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Degree of controllability and degree of observability of a monovariable system

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Présenté par : SAOULI Khaled

Dirigé par : ABADA Khaled

Résumés (Arabe, Anglais et français)

الملخص

الهدف الأساسي من هذه المذكرة هو ايجاد طريقة ما لمعرفة مدى قدرتنا على السيطرة والتحكم في نظام ديناميكي من جهة ومدى قدرتنا على رؤية وملاحظة هذا النظام بصفة عامة، حيث أن البحث يتركز حول طريقة حسابية تجعلنا نحيط بهذه المعرفة لايجاد قيم عددية تعبر على المطلوب. الأمر مبني على أن النظام الديناميكي العام مكون من أنظمة جزئية، حيث قد نستطيع السيطرة وملاحظة بعض من هذه الأنظمة وعدم القدرة على ذلك مع الأخرى، فبناءا على عدد الأنظمة الجزية المتحكم بها نعرف درجة تحكمنا في النظام الديناميكي عامة، نفس الأمر فيما يخص قدرتنا على رؤية وملاحظة النظام الديناميكي على وجه العموم.

Abstract

The main goal of this thesis is to find a way to know the extent of our ability to control and observe a dynamic system, as the research is focused on a computational method that makes us surround this knowledge to find numerical values that express the required. However, the general dynamic system is composed of partial systems, where we may be able to control and observe some of these systems and not be able to do so with others, based on the number of controlled and observed systems we know the degree of the controllability and the degree of the observability of a dynamic system in general.

Résumé

L'objectif principal de cette mémoire est de trouver un moyen de connaître l'étendue de notre capacité à contrôler et à observer un système dynamique, car la recherche se concentre sur une méthode de calcul qui nous fait entourer ces connaissances pour trouver des valeurs numériques qui expriment le besoin. Cependant, le système dynamique général est composé de systèmes partiels, où nous pouvons être en mesure de contrôler et d'observer certains de ces systèmes sans pouvoir le faire avec d'autres, en fonction du nombre de systèmes contrôlés et observés, nous connaissons le degré de la contrôlabilité et degré d'observabilité d'un système dynamique en général.

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This project would not have been possible without the support of many people.

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Thanks to the University of Mohamed Khider.

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Dedications

to my father,

MABROUK

to my mother,

to my sister,

To my brothers and all my family,

To All my friends.

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Notations and symbols

$u(t)$ the system's input in time domain

$y(t)$ the system's output time domain

$x(t)$ the state variable of a system time domain

$U(s)$ the system's input Laplace domain

$Y(s)$ the system's output Laplace domain

$X(s)$ the state variable of a system Laplace domain

$F(s)$ the system's transfer function

$M(s)$ the system's transfer matrix

x(k) the state variable of discrete system

u(k) the input of discrete system

x_f the state variable of a system for the last time t_f

M_c the controllability's matrix

M_o the observability's matrix

General introduction

In the study of the dynamic systems, there are a many of stations those we must stop for them, some of these stations are the controllability and the observability, and this is what our topic revolves around, there are a controllable systems and uncontrollable systems, and there are an observable systems and an unobservable systems. Now, between the controllability and the uncotrollability of a system, there is a question about how much we could control this system ? also, between the observability and the unobservability of a system, there is a question about how much we could observe this system ? Those are the main questions.

On another side, some one may consider that the pecedent questions are coming from an inexperience work in the past, or a little knowledge, as we didn't pose the questions those could be enough for our subject, because we didn't ask if there are a conditions to know how much control or observe a system, especially if the method that lead us to that, is indirectly method.

The controllable system is a system that capable to be controled, so, we ask about how deep the capability to control this system ? in the light of this question, we must look for the elements those control the capability to control a system. Controlling a system is an act we do, so, the elements those control the capability to control a system are an elements those we make and we create, may these elements exist when the system starts its operation.

On an other side the observable system is a system that capable to be observed, it is clearly that we ask about how deep the capability to observe this system ? this is the question that leads us to go looking for the elements those control the capability to observe a system. Here, observing a system is an act based on what we see in the system as a reactions of the system operations, so, we must study these reactions to discover the elements those control the capability to observe a system.

Somethings push us to ask another question, if we found an answers for all previous questions, and we found a quantities of the controllability and the observability, how much we can say that the quantities are accurate, and what if this precision is related with representation of the system and we must work more to obtain a good precision.

We know that every system consists of inputs and output, so, we must deeply study everything about these two things, but we must not forget that the system consists of more than the input and the output, because there is no doubt the relation between the input and the output, create an elements, without them the relation can not be exist .

In the end, we say that we must study most of the things those related with the dynamic systems, may this what allow us to find an answers for every previous question.

Chapter I: representation of systems

INTRODUCTION

We want in this chapter to reminder of some fundamental concepts about the dynamical systems and some mathematical tools, those tools show us how to understand and manage the mathematic model of a dynamic system. Firstly, we will talk about some types of systems through define every one of them, and explain the general concept, the explaining will be mathematically, or it will be by figures. Note that, we will focus in the explication of the monovariable system and the multivariable system, through noting some details about the internal and the external representation of both of them.

I.1 Definition

I.1.1 System

The system is a set of tools which are connected to each other in order to achieve a specific task. Everything outside that system connects with the system itself, this connection is making through the intermediary of magnitudes, those magnitudes are presented as function of time which are called signals.

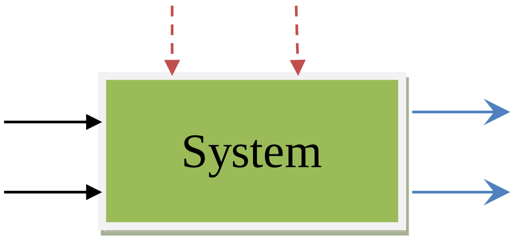

Figure I.1: Block diagram of a system

The black fleche is controllable input
The dotted red line is the uncontrollable input
The blue line is output
The controllable input is manageable; we can manage the system through the input.
The uncontrollable input is coming out from the factors which are out of our control, it affects on system.
The output is the result of the system's function.

I.1.2 the static system

This type of systems is written by algebraic equations .The response of these systems to an external excitation is instantaneous. In the operation of the system, the time t cannot be intervened.

Example 1.1: Pure electrical resistance $v(t) = Ri(t)$ so $i(t) = \frac{1}{R}i(t)$ (I.1)

The input in this system is $v(t)$: voltage across the resistor R.
The output in this system is $i(t)$: current flowing through the resistor R.

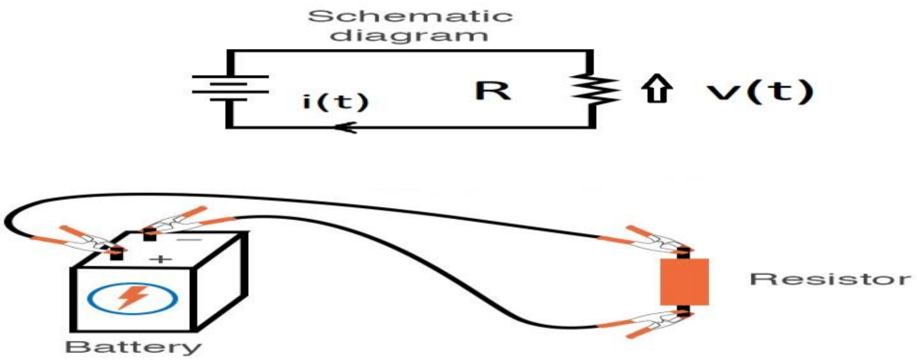

Figure I.2: Pure electrical resistance

I.1.3 Dynamic systems

This kind of systems is written or described by differential equations. These systems have a memory, in which the time response depends on the present input, and also depends on the past input.

Example 1.2: electric capacitor of capacity C.

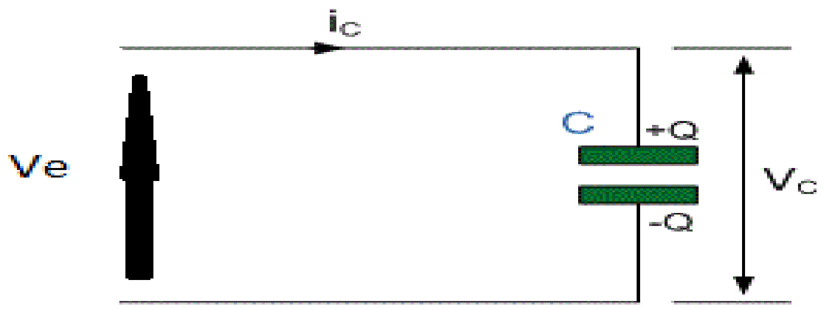

Figure I.3: electric capacitor of capacity C

The input in this system is $v(t)$: voltage accrorss the capacitor C.
The output in this system is $q(t)$: charge of the capacitor. $i(t)$ is the current intensity.

$$v(t) = \frac{1}{C}\int i(t)dt$$

so (I.2)

$$i(t) = \frac{dq}{dt}$$

the response in this system is : $q(t) = q_0 + Cv(t)$
$q_0 = q(t = 0)$ is the charge of the capacitor in the initial instant.

I.1.4 linear system

We can say that the system is linear system; if and only if the relations who connect its inputs and its outputs can be in the form of a set of differential equations with constant coefficients. Linear systems are characterized by the following two properties:

- Proportionality:

If $y(t)$ is the system response to the input $u(t)$, then $\alpha y(t)$ is the system response to the input $\alpha u(t)$, where α is a scalar.

Where α is scalar, so it’s not a vector or matrix.

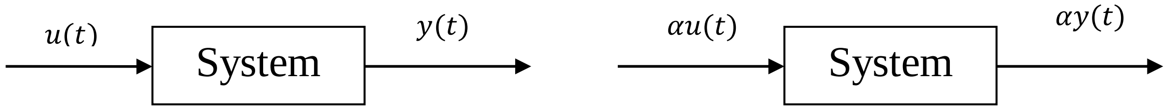

Figure I.4: Principle of proportionality

- Additively or superposition:

Let’s suppose that $y_1(t)$ is the system response to the input $u_1(t)$and let say that $y_2(t)$ is the system response to the input $u_2(t)$, then $y_1(t) + y_2(t)$ is the system response to input $u_1(t) + u_2(t)$.

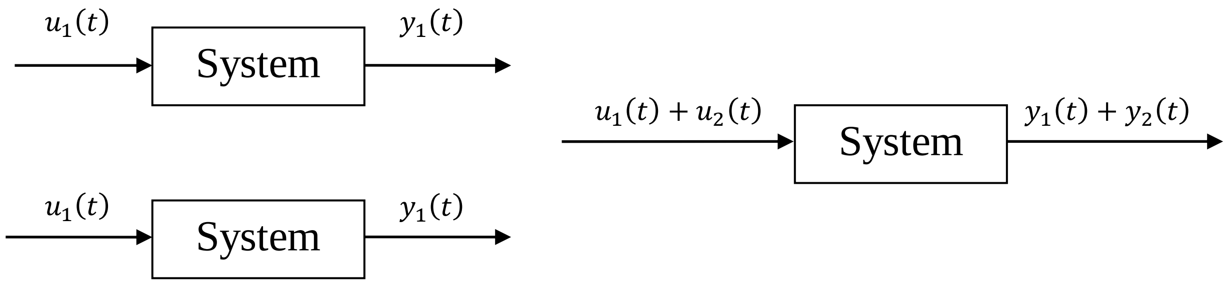

Figure I.5: Principle of superposition.

We contrast linear systems with non-linear systems. In practical reality there is no linear system and the physical system is not linear as well, but we can consider non-linear systems as linear in a certain operating area.

I.1.5 Causal system

The meaning of the causation is that the effect can never precede the cause. The cause is represented by the input signal from a system, and the system response represents the effect. If we want to make a system as causal system we must make its temporal response only depends on the present and past values of its input quantity, the response (t) at the instant t only depends on the values of the input $u(\tau)$ where $\tau \leq t$. These systems are also called physically feasible systems

Example:

Memoryless system:

$$y(t) = 1 - x(t)\cos(\omega t) \qquad \text{(I.3)}$$

Autoregressive filter :

$$y(t) = x(t - \tau)\mathrm{e}^{-\beta i} \qquad \text{(I.4)}$$

I.1.6 Invariant system

We call a system as an invariant system if the relation between the input and the output is independent of time.

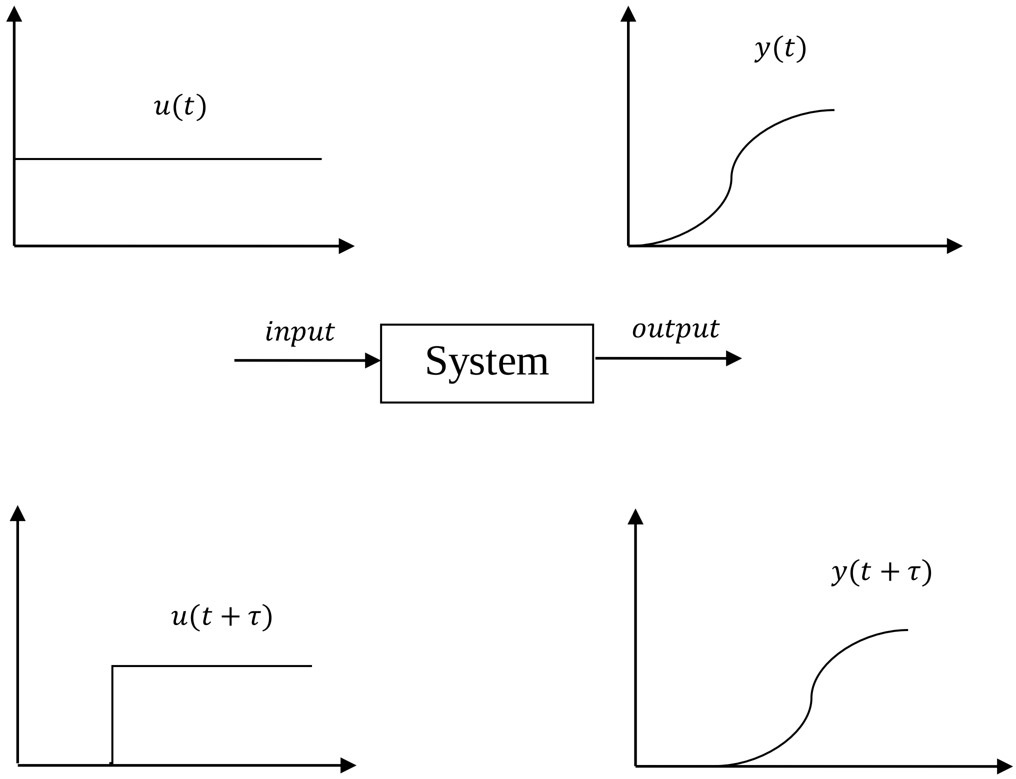

Figure I.6: Behavior of an invariant system

I.1.7 Continuous system / sampled system

If the variations in the quantities characterizing it are functions of type $f\ (t)$ where t is a continuous variable, we can say that the system continuous. Continuous systems are opposed to discrete (sampled) systems.

I.1.8 Monovariable system / multivariable system

We call the system that have only one input and only one output a monovariable system, (single input and a single output) SISO. however, We call the system that have many of inputs and many of outputs a multivariable system (Multiple Input Multiple Output) MIMO.

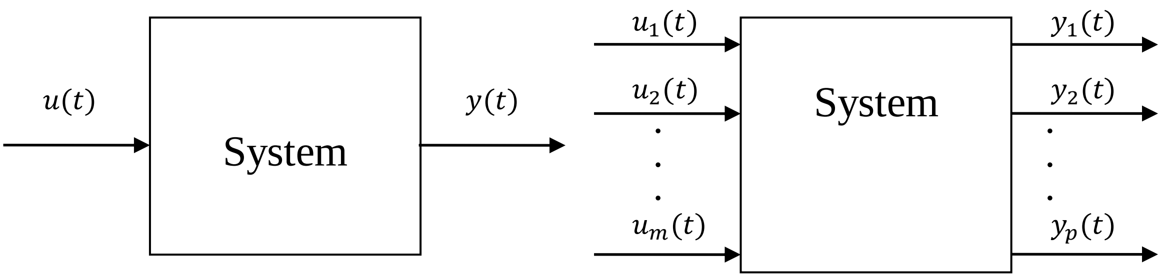

Figure I.7: Monovariable system (left), multivariable system (right).

I.1.9 Deterministic system / Random system

We say that a system is deterministic if and only if for each input there is only one possible output. On another side, we note that a system is a random for any input it exists more than one output, every output represent a probability.

I.2. representation of monovariable system

I.2.1 External representation

In this representation we uses directly the input / output relation considering the system as a black box**.**

I.2.1.1 Differential equation

We consider a linear invariant monovariable system of order n.

Figure I.8: Block diagram of a monovariable system

So, a system is written or described by a linear differential equation with constant coefficients of the form:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_0 u(t) \quad \text{(I.5)}$$

Where $u\ (t)$ and $y\ (t)$ are the input and output of the system, respectively, and $n \geq m$ (causal system).

I.2.1.2 Transfer function

The transfer function of the system that we study is the relation between its output and input. We apply the Laplace transform on the equation in order to find the system's transfer function.

With taking the initial conditions zero, we obtain:

$$Y(s)[s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0] = U(s)[b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0] \quad \text{(I.6)}$$

So

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad \text{(I.7)}$$

I.2.2 Internal (or state) representation

Many students think that the general principle of state representation is just about the relation between its input and its output, yet that's not totally true, because it's also about describing the system by considering its internal dynamic. so, restoring importance to magnitudes that are neither input nor output is necessary, while taking into account all of the dynamic and static phenomena which give the system its behavior. According to that, we jump to the following definitions.

State: is one thing that studies the evolution of a system of order n, of n information. The evolution must be in every moment, from the moment t_0, and it has to be studyed through the knowledge of the inputs.

State variables: every information can lead us to the state variables at the moment t_0 so we obtain clearly:

$$x_1(t_0), x_2(t_0), \dots, x_n(t_0) \tag{I.9}$$

State vector: always the state variables are at form a vector x called state vector, so when $t = t_0$, we obtain :

$$x(t_0) = [x\_1\ (t\_0\ ), x\_2\ (t\_0\ ), \dots, x\_n\ (t\_0\ )]^T \tag{I.10}$$

We can say that the state variables represent the initial conditions evolutions of the system; we can say that the state variables are the memory of the past in the system.

State space: simply, we can say that the state space is the general mathematical describe of the system, note that, it is like next:

$$\begin{aligned} \dot{x}(t) &= f(x,u,t) \\ y(t) &= g(x,u,t) \end{aligned} \tag{I.11}$$

When we consider that system is linear, we represent the state at form :

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{I.12}$$

If A B C and D are constants, it comes:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{I.13}$$

And we note :
$x(t) \in R^n$: State vector.
$y(t) \in R$: Output (it is a scalar).
$u(t) \in R$: Input (it is a scalar).
$A \in R^{n\times n}$: Evolution matrix.
$B \in R^{n\times 1}$: Command vector (input vector).
$C \in R^{1\times n}$ Observation vector (output vector).
$D \in R$: Direct transmission constant (often zero).

I.3. Representation of multivariable systems (MIMO)

Generally, we use a technics of the monovariable case to represent the multivariable systems.

I.3.1 External representation

I.3.1.1 System of differential equations

We can describe a multivariable linear invariant system by system of linear differential with constant coefficients, and the system have more than one input and more than one output(we consider m inputs and p outputs).when we apply a physics laws, we obtain a description represented in algebraic equations at the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ \dot{x}_n = f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \end{cases}$$
$$\begin{cases} y_1 = h_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ \quad\quad\quad\quad\quad . \\ y_p = h_p(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \end{cases} \quad (I.14)$$

With taking $u_i(t)$ as an input, with $i = 1, \dots, m$ and $y_p(t)$ as an output, $j = 1, \dots, p$. Note that f_i and h_i are a linear mathematic functions.

I.3.1.2 Transfer matrix

In a monovariable system, note that we have a single input and a single output, we connect both of them by the transfer function, on another side, in a multivariable system, we know that exist more than one input and more than one output, so, we can represent the relation between these inputs and these outputs in a transfer matrix, this matrix is some transfer functions arranged in a matrix.

SISO systems:

$$F\ (S) = Y\ (s)\ /\ U\ (s) \quad (I.15)$$

MIMO systems:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \quad (I.16)$$

I.3.2 Internal (or state) representation

As the internal representation of the monovariable system, we need here the same mathematic tools for represent what inside the black box of a multivariable system, with some different details, so:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (I.17)$$

And we note:
$x(t) \in R^n$: State vector.
$u(t) \in R^m$: Output (it is a matrix).
$y(t) \in R^p$: Input (it is a matrix).
$A \in R^{n \times n}$: Evolution matrix.
$B \in R^{n \times m}$: Command matrix (input matrix).
$C \in R^{p \times n}$: Observation vector (output matrix).
$D \in R^{p \times m}$: Direct trans0mission matrix.

I.4. Passage from the transfer matrix to the state representation

In this section we will show how to turn from the transfer matrix to the state representation, using the method of Gilbert, where the poles of the transfer matrix are simple and real, seeking the help from an example explain the method.
We consider that λ_i are the poles of the transfer matrix, where $i = 1 \dots n$.
By writing $M(s)$ in a simple way:

$$M(s) = \sum_{i=1}^{n} \frac{M_i}{s-\lambda_i} \tag{I.18}$$

With the expression of M_i as next :

$$M_i = \lim_{s\to\lambda_i} (s_i - \lambda_i)M(s) \tag{I.19}$$

Note that, to find the matrices A, B and C we need the next laws :

$$A = bloc\ diag(\lambda_i I_n), M_i = C_i B_i \tag{I.20}$$

Let explain the method by taking an exemple

$$M(s) = \frac{1}{(s-2)(s-1)(s+1)} \begin{bmatrix} -s & 3s \\ s & -3s \end{bmatrix}$$

Firstly, the poles of the transfer matrix are:

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$$

So, to find the matrix M_i, we apply:

$$M_1 = \lim_{s\to 2}(s-2)M(s) = \frac{1}{3}\begin{bmatrix} -2 & 6 \\ 2 & -6 \end{bmatrix}$$
$$M_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \end{bmatrix}\begin{bmatrix} -1 & 3 \end{bmatrix}$$

$$M_2 = \lim_{s\to 1}(s-1)M(s) = -\frac{1}{2}\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$
$$M_2 = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \end{bmatrix}$$

$$M_3 = \lim_{s\to -1}(s+1)M(s) = \frac{1}{6}\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$$

$$M_3 = \frac{1}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \end{bmatrix}$$

So

$$B_1 = \begin{bmatrix} -1 & 3 \end{bmatrix}, C_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -1 & 3 \end{bmatrix}, C_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -1 & 3 \end{bmatrix}, C_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus, the state representation is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ -1 & 3 \\ -1 & 3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Conclusion

As a result to the mentioned information we can say that we already have all what we need to start our trip of researching and studying the next few chapters. In this chapter we did some digging about various systems, and we have shown the way how analyze a system in the mathematical side, we discovered the relation between the state space and the transfer function or the transfer matrix, and we saw the role of the differential equation in a system, and we have shown the different between the monovariable systems and the multivariable systems. We don't forget that in the beginning of the chapter we represented a types of systems in a simple way, the way that make everyone understand the concept of a system, and see deeply a physical parameters and quantities of system as a mathematic values.

Chapter II: the controllability

Introduction

Controllability is one of things those represent the major concepts of modern control system theory. R.Kalman introduced these Concepts in 1960. In order to be able to do everything we want and we give a dynamic system under control input, this system has to be controllable. Let's say that in this lecture we define that the concept of controllability is related to linear systems of algebraic equations. If and only if the rank of the system Matrix is full we can consider that the linear algebraic system is solvable. It's well-known that controllability is related with the rank of the system Matrix.

II.1 Definition of controllability:

When we have a controllable system let it be in our knowledge that for any state x_f of the state vector it exists an input signal u(t) of finite energy that make the system able to pass from the initial state to the state x_f in finite time. We say that our system is fully controllable, if and only if it is controllable at any points in the state space.

II.2 Controllability of Discrete Systems

The linear discrete time invariant system can be written as a function as next:

$$x(k+1) = A_d x(k) + B_d u(k) \qquad \text{(II.1)}$$

We start with a simplified problem, in the same time let us consider that the input $u(k)$ is a scalar, so let replace B_d by b_d. So, we have the next function:

$$x(k+1) = A_d x(k) + b_d u(k) \qquad \text{(II.2)}$$

We supose

$$x(0) = x_0$$

We take $k = 0,1,2, \dots . n$, so w obtain the next equations:

$x(1) = A_d x(0) + b_d u(0)$

$x(2) = A_d x(1) + b_d u(1) = A_d^2 x(0) + A_d b_d x(0) + b_d u(1)$

.

. (II.3)

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$x(n) = A_d^n x(0) + A_d^{n-1} b_d x(0) + \cdots + b_d u(n-1)$

$$x(n) - A_d^n x(0) = [b_d \quad A_d b_d \quad A_d^{n-1} b_d] \begin{bmatrix} u(n-1) \\ u(n-2) \\ . \\ . \\ . \\ u(2) \\ u(1) \end{bmatrix} \qquad (II.4)$$

We don't forget that the matrix $[b_d \quad A_d b_d \quad A_d^{n-1} b_d]$ is square matrix. So this matrix is called the controllability matrix, it is denoted M_c.

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ . \\ . \\ . \\ u(2) \\ u(1) \end{bmatrix} = C^{-1}(x(n) - A_d^n x(0)) \qquad (II.5)$$

When the input u(k) is a vector of dimention:

$$x(n) - A_d^n x(0) = [B_d \quad A_d B_d \quad A_d^{n-1} B_d] \begin{bmatrix} u(n-1) \\ u(n-2) \\ . \\ . \\ . \\ u(2) \\ u(1) \end{bmatrix} \qquad (II.6)$$

The controllability matrix is defined by the next expression:

$$M_c(A_d, B_d) = [B_d \quad A_d B_d \quad A_d^{n-1} B_d] \qquad (II.7)$$

So, we have n linear algebraic equations and $u(k)$ is vector of dimension r. so, we have the next expression:

$$M_c^{\,n\times nr} \begin{bmatrix} u(n-1) \\ u(n-2) \\ . \\ . \\ . \\ u(2) \\ u(1) \end{bmatrix}^{nr\times 1} = x(n) - A_d^n x(0) = x_f - A_d^n x(0) \qquad (II.8)$$

If and only the rank of the matrix M_c is full (means the rank=n), we say that our linear algebraic system has solution for any x_f.

We can consider that our linear discrete-time system is controllable, only if the rankM_c=n.

II.3 Controllability of Continuous Systems

It is well known that the challenge of studying the controllability concept in the continuous-time domain is more than studying it in the discrete-time domain. Firstly, in this part of this study we will do the same of what we did in the precedent section (the same strategy that we use in the section of controllability of discrete system), this is making us see the difficulties that we face in the continuous-time. So, for transferring of our system from any initial state to any final state, we will write the equations that will show how to find a control input that can do this transferring. For a scalar input, the linear continuous-time system described by:

$$\dot{x}=Ax+bu \tag{II.9}$$

We supose

$$x(0)=x_0$$

Using the same strategy of the precedent section (controllability of discrete system), with scalar input, we obtain the next equations:

$\dot{x}=\frac{d}{dt}x= Ax+bu$

$\ddot{x}=\frac{d^2}{dt^2}x=A^2x+Abu+b\dot{u}$

.

. (II.10)

.

.

$x^{(n)}=\frac{d^n}{dt^n}x=A^nx+A^{n-1}bu+A^{n-2}b\dot{u}+\cdots+bu^{(n-1)}$

$$x^{(n)}(\mathrm{t})-A^nx=M_c\begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ . \\ . \\ . \\ \dot{u} \\ u \end{bmatrix} \tag{II.11}$$

We don't forget that t belong to the time domain $[t_0,t_f]$. it is well known that t_f is free but finite.

If the input is a vector of dimension m, we obtain :

$$M_c{}^{n\times mn}\begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ . \\ . \\ . \\ \dot{u} \\ u \end{bmatrix}^{nm\times 1} = x^{(n)}(\text{t}) - A^n x = \gamma(t) \quad \text{(II.12)}$$

From these algebraic calculations we found that the condition that makes our system a solvable system, is represented by:

$$Rank M_c = rank[M_c, \psi(t)] \quad \text{(II.13)}$$

So, we say that the condition means and only means that:

$$Rank M_c = n$$

In another side, when we talk about the solution of the state space equation, we suppose:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t1} e^{A(t-t_0)}Bu(\tau)d\tau \quad \text{(II.14)}$$

$t1$ is the final time, at this time we hame :

$$x(t_1) = x_f = e^{A(t_1-t_0)}x(t_0) + \int_{t_0}^{t1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

So

$$e^{-At_1}x_f - e^{-At_0}x(t_0) = \int_{t_0}^{t1} e^{-A\tau}Bu(\tau)d\tau \quad \text{(II.15)}$$

When we use the theorem of Cayley–Hamilton, we obtain:

$$e^{-A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)A^i \quad \text{(II.16)}$$

Using $\alpha_i(\tau)$, with taking $i = 0,1,2, \dots, n-1$, as scalar time functions, we obtain:

$$e^{-At_1}x_f - e^{-At_0}x(t_0) = \sum_{i=0}^{n-1} A^i\, B \int_{t_0}^{t1} \alpha_i(\tau)u(\tau)d\tau \quad \text{(II.17)}$$

$$e^{-At_1}x_f - e^{-At_0}x(t_0) = [B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B] \begin{bmatrix} \int_{t_0}^{t1} \alpha_0(\tau)u(\tau)d\tau \\ \int_{t_0}^{t1} \alpha_1(\tau)u(\tau)d\tau \\ \int_{t_0}^{t1} \alpha_2(\tau)u(\tau)d\tau \\ . \\ . \\ . \\ \int_{t_0}^{t1} \alpha_{n-1}(\tau)u(\tau)d\tau \end{bmatrix} \quad \text{(II.18)}$$

In this equation, we have $e^{-At_1}x_f - e^{-At_0}x(t_0)$ is constant vector, because we compute x in one moment t_0 and t_1. On another side, we have the matrix of the controllability M_c and a vector that composed by functions of the control input that we require (M_c and the vector are multiplied to each other). So, this is our functional equation in the new form:

$$const^{n\times1} = C(A,B)^{n\times rn}\begin{bmatrix} f_1(u(\tau)) \\ f_2(u(\tau)) \\ f_3(u(\tau)) \\ . \\ . \\ . \\ f_{n-1}(u(\tau)) \end{bmatrix}^{rn\times1} \quad \text{(II.19)}$$

With taking

$$\tau \in (t_0, t_1)$$

The existence of the equation's solution is related by the rank of M_c, as we say there is no solution if rankM_c (A,B)=n, an another expression the rank of M_c must be full.

II.4 The completely controllable system:

We consider linear system state space:

$$\dot{x} = Ax + bu$$

We say that a system is completely controllable; if and only if the rank of the controllability matrix equal n, where n is the number of the state variables (n is the system order).

As we say, it must be like that :

$$Rank([B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B]) = n \quad \text{(II.20)}$$

Exemple :

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 2]x$$

Where M_c is controllability matrix

$$M_c = [B \;\; AB] = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \text{ so } \det(M_c) = 16 \neq 0$$

The system is completely controllable.

Exemple :

$\dot{x} = \begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} x + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u$

$y = \begin{bmatrix} 2 & 0 \end{bmatrix} x$

Where M_c is controllability matrix

$M_c = [B \ \ AB] = \begin{bmatrix} 2 & -8 \\ -4 & 16 \end{bmatrix}$ so $\det(M_c) = 0$

The system is not completely controllable.

II.5 The controllability on Laplace domain:

In this section we will explain the relationship between Laplace domain and the controllability, by an example of state space system:

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$

A is a diagonal matrix, so, it is clear that the system is not completely controllable because of the matrix B, its second line is zero, but there is a partial controllability, we will explain this by next :

The state variable x_1 is controllable because we have:

$$\dot{x}_1 = -x_1 + u$$

And by convert this equation to the Laplace domain we obtain:

$$sX_1 = -X_1 + U$$

$$sX_1 + X_1 = U$$

$$X_1(s+1) = U$$

$$X_1 = \frac{U}{(s+1)}$$

So, in this state we note that we can control the variable x_1 through the input u, so, x_1 is controllable.

The variable x_2 is not controllable (NC), because we have:

$$\dot{x}_2 = -4x_2$$

And by convert this equation to Laplace domain we obtain:

$$sX_2 = -4X_2$$

$$sX_2 + 4X_2 = 0$$

$$X_2(s + 4) = 0$$

So, it is clear that we cannot control x_2 through the input u, so x_2 is not controllable, in this state we note clearly that the zero in the second line of the matrix B is the main cause.

All this explaining about the relationship between Laplace domain and the controllability can be represented by the next bloc diagram:

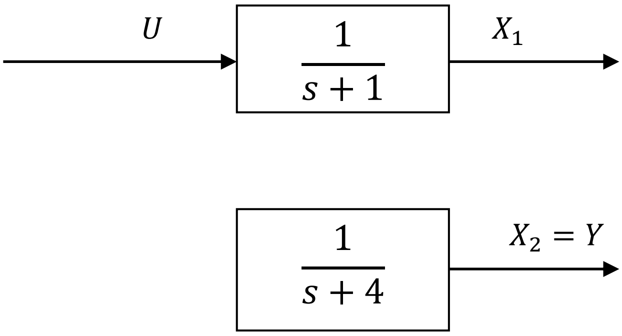

Figure: bloc diagram

Thus, through this bloc diagram we note that there is no effect to X_2 by U, on the other side, it is clearly that the consequence X_1 is related mainly by U.

II.6 The companion form of controllability

In fact, if a system is controllable, it means that we can write it at the companion form of controllability. Thus, we will show how to do this, by an example as next:

$$\frac{Y(s)}{U(s)} = \frac{32}{s^2 + 16}$$

So, we have a system described by a transfer function, the function can be written like that:

$$\frac{Y(s)V(s)}{U(s)V(s)} = \frac{32}{s^2 + 16}$$

Where $V\ (s)$ corresponds to a variable internal to the system, such as:

$$\frac{Y(s)}{V(s)} = 32$$

So

$$Y(s) = 32V(s)$$

And

$$\frac{V(s)}{U(s)} = \frac{1}{s^2 + 16}$$

So

$$U(s) = s^2V(s) + 16V(S)$$

$$s^2V(s) = -U(s) + 16V(s)$$

We suppose

$$sV(s) = X_2 \text{ and } V(s) = X_1$$

$$X_2 = sX_1$$

$$sX_2 = -U(s) + 16V(s)$$

So, by converting the precedent expressions from Laplace domain to the time domain, we obtain:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 16x_1 - u(t)$$

$$y = 32x_1$$

So, through what we have, we obtain the companion form of controllability:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u \\ y = [32 \quad 0]x_1 \end{cases}$$

Let verify the controllability of the system at this form:

$$M_c = [B \quad AB]$$

$$AB = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -16 \end{bmatrix}$$

$$M_c = \begin{bmatrix} -1 & 0 \\ 0 & -16 \end{bmatrix}$$

$$\det(M_c) = 16$$

$$rank(M_c) = 2$$

So, we note that the system at the companion form is controllable.

Conclusion

By studying five section in this chapter, based mainly on the definition of the controllability, we conclude by saying that indeed, the question of the controllability is if we can control a system through the input or not. We found in section of studying the controllability of discrete system that we define a system by equation represent a relation of recurrence, on another side, we note that on continuous system, we define a system by differential equation, but we need in both of sections to the same method to determine the controllability matrix. We found that in the mathematical meaning of the controllability, mainly, the question that we need to ask is about the rank; if it is full or not, as we say if the rank is equal to n, note that n is number of the state variables (n is the system order). When we study the completely controllable system, we use the theory of R.Kalman, like discovering the general mathematic idea of the controllability. When we study the controllability on Laplace domain, we discover the partial controllability, and we find that when a system is not completely controllable, it can be partially controllable. In the last section, we can say that we learn how to verify controllability of the system throughout going to finding the companion form of the controllability. Finally, we can consider that we the knowledge of this chapter can qualify us to make more progress in the controllability studying.

Chapter III: the observability

Introduction

The observability is a major concept of modern control system; it is introduced by R.Kalman in 1960. It's well known that in order to see everything happening inside a system by observation, this system have to be observable. We don't forget that in this lecture we Define that the observability's concept is related to linear systems of algebraic equations. If the rank of matrix system is full, we can say that the solvability of the linear algebraic system is able to be. We have to know that the rank of the matrix system and the observability are related to each other.

III.1 Definition of observability

If we can identify a state $x(t_0)$ through the knowledge of the input $u(t)$ and the output $y(t)$ over a finite time interval $[t_0, t_f]$, we can that this state is observable. If we can restore or identify a system value from the mere knowledge of the input $u(t)$ and the output $y(t)$ for all $x(t_0)$ belong in the state space, it's well known that our system is completely observable.

III.2 Observability of Discrete Systems

We consider a linear invariant system at a discrete-time in state space at the next form:

$$x(k+1) = A_d x(k) \qquad \text{(III.1)}$$

With

$$x(0) = x_0$$

And also the measurements of the output:

$$x(k) = C_d y(k) \qquad \text{(III.2)}$$

Note: x_0 is unknown.

We must know that C_d is a constant matrices, and $x(k) \in R^n$, and $y(k) \in R^p$, n and p are natural nombers. If we know x_0, we can use this knowledge to know the state variation in every discrete-time instant. We just need to determine the initial vector $x(0) = x_0$ from the state measerments.

When we take k=0,1,2,…n-1 we obtain :

$y(0) = C_d x(0)$

$y(1) = C_d x(1) = C_d A_d x(0)$

$y(2) = C_d x(2) = C_d A_d x(1) = C_d A_d^2 x(0)$

. (III.3)

.

.

$$y(n-1) = C_d x(n-1) = C_d A_d^{n-1} x(0)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ . \\ . \\ . \\ y(n-1) \end{bmatrix}^{np\times 1} = \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ . \\ . \\ . \\ C_d A_d^{n-1} \end{bmatrix}^{np\times n} \times x(0) \qquad \text{(III.4)}$$

The knowledge from the linear algebra make us say that the linear algebraic equations system has a unique solution if and only the rank of system matrix is full, it means that the rank=n.

$$\text{Rank} \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ . \\ . \\ . \\ C_d A_d^{n-1} \end{bmatrix} = n \qquad \text{(III.5)}$$

The observability matrix:

$$M_o\ (A_d\,, C_d) = \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ . \\ . \\ . \\ C_d A_d^{n-1} \end{bmatrix}^{np\times n} \qquad \text{(III.6)}$$

So it must be:

Rank M_o = n

Let's say that if we want to make an observable linear discrete-time system with measurements, in any way the rank of observability matrix must be equal to n, like it must be full.

III.3 Observability of Continuous Systems

We want to study the observability of a system, so we consider that the input of this system is the next expression:

$$\dot{x} = Ax(t)$$

We supose

$$x(t_0) = x_0$$

With the output measurements:

$$y = Cx(t)$$

Note: x_0 is unknown.

It should be known that C_d is a constant matrices, with $x(k) \in R^n$ and $y(k) \in R^p$, n and p are natural nombers. We will follow the same strategy of the precedent section (previously we study the observability at discrete-time). We can understand that the knowledge of x_0 is enough to find and determine $x(t)$ in every time instant, we consider the solution of the differential equation is the next expression:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad \text{(III.7)}$$

We should know that we face the problem of finding $x(t_0)$ from the available measurements. In discrete-time system, we fixed the problem by taking k=0,1,2,…,n-1, so we generated a sequence of measurements in every time instant. in the continuous-time domain, we apply An analogical technics, by taking the continuous-time measurements derivatives, we obtain :

$$\begin{aligned}
&y(t_0) = Cx(t_0)\\
&\dot{y}(t_0) = C\dot{x}(t_0) = CAx(t_0)\\
&\ddot{y}(t_0) = C\ddot{x}(t_0) = CA^2x(t_0)\\
&.\\
&.\\
&.\\
&.\\
&y^{(n-1)}(t_0) = \mathrm{C}x^{n-1}(t_0) = CA^{n-1}x(t_0)
\end{aligned} \qquad \text{(III.8)}$$

There are np linear algebric equations. We put the equations in matrices as the next form:

$$\begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \ddot{y}(t_0) \\ . \\ . \\ . \\ . \\ y^{(n-1)}(t_0) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}^{(np)\times n} \times\ x(t_0) = M_o x(t_0) = Y(t_0) \qquad \text{(III.9)}$$

M_o is matrix of the observability. We can say that we can determine the initial condition, only if the rank of the observability matrix is full. It means rank(M_o)=n.

So, we can say that, with measurements a linear continuous-time system is observable, if and only if the rank of the observability matrix is full.

III.4 The completely observable system

Let's consider a linear system state space :

$\dot{x} = Ax + bu$

We note that a system is completely observable; if and only if the rank of the observability matrix equal n, where n is the system order (n is the number of the state variables).

Note that, it must be as next:

$$Rank \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} = n \qquad \text{(III.10)}$$

Exemple :

$\dot{x} = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$

$y = [0 \quad 2]x$

Where M_o is observability matrix

$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix}$ so $\det(M_o) = -16 \neq 0$

The system is completely observable.

Exemple :

$\dot{x} = \begin{bmatrix} 2 & 0 \\ -4 & -6 \end{bmatrix} x + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u$

$y = [2 \quad 0]x$

Where M_o is observability matrix

$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$ so $\det(M_o) = 0$

The system is not completely observable.

III.5 The observability on Laplace domain:

In this section we will explain the relation between Laplace domain and the controllability, by an example of state space system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1]x$$

As A is a diagonal matrix, so, we note that the system is not completely observable because of the matrix C, its first column is equal to zero, but there is a partial observability, we will explain this by next :

We have

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = -4x_2$$

$$y = x_2$$

The state variable x_1 is not observable, because, about x_1, we have only:

$$\dot{x}_1 = -2x_1 + u$$

Note that, there is no relation between x_1 and y

And by convert this equations to the Laplace domain, we obtain:

$$sX_1 = -2X_1 + U$$

$$sX_1 + 2X_1 = U$$

$$X_1(s + 2) = U$$

$$X_1 = \frac{U}{(s + 2)}$$

So, in this state we note that we cannot observe the variable x_1 through the output y, so, x_1 is observable.

The variable x_2 is observable because we have $y = x_2$, and by convert this equation to Laplace domain we obtain:

$$Y = X_2$$

Thus, we note clearly that we can observe x_2 through the output y, so x_2 is observable. in this state, it seems clearly that the zero in the first column of the matrix C is the main cause.

All this explaining about the relation between Laplace domain and the observability can be represented by the next bloc diagram:

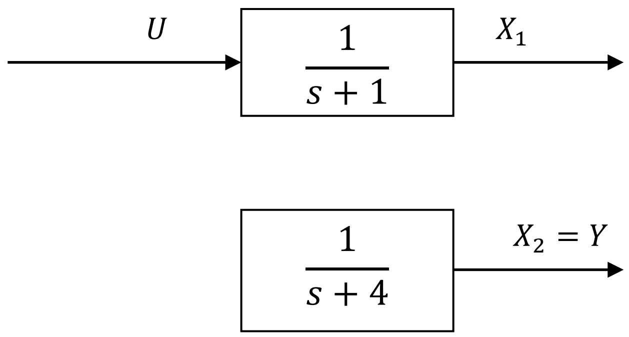

Figure: bloc diagram

so, the bloc diagram show and explain the effect from X_2 to Y, on an other side, we note clearly that X_1 is not related absolutely with Y.

III.6 The companion form of observability

In fact, if a system is observable, we note that we can write it at the companion form of observability. Thus, we will explain how to do this, with the help of the next example:

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+1}$$

Thus, this transfer function describes a system, we can write the function on another form, as next:

$$\frac{Y(s)}{U(s)} = \frac{s^{-2}}{1+s^{-2}}$$

So

$$Y(s)(1+s^{-2}) = U(s)s^{-2}$$

$$Y(s) = -Y(s)s^{-2} + U(s)s^{-2}$$

$$Y(s) = s^{-2}(-Y(s) + U(s))$$

$$Y(s) = s^{-1}(s^{-1}(-Y(s) + U(s)))$$

We supose

$$X_1 = s^{-1}(-Y(s) + U(s))$$

$$X_2 = Y(s)$$

So, by converting the precedent expressions from Laplace domain to the time domain, we obtain:

$$\dot{x}_1 = -x_2 + u$$

$$\dot{x}_2 = x_1$$

$$y(t) = x_2$$

So, through what we have, we obtain the companion form of observability:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let verify the controllability of the system at this form:

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$M_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(M_o) = -1$$

$$rank(M_o) = 2$$

So, we note that the system at the companion form is observable.

Conclusion

After the study of five sections here, and in the light of observability definition, we discovered that the main question is if we can observe a system through their outputs or not. When we studied the observability in discrete system, we define a system by equation represent a relation of recurrence, on another side, we note that on continuous system, we define a system by differential equation, but we need in both of sections to the same method to determine the observability matrix. We found that in the mathematical meaning of the observability, mainly, the question that we need to ask is about the rank; if it is full or not, as we say if the rank is equal to n, note that n is number of the state variables (n is the system order). When we study the completely controllable system, we use the theory of R.Kalman, like discovering the general mathematic idea of the observability. When we study the observability on Laplace domain, we discover the partial observability, and we find that when a system is not completely observable, it can be partially observable. In the last section, we can say that we learn how to verify observability of the system throughout going to finding the companion form of the observability. In the end we note that the knowledge of this chapter can qualify us to make more progress in the observability studying.

Chapter IV: measure of controllability

Introduction

Generally, when we hear the controllability subject the first thing that we think about is a binary result, a controllable system or uncontrollable system as there is nothing more, but in fact, the controllability is more than that, because when we have an uncontrollable system, we must answer the question if we can control a system partially, note that, the system that we study is not fully controllable. previously, we used to demonstrate and verify only if a system is controllable or uncontrollable, but , in this chapter, we will measure a controllability of a system, note that, the study is based on a systems those are not fully controllable. Indeed, the main question is how to measure a system controllability, surely, the methods is basis on A and B matrices, and probably we need numerical analysis, because the precision of any measurement have an error, but, maybe there are a methods give us a measurement with a good precision. Mainly, we need to find how much we can control a system first, it must be a simple method for show us how to measure a system controllability in simple way.

IV.1 The partial controllability and the state space form

Let say that the partial controllability is the opposite of the fully controllability, in this section we take an example that show us what we can call it the partial controllability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By the Laplace analysis of the state space we found the following:

Firstly, about the state variable x_1 :

$$sX_1 = -X_1 + U$$

$$sX_1 + X_1 = U$$

$$X_1(s+1) = U$$

$$X_1 = \frac{U}{(s+1)}$$

Secondly, about the state variable x_2 :

$$sX_2 = -3X_2$$

$$sX_2 + 3X_2 = 0$$

$$X_2(s+3) = 0$$

$$y = X_2$$

This is what lead us to say that the system is controllable in the state variable x_1, an another side, it is not controllable in in the state variable x_2.

We know that we have a zero in the second line of the matrix B and this is the cause of the uncontrollability in the state variable x_2, but, we could not say that, if the matrix A was not a diagonal matrix, so, we can note that if the matrix A is diagonal matrix the controllability will be seen.

So, in the light of those Conclusions, we can note that the converting to the diagonal form of the matrix A is a main step to see clearly the controllability.

So, in the light of the accumulation of previous information and conclusions, and based on what we obtain of our example, we can say that the system that represented by our example is controllable 50%, and this is according on existance of two state variables x_1 and x_2, the first is controllable and the second is uncontrollable, and according on that, also we can consider $1/2$ as a controllability measurement value of the system that represented by our example.

IV.2 The diagonal form and the controllability

In the light of what we talk in the previous section, we saw the Benefit of the diagonal form for measure a system’s controllability. So, we consider the following state space:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

We can write it at diagonal form like the following:

$$\dot{z} = \hat{A}z + \hat{B}u$$

$$y = \hat{C}z$$

With

$$\hat{A} = T^{-1}AT, \hat{B} = T^{-1}B, \hat{C} = CT, z = T^{-1}x$$

So, we need do find the matrix T, then, we will take a state space example of a system, and show how to obtain a diagonal form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, now we calculate $A - pI$:

$$\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} -3-p & -4 \\ 5 & 6-p \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -3-p & -4 \\ 5 & 6-p \end{bmatrix}\right) = (-3-p)(6-p) - 5(-4)$$

$$= -18 + 3p - 6p + p^2 + 20$$

$$= p^2 - 3p + 2$$

$$= (p-1)(p-2)$$

In solving the eqution :

$$\det\left(\begin{bmatrix} -3-p & -4 \\ 5 & 6-p \end{bmatrix}\right) = 0$$

We find $p_1 = 1$ and $p_2 = 2$ as a solutions.

Now, we solve the equation :

$$(A - pI)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

One time we take $p = 1$, and other we take $p = 2$:

So, firstly we take $p = 1$

$$(A - (1)I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3-1 & -4 \\ 5 & 6-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-4x_1 - 4x_2 = 0$$

$$5x_1 + 5x_2 = 0$$

So

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

We take $x_1 = -1$, we obtain $x_2 = 1$:

So, secondly we take $p = 2$

$$(A - (2)I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3-2 & -4 \\ 5 & 6-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We take $x_2 = 1$:

$$-5x_1 - 4x_2 = 0$$

$$5x_1 + 4x_2 = 0$$

So, we obtain $x_1 = -\frac{4}{5}$, and $x_2 = 1$.

So, now we find the matrix T:

$$T = \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix}$$

So, we go now to calculate T^{-1}:

$$T^{-1} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} \frac{1}{(-1)(1) - 1(\frac{-4}{5})} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} \frac{1}{\frac{1}{5}} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} (-5)$$

$$T^{-1} = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$$

Now, we compute $\hat{A}$:

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, let calculate $\hat{B}$:

$$\hat{B} = T^{-1}B = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -5(4) - 4(-5) \\ 5(4) + 5(-5) \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

Now, we compute $\hat{C}$:

$$\hat{C} = CT = [4 \quad 1] \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix} = \left[4(-1) + 1(1) \quad 4\left(\frac{-4}{5}\right) + 1(1)\right]$$

$$\hat{C} = \left[-3 \quad \frac{-11}{5}\right]$$

After all these calculation, we obtain a state space at the diagonal form:

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ -5 \end{bmatrix} u$$

$$y = \begin{bmatrix} -3 & \frac{-11}{5} \end{bmatrix} z$$

We have the matrix $\hat{B} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$ that shows us the degree of controllability; here we have a controllable system in fifty percent.

The diagonalization is indirectly method to discover the degree of the controllability.

We found that the matrix T must be reversible.

Now, let's search for the general rule for any quantities values of a system based on the idea that prove that diagonalization is the way to measure a system controllability, in this case we are looking for the conditions of the diagonalization as a conditions qualify us to measure controllability.

We consider following state space:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

And a state space at the diagonal form:

$$\dot{z} = \hat{A}z + \hat{B}u$$

$$y = \hat{C}z$$

With :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, C = [\, c_1 \quad c_2]$$

Where a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1 and c_2 belong to $\mathfrak{R}$.

So

$$A - pI = \begin{bmatrix} a_{11} - p & a_{12} \\ a_{21} & a_{22} - p \end{bmatrix}$$

So

$$\det(A - pI) = (a_{11} - p)(a_{22} - p) - a_{21}a_{12}$$

$$= a_{11}a_{22} - a_{11}p - a_{22}p + p^2 - a_{21}a_{12}$$

$$= p^2 - (a_{11} + a_{22})p + a_{11}a_{22} - a_{21}a_{12}$$

$$= p^2 - (a_{11} + a_{22})p + \det(A)$$

Now, we go to find a solutions for the equation $\det(A - pI) = 0$:

$$\Delta = (a_{11} + a_{22})^2 - 4\det(A)$$

$$p_1 = \frac{(a_{11} + a_{22}) - \sqrt{\Delta}}{2}$$

$$p_2 = \frac{(a_{11} + a_{22}) + \sqrt{\Delta}}{2}$$

So, as a condition:

$$(a_{11} + a_{22})^2 - 4\det(A) \geq 0$$

Now, we solve the equation:

$$(A - pI)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

One time we take $p = p_1$, and other we take $p = p_2$:

So, firstly we take $p = p_1$:

$$(A - (p_1)I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} - p_1 & a_{12} \\ a_{21} & a_{22} - p_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} - p_1 & a_{12} \\ a_{21} & a_{22} - p_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(a_{11} - p_1)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - p_1)x_2 = 0$$

We take $x_1 = -1$, we obtain $x_2 = \frac{a_{11} - p_1}{a_{12}}$:

So, secondly we take $p = p_2$

$$(A - (2)I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} - p_2 & a_{12} \\ a_{21} & a_{22} - p_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} - p_2 & a_{12} \\ a_{21} & a_{22} - p_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(a_{11} - p_2)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - p_2)x_2 = 0$$

So, we take $x_2 = 1$, we obtain $x_1 = -\frac{a_{22} - p_2}{a_{21}}$.

So

$$T = \begin{bmatrix} -1 & -\frac{a_{22}-p_2}{a_{21}} \\ \frac{a_{11}-p_1}{a_{12}} & 1 \end{bmatrix}$$

As a first condition $a_{12} \neq 0$ and $a_{21} \neq 0$.

As a condition to be able to make the diagonalization; T must be reversible, it means that $\det(T)$ mustn't equal to zero, as follow:

$$-1 \times 1 - \left(\frac{a_{11}-p_1}{a_{12}}\right)\left(-\frac{a_{22}-p_2}{a_{21}}\right) \neq 0$$

$$\frac{(a_{11}-p_1)(a_{22}-p_2)}{a_{12}a_{21}} \neq -1$$

$$(a_{11}-p_1)(a_{22}-p_2) \neq a_{21}a_{12}$$

$$(a_{11}-p_1)(a_{22}-p_2) - a_{21}a_{12} \neq 0$$

So, $p_1 \neq p_2$ is a condition, because we have:

$$\det(A - pI) = (a_{11}-p)(a_{22}-p) - a_{21}a_{12} = 0$$

We have :

$$\Delta = (a_{11}+a_{22})^2 - 4\det(A) \geq 0$$

It goes to be :

$$\Delta = (a_{11}+a_{22})^2 - 4\det(A) > 0$$

So, next we need to find T^{-1}:

$$T^{-1} = \frac{1}{-1 + \frac{(a_{11}-p_1)(a_{22}-p_2)}{a_{12}a_{21}}} \begin{bmatrix} 1 & \frac{a_{22}-p_2}{a_{21}} \\ -\frac{a_{11}-p_1}{a_{12}} & -1 \end{bmatrix}$$

$$T^{-1} = \frac{a_{12}a_{21}}{-a_{12}a_{21} + (a_{11}-p_1)(a_{22}-p_2)} \begin{bmatrix} 1 & \frac{a_{22}-p_2}{a_{21}} \\ -\frac{a_{11}-p_1}{a_{12}} & -1 \end{bmatrix}$$

$$T^{-1} = \frac{1}{-a_{12}a_{21} + (a_{11}-p_1)(a_{22}-p_2)} \begin{bmatrix} a_{12}a_{21} & a_{12}(a_{22}-p_2) \\ -a_{21}(a_{11}-p_1) & -a_{12}a_{21} \end{bmatrix}$$

$$T^{-1}=\begin{bmatrix}\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)} & \frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}\\ -\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)} & -\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}\end{bmatrix}$$

Now, we can find the matrix $\hat{B}=T^{-1}B$ to measure the controllability:

$$\hat{B}=\begin{bmatrix}\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)} & \frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}\\ -\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)} & -\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$\hat{B}=\begin{bmatrix}\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1+\frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2\\ -\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1-\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2\end{bmatrix}$$

Now, we can note that:

a)

If

$$\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1+\frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}-(a_{11}-p_1)(a_{22}-p_2)}b_2=0$$

And

$$-\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1-\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2=0$$

The system is completely uncontrollable (0%).

b)

If

$$\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1+\frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2\neq 0$$

And

$$-\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1-\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2=0$$

Or

$$\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1+\frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2=0$$

And

$$-\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1-\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2 \neq 0$$

The system is controllable until 50%.

a)

If

$$\frac{a_{12}a_{21}}{a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1+\frac{a_{12}(a_{22}-p_2)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2 \neq 0$$

And

$$-\frac{a_{21}(a_{11}-p_1)}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_1-\frac{a_{12}a_{21}}{-a_{12}a_{21}+(a_{11}-p_1)(a_{22}-p_2)}b_2 \neq 0$$

The system is completely controllable (100%).

Conclusion

Before discovering the idea of the diagonalization, it cannot be able to see a controllability degree of a system, but after that, it is clearly that the degree of the controllability was seen. Now, the way to measure a system's controllability is shown, it means that we must take a help through the mathematical analysis based on the diagonalization method. We cannot deny that the method that we follow to measure a system controllability is a limited method, because there are a conditions to use this method, so we are not denying that we have many of systems those we could not see their controllability degree through this method, but the method that we follow pose a many of questions about progressing the way to measure a systems controllability.

On another side some can tell that method is related only with the example that it's based on, but he can't deny that our work open the way to measure the controllability of other examples related with the used example.

Chapter V: measure of observability

Introduction

There are a many of students think that the concept of the observability is limited in two principals, if a system is observable or unobservable, forgetting the question that they must find an answer for it, and this is question is about how much we can observe a system through their outputs. Surely, we go to measure the observability of a system that we know that it is not a completely observable. In the chapter number three when we talk about the observable generally, we used to make a demonstrations and verifications only about if a system is observable or unobservable, but in this chapter we will measure an observability of a systems those are not completely observable. We need to find an answer about how to measure a system observability, there is no doubt that the method to measure a system observability is based on A and C matrices, maybe we need the numerical analysis, the cause that lead us to use this analysis is the need obtain the best precision measurements, but nothing sure about that. Before everything we need to find the simple way to measure a system observability, answering to a main question about how much we can observe a system.

V.1 the partial observability and the state space form

Let say that the opposite of the completely observability is the partial observability is, in this section the example that we will take will represent what we can call it the partial observability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By the Laplace analysis of the state space we found the following:

Firstly, about the state variable x_1:

$$sX_1 = -5X_1 + 2U$$

$$sX_1 + 5X_1 = 2U$$

$$X_1(s+5) = 2U$$

$$X_1 = \frac{2U}{(s+5)}$$

Secondly, about the state variable x_2:

$$sX_2 = -7X_2 + 4U$$

$$sX_2 + 7X_2 = 4U$$

$$X_2(s+7) = 4U$$

$$X_2 = \frac{4U}{(s+7)}$$

$$y = 5X_2$$

After that we can note that the system is not observable in the state variable x_1, bu, it is observable in the state variable x_2.

It is well known that the zero in the first column of the matrix C is the cause of the unobservability in the state variable x_1, but, if the matrix A was not a diagonal matrix, we could not say that, so, we can note that if the matrix A is diagonal matrix the observability will be seen.

So, in the light of those Conclusions, we can say that, it is clearly that the main step to see the observability is to turn the matrix A to the diagonal form.

So, after the accumulation of the precedent conclusions and informations, and based on what we obtain of our example, we can say that the system that represented by our example is observable 50%, and this is according on existance of two state variables x_1 and x_2, the first is unobservable, and the second is observable, and according on that, also we can consider $1/2$ as a obsevability measurement value of the system that represented by our example.

V.2 The diagonal form and the observability

After the first section in this chapter, we note that we saw the benefit of the diagonal form for measure a system's observability. Let's consider the following state space:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

We can write it at diagonal form as the following:

$$\dot{z} = \hat{A}z + \hat{B}u$$

$$y = \hat{C}z$$

With

$$\hat{A} = T^{-1}AT, \hat{B} = T^{-1}B, \hat{C} = CT, z = T^{-1}x$$

So, we must find the matrix T, then, we will take a state space exemple of a system, and it will be shown how to obtain a diagonal form :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, we go to find the matrix T like the previous section:

$$T = \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix}$$

So, we go now to calculate T^{-1}:

$$T^{-1} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} \frac{1}{(-1)(1) - 1(\frac{-4}{5})} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} \frac{1}{\frac{1}{5}} = \begin{bmatrix} 1 & \frac{4}{5} \\ -1 & -1 \end{bmatrix} (-5)$$

$$T^{-1} = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$$

Now, we compute $\hat{A}$:

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, let calculate $\hat{B}$:

$$\hat{B} = T^{-1}B = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5(1) - 4(2) \\ 5(1) + 5(2) \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} -13 \\ 15 \end{bmatrix}$$

Now, we compute $\hat{C}$:

$$\hat{C} = CT = [4 \quad 4] \begin{bmatrix} -1 & \frac{-4}{5} \\ 1 & 1 \end{bmatrix} = \left[4(-1) + 4(1) \quad 4\left(\frac{-4}{5}\right) + 4(1) \right]$$

$$\hat{C} = \begin{bmatrix} 0 & \frac{4}{5} \end{bmatrix}$$

After all these calculation, we obtain a state space at the diagonal form:

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} -13 \\ 15 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \frac{4}{5} \end{bmatrix} z$$

Here it is clearly that the degree of the observability is fifty percent, based on the matrix $\hat{C} = \begin{bmatrix} 0 & \frac{4}{5} \end{bmatrix}$.

Discovered the degree of the observability throughout the diagonalization indirectly.

After all that, we go find the general rule for any quantities values of a system, in this way we follow the same steps of the previous chapter, not forgetting that the conditions the diagonalization here is a conditions to discover the observability.

We consider :

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

And a state space at the diagonal form:

$$\dot{z} = \hat{A}z + \hat{B}u$$

$$y = \hat{C}z$$

With:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, C = [\, c_1 \quad c_2 \,]$$

Where a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1 and c_2 belong to $\mathfrak{R}$**.**

By following the same steps of precedent chapter, we find:

$$T = \begin{bmatrix} -1 & -\dfrac{a_{22} - p_2}{a_{21}} \\ \dfrac{a_{11} - p_1}{a_{12}} & 1 \end{bmatrix}$$

With

$$p_1 = \frac{(a_{11} + a_{22}) - \sqrt{\Delta}}{2}$$

$$p_2 = \frac{(a_{11} + a_{22}) + \sqrt{\Delta}}{2}$$

Now, we can find $\hat{C} = CT$:

$$\hat{C} = [\, c_1 \quad c_2 \,] \begin{bmatrix} -1 & -\dfrac{a_{22} - p_2}{a_{21}} \\ \dfrac{a_{11} - p_1}{a_{12}} & 1 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} -c_1 + \dfrac{a_{11} - p_1}{a_{12}} c_2 & -\dfrac{a_{22} - p_2}{a_{21}} c_1 + c_2 \end{bmatrix}$$

Now, we can note that:

a)

If

$$-c_1 + \frac{a_{11} - p_1}{a_{12}} c_2 = 0$$

And

$$-\frac{a_{22} - p_2}{a_{21}} c_1 + c_2 = 0$$

The system is completely uncontrollable (0%).

b)

If

$$-c_1 + \frac{a_{11} - p_1}{a_{12}} c_2 \neq 0$$

And

$$-\frac{a_{22} - p_2}{a_{21}} c_1 + c_2 = 0$$

Or

$$-c_1 + \frac{a_{11} - p_1}{a_{12}} c_2 = 0$$

And

$$-\frac{a_{22} - p_2}{a_{21}} c_1 + c_2 \neq 0$$

The system is controllable 50%.

a)

If

$$-c_1 + \frac{a_{11} - p_1}{a_{12}} c_2 \neq 0$$

And

$$-\frac{a_{22} - p_2}{a_{21}} c_1 + c_2 \neq 0$$

The system is comletely controllable (100%).

Conclusion

After the appearance on diagonalization idea, the degree of the observability is clearly seen, on another side before this appearance, we could not see the degree of the observability. It is well known, that if we want to measure a system's observability, the way to do that is not hidden, throughout taking the mathematical analysis as a leader in this way, basing on the diagonalization. We cannot hide the reality about our method to measure a system's observability, it is a limit method as we could not measure the observability in many of systems; because there are a conditions to use this method, so, the observability in those systems could not be seen, on another side, maybe it is the beginning of a study that makes a progressing the way to measure a system's observability.

If some told us that our method is very limited because it's based on the used example, he mustn't deny that our work open the way to measure the observability of other examples related with the used example.

Chapter VI: simulation and interpretation

Introduction

In this chapter we will write the program on Matlab that allows us to know the degree of controllability, than we will write other for the observability. Mainly, we ask about what we need to write it, as algorithmic functions, it is clearly that we have to make what we obtain in the last two chapters to a programs, we can measure through them the controllability and the observability.

VI.1 The matlab's work:

In this section we will represent the program that allows us to measure the controllability and the observability, based on the previous information:

VI.1.1 A program to measure the controllability:

```
%program measure controllability
a11=;
a12=;
a21=;
a22=;
b1=;
b2=;
p1=((a11+a22)-sqrt((a11+a22)^2-4*(a11*a22-a12*a21)))/2;
p2=((a11+a22)+sqrt((a11+a22)^2-4*(a11*a22-a12*a21)))/2;
d=(-a12*a21+(a11-p1)*(a22-p2));
e=a12*a21;
f=(a11+a22)^2-4*(a11*a22-a12*a21);
B1=(b1*(a12*a21))/(-a12*a21+(a11-p1)*(a22-p2))+(b2*a12*(a22-p2))/(-
a12*a21+(a11-p1)*(a22-p2));
B2=-(b1*a21*(a11-p1))/(-a12*a21+(a11-p1)*(a22-p2))-(b2*(a12*a21))/(-
a12*a21+(a11-p1)*(a22-p2));
if d==0 || e==0 || f<=0
    disp('we can not measure the controllability in this case')
elseif B1==0 && B2==0
    disp('the system is completely uncontrollable(0%)')
elseif (B1~=0 && B2==0) || (B1==0 && B2~=0)
    ('the system is controllable until 50%')
else
    disp('the system is completely controllable(100%)')
end
```

Taking $a_{11} = -3, a_{12} = -4, a_{21} = 5\ a_{22} = 6$ for any case.

For the first case we take $b_1 = 4, b_2 = -5$, we obtain that the system is controllable until 50%.

For another time we take $b_1 = 4, b_2 = 4$, we obtain that the system is competly controllable (100%).

VI.1.2 A program to measure the observability:

```
%program measure observability
a11=-3;
a12=-4;
a21=5;
a22=6;
c1=4;
c2=4;
p1=((a11+a22)-sqrt((a11+a22)^2-4*(a11*a22-a12*a21)))/2;
p2=((a11+a22)+sqrt((a11+a22)^2-4*(a11*a22-a12*a21)))/2;
d=(-a12*a21+(a11-p1)*(a22-p2));
e=a12*a21;
f=(a11+a22)^2-4*(a11*a22-a12*a21);
O1=-c1+c2*(a11-p1)/a12;
O2=-c1*((a22-p2)/a21)+c2;
if d==0 || e==0 || f<=0
    disp('we can not measure the observability in this case')
elseif O1==0 && O2==0
    disp('the system is completely unobservable(0%)')
elseif (O1~=0 && O2==0) || (O1==0 && O2~=0)
    ('the system is observable until 50%')
else
    disp('the system is completely observable(100%)')
end
```

Taking $a_{11} = -3, a_{12} = -4, a_{21} = 5\ a_{22} = 6$ for any case.

For the first case we take $c_1 = 4, c_2 = 4$, we obtain that the system is observable until 50%.

For another time we take $c_1 = 4, c_2 = 1$, we obtain that the system is competly observable (100%).

VI.2 The influence of the controllability and the observability in graphical vision:

VI.2.1 A controllability's influence

The first system is controllable until 50%, and it is represented by the next state space :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The second system is completely controllable, and it is represented by the next state space :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The next program allow us to see the graphical responses of a systems:

```
% the controllability in a graphical
A=[-3 -4;5 6];
B1=[4;-5];
B2=[4;4];
C=[4 1];
D=0;
sys1=ss(A,B1,C,D);
sys2=ss(A,B2,C,D);
figure (1)
step(sys1)
grid on
figure (2)
step(sys2)
grid on
```

The graphical response:

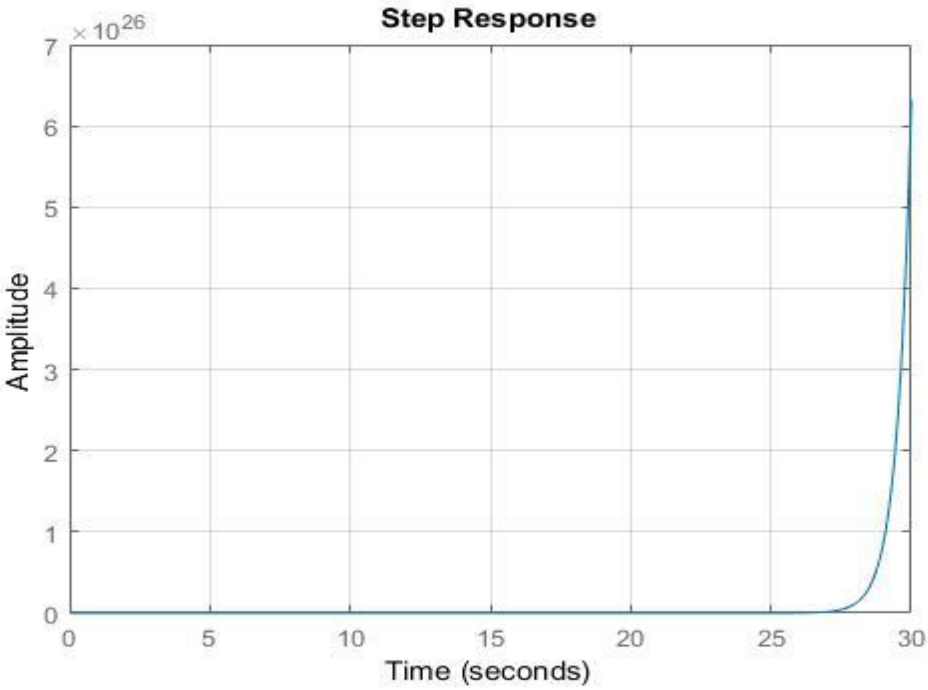

Figure VI.1: A graphic curve of 50% controllability

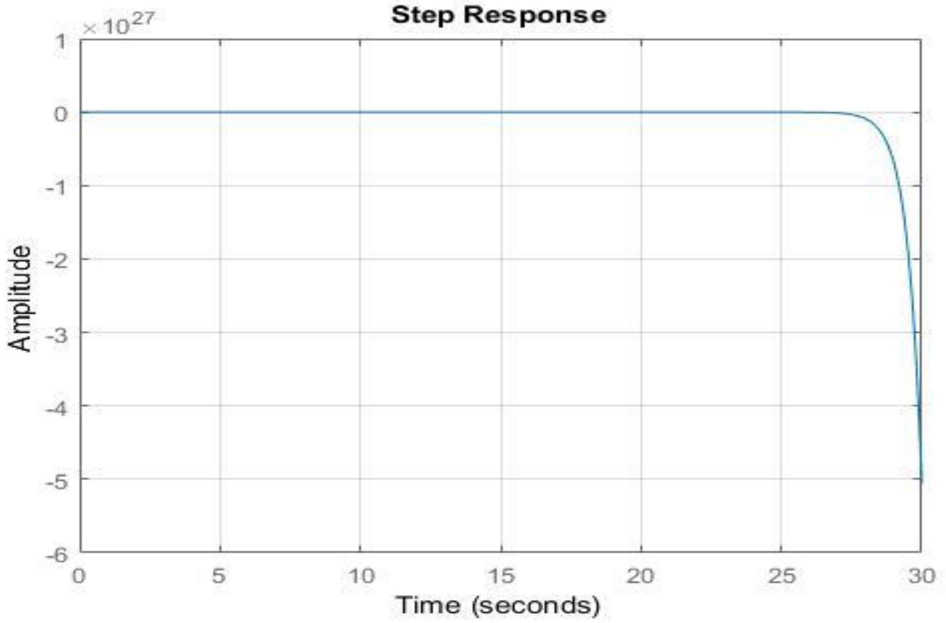

Figure VI.2: A graphic curve of 50% controllability

VI.2.2 An observability's influence

The first system is observable until 50%, and it is represented by the next state space:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-3 & -4\\ 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The second system is completely observable, and it is represented by the next state space:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-3 & -4\\ 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The next program allows us to see the graphical responses of systems:

```
% the observability in a graphical vision
A=[-3 -4;5 6];
B=[1;1];
C1=[4 4];
C2=[4 1];
D=0;
sys1=ss(A,B,C1,D);
sys2=ss(A,B,C2,D);
figure (1)
step(sys1)
grid on
```

```
figure (2)
step(sys2)
grid on
```

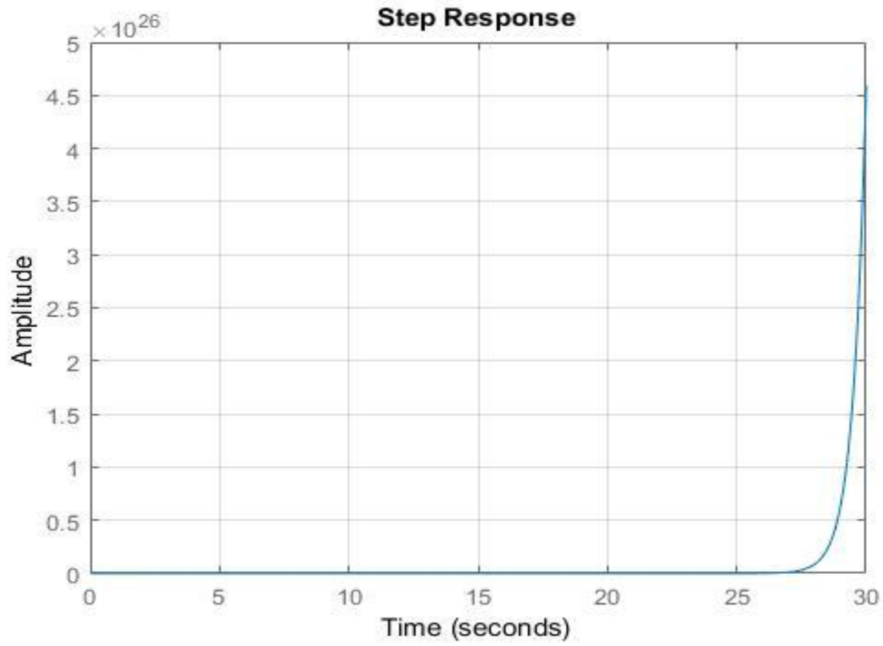

Figure VI.3: A graphic curve of 50% Observability

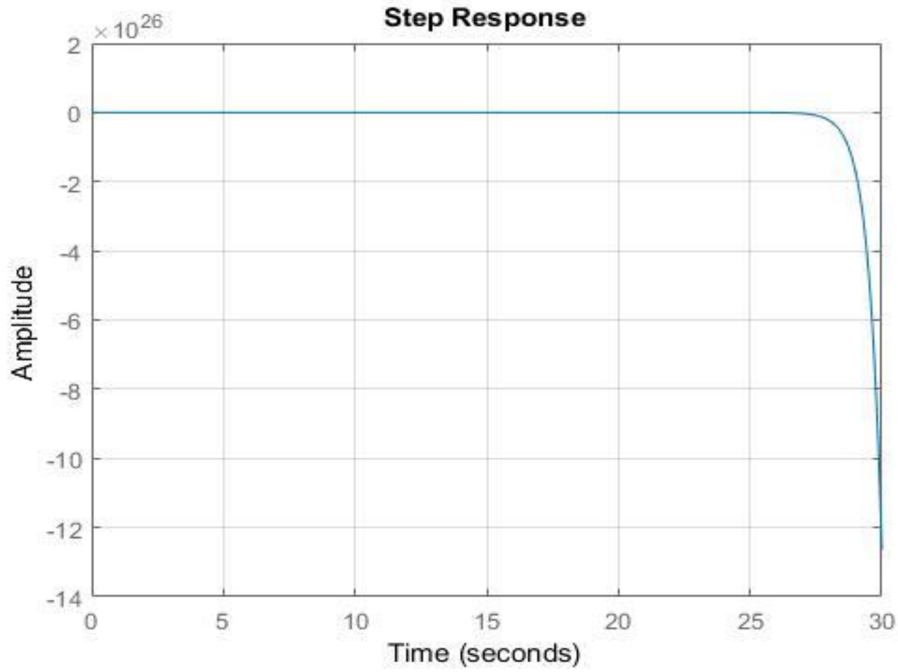

Figure VI.4: A graphic curve of 100% Observability

VI.3 Interpretations

A) The measure of the controllability and the observability is related with three conditions represented by two equations and one Inequality, there are:

$$d \neq 0, e \neq 0, f > 0$$

There are what make measure possible to be.

B) about the influence of the controllability, we see that the only deferent between the examples those we take, is the second line of the matrix B, this deferent creates a completely controllable system and another system just controllable until 50%.

For the influence of the observability, it is the same thing, with deferent in the second column, and this is what creates a completely observable system and another system just observable until 50%.

Graphically, in the controllability and the observability, the systems those are completely controllable or observable the response of it, is diverge at $-\infty$, on another side those are not completely controllable or observable (50%), the divergent is to $+\infty$, note that the deferant is just about number in line or column.

Conclusion

The only way to use the method that we represented is to considerate and respect the conditions that we give. A little deferent of numbers in lines or columns could create a considerable influence, this influence can change the degree of the controllability and the observability, and going until the graphical response, exactly in curve that can be diverge at the positive infinity, or the negative infinity.

General conclusion

In many cases in the conclusion closes the subjects those were opened before, and create a general rules for these subjects, like there is no way to progress the scientific research regarding these subjects, or the progression cannot be large. On another side, the conclusion or maybe all the thesis could be a station to start a large progression and considerable scientific research, rather, this is the way that we were lead to, in this thesis, it can be a beginning for another scientific research.

We discovered that if we want to measure the controllability and the observability, we must respect a conditions came from the capability of the diagonalization. The way to get a method to measure the controllability and the observability is a trip consisting of two ways, the first is the diagonalization, the second is the calculations.

In this scientific research, the operation to measure the controllability and the observability mentioned that the controllability and the observability of the systems those could be measurable is complet or it is until fifty percent or it doesn't exist completely, this was coming from the number of the state variables, note that, there is two variables, if the number of the state variables was three, the measurements of the controllability and the observability were going to be zero for the completely uncontrollability or unobservability ,thirty three, sixty six, or hundred percent, and if the number of the state variables is four the measurements were going to be zero, twenty five, fifty, seventy five, or hundred percent for the completely controllability or observability.

In the light of the previous paragraph, we conclude that the number of the probable measurements values is related with the number of the state variables. On another side, when we have a many of measurements values, the measure operation is characterized by the Precision, then, when we want a massively precise measurements we need to represent a system by a state space consisting by a considerable state space number. Thus, the opened subject now is how to find another state space representation of a system consisting by a state variables number enable us to obtain the precision that we want.

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