$

Now, from (3.19) and *Lemma 1*, we obtain

$$E\int_0^T[\Phi^*(t)b_v(t)+\sum_{j=1}^d Q^{j*}(t)\sigma_v^j(t)+\sum_{j=1}^d G^{j*}(t)g_v^j(t)+K(t)f_v(t)+\ell_v(t)]\beta(t)dt=0,$$

From (3.5), we obtain

$$E\int_0^T\mathcal{H}_v(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t))\beta(t)dt=0. \tag{3.20}$$

Now, fix $t \in [0,T]$ and apply the above to $\beta(s) = (0,...,\beta_i(s),...,0)$, where $\beta_i(s) = \alpha_i 1_{[t,t+r]}(s)$, $s \in [0,T]$, $t + r \leq T$ and $\alpha_i = \alpha_i(w)$ is bounded, $\mathcal{G}_t-$measurable. Then from (3.20), we get

$$E\int_t^{t+r} \mathcal{H}_{v_i}(s,\psi^*(s),E(\psi^*(s)),v^*(s),\Phi^*(s),Q^*(s),G^*(s),K^*(s))\alpha_i(w)ds = 0,$$

by differentiating (3.21) with respect to r at $r = 0$ we have

$$E[\mathcal{H}_{v_i}(s,\psi^*(s),E(\psi^*(s)),v^*(s),\Phi^*(s),Q^*(s),G^*(s),K^*(s))\alpha_i] = 0. \tag{3.21}$$

Since (3.21) holds for all bounded $\mathcal{G}_t-$measurable α_i, we have

$$E[\mathcal{H}_v(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t)) \mid \mathcal{G}_t] = 0. \ a.e., \ t \in [0,T].$$

This completes the proof of *Theorem 1* □

3.4 Sufficient conditions for optimal control of MF-FBSDEs with Teugels martingale

Our purpose of this section is to derive partial information sufficient conditions for optimal control, where the system is governed by the MF-FBSDEs-(2.1) driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion. We prove that under some additional assumptions, the necessary condition (3.11) is a sufficient condition for optimality.

Assumptions (H4). We assume :

(1) The functional $\mathcal{H}(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\Phi^*(t),Q^*(t),\ G^*(t),\ K^*(\cdot))$ is convex with respect to $(x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},q,\widetilde{q},v)$ for $a.e.t \in [0,T]$, $P-a.s.$

(2) The functions $\phi(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$ are convex with respect to $(x,\widetilde{x})$ and $h(\cdot,\cdot)$ is concave

with respect to $(x, \widetilde{x})$.

Now we are able to state and prove the sufficient conditions for optimality for our control problem (2.1)−(3.2), which is the second main result of this paper.

Let $v^*(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])$ be a given admissible control. Let $(x^*(\cdot), y^*(\cdot), z^*(\cdot), q^*(\cdot))$ and $(\Phi^*(\cdot), Q^*(\cdot), G^*(\cdot), K^*(\cdot))$ be the solution to (2.1) and (3.4) respectively, associated with $v^*(\cdot)$.

Theorem 2. *Let conditions* (H1)-(H4) *hold. If for any admissible control* $v^*(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])$ *the following relation holds*

$$\{E[\mathcal{H}_v(t, \psi^*(t), E(\psi^*(t)), v^*, \Phi^*(\cdot), Q^*(\cdot), G^*(\cdot), K^*(\cdot))) \mid \mathcal{G}_t] = 0, \ a.e., \ t \in [0,T], \tag{3.22}$$

then we have

$$J(v^*(\cdot)) = \inf_{v(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])} J(v(\cdot)). \tag{3.23}$$

i.e., the admissible control $v^*(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])$ *is an optimal control for the problem* (2.1)-(3.2).

To prove *Theorem 2*, we need the following auxiliary result, which deals with the duality relations between $\Phi^*(t)$, $[x(t) - x^*(t)]$, and between $K^*(t)$, $[y(t) - y^*(t)]$. This Lemma is important for establishing our sufficient optimality conditions.

Lemma 2. Let $(x(\cdot), y(\cdot), z(\cdot), q(\cdot))$ be the solution of MF-FBSDEs-(2.1) corresponding to any admissible control $v(\cdot)$. We have

$$\begin{aligned}
&E[\Phi^*(T)(x(T) - x^*(T))] \\
&= E\int_0^T \Phi^*(t)[b(t, x(t), E(x(t)), v(t)) - b(t, x^*(t), E(x^*(t)), v^*(t))]\,dt \\
&+E\int_0^T \mathcal{H}_x^*(t)(x(t) - x^*(t))\,dt + E\int_0^T E[\mathcal{H}_{\widetilde{x}}^*(t)](E(x(t)) - E(x^*(t)))dt \\
&+E\int_0^T \sum_{j=1}^d Q^{j*}(t)[\sigma^j(t, x(t), E(x(t)), v(t)) - \sigma^j(t, x^*(t), E(x^*(t)), v^*(t))]\,dt \\
&+E\int_0^T \sum_{j=1}^\infty G^{j*}(t)[g^j(t, x(t), E(x(t)), v(t)) - g^j(t, x^*(t), E(x^*(t)), v^*(t))]\,dt
\end{aligned} \tag{3.24}$$

Similarly

$$
\begin{aligned}
& E\left[K^*(T)\left(y(T)-y^*(T)\right)\right] \\
& =-E\left(\varphi_y\left(y(0),E\left(y(0)\right)\right)\left(y^*(0)-y(0)\right)\right)-E\left(\varphi_{\widetilde{y}}\left(y(0),E\left(y(0)\right)\right)\right)\left(E\left(y^*(0)\right)-E\left(y(0)\right)\right) \\
& +E\int_0^T K^*(t)\{f(t,\psi(t),E(\psi(t)),v(t))-f(t,\psi^*(t),E(\psi^*(t)),v^*(t))\}dt \\
& +E\int_0^T \mathcal{H}_y^*(t)\left(y(t)-y^*(t)\right)dt+E\int_0^T E(\mathcal{H}_{\widetilde{y}}^*(t))\left(E(y(t))-E(y^*(t))\right))dt \\
& +E\int_0^T \sum_{j=1}^d \mathcal{H}_{z^j}^{j*}(t)\left(z^j(t)-z^{j*}(t)\right)dt+E\int_0^T \sum_{j=1}^d E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t))\left(E(z^j(t))-E(z^{j*}(t))\right)dt \\
& +E\int_0^T \sum_{j=1}^\infty \mathcal{H}_{q^j}^{j*}(t)\left(q^j(t)-q^{j*}(t)\right)dt+E\int_0^T \sum_{j=1}^\infty E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))\left(E(q^j(t))-E(q^{j*}(t))\right)dt,
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
& E\left[\Phi^*(T)\left(x(T)-x^*(T)\right)\right]+E\left[K^*(T)\left(y(T)-y^*(T)\right)\right] \\
& +E\left(\varphi_y\left(y(0),E\left(y(0)\right)\right)\left(y^*(0)-y(0)\right)\right)+E[\varphi_{\widetilde{y}}\left(y(0),E\left(y(0)\right)\right)]\left(E\left(y^*(0)\right)-E\left(y(0)\right)\right) \\
& =E\int_0^T \Phi^*(t)(b(t,x(t),E(x(t)),v(t))-b(t,x^*(t),E(x^*(t)),v^*(t)))dt \\
& +E\int_0^T \sum_{j=1}^d Q^{*,j}(t)[\sigma^j(t,x(t),E(x(t)),v(t))-\sigma^j(t,x^*(t),E(x^*(t)),v^*(t))]dt \\
& +E\int_0^T \sum_{j=1}^\infty G^{*,j}(t)\left[g^j(t,x(t),E(x(t)),v(t))\ -\ g^j(t,x^*(t),E(x^*(t)),v^*(t))\right]dt \\
& +E\int_0^T K^*(t)[f(t,\psi(t),E(\psi(t)),v(t)))-f(t,\psi^*(t),E(\psi^*(t)),v^*(t))]dt \\
& +E\int_0^T \mathcal{H}_x^*(t)\left(x(t)-x^*(t)\right)dt+E\int_0^T E\left[\mathcal{H}_{\widetilde{x}}^*(t)\right]\left(E(x(t))-E(x^*(t))\right)dt \\
& +E\int_0^T \mathcal{H}_y^*(t)\left(y(t)-y^*(t)\right)dt+E\int_0^T E(\mathcal{H}_{\widetilde{y}}^*(t))\left(E(y(t))-E(y^*(t))\right))dt \\
& +E\int_0^T \sum_{j=1}^d \mathcal{H}_{z^j}^{j*}(t)\left(z^j(t)-z^{j*}(t)\right)dt+E\int_0^T \sum_{j=1}^d E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t))\left(E(z^j(t))-E(z^{j,*}(t))\right)dt \\
& +E\int_0^T \sum_{j=1}^\infty \mathcal{H}_{q^j}^{j*}(t)\left(q^j(t)-q^{j*}(t)\right)dt+E\int_0^T \sum_{j=1}^\infty E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))\left(E(q^j(t))-E(q^{j*}(t))\right)dt.
\end{aligned}
\tag{3.26}
$$

Proof. First, by simple computations, we get

$$
\begin{aligned}
&d\left(x(t)-x^*(t)\right)=[b(t,x(t),E(x(t)),v(t))-b(t,x^*(t),E(x^*(t)),v^*(t))]dt \\
&+[\textstyle\sum_{j=1}^{d}\sigma^j(t,x(t),E(x(t)),v(t))-\sigma^j(t,x^*(t),E(x^*(t)),v^*(t)]dW^j(t) \\
&+[\textstyle\sum_{j=1}^{\infty}g^j(t,x(t),E(x(t)),v(t))-g^j(t,x^*(t),E(x^*(t)),v^*(t)]dH^j(t).
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&d\left(y(t)-y^*(t)\right)=[f(t,\psi(t),E(\psi(t)),v(t))-f(t,\psi^*(t),E(\psi^*(t)),v^*(t))]dt \\
&+\textstyle\sum_{j=1}^{d}\left(z^j(t)-z^{j*}(t)\right)dW^j(t)+\sum_{j=1}^{\infty}\left(q^j(t)-q^{j*}(t)\right)dH^j(t).
\end{aligned}
\tag{3.28}
$$

By applying integration by parts formula to $\Phi^*(t)\left(x(t)-x^*(t)\right)$ and the fact that $x(0)-x^*(0)=0$, we get

$$
\begin{aligned}
&E\left\{\Phi^*(T)\left(x(T)-x^*(T)\right)\right\}=E\textstyle\int_0^T\Phi^*(t)d\left(x(t)-x^*(t)\right) \\
&+E\textstyle\int_0^T\left(x(t)-x^*(t)\right)d\Phi^*(t)+E\int_0^T\sum_{j=1}^{d}Q^{j*}(t)[\sigma^j(t,x(t),E(x(t)),v(t)) \\
&-\sigma^j(t,x^*(t),E(x^*(t)),v^*(t))]dt+E\textstyle\int_0^T\sum_{j=1}^{\infty}G^{j*}(t)[g^j(t,x(t),E(x(t)),v(t)) \\
&-g^j(t,x^*(t),E(x^*(t)),v^*(t))]dt \\
&=A_1+A_2+A_3+A_4.
\end{aligned}
\tag{3.29}
$$

From (3.27), we obtain

$$
\begin{aligned}
A_1&=E\textstyle\int_0^T\Phi^*(t)d\left(x(t)-x^*(t)\right) \\
&=E\textstyle\int_0^T\Phi^*(t)[b(t,x(t),E(x(t)),v(t))-b(t,x^*(t),E(x^*(t)),v^*(t))]dt,
\end{aligned}
\tag{3.30}
$$

similarly, by applying (3.6), we get

$$
\begin{aligned}
&A_2=E\textstyle\int_0^T\left(x(t)-x^*(t)\right)d\Phi^*(t) \\
&=E\textstyle\int_0^T\left(x(t)-x^*(t)\right)[\mathcal{H}_x^*(t)+E(\mathcal{H}_{\widetilde{x}}^*(t))]dt \\
&=E\textstyle\int_0^T\mathcal{H}_x^*(t)\left(x(t)-x^*(t)\right)dt+\int_0^T E(\mathcal{H}_{\widetilde{x}}^*(t))\left(E(x(t))-E(x^*(t))\right)dt.
\end{aligned}
\tag{3.31}
$$

By standard arguments, we obtain

$$A_3 = E\int_0^T \sum_{j=1}^{d} Q^{j*}(t)[\sigma^j(t,x(t),E(x(t)),v(t)) - \sigma^j(t,x^*(t),E(x^*(t)),v^*(t))]dt, \quad (3.32)$$

and

$$A_4 = E\int_0^T \sum_{j=1}^{\infty} G^{j*}(t)[g^j(t,x(t),E(x(t)),v(t)) - g^j(t,x^*(t),E(x^*(t)),v^*(t))]dt, \quad (3.33)$$

the duality relation (3.24) follows from combining (3.30)∼(3.33) together with (3.29).
Let us turn to second duality relation (3.25). By applying *integration by parts formula to* $K^*(t)\left[y^*(t) - y(t)\right]$, we get

$$\begin{aligned}
&E\left(K^*(T)\left(y^*(T) - y(T)\right)\right) = E\left\{K^*(0)\left(y^*(0) - y(0)\right)\right\} \\
&+E\int_0^T K^*(t)d\left(y(t) - y^*(t)\right) + E\int_0^T \left(y(t) - y^*(t)\right)dK^*(t) \\
&+E\int_0^T \sum_{j=1}^d \left(z^j(t) - z^{j*}(t)\right)[\mathcal{H}_{z^j}^{j*}(t) + E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t))]dt \quad (3.34)\\
&+E\int_0^T \sum_{j=1}^\infty \left(q^j(t) - q^{j*}(t)\right)[\mathcal{H}_{q^j}^{j*}(t) + E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))]dt \\
&= B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}$$

Let us turn to the second term B_2. From (3.28), we get

$$\begin{aligned}
B_2 &= E\int_0^T K^*(t)d\left(y(t) - y^*(t)\right) \\
&= E\int_0^T K^*(t)[f(t,\psi(t),E(\psi(t)),v(t)) - f(t,\psi^*(t),E(\psi^*(t)),v^*(t))]dt,
\end{aligned} \quad (3.35)$$

from (3.6), we obtain

$$\begin{aligned}
B_3 &= E\int_0^T \left(y(t) - y^*(t)\right)dK^*(t) \quad (3.36)\\
&= E\int_0^T \left(y(t) - y^*(t)\right)(\mathcal{H}_y^*(t) + E(\mathcal{H}_{\widetilde{y}}^*(t)))dt,
\end{aligned}$$

$$B_4 = E\int_0^T \sum_{j=1}^{d} \left(z^j(t) - z^{j,*}(t)\right) [\mathcal{H}_{z^j}^{j*}(t) + E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t))]dt \tag{3.37}$$

and

$$B_5 = E\int_0^T \sum_{j=1}^{\infty} \left(q^j(t) - q^{j,*}(t)\right) [\mathcal{H}_{q^j}^{j*}(t) + E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))]dt. \tag{3.38}$$

From (3.4) and the fact that

$$\begin{aligned} & B_1 = E\left\{K^*(0)\left(y^*(0) - y(0)\right)\right\} \\ & = -E\Big\{ \left[\varphi_y\left(y(0), E\left(y(0)\right)\right) + E(\varphi_{\widetilde{y}}(y(0), E\left(y(0)\right))\right] \\ & \times (y^*(0) - y(0))\Big\}, \end{aligned} \tag{3.39}$$

the duality relation (3.25) follows immediately by combining (3.35)$\sim$(3.39) together with (3.34). Finally, inequality (3.26) follows from combining (3.24) and (3.25). $\square$

Proof of Theorem 2. Let $(x(\cdot), y(\cdot), z(\cdot), q(\cdot))$ be the solution of the state equation (2.1) and $(\Phi(\cdot), Q(\cdot), G(\cdot), K(\cdot))$ be the solution of the adjoint equation (3.4), corresponding to $v(\cdot) \in \mathcal{U}_{\mathcal{G}}([0,T])$.

$$\begin{aligned} & J\left(v^*(\cdot)\right) - J\left(v(\cdot)\right) \\ & = E\Big\{ \textstyle\int_0^T [\ell(t, \psi^*(t), E(\psi^*(t)), v^*(t)) \\ & -\ell(t, \psi(t), E(\psi(t)), v(t))]dt \\ & + \left[\phi\left(x^*(T), E(x^*(T))\right) - \phi\left(x(T), E(x(T))\right)\right] \\ & + \left[\varphi\left(y^*(0), E\left(y^*(0)\right)\right) - \varphi\left(y(0), E\left(y(0)\right)\right)\right] \Big\}, \end{aligned}$$

By the convexity condition on $\phi(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$, we get

$$
\begin{aligned}
& J\left(v^*(\cdot)\right)-J\left(v(\cdot)\right) \\
& \leq E[(\phi_x\left(x^*(T),E(x^*(T))\right)+E(\phi_{\widetilde{x}}(x^*(T),E(x^*(T)))) \\
& \times\left(x^*(T)-x(T)\right)] \\
& +E[(\varphi_y\left(y^*(0),E\left(y^*(0)\right)\right)+\varphi_{\widetilde{y}}\left(y(0),E\left(y(0)\right)\right)) \\
& \times\left(y^*(0)-y(0)\right)] \\
& +E\Big\{\int_0^T[\ell(t,\psi^*(t),E(\psi^*(t)),v^*(t)) \\
& -\ell(t,\psi(t),E(\psi(t)),v(t))]dt\Big\},
\end{aligned}
$$

By applying *Lemma 2*, we get

$$
\begin{aligned}
& J\left(v^*(\cdot)\right)-J\left(v(\cdot)\right) \\
& \leq E\int_0^T\Big\{(\mathcal{H}(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t), \\
& \qquad G^*(t),K^*(t)) \\
& -\mathcal{H}(t,\psi(t),E(\psi(t)),v(t),\Phi^*(t),Q^*(t),G^*(t), \\
& \qquad K^*(t)))\Big\}dt \\
& -E\int_0^T(\mathcal{H}_x^*(t)+E(\mathcal{H}_{\widetilde{x}}^*(t)))(x(t)-x^*(t))dt \\
& -E\int_0^T(\mathcal{H}_y^*(t)+E(\mathcal{H}_{\widetilde{y}}^*(t)))(y(t)-y^*(t))dt \\
& -E\int_0^T\sum_{j=1}^d(\mathcal{H}_{z^j}^{j*}(t)+E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t)))(z^j(t)-z^{j*}(t)))dt \\
& -E\int_0^T\sum_{j=1}^d(\mathcal{H}_{q^j}^{j*}(t)+E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t)))(q^j(t)-q^{j*}(t)))dt.
\end{aligned}
\tag{3.40}
$$

By the convexity of the functional $\mathcal{H}\left(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\Phi^*(t),Q^*(t),G^*(t),K^*(\cdot)\right)$, in the sense of Clarke's generalized gradient, the following holds

$$
\begin{aligned}
&E\int_0^T[\mathcal{H}(t,\psi(t),E(\psi(t)),v(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t)) \\
&-\mathcal{H}(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t))]dt \\
&\geq E\int_0^T\Big\{\mathcal{H}_x^*(t)(x(t)-x^*(t)) \\
&+E(\mathcal{H}_{\widetilde{x}}^*(t))(E(x(t)-x^*(t)))+\mathcal{H}_y^*(t)(y(t)-y^*(t)) \\
&+E(\mathcal{H}_{\widetilde{y}}^*(t))(E(y(t)-y^*(t))) \\
&+\sum_{j=1}^d\mathcal{H}_{z^j}^{*,j}(t)(z^j(t)-z^{j*}(t))) \\
&+\sum_{j=1}^d E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t))(E(z^j(t)-z^{*,j}(t))) \\
&+\sum_{j=1}^\infty\mathcal{H}_{q^j}^{j*}(t)(q^j(t)-q^{j*}(t))) \\
&+\sum_{j=1}^\infty E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))(E(q^j(t)-q^{j*}(t))) \\
&+\mathcal{H}_v^*(t)(v(t)-v^*(t))\Big\}dt.
\end{aligned}
\tag{3.41}
$$

Since the conditional expectation $E[\mathcal{H}_v(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t))\mid\mathcal{G}_t]$, $v(\cdot)$ and $v^*(\cdot)$ are $\mathcal{G}_t-$measurable, we have

$$
\begin{aligned}
&E[\mathcal{H}_v(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t), \\
&K^*(t))\mid\mathcal{G}_t](v(t)-v^*(t)) \\
&=E[\mathcal{H}_v(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t)) \\
&\times(v(t)-v^*(t))\mid\mathcal{G}_t]
\end{aligned}
\tag{3.42}
$$

Using condition (3.41), (3.42) and (3.22), we obtain

$$\begin{aligned}
&E\int_0^T[\mathcal{H}(t,\psi^*(t),E(\psi^*(t)),v^*(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t))\\
&-\mathcal{H}(t,\psi(t),E(\psi(t)),v(t),\Phi^*(t),Q^*(t),G^*(t),K^*(t))]dt\\
&-E\int_0^T\Big\{[\mathcal{H}_x^*(t)+E(\mathcal{H}_{\widetilde{x}}^*(t))](x^*(t)-x(t))\\
&+[\mathcal{H}_y^*(t)+E(\mathcal{H}_{\widetilde{y}}^*(t))](y^*(t)-y(t))\\
&+\sum_{j=1}^d[\mathcal{H}_{z^j}^{j*}(t)+E(\mathcal{H}_{\widetilde{z}^j}^{j*}(t)))(z^{j*}(t)-z^j(t)))\\
&+\sum_{j=1}^\infty[\mathcal{H}_{q^j}^{j*}(t)+E(\mathcal{H}_{\widetilde{q}^j}^{j*}(t))](q^{j*}(t)-q^j(t)))\Big\}dt\leq 0,
\end{aligned}\tag{3.43}$$

from (3.40) and (3.43), we get

$$J\left(v^*(\cdot)\right)\leq J(v(\cdot)),\ \text{for any control } v(\cdot)\in\mathcal{U}_{\mathcal{G}}([0,T]).$$

Finally, we observe that since $v(\cdot)$ is an arbitrary admissible control of $\mathcal{U}_{\mathcal{G}}([0,T])$, the desired result (3.23) follows. This completes the proof of *Theorem 2.* □

3.5 Application : Optimal portfolio strategy driven by Teugels martingales associated with Gamma Process

In this section, we will apply our necessary and sufficient maximum principle of optimality to study mean-variance portfolio selection problem driven by Teugels martingales associated to Gamma processes. Let $\mathcal{G}_t$ be a given subfiltration of $\mathcal{F}_t$, $t\geq 0$,. For example, $\mathcal{G}_t$ could be the $\gamma-$delayed information defined by

$$\mathcal{G}_t=\mathcal{F}_{(t-\gamma)^+}:t\geq 0,\tag{3.44}$$

where γ is a given constant delay. Suppose that we are given a mathematical market consisting of two investment possibilities. The first asset is a risk-free security whose price $R_0(t)$ evolves according to the ordinary differential equation

$$dR_0\left(t\right) = R_0\left(t\right)\rho(t)dt,\ R_0\left(0\right) > 0, \tag{3.45}$$

where $\rho\left(\cdot\right) : [0, T] \to \mathbb{R}_+$ is a locally bounded continuous deterministic function. The second asset is a risky security (*Stock)* where the price $R_1\left(t\right)$ at time t is given by

$$\begin{aligned} dR_1\left(t\right) &= \tau(t)R_1\left(t\right)dt + \pi(t)R_1\left(t\right)dW(t) \\ &+ \textstyle\sum_{j=1}^{\infty} G^j(t)H^j(t),\ R_1\left(0\right) > 0, \end{aligned} \tag{3.46}$$

where $H^j(t)$ the orthogonal Teugels martingales associated with *Gamma processes* as Lévy process of bounded variation $X = \{X(t) : t \geq 0\}$ with Lévy measure given by

$$\mu(dx) = \frac{e^{-x}}{x} I_{\{x>0\}} dx.$$

We denote by $X^j(t) = \sum_{0 \leq s \leq t} \left(\triangle X(s)\right)^j : j \geq 1$ the power jump processes of X. By applying *exponential formula* proved in Bertoin [9], we obtain

$$\begin{aligned} &E\left(\exp(i\theta X^j(t))\right) \\ &= \exp\left(t\int_{\mathbb{R}_+}(\exp(j\theta z) - 1)\frac{\exp(-z^{\frac{1}{j}})}{jz}dz\right), \end{aligned}$$

which means that the Lévy measure of X^j is $\frac{\exp(-z^{\frac{1}{j}})}{jz}dz$. Since

$$\begin{aligned} E\left(X^j(t)\right) &= E\left[\sum_{0\leq s\leq t} (\triangle X(s))^j\right] \\ &= t\int_0^{+\infty} x^j \frac{e^{-x}}{x} dx : j \geq 1, \\ &= t\Gamma(j) = (j-1)!t : j \geq 1, \end{aligned} \tag{3.47}$$

and thus $\widehat{X}^j(t) = X^j(t) - (j-1)!t : j \geq 1$ is the Teugels martingales of order j of the Gamma processes. Now, we orthogonalize the set $\left\{\widehat{X}^j(\cdot) : j \geq 1\right\}$ of martingales, then we have a set of orthogonal Teugels martingales of the form

$$H^i(t) = \sum_{1\leq j\leq i-1} a_{ij}\widehat{X}^j(t) : i \geq 1. \tag{3.48}$$

In order to ensure that $R_1(t) > 0$ for all $t \in [0,T]$ we assume :

(i) The functions $\tau(\cdot) : [0,T] \to \mathbb{R}$, $\pi(\cdot) : [0,T] \to \mathbb{R}$ are bounded continuous deterministic maps such that $\tau(t)$, $\pi(t) \neq 0$ and $\tau(t) - \rho(t) > 0,\ \forall t \in [0,T]$.

(ii) For any $t \in [0,T] : g(t) > 0$.

By combining (3.45), (3.46) and (3.48), we introduce the wealth dynamics

$$\begin{cases} dx^v(t) = \left[\rho(t)x^v(t) + (\tau(t) - \rho(t))v(t)\right] dt \\ \quad + \pi(t)v(t)dW(t) + \sum_{j=1}^{\infty} g^j(t)H^j(t), \\ -dy^v(t) = \left[\rho(t)x^v(t) + (\tau(t) - \rho(t))v(t) - cy^v(t)\right] dt \\ \quad - z^v(t)dW(t) - \sum_{j=1}^{\infty} q^{v,j}(t)H^j(t), \\ x^v(0) = a, \quad y^v(T) = x^v(T), \\ H^i(t) = \sum_{1\leq j\leq i-1} a_{ij}(X^j(t) - (j-1)!t) : i,j \geq 1. \end{cases} \tag{3.49}$$

More precisely, for any admissible control $v(\cdot)$ the utility functional is given by

$$J\left(v(\cdot)\right) = \frac{\delta}{2} Var(x^v(T)) - E(x^v(T)) + y^v(0). \tag{3.50}$$

Let $\mathbb{U}$ be a compact convex subset of $\mathbb{R}$. We denote $\mathcal{U}_{\mathcal{G}}([0,T])$, the set of admissible $\mathcal{G}_t-$predictable portfolio strategies $v\left(\cdot\right)$ valued in $\mathbb{U}$.

The Hamiltonian functional (3.5) gets the form

$$\begin{aligned}
&\mathcal{H}\left(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, q, \widetilde{q}, v, \Phi(\cdot), Q(\cdot), G(\cdot), K(\cdot)\right) \\
&= \left[\rho(t)x(t) + (\tau(t) - \rho(t))v(t)\right]\left(\Phi(t) + K(t)\right) \\
&+ \pi(t)v(t)Q(t) - cK(t)y(t) + \textstyle\sum_{j=1}^{\infty} G^j(t)g^j\left(t\right)
\end{aligned}$$

According to the maximum condition ((3.11), *Theorem 1),* and since $v^*(\cdot)$ is optimal, we immediately get

$$E\left[\left(\tau(t) - \rho(t)\right)\left(\Phi^*(t) + K^*(t)\right) + \pi(t)Q^*(t) \mid \mathcal{G}_t\right] = 0. \tag{3.51}$$

The adjoint equation (3.4) has the form :

$$\begin{cases}
d\Phi^*(t) = -\rho(t)\left(K^*(t) + \Phi^*(t)\right)dt \\
\qquad + Q^*(t)dW(t) + \sum_{j=1}^{\infty} G^{*,j}(t)dH^j(t), \\
\Phi^*(T) = \delta\left(x^*(T) + E(x^*(T))\right) - 1 - K^*(T), \\
dK^*(t) = -cK^*(t)dt,\ \ K^*(0) = -1,\ \ t \in \left[0, T\right].
\end{cases} \tag{3.52}$$

In order to solve the above equation (3.52) and to find the expression of optimal portfolio strategy $v^*(\cdot)$ we conjecture a process $\Phi^*(\cdot)$ of the form

$$\Phi^*(t) = V_1(t)x^*(t) + V_2(t)E\left(x^*(t)\right) + V_3(t), \tag{3.53}$$

where $V_1(\cdot), V_2(\cdot)$ and $V_3(\cdot)$ are deterministic differentiable functions. From last equation in (3.52), which is a simple ordinary differential equation, we get immediately

$$K^*(t) = -\exp(-ct), \ t \in [0, T]. \tag{3.54}$$

Noting that from (3.49), we get

$$d(E(x^*(t)) = \{\rho(t)E(x^*(t)) + (\tau(t) - \rho(t))E(v^*(t))\} dt.$$

Applying Itô's formula to (3.53) in virtue of SDE-(3.49), we get

$$\begin{cases} d\Phi^*(t) = \{V_1(t)[\rho(t)x^*(t) + (\tau(t) - \rho(t))v^*(t)] \\ +x^*(t)\dot{V}_1(t) \\ +V_2(t)[\rho(t)E(x^*(t)) + (\tau(t) - \rho(t))E(v^*(t))] \\ +\dot{V}_2(t)E(x^*(t)) + \dot{V}_3(t)\Big\} dt \\ +V_1(t)\pi(t)v^*(t)dW(t) + \sum_{j=1}^{\infty} V_1(t)g^j(t)H^j(t), \\ \Phi^*(T) = V_1(T)x^*(T) + V_2(T)E(x^*(T)) + V_3(T), \end{cases} \tag{3.55}$$

where $\dot{V}_1(t)$, $\dot{V}_2(t)$, and $\dot{V}_3(t)$ denotes the derivatives with respect to t.
Next, comparing (3.55) with (3.52), we get

$$\begin{aligned} &-\rho(t)(K^*(t) + \Phi^*(t)) \\ &= V_1(t)[\rho(t)x^*(t) + (\tau(t) - \rho(t))v^*(t)] + x^*(t)\dot{V}_1(t) \\ &+V_2(t)[\rho(t)E(x^*(t)) + (\tau(t) - \rho(t))E(v^*(t))] \\ &+\dot{V}_2(t)E(x^*(t)) + \dot{V}_3(t), \end{aligned} \tag{3.56}$$

$$Q^*(t) = V_1(t)\pi(t)v^*(t). \tag{3.57}$$

$$G^*(t) = V_1(t)g(t). \tag{3.58}$$

By looking at the terminal condition of $\Phi^*(t)$, in (3.55), it is reasonable to get

$$V_1(T) = \delta,\ V_2(T) = -\delta,\ V_3(T) = -1 - K^*(T). \tag{3.59}$$

Combining (3.56) and (3.53) we deduce that $V_1(\cdot), V_2(\cdot)$ and $V_3(\cdot)$ satisfying the following ordinary differential equation

$$\begin{cases} \dot{V}_1(t) = -2\rho(t)V_1(t),\ V_1(T) = \delta, \\ \dot{V}_2(t) = -2\rho(t)V_2(t),\ V_2(T) = -\delta, \\ \dot{V}_3(t) + \rho(t)V_3(t) = \rho(t)\exp\{-ct\}, \\ V_3(T) = \exp\{-cT\} - 1. \end{cases} \tag{3.60}$$

By solving the first two ordinary differential equations in (3.60), we obtain

$$V_1(t) = -V_2(t) = \delta \exp\left\{2\int_t^T \rho(s)ds\right\}. \tag{3.61}$$

3.6 Conclusions and future works

In this paper, necessary and sufficient conditions for optimal control for MF-FBSDEs driven by Teugels martingale associated to Lévy processes have been discussed. An interesting observation is that by adding the tool of the derivatives with respect to measures, we can treat more general mean-field cases. As an illustration, mean-variance portfolio selection problem driven by Teugels martingales associated with Gamma process as Lévy process with bounded variation has been studied. An open question is to derive optimality conditions for this control problem for general (non-convex) control domain. Apparently, there are many problems left unsolved and one possible problem is to study the the general maximum principle for fully coupled MF-FBSDEs driven by Teugels martingales, and to study some applications in finance models governed by orthogonal Teugels martingales associated with *Meixner process* as a Lévy process with bounded variation, using the orthogonality of the *Meixner-Pollaczek polynomials.*

Appendix

The following result gives a case of the Itô formula for jump diffusions of mean-field type.

Lemma A1. Suppose that the processes $x_1(t)$ and $x_2(t)$ are given by : for $j = 1, 2$, $t \in [s, T]$:

$$\begin{aligned} dx_j(t) &= f\left(t, x_j(t), E(x_j(t)), u(t)\right) dt + \sigma\left(t, E(x_j(t)), u(t)\right) dW(t) \\ &\quad + \int_\Theta g\left(t, x_j(t^-), E(x_j(t)), u(t), \theta\right) N\left(d\theta, dt\right), \\ x_j(s) &= 0, \end{aligned}$$

then we get

$$\begin{aligned} E\left(x_1(T)x_2(T)\right) &= E\left[\int_s^T x_1(t)dx_2(t) + \int_s^T x_2(t)dx_1(t)\right] \\ &\quad + E\int_s^T \sigma^*\left(t, x_1(t), E(x_1(t)), u(t)\right)\sigma\left(t, x_2(t), E(x_2(t)), u(t)\right) dt \\ &\quad + E\int_s^T \int_\Theta g^*\left(t, x_1(t), E(x_1(t)), u(t), \theta\right) g\left(t, x_2(t), E(x_2(t)), u(t), \theta\right) \mu(d\theta)dt. \end{aligned}$$

Applying a similar method as in [23, Lemma 2.1], for the proof of the above Lemma.

Proposition A2. [11, Appendix] Let $\mathcal{G}$ be the predictable $\sigma-$field on $\Omega \times [0, T]$, and f be a $\mathcal{G} \times \mathcal{B}\left(\Theta\right) -$measurable function such that

$$E\left\{\int_0^T \int_\Theta \left|f\left(w, s, \theta\right)\right|^2 \mu\left(d\theta\right) ds\right\} < +\infty.$$

Then for all $p \geq 2$ there exists a positive constant $C = C\left(p, T, \mu\left(\Theta\right)\right)$ such that

$$E\left\{\sup_{t\in[0,T]}\left|\int_0^t\int_\Theta f\left(w,s,\theta\right)N\left(ds,d\theta\right)\right|^p\right\} \leq CE\left\{\int_0^T\int_\Theta \left|f\left(w,s,\theta\right)\right|^p \mu\left(d\theta\right)ds\right\}.$$

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